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CENTRE-BY-METABELIAN GROUPS WITH A CONDITION ON INFINITE SUBSETS

Nadir Trabelsi

Abstract

In this note, we consider some combinatorial conditions on infinite subsets of groups and we obtain in terms of these conditions some characterizations of the classes $\mathcal{L}(\mathcal{N}_k)\mathcal{F}$ and $\mathcal{F}\mathcal{L}(\mathcal{N}_k)$ for the finitely generated centre-by-metabelian groups, where $\mathcal{L}(\mathcal{N}_k)$ (respectively, $\mathcal{F}$) denotes the class of groups in which the normal closure of each element is nilpotent of class at most k (respectively, finite groups).

1. Introduction and results

Following a question of Erdös, B. H. Neumann proved in [**13**] that a group is centre-by-finite if, and only if, every infinite subset contains a commuting pair of distinct elements. Since this result, problems of similar nature have been the object of many papers (for example [**1**], [**2**], [**3**], [**4**], [**5**], [**9**], [**11**], [**10**], [**16**], [**17**]). We present here some further results of the same type.

Let k be a fixed positive integer. Denote by E_k^* the class of groups such that for every infinite subset X there exist two distinct elements x, y in X, and integers $t_0, t_1, \dots, t_k$ depending on x, y, and satisfying $\left[z_0^{t_0}, z_1^{t_1}, \dots, z_k^{t_k}\right] = 1$, where $z_i \in \{x, y\}$ for every $i \in \{0, 1, \dots, k\}$ and $z_0 \neq z_1$. Denote also by $E_k^{\#}$ the class of groups $G \in E_k^*$ for which the integers $t_0, \dots, t_k$ belong to $\{-1, 1\}$. In [**3**], it is proved that if G is a finitely generated soluble group in the class E_k^* (respectively $E_k^{\#}$), then there is an integer c, depending only on k, such that G is in $\mathcal{N}_c\mathcal{F}$ (respectively $\mathcal{F}\mathcal{N}_c$); where $\mathcal{N}_c$ and $\mathcal{F}$ denote respectively the class of nilpotent groups of class at most c and the class of finite groups. In [**3**], it is also proved that a finitely generated metabelian group G is in E_k^* (respectively $E_k^{\#}$) if, and only if, G belongs to $\mathcal{N}_k\mathcal{F}$ (respectively $\mathcal{F}\mathcal{N}_k$); and it is

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observed that these results are not true if the derived length of G is ≥ 3. Among the examples cited, which are due to Newman [**14**] (see also [**2**]), there is a finitely generated torsion-free nilpotent group G of class 4, of derived length 3, and whose 2-generated subgroups are nilpotent of class at most 3. So G is a finitely generated centre-by-metabelian group which belongs to E_3^* (respectively $E_3^\#$) and such that $G \notin \mathcal{N}_3\mathcal{F}$ (respectively $G \notin \mathcal{F}\mathcal{N}_3$). Note that if a group belongs to $\mathcal{N}_k$, then it is in $\mathcal{L}(\mathcal{N}_{k-1})$, where $\mathcal{L}(\mathcal{N}_{k-1})$ denotes the class of groups in which the normal closure of each element is nilpotent of class at most $k-1$. Considering this weaker condition we are able to prove the following results:

Theorem 1.1. *A finitely generated centre-by-metabelian group G is in E_{k+1}^* if, and only if, G belongs to $\mathcal{L}(\mathcal{N}_k)\mathcal{F}$.*

Theorem 1.2. *A finitely generated centre-by-metabelian group G is in $E_{k+1}^\#$ if, and only if, G belongs to $\mathcal{F}\mathcal{L}(\mathcal{N}_k)$. In particular, a torsion-free centre-by-metabelian group G is in $E_{k+1}^\#$ if, and only if, G belongs to $\mathcal{L}(\mathcal{N}_k)$.*

In [**7**], it is proved that a metabelian group G is $(k+1)$-Engel if, and only if, G belongs to $\mathcal{L}(\mathcal{N}_k)$. Morse [**12**] extended this result to a certain class of soluble groups of derived length ≤ 5 which contains the centre-by-metabelian groups. So our theorems improve Morse's result for the centre-by-metabelian groups.

Denote by $\mathcal{B}_k^*$ the class of groups such that every infinite subset contains an element x such that $\langle x \rangle$ is subnormal of defect k. It is proved in [**8**, Corollary 2.5] that a metabelian non-torsion group is a k-Baer group (that is every cyclic subgroup of G is subnormal of defect k) if, and only if, G is a k-Engel group. Here, using Theorem 1.2, we shall improve this result with the following:

Theorem 1.3. *Let G be a finitely generated centre-by-metabelian group. If G is in $\mathcal{B}_k^*$, then G is finite-by-(k-Engel). In particular, a torsion-free centre-by-metabelian group G belongs to $\mathcal{B}_k^*$ if, and only if, G is k-Engel.*

2. Proof of the results

Lemma 2.1. *Let G be a finitely generated torsion-free nilpotent group of class at most $k+1$. If G belongs to E_k^*, then G is a k-Engel group.*

Proof: Let G be a group in E_k^* and assume that G is not k-Engel. Therefore there exist x, y in G such that $[x,{}_k y] \neq 1$. The group G, being a finitely generated torsion-free nilpotent group, is a residually finite p-group for every prime p. So G has a normal subgroup N such that $[x,{}_k y] \notin N$ and $|G/N| = p^r$ for some positive integer r. Considering the infinite subset $\left\{x^{p^{r+i}} y : i \text{ integer}\right\}$, there are integers $n, m, t_0, t_1, \ldots, t_k$ such that $\left[z_0^{t_0}, z_1^{t_1}, \ldots, z_k^{t_k}\right] = 1$, where $z_i \in \left\{x^{p^{r+n}} y, x^{p^{r+m}} y\right\}$, $n \neq m$ and $z_0 \neq z_1$. Since G is nilpotent of class at most $k+1$, the commutator $\left[z_0^{t_0}, z_1^{t_1}, \ldots, z_k^{t_k}\right]$ is linear in each argument [**1**, Lemma 1], so we get that $[z_0, z_1, \ldots, z_k]^{t_0 t_1 \ldots t_k} = 1$, and therefore $[z_0, z_1, \ldots, z_k] = 1$ since G is torsion-free. Put $z_0 = x^{p^{r+s_0}} y$ and $z_1 = x^{p^{r+s_1}} y$, where $s_0 \neq s_1 \in \{m, n\}$. So

$$1 = [z_0, z_1, \ldots, z_k] = \left[\left[x^{p^{r+s_0}} y, x^{p^{r+s_1}} y\right], z_2, \ldots, z_k\right]$$

$$= \left[\left[x^{(p^{r+s_0} - p^{r+s_1})}, y\right]^{z_1}, z_2, \ldots, z_k\right] = \left[x^{(p^{r+s_0} - p^{r+s_1})}, y, z_2, \ldots, z_k\right]^{z_1}.$$

Hence

$$1 = \left[x^{(p^{r+s_0} - p^{r+s_1})}, y, z_2, \ldots, z_k\right] = [x, y, z_2, \ldots, z_k]^{(p^{r+s_0} - p^{r+s_1})}.$$

Thus $[x, y, z_2, \ldots, z_k] = 1$ as G is torsion-free and $s_0 \neq s_1$. Consequently $[x, y, z_2, \ldots, z_k]\, N = N$. Now $x^{p^{r+n}}, x^{p^{r+m}} \in N$, so $z_i N = yN$. It follows that $[x,{}_k y]\, N = N$; this means that $[x,{}_k y] \in N$, a contradiction which completes the proof. □

It is proved in [**12**, Theorem 1] that if G is nilpotent of class at most $k+2$, then G is $(k+1)$-Engel if and only if $G \in \mathcal{L}(\mathcal{N}_k)$. So combining this result and Lemma 2.1, we have the following consequence:

Lemma 2.2. *Let G be a finitely generated nilpotent group of class at most $k+2$. If G is in E_{k+1}^*, then G belongs to $\mathcal{FL}(\mathcal{N}_k)$. In particular, a torsion-free nilpotent group G of class at most $k+2$ is in E_{k+1}^* if, and only if, G belongs to $\mathcal{L}(\mathcal{N}_k)$.*

Proof: Let G be a finitely generated nilpotent group of class at most $k+2$ and suppose that G is in E_{k+1}^*. Then T, the torsion subgroup of G, is finite and G/T is a finitely generated torsion-free group of nilpotency class at most $k+2$ which belongs to E_{k+1}^*. It follows, from Lemma 2.1, that G/T is a $(k+1)$-Engel group, and by [**12**, Theorem 1], G/T belongs to $\mathcal{L}(\mathcal{N}_k)$. Hence, G is in $\mathcal{FL}(\mathcal{N}_k)$; as claimed.

Now, we suppose that G is a torsion-free group of nilpotency class at most $k+2$ which belongs to E^*_{k+1} and let $x, y_1, \ldots, y_{k+1} \in G$. Then $H = \langle x, y_1, \ldots, y_{k+1} \rangle$ is a finitely generated group of nilpotency class at most $k+2$ which belongs to E^*_{k+1}. It follows, from the first part of the proof, that H is in $\mathcal{FL}(\mathcal{N}_k)$. So H is in $\mathcal{L}(\mathcal{N}_k)$ since it is torsion-free. Hence, $[x^{y_1}, \ldots, x^{y_{k+1}}] = 1$, and this means that G belongs to $\mathcal{L}(\mathcal{N}_k)$.

Clearly, any group in $\mathcal{L}(\mathcal{N}_k)$ is $(k+1)$-Engel, so it belongs to E^*_{k+1}. $\square$

Proof of Theorem 1.1: Let G be a finitely generated centre-by-metabelian group in E^*_{k+1}. So $G/Z(G)$ is a finitely generated metabelian group in E^*_{k+1}. Therefore, by [**3**, Theorem 1.3], $G/Z(G)$ is in $\mathcal{N}_{k+1}\mathcal{F}$. Hence, G belongs to $\mathcal{N}_{k+2}\mathcal{F}$. Since finitely generated nilpotent groups are (torsion-free)-by-finite [**15**, 5.4.15(i)], G has a normal subgroup H, of finite index such that H is a torsion-free nilpotent group of class at most $k+2$ which belongs to E^*_{k+1}. It follows, by Lemma 2.2, that H is in $\mathcal{L}(\mathcal{N}_k)$; so G belongs to $\mathcal{L}(\mathcal{N}_k)\mathcal{F}$.

Conversely, suppose that G is in $\mathcal{L}(\mathcal{N}_k)\mathcal{F}$. Therefore there is a positive integer n and a normal subgroup H such that $H \in \mathcal{L}(\mathcal{N}_k)$ and $|G/H| = n$. So H is a $(k+1)$-Engel group and $x^n, y^n \in H$ for any x, y in G. Hence, $[x^n, {}_{k+1}y^n] = 1$ and consequently G belongs to E^*_{k+1}. $\square$

Proof of Theorem 1.2: Let G be a finitely generated centre-by-metabelian group in $E^\#_{k+1}$. So $G/Z(G)$ is a finitely generated metabelian group which belongs to $E^\#_{k+1}$. Therefore, by [**3**, Theorem 1.6], $\frac{G/Z(G)}{Z_{k+1}(G/Z(G))}$ is finite; so $G/Z_{k+2}(G)$ is finite. It follows, by [**6**, Theorem 1], that G is in the class $\mathcal{FN}_{k+2}$. Let H be a finite normal subgroup such that G/H is nilpotent of class at most $k+2$. If T/H is the torsion subgroup of G/H, then T/H is finite; so T is finite and G/T is a torsion-free finitely generated nilpotent group of class at most $k+2$ which belongs to $E^\#_{k+1}$. It follows, by Lemma 2.2, that G/T is in $\mathcal{L}(\mathcal{N}_k)$; so G belongs to $\mathcal{FL}(\mathcal{N}_k)$; as required.

Conversely, suppose that G is in the class $\mathcal{FL}(\mathcal{N}_k)$. Therefore there is a finite normal subgroup H such that G/H is $(k+1)$-Engel. Since G is a finitely generated soluble group, G/H is therefore nilpotent. It follows that G is finite-by-nilpotent, so G is residually finite. Consequently, there is a normal subgroup N of finite index such that $H \cap N = 1$. Since G/N is finite, if X is an infinite subset of G, then there are $x, y \in X$ such that $x \neq y$ and $xN = yN$. We have $[x, {}_{k+1}y] \in H$ and $\frac{\langle x,y \rangle N}{N}$ is cyclic, since G/H is $(k+1)$-Engel and $xN = yN$. Thus, $[x, {}_{k+1}y] \in H \cap N$. It follows that $[x, {}_{k+1}y] = 1$ and, therefore, G belongs to $E^\#_{k+1}$.

Now we suppose that G is a torsion-free centre-by-metabelian group in the class $E^{\#}_{k+1}$ and let $x, y_1, \dots, y_{k+1} \in G$. Then $H = \langle x, y_1, \dots, y_{k+1}\rangle$ is a torsion-free finitely generated centre-by-metabelian group. It follows, from the first part of the proof, that H belongs to $\mathcal{FL}(\mathcal{N}_k)$, and consequently $H \in \mathcal{L}(\mathcal{N}_k)$ since it is torsion-free. Hence, $[x^{y_1}, \dots, x^{y_{k+1}}] = 1$ and, therefore, G belongs to $\mathcal{L}(\mathcal{N}_k)$. □

For the proof of Theorem 1.3, we need further lemmas. Note that it is proved in [**8**, Theorem 2.3] that every non-torsion k-Baer group is a k-Engel group. But the converse is shown only in the metabelian case. As a consequence of Morse's result [**12**], we will extend this result with the following lemma:

Lemma 2.3. *Let G be a non-torsion centre-by-metabelian group. Then, G is a k-Baer group if, and only if, G is a k-Engel group.*

Proof: Let G be a non-torsion centre-by-metabelian group, and suppose that G is a k-Engel group. From [**12**, Theorem 2], G is in $\mathcal{L}(\mathcal{N}_{k-1})$. Let x in G; then x^G, the normal closure of x in G, is in $\mathcal{N}_{k-1}$. Now, it is well known that subgroups of a group of nilpotency class at most $k-1$ are subnormal of defect $k-1$. Thus, $\langle x\rangle$ is $(k-1)$-subnormal in x^G, so $\langle x\rangle$ is k-subnormal in G. It follows that G is a k-Baer group. □

Lemma 2.4. *Let G be a torsion-free group in $\mathcal{L}(\mathcal{N}_k)$. If G belongs to $\mathcal{B}^*_k$, then G is a k-Engel group.*

Proof: Let x, y in G; since G is torsion-free, the subset $\{x^i : i \text{ positive integer}\}$ is infinite. Therefore there is a positive integer i such that $\langle x^i\rangle$ is k-subnormal in G. Thus, $\left[x^i, \left[y, {}_{k-1}x^i\right]\right] \in \langle x^i\rangle$, so $\left[x^i, \left[y, {}_{k-1}x^i\right]\right] = x^r$ for some integer r. Since G belongs to $\mathcal{L}(\mathcal{N}_k)$, we have that G is a $(k+1)$-Engel group. Hence, $1 = \left[x^i, {}_{k+1}\left[y, {}_{k-1}x^i\right]\right] = x^{r^{k+1}}$; and this gives that $r = 0$ as G is torsion-free. It follows that $\left[x^i, \left[y, {}_{k-1}x^i\right]\right] = 1$, so $\left[y, {}_kx^i\right] = 1$. Now, because x^G is in $\mathcal{N}_k$, we have that every commutator in x^G of length k is multilinear. Thus $1 = \left[y, {}_kx^i\right] = \left[\left[y, x^i\right], {}_{k-1}x^i\right] = \left[y, {}_kx\right]^{i^k}$. Once again, as G is torsion-free, we obtain that $[y, {}_kx] = 1$; this means that G is a k-Engel group. □

Proof of Theorem 1.3: Let G be a finitely generated centre-by-metabelian group in the class $\mathcal{B}^*_k$. So every infinite subset of G contains an element x such that $\langle x\rangle$ is k-subnormal in G. Hence, for any y in G we have $[y, {}_{k+1}x] = 1$. Thus, G belongs to $E^{\#}_{k+1}$. It follows, from [**11**, Theorem 1], that G is finite-by-nilpotent. Therefore there is a finite normal subgroup T such that G/T is a torsion-free centre-by-metabelian group which belongs to $E^{\#}_{k+1}$. It follows from Theorem 1.2 that G/T is in $\mathcal{L}(\mathcal{N}_k)$, and by Lemma 2.4, we obtain that G/T is a k-Engel group. Therefore, G is finite-by-(k-Engel); as claimed.

Now, assume that G is a torsion-free centre-by-metabelian group in $\mathcal{B}^*_k$ and let x, y in G. Then, from the first part of the proof, $H = \langle x, y\rangle$ is finite-by-(k-Engel). Since G is torsion-free we deduce that H is k-Engel. Hence, $[y, {}_k x] = 1$, so G is a k-Engel group.

Conversely, suppose that G is a torsion-free centre-by-metabelian and a k-Engel group. From Lemma 2.3 we get that G is a k-Baer group, so G is in $\mathcal{B}^*_k$. □

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Département de Mathématiques
Faculté des Sciences
Université Ferhat Abbas
Sétif 19000
Algérie
E-mail address: trabelsi_dz@yahoo.fr

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