Publ. Mat. **47** (2003), 489–496

ON THE TOËPLITZ CORONA PROBLEM

E. Amar

Abstract

The aim of this note is to characterize the vectors $g = (g_1, \dots, g_k)$ of bounded holomorphic functions in the unit ball or in the unit polydisk of $\mathbb{C}^n$ such that the Corona is true for them in terms of the H^2 Corona for measures on the boundary.

Let D be a bounded domain in $\mathbb{C}^n$, the Corona problem is: given functions $g_1, \dots, g_N$ holomorphic and bounded in D such that:

$$\forall\, z \in D, \quad \sum_{i=1}^{N} |g_i(z)|^2 \geq \delta^2 > 0,$$

find $f_1, \dots, f_N$ still holomorphic and bounded in D such that $\sum_{i=1}^{N} f_i g_i = 1$ in D. This was solved for $D = \mathbb{D}$, the unit disk in $\mathbb{C}$ by L. Carleson [**8**] and it is still open for $n > 1$ for the basic domains namely the unit ball $\mathbb{B}_n$ and the unit polydisk $\mathbb{D}^n$.

We shall link this question to a question on Toëplitz operators via the $H^p(\mu)$ Corona.

1. Notations

We are interested by the basic domains, the unit ball in $\mathbb{C}^n$, $D = \mathbb{B}_n$, in fact any bounded convex domain with smooth boundary D, or the unit polydisc $D = \mathbb{D}^n$.

If $D = \mathbb{D}^n$ we set $bD = \mathbb{T}^n$, the distinguished boundary; if D is a bounded convex domain with smooth boundary, $bD = \partial D$ the topological boundary.

Recall that:

$$H^\infty(D) := \left\{ f \text{ holomorphic in } D / \, \|f\|_\infty := \sup_{z \in D} |f(z)| < \infty \right\}.$$

2000 *Mathematics Subject Classification.* 32A35, 32A36, 47B35.

Key words. Corona problem, Toëplitz operators, Von Neumann Minimax theorem.

Let $\mathcal{M}$ be the set of all probability measures on bD and for $\mu \in \mathcal{M}$ and $1 \leq p < \infty$ let $H^p(\mu)$ be the closure in $L^p(\mu)$ of the holomorphic polynomials.

If $\mu \in \mathcal{M}$ and $f \in H^\infty(D)$ then, with the assumption that $0 \in D$, for any $r < 1$, $f_r(z) := f(rz)$ is such that $f_r \in A(D) := H^\infty(D) \cap \mathcal{C}(\overline{D})$. There is a subsequence of $\{f_r,\ r < 1\}$ which converges in $(L^1(\mu), L^\infty(\mu))$ topology and uniformly on compact sets of D to a $\tilde{f} \in H^\infty(\mu) \cap H^\infty(D)$. Hence for a fixed $\mu \in \mathcal{M}$ we can assume that a $f \in H^\infty(D)$ is in $H^\infty(\mu) \cap H^\infty(D)$.

Now suppose that the Corona problem is solvable, i.e.

$g_1, \ldots, g_N \in H^\infty(D)$ are such that $\exists\ f_1, \ldots, f_N \in H^\infty(D)$ with $1 = f_1 g_1 + \cdots + f_N g_N$; we have, for any polynomial P:

$$P = P f_1 g_1 + \cdots + P f_N g_N.$$

Let $h \in H^p(\mu)$. Then there is a sequence $\{P_k\}_{k\in\mathbb{N}}$ of polynomials such that $P_k \to h$ in $H^p(\mu)$, hence:

$$P_k = \sum_{j=1}^{N} P_k f_j g_j;$$

but then $P_k f_j \to h_j$ in $H^p(\mu)$, because the f_j can be seen as in $H^\infty(\mu) \cap H^\infty(D)$.

So if the Corona is true then the $H^p(\mu)$ Corona is also true for any $\mu \in \mathcal{M}$:

$$\boxed{CH^p(\mu) : \forall\ h \in H^p(\mu), \quad \exists\ k_1, \ldots, k_N \in H^p(\mu) \text{ s.t. } h = \sum_{j=1}^{N} g_j k_j}.$$

The aim of this paper is to show the converse.

If $f := (f_1, \ldots, f_N)$ we set $|f|^2(z) := \sum_{j=1}^{N} |f_j(z)|^2$ and $\|f\|_p := \||f(\cdot)|\|_p$, where $\|\cdot\|_p$ is the $L^p(bD, \mu)$ norm and $\|\cdot\|_\infty$ is the sup norm in D.

Theorem 1.1. *Let D be a bounded convex domain D containing O and with a smooth boundary or the unit polydisk $\mathbb{D}^n$ of $\mathbb{C}^n$, $n \geq 1$. Let: $g_1, \dots, g_N \in H^\infty(D)$ and $\delta > 0$. The following are equivalent:*

(i) *There exist functions $f_1, \dots, f_N$ in $H^\infty(D)$ such that $\sum_{i=1}^{N} f_i g_i = 1$ and $\|f\|_\infty^2 \leq \frac{1}{\delta^2}$.*

(ii) *For all measures μ on bD,*

$\forall\, h \in H^2(\mu)$,

$$\exists\, k_1, \dots, k_N \in H^2(\mu) \text{ s.t. } h = \sum_{j=1}^{N} g_j k_j \text{ and } \|k\|_2^2 \leq \frac{1}{\delta^2} \|h\|_2^2 .$$

Let $g_1, \dots, g_N \in H^\infty(D)$ be such that

$$\forall\, z \in D, \quad |g|(z)^2 := \sum_{j=1}^{N} |g_j(z)|^2 \geq \delta^2 > 0,$$

we already know that:

if $D = \mathbb{B}_n$, μ the Lebesgue's measure on $\partial\mathbb{B}_n$ and $2 \leq p < \infty$, then $CH^p(\mu)$ is true [**2**];

if $D = \mathbb{D}^n$, μ the Lebesgue's measure on $\mathbb{T}^n$ and $1 \leq p < \infty$, then $CH^p(\mu)$ is true [**10**], [**11**];

if D is strictly pseudo-convex, μ the Lebesgue's measure on ∂D and $2 \leq p < \infty$, then $CH^p(\mu)$ is true [**5**];

if D is a bounded pseudo-convex domain with smooth boundary, μ the Lebesgue's measure on ∂D, then $CH^2(\mu)$ is true [**4**].

In the case $n = 1$, $D = \mathbb{D}$ the unit disc in $\mathbb{C}$, μ the Lebesgue's measure on $\mathbb{T}$, then $CH^2(\mu) \Rightarrow CH^\infty(\mathbb{D})$ [**12**], by an operator method: the commutant lifting theorem of Nagy-Foias.

This means that the Corona theorem in one variable can be proved this way, hence there is some hope to prove a general version of the Corona theorem also by this way.

2. Proof of the theorem

We already seen that i) $\Rightarrow$ ii); to prove that ii) $\Rightarrow$ i) we shall use the minimax theorem of Von Neuman. The minimax theorem was already used by Berndtsson [**6**], [**7**] in order to get estimates on solutions of the $\overline{\partial}$-equation; here the situation and the method are quite different.

We shall work with $N = 2$ in order to simplify notations. Because D is always convex containing 0, we may assume by dilation that the data $g := (g_1, g_2)$ are continuous up to the boundary, provided that the estimates do not depend on it.

Let Ω be an open set in D such that $\overline{\Omega} \subset D$, $0 \in \Omega$ and let, for $\epsilon > 0$, $\mathcal{C}_\epsilon$ be:

$$\mathcal{C}_\epsilon := \{(f = (f_1, f_2) \in A(D)^2, \text{ s.t. } \|1 - f \cdot g\|_\Omega \leq \epsilon\} \qquad \text{where } \|f\|_\Omega := \sup_{z\in\Omega} |f(z)|\,;$$

this set is clearly convex in $A(D)^2$. Let $\mathcal{M}$ be the set of probability measures on bD and for $0 < \eta \leq 1$ let $\mathcal{M}_\eta = \eta m + (1-\eta)\mathcal{M}$, where m is the Lebesgue measure on bD; this is a convex weakly compact set.

Let us define N as

$$\forall\, f \in \mathcal{C}_\epsilon,\ \forall\, \mu \in \mathcal{M}_\eta, \quad N(f, \mu) := \|f\|_\mu^2 := \|f_1\|_{L^2(\mu)}^2 + \|f_2\|_{L^2(\mu)}^2\,.$$

Then N is convex on $\mathcal{C}_\epsilon$ for μ fixed in $\mathcal{M}_\eta$ and concave, in fact affine, and continuous on $\mathcal{M}$ for f fixed in $\mathcal{C}_\epsilon$, hence we can apply the minimax theorem [**9**]:

$$(*) \qquad \sup_{\mu\in\mathcal{M}_\eta} \inf_{f\in\mathcal{C}_\epsilon} N(f, \mu) = \inf_{f\in\mathcal{C}_\epsilon} \sup_{\mu\in\mathcal{M}_\eta} N(f, \mu);$$

by (ii) with $h = 1$ we have $\exists\, k = (k_1, k_2) \in (H^2(\mu))^2$, $g \cdot k = 1$, $\|k\|_\mu \leq \frac{1}{\delta}$; because $\mu = \eta m + (1-\eta)\nu$ we get $\|k\|_m \leq \frac{1}{\delta\sqrt{\eta}}$ hence $k \in (H^2(m))^2$; by the very definition of $H^2(\mu)$ there is a sequence $f_n \in (A(D))^2$ such that $f_n \to k$ in $(H^2(\mu))^2$ hence also in $(H^2(m))^2$ hence $f_n \to k$ uniformly on compact sets of D; so for $\epsilon' \leq \epsilon$ there is a $f \in A(D)^2$ with $\|f - k\|_\Omega \leq \frac{\epsilon}{\|g\|_\infty}$ and $\|f - k\|_{H^2(\mu)} \leq \epsilon'$. Hence we have

$$\|1 - f \cdot g\|_\Omega = \|k \cdot g - f \cdot g\|_\Omega \leq \|g\|_\infty \|f - k\|_\Omega \leq \epsilon$$

which means that $f \in \mathcal{C}_\epsilon$. We deduce that the left side of $(*)$ is bounded by $\frac{1}{\delta^2}$ hence for any $\epsilon > 0$, $\eta > 0$, $\gamma > 0$ there is a $f_{\epsilon,\eta,\gamma} \in \mathcal{C}_\epsilon$ with $\sup_{\mu\in\mathcal{M}_\eta} N(f_{\epsilon,\eta,\gamma}, \mu) \leq \frac{1}{\delta^2} + \gamma$.

Now let $a \in D$ and ν_a a representing measure for a supported by bD, then we have with $\mu := \eta m + (1-\eta)\nu_a$:

$$|\eta f_{\epsilon,\eta,\gamma}(0) + (1-\eta) f_{\epsilon,\eta,\gamma}(a)| = \left|\int f_{\epsilon,\eta,\gamma}\, d\mu\right| \leq \frac{1}{\delta} + \gamma$$

and with $\mu = m$,

$$|f_{\epsilon,\eta,\gamma}(0)| = \left|\int f_{\epsilon,\eta,\gamma}\, dm\right| \leq \frac{1}{\delta} + \gamma;$$

hence

$$|f_{\epsilon,\eta,\gamma}(a)| \leq \frac{(1+\eta)(\frac{1}{\delta}+\gamma)}{1-\eta};$$

because this is true for any $a \in D$ we get

$$\|f_{\epsilon,\eta,\gamma}\|_\infty \leq \frac{(1+\eta)(\frac{1}{\delta}+\gamma)}{1-\eta}.$$

Using Montel property we get that there is a $f \in (H^\infty(D))^2$ bounded by $\frac{1}{\delta}$ and such that $g \cdot f = 1$ on Ω hence, because Ω is open and $f \cdot g$ is holomorphic in D, $f \cdot g = 1$ in D. □

3. Operator version

We shall give an operator version of the previous result strongly inspired by [**3**], but first we need some definitions. Let D be as before and $\mu \in \mathcal{M}$; for any function f in $L^\infty(\mu)$ define the Toëplitz operator T_f^μ on the Hilbert space $H^2(\mu)$ by

$$\forall\, g \in H^2(\mu), \quad T_f^\mu g := P_\mu(fg),$$

where P_μ is the orthogonal projection from $L^2(\mu)$ on $H^2(\mu)$. We can state:

Corollary 3.1. *Let D be a bounded convex domain containing 0 and with a smooth boundary or the unit polydisk $\mathbb{D}^n$ of $\mathbb{C}^n$, $n \geq 1$. Let: $g_1, \dots, g_N \in H^\infty(D)$ and $\delta > 0$. The following are equivalent:*

(i) *There exist functions $f_1, \dots, f_N$ in $H^\infty(D)$ such that $\sum_{i=1}^{N} f_i g_i = 1$ and $\|f\|_\infty^2 \leq \frac{1}{\delta^2}$.*

(ii) *For all measures μ on bD, $\sum_{j=1}^{N} T_{g_j}^\mu (T_{g_j}^\mu)^* \geq \delta^2 \mathbb{1}$.*

For $D = \mathbb{D}^2$, this was proved in [**1**]; they used a method specific to the bidisc which explicitly cannot work even for $\mathbb{D}^3$.

Proof: We shall prove that (ii) is equivalent to:

(iii) *For all measures μ on bD,*

$\forall\, h \in H^2(\mu)$,

$$\exists\, k_1, \dots, k_N \in H^2(\mu) \text{ s.t. } h = \sum_{j=1}^{N} g_j k_j \text{ and } \|k\|_2^2 \leq \frac{1}{\delta^2} \|h\|_2^2,$$

and then we apply the theorem to be done.

(ii) $\Rightarrow$ (iii) (same proof as in [**1**]): Let μ be a probability measure on bD and set $G_i := T^{\mu}_{g_i}$; by (ii) we get that the operator $Q := G_1G_1^* + \cdots + G_NG_N^*$ is invertible and $\left\|Q^{-1}\right\| \leq \frac{1}{\delta^2}$. We can define:

$$F_i := G_i^* Q^{-1}, \quad i = 1, \ldots, N;$$

these are bounded operators on $H^2(\mu)$ and clearly we get:

$$G_1F_1 + \cdots + G_NF_N = \mathbb{1}. \tag{1}$$

Now take $k_i = F_i h$, $k := (k_1, \ldots, k_N)$; we have

$$\|k\|_2^2 = \left\|G_1^* Q^{-1} h\right\|^2 + \cdots + \left\|G_N^* Q^{-1} h\right\|^2,$$

but

$$\left\|G_1^* Q^{-1} h\right\|^2 = \left\langle G_1^* Q^{-1} h, G_1^* Q^{-1} h\right\rangle = \left\langle G_1 G_1^* Q^{-1} h, Q^{-1} h\right\rangle$$

hence

$$\|k\|_2^2 = \left\langle h, Q^{-1} h\right\rangle \leq \frac{1}{\delta^2} \|h\|^2,$$

because $(G_1G_1^* + \cdots + G_NG_N^*)Q^{-1} = \mathbb{1}$.

Together with equation (1) this means precisely that the $H^2(\mu)$ Corona is true, i.e.

(iii) $\forall\, h \in H^2(\mu)$,

$$\exists\, k_1, \ldots, k_N \in H^2(\mu) \;\; s.t. \;\; h = \sum_{j=1}^{N} g_j k_j, \;\; and \;\; \|k\|_2^2 \leq \frac{1}{\delta^2} \|h\|^2. \qquad \square$$

(iii) $\Rightarrow$ (ii): Let $\mu \in \mathcal{M}$, then by (iii) we have:

$\forall\, h \in H^2(\mu)$,

$$\exists\, k_1, \ldots, k_N \in H^2(\mu) \text{ s.t. } h = \sum_{j=1}^{N} g_j k_j \text{ and } \|k\|_2^2 \leq \frac{1}{\delta^2} \|h\|_2^2,$$

then $S_h := \{k = (k_1, \ldots, k_N) \in (H^2(\mu))^N : \sum_{j=1}^{N} G_j k_j = h\}$ is not empty and it has elements of norm less than $\frac{1}{\delta^2} \|h\|_2^2$; S_0 is a subspace of the Hilbert space $(H^2(\mu))^N$ hence there is a unique element $k = (k_1, \ldots, k_N)$ in S_h which is orthogonal to S_0 and hence of minimal norm. Then we get: $\|k\|_2^2 \leq \frac{1}{\delta^2} \|h\|_2^2$ and, defining F_j by $F_j h := k_j$, $j = 1, \ldots, N$, we have:

$$\sum_{j=1}^{N} \|F_j h\|_2^2 \leq \frac{1}{\delta^2} \|h\|_2^2 \tag{2}$$

$$\forall\, h \in H^2(\mu), \quad \sum_{j=1}^{N} G_j F_j h = h. \tag{3}$$

From equation (3) we get:

$$\forall\, h \in H^2(\mu), \quad \left\langle \sum_{j=1}^{N} G_j F_j h, h \right\rangle = \|h\|_2^2\,,$$

hence $\forall\, h \in H^2(\mu)$, $\sum_{j=1}^{N} \langle F_j h, G_j^* h \rangle = \|h\|_2^2$,

$$\forall\, h \in H^2(\mu), \quad \|h\|_2^2 \leq \sum_{j=1}^{N} \|F_j h\| \left\|G_j^* h\right\|$$

$$\leq \left(\sum_{j=1}^{N} \|F_j h\|^2\right)^{1/2} \left(\sum_{j=1}^{N} \left\|G_j^* h\right\|^2\right)^{1/2}.$$

Using equation (2) we get:

$$\forall\, h \in H^2(\mu), \quad \|h\|_2^2 \leq \frac{1}{\delta} \|h\|_2 \left(\sum_{j=1}^{N} \left\|G_j^* h\right\|^2\right)^{1/2},$$

hence $\forall\, h \in H^2(\mu)$, $\sum_{j=1}^{N} \left\|G_j^* h\right\|^2 \geq \delta^2 \|h\|_2^2$ and the corollary. □

References

[1] J. Agler and J. E. McCarthy, Nevanlinna-Pick interpolation on the bidisk, *J. Reine Angew. Math.* **506** (1999), 191–204.

[2] E. Amar, On the corona problem, *J. Geom. Anal.* **1(4)** (1991), 291–305.

[3] E. Amar and C. Menini, On an operator theory approach to the corona problem, *Bull. London Math. Soc.* **34(3)** (2002), 369–373.

[4] M. Andersson, The H^2 corona problem and $\overline{\partial}_b$ in weakly pseudoconvex domains, *Trans. Amer. Math. Soc.* **342(1)** (1994), 241–255.

[5] M. ANDERSSON AND H. CARLSSON, Wolff type estimates and the H^p corona problem in strictly pseudoconvex domains, *Ark. Mat.* **32(2)** (1994), 255–276.
[6] B. BERNDTSSON, $\overline{\partial}_b$ and Carleson type inequalities, in: *"Complex analysis, II"* (College Park, Md., 1985-86), Lecture Notes in Math. **1276**, Springer, Berlin, 1987, pp. 42–54.
[7] B. BERNDTSSON, Uniform estimates with weights for the $\overline{\partial}$-equation, *J. Geom. Anal.* **7(2)** (1997), 195–215.
[8] L. CARLESON, Interpolations by bounded analytic functions and the corona problem, *Ann. of Math. (2)* **76** (1962), 547–559.
[9] T. W. GAMELIN, *"Uniform algebras"*, Prentice-Hall, Inc., Englewood Cliffs, N. J., 1969.
[10] S.-Y. LI, Corona problem of several complex variables, in: *"The Madison Symposium on Complex Analysis"* (Madison, WI, 1991), Contemp. Math. **137**, Amer. Math. Soc., Providence, RI, 1992, pp. 307–328.
[11] K. C. LIN, The H^p-corona theorem for the polydisc, *Trans. Amer. Math. Soc.* **341(1)** (1994), 371–375.
[12] C. F. SCHUBERT, The corona theorem as an operator theorem, *Proc. Amer. Math. Soc.* **69(1)** (1978), 73–76.

Département de Mathématiques
Université de Bordeaux I
33405 Talence
France
E-mail address: Eric.Amar@math.u-bordeaux.fr

Primera versió rebuda el 13 de desembre de 2002,
darrera versió rebuda el 24 d'abril de 2003.