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THE LINEAR INVARIANTS ($\underline{DN}$) AND ($\overline{\Omega}$) FOR SPACES OF GERMS OF HOLOMORPHIC FUNCTIONS ON COMPACT SUBSETS OF $\mathbf{C}^n$

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Abstract

For a compact subset K of $\mathbf{C}^n$, we give necessary and sufficient conditions for $[\mathcal{H}(K)]'$ to have the property ($\underline{DN}$), and similarly for the property ($\overline{\Omega}$). We also show that $\mathcal{H}(\overline{D})$ is isomorphic to $\mathcal{H}(\overline{\Delta}^n)$, where Δ^n is the unit polydisc in $\mathbf{C}^n$ and D is any bounded Reinhardt domain in $\mathbf{C}^n$. This last result requires a generalization of the classical Hartogs phenomenon.

1. Introduction

The linear topological invariants of types (DN) and (Ω) were introduced and investigated by D. Vogt in the early 80's. These properties play a central role in the modern theory of Fréchet spaces. They have been applied successfully to the problem of classifying Fréchet spaces, in particular, to investigating the structure of holomorphic functions on open subsets, and compact subsets, of $\mathbf{C}^n$, as well as of more general objects, such as complex manifolds, Fréchet spaces, etc. In this direction, we could mention an interesting result due to Aytuna (see [**Ay1**], [**Ay2**]) which says that if M is a Stein manifold, and $n = \dim M$, and Δ^n denotes the open unit polydisk in $\mathbf{C}^n$, then $\mathcal{H}(M) \cong \mathcal{H}(\Delta^n)$ if and only if $\mathcal{H}(M) \in (\overline{\Omega})$, or equivalently M is hyperconvex, meaning that there exists a negative, continuous, plurisubharmonic, exhaustion function on M. A slightly weaker form of this statement was proved by Zaharjuta, in [**Za**]. For a compact K (in a Stein manifold), Zaharjuta also gave a characterization, in terms of pluripotential theory, for $\mathcal{H}(K)$ to be isomorphic to $\mathcal{H}(\overline{\Delta}^n)$. Later, Aytuna showed, in [**Ay2**], that for a general complex manifold M, $\mathcal{H}(M) \cong \mathcal{H}(\mathbf{C}^n)$ if and only if $\mathcal{H}(M) \in (DN)$, or

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equivalently M admits no non-constant, bounded-from-above, plurisubharmonic function. More recently, it was shown in [**HT**], that, if K is a compact subset of a Stein complex space (possibly with singularities), then $[\mathcal{H}(K)]' \in (\tilde{\Omega})$ if and only if $K \cap Z$ is either empty or non-pluripolar in Z, for every irreducible branch Z of every neighbourhood U (in X) of K.

The aim of the present paper is to study the structure of the space $\mathcal{H}(K)$ in terms of invariants $(\underline{DN})$, $(\overline{\Omega})$, where K is a compact set in $\mathbf{C}^n$. We now briefly outline the principal results in this work. In Section 3, we show in Theorem 3.1 that $[\mathcal{H}(K)]' \in (\underline{DN})$ if and only if K has a Runge neighbourhood. The proof of this theorem uses a standard application of the L^2-estimates of Hörmander, as was presented in [**Ay1**]. A characterization for the property $(\overline{\Omega})$ of $[\mathcal{H}(K)]'$ is given in Section 4. Namely it is proved there that if K has a Stein neighbourhood basis then $[\mathcal{H}(K)]' \in (\overline{\Omega})$ if and only if K is pluriregular in every neighbourhood of K. As a consequence to this, we show that for every locally pluriregular compact Stein set K in $\mathbf{C}^n$ we have $[\mathcal{H}(\Omega_\varphi(K))]' \in (\overline{\Omega})$ where φ is a (real valued) continuous, plurisubharmonic function on a neighbourhood of K and $\Omega_\varphi(K)$ is the compact Hartogs set defined by

$$\Omega_\varphi(K) = \{(z, \lambda) \in K \times \mathbf{C} : |\lambda| \leq e^{-\varphi(z)}\}.$$

We also display an example of a compact set K in $\mathbf{C}^2$ such that $[\mathcal{H}(K)]' \in (\overline{\Omega})$ but $[\mathcal{H}(\Omega_\varphi(K))]' \notin (\overline{\Omega})$ for *some* real valued continuous function φ on K. This contrasts sharply with Corollary 3.4 in Section 3.

Finally, in Section 5 we consider the case where $K = \overline{D}$ where D is a bounded Reinhardt domain in $\mathbf{C}^n$. The main result is that $\mathcal{H}(\overline{D}) \cong H(\overline{\Delta}^n)$. The main ingredient is the following observation: *There exists a bounded hyperconvex Reinhardt domain $\tilde{D}$ in $\mathbf{C}^n$ such that $\mathcal{H}(\overline{D}) \cong \mathcal{H}(\overline{\tilde{D}})$.* This observation should be considered as a generalization of the well known fact that every holomorphic function on a neighbourhood of the Hartogs triangle (in $\mathbf{C}^2$) extends holomorphically to a neighbourhood of the closed unit bidisk.

2. Preliminaries

2.1. We first recall the definitions of linear topological invariants (DN), (Ω). These properties were introduced and investigated by Vogt in the early 80's.

Let E be a Fréchet space with an increasing fundamental system of seminorms

$$||\cdot||_1 \leq ||\cdot||_2 \leq \cdots \leq ||\cdot|| \leq \cdots$$

For each $k \geq 1$, define a generalized semi-norm $||\cdot||_k^*$ on E', the dual space of E, by

$$||u||_k^* = \sup\{|u(x)| : ||x||_k \leq 1\}.$$

We say that E has the property

$$||\cdot||_q^{1+d} \leq C||\cdot||_k||\cdot||_p^d \begin{cases} (DN) & \exists\, p \geq 1\, \forall\, d > 0\, \forall\, q \geq p\, \exists\, k \geq q,\, C > 0 \\ (\underline{DN}) & \exists\, p \geq 1\, \forall\, q \geq p\, \exists\, k, d, C > 0 \end{cases}$$

$$||\cdot||_q^{*1+d} \leq C||\cdot||_k^*||\cdot||_p^{*d} \begin{cases} (\overline{\Omega}) & \forall\, p \geq 1\, \forall\, d > 0\, \exists\, q \geq p\, \forall\, k \geq q\, \exists\, C > 0 \\ (\tilde{\Omega}) & \forall\, p \geq 1\, \exists\, q \geq p\, \exists\, d > 0\, \forall\, k \geq q\, \exists\, C > 0. \end{cases}$$

For more details concerning these topological invariants, we refer the reader to the book [**MV**].

2.2. We next recall the definition of germs of holomorphic functions on compact sets. For K a compact set in $\mathbf{C}^n$, denote by $\mathcal{H}(K)$ the space of germs of holomorphic functions on K, equipped with the inductive limit topology. More precisely

$$\mathcal{H}(K) = \operatorname*{limind}_{U \supset K} \mathcal{H}^\infty(U),$$

where $\mathcal{H}^\infty(U)$ is the Banach space of bounded holomorphic functions on U and U runs over all neighbourhoods of K.

2.3. Finally we also require some elements of pluripotential theory that will be used further on. For more details on this material, we refer to the surveys [**Sa**] and [**Sic**].

Let U be a domain in $\mathbf{C}^n$ and K a subset of U. We define the relative extremal function (or plurisubharmonic measure) $\omega^*(U, K, \cdot)$ as follows

$$\omega^*(U, K, z) = \overline{\lim_{z' \to z}} \sup\{u(z') : u \in PSH(U),\, u \leq 1,\, u\big|_K \leq 0\},$$

where $PSH(U)$ denotes the cone of plurisubharmonic functions on U.

We say that K is pluriregular in U if $\omega^*(U, K, \cdot)\big|_K \equiv 0$, or equivalently $\omega^*(U, K, \cdot)$ is continuous on U. A point $a \in K$ is called a pluriregular point of K if for every neighbourhood U of a we have $\omega^*(U, K \cap U, \cdot)\big|_{K \cap U} \equiv 0$. K is called locally pluriregular if every point of K is a pluriregular point.

3. The property $(\underline{DN})$ for $[\mathcal{H}(K)]'$

The main result in this section is the following

Theorem 3.1. *Let K be a compact subset in $\mathbf{C}^n$. Then the following assertions are equivalent*

(i) $[\mathcal{H}(K)]' \in (\underline{DN})$.
(ii) $[\mathcal{H}(K)]'$ *has a continuous norm.*
(iii) K *has a Runge neighbourhood.*

Here by a Runge neighbourhood of K we mean an open set D in $\mathbf{C}^n$ containing K and satisfies the following requirement: For every holomorphic function f on a neighbourhood U of K in D there exists a (possibly smaller) neighbourhood $\tilde{U}$ of K such that for every $\varepsilon > 0$ we can find a holomorphic function g on D satisfying

$$\sup\{|f(z) - g(z)| : z \in \tilde{U}\} < \varepsilon.$$

Proof: (i) $\Rightarrow$ (ii) is trivial.

(ii) $\Rightarrow$ (iii). Assume that $||\cdot||$ is a continuous norm of $[\mathcal{H}(K)]'$. Take a bounded subset B in $\mathcal{H}(K)$ such that

$$||\mu|| \leq C||\mu||_B^*, \quad \forall\, \mu \in [\mathcal{H}(K)]'$$

for some constant $C > 0$.

Since $\mathcal{H}(K) = \liminf_{U \supset K} \mathcal{H}^\infty(U)$ is regular, we can find a neighbourhood U of K such that B is contained and bounded in $\mathcal{H}^\infty(U)$. We claim that U is a Runge neighbourhood of K. Indeed, it suffices to show that $\mathcal{H}^\infty(U)$ is dense in $\mathcal{H}(K)$. Otherwise, by the Hahn-Banach theorem there exists $\mu \in [\mathcal{H}(K)]'$, $\mu \neq 0$ such that $\mu\big|_{\mathcal{H}^\infty(U)} = 0$. This implies that $\mu = 0$. A contradiction!

(iii) $\Rightarrow$ (i). The proof relies on the following lemma which is of independent interest.

Lemma 3.2. *Let D be a Runge neighbourhood of K. Then there exists a compact holomorphically convex subset $\tilde{K}$ of $\hat{D}$, the envelope of holomorphy of D, such that $\mathcal{H}(K)$ is isomorphic to $\mathcal{H}(\tilde{K})$.*

We also need methods and the following result taken from the proof of Theorem 2 in [**Ay1**].

Lemma 3.3. *Let M be a Stein manifold and dV the volume form of a Hermitian metric on M. Then there exists a measure μ equivalent to dV and a strictly positive continuous function c on M with the following properties: For any covering $\{W_+, W_-\}$ of M with $W_+ \cap W_-$ relatively compact, there exists a constant $C > 0$, such that for any plurisubharmonic function ρ on M and for any holomorphic function f on $W_+ \cap W_-$, there exist holomorphic functions f_+ on W_+ and f_- on W_- such that $f_+ - f_- = f$ on $W_+ \cap W_-$ and the following estimates hold*

$$\int_{W_+} |f_+|^2 e^{-\rho} c \, d\mu \le C \int\limits_{W_+ \cap W_-} |f|^2 e^{-\rho} \, d\mu$$

$$\int_{W_-} |f_-|^2 e^{-\rho} c \, d\mu \le C \int\limits_{W_+ \cap W_-} |f|^2 e^{-\rho} \, d\mu.$$

Assuming Lemma 3.2 for the moment, we continue the proof of (iii) $\Rightarrow$ (i). It follows from Lemma 3.2 that $[\mathcal{H}(K)]'$ is isomorphic to $[\mathcal{H}(\tilde{K})]'$. It is therefore sufficient to show that $[\mathcal{H}(\tilde{K})]' \in (\underline{DN})$. Since $\tilde{K}$ is holomorphically convex in the Stein Riemann domain $\hat{D}$, there exists a non negative plurisubharmonic function ρ on $\hat{D}$ such that $\tilde{K}$ is the zero set of ρ (see [**Sib**, p. 203], the proof given there is valid without changes to the case of Stein manifolds). We may assume that the open sets

$$U_k = \{z \in \hat{D} : \rho(z) < 1/k\}, \quad k \ge 1$$

form a Stein neighbourhood basis for $\tilde{K}$.

Now let $d\lambda = c \, d\mu$, where c, μ are as in Lemma 3.3, we define

$$V_k = \left\{ f \in \mathcal{H}(U_k) : \int\limits_{U_k} |f|^2 \, d\lambda \le 1 \right\}.$$

Then $\{V_k\}_{k \ge 1}$ is a basis of bounded subsets of $H(\tilde{K})$. In order to prove that $[\mathcal{H}(\tilde{K})]' \in (\underline{DN})$, it suffices to verify the following assertion

$$\forall\, q \ge 2 \quad \exists\, k, d, M > 0 \quad \forall\, r > 0 : V_q \subset M r^d V_2 + \frac{1}{r} V_k. \tag{1}$$

Indeed, given $q \ge 2$, from Theorem 2.6.11 in [**Hö**] we deduce that there exists a continuous plurisubharmonic function $\tilde{\rho}$ on U_1 such that

$$\tilde{K} \subset W_- := \{z \in U_1 : \tilde{\rho}(z) < 1\} \subset U_q \subset U_2.$$

Choose δ_1, δ_2 such that

$$\sup_{z\in\tilde{K}} \tilde{\rho}(z) < \delta_2 < \delta_1 < 1.$$

Let $\alpha = \sup\limits_{z\in U_2} \tilde{\rho}(z) < \infty$. Fix $L > 0$, and let φ_L be the convex function on $(-\infty, \alpha)$ given by

$$\varphi_L(t) = \begin{cases} \dfrac{L}{\delta_2} t - L & \text{for } \delta_2 < t < \alpha \\ 0 & \text{for } t \leq \delta_2. \end{cases}$$

It follows that $\rho_L = \varphi_L \circ \tilde{\rho}$ is continuous plurisubharmonic on U_2. Take $k > q$ sufficiently large such that $U_k \subset \{z \in U_1 : \tilde{\rho}(z) < \delta_2\}$. Set

$$W_+ = U_2 \backslash \overline{\{z \in U_2 : \tilde{\rho}(z) < \delta_1\}}, \quad W = W_- \cap W_+.$$

For each $f \in V_q$, we have

$$\int\limits_W |f|^2 e^{-\rho_L}\, d\lambda \leq \sup_W e^{-\rho_L} = e^{L(1-\frac{\delta_1}{\delta_2})}. \tag{2}$$

By Lemma 3.3 there exist $f_+ \in \mathcal{H}(W_+)$ and $f_- \in \mathcal{H}(W_-)$ such that

$$f \equiv f_+ - f_- \text{ on } W$$

and

$$\begin{aligned} &\int\limits_{W_+} |f_+|^2 e^{-\rho_L}\, d\lambda \leq C e^{L(1-\frac{\delta_1}{\delta_2})}; \\ &\int\limits_{W_-} |f_-|^2 e^{-\rho_L}\, d\lambda \leq C e^{L(1-\frac{\delta_1}{\delta_2})} \end{aligned} \tag{3}$$

for some constant $C > 0$ independent of f and L.

As $\rho_L \equiv 0$ on $W_{\delta_2} = \{z \in U_1 : \tilde{\rho}(z) < \delta_2\}$, from (3) we easily get the following estimates

$$\int\limits_{W_{\delta_2}} |f_-|^2\, d\lambda = \int\limits_{W_{\delta_2}} |f_-|^2 e^{-\rho_L}\, d\lambda \le Ce^{L(1-\frac{\delta_1}{\delta_2})} \tag{4}$$

$$\int\limits_{W_+} |f_+|^2\, d\lambda \le \sup_{W_+} e^{\rho_L} \int\limits_{W_+} |f_+|^2 e^{-\rho_L}\, d\lambda \le Ce^{\frac{L}{\delta_2}(\alpha-\delta_1)} \tag{5}$$

$$\int\limits_{W_-\setminus W_+} |f_-|^2\, d\lambda \le \sup_{W_-\setminus W_+} e^{\rho_L} \int\limits_{W_-} |f_-|^2 e^{-\rho_L}\, d\lambda \le Ce^{\frac{L}{\delta_2}(1-\delta_1)}. \tag{6}$$

From (6) and the fact that $\int\limits_{U_q} |f|^2\, d\lambda \le 1$ we get

$$\begin{aligned} \int\limits_{W_-\setminus W_+} |f+f_-|^2\, d\lambda &\le \left[\left\{\int\limits_{W_-\setminus W_+} |f|^2\, d\lambda\right\}^{1/2} + \left\{\int\limits_{W_-\setminus W_+} |f_-|^2\, d\lambda\right\}^{1/2}\right]^2 \\ &\le (1+C^{1/2})^2 e^{\frac{L}{\delta_2}(\alpha-\delta_1)}. \end{aligned} \tag{7}$$

Since $f_+ - f_- \equiv f$ on W and $W_- \bigcup W_+ = U_2$ then we can define $g \in \mathcal{H}(U_2)$ by

$$g(z) = \begin{cases} f_+(z) & \text{for } z \in W_+ \\ f(z) + f_-(z) & \text{for } z \in W_-. \end{cases}$$

It now follows from (4) that

$$\int\limits_{W_{\delta_2}} |h|^2\, d\lambda = \int\limits_{W_{\delta_2}} |f_-|^2\, d\lambda \le Ce^{L(1-\frac{\delta_1}{\delta_2})}, \quad h = f - g.$$

Moreover, since $U_k \subset W_{\delta_2}$ then

$$\int\limits_{U_k} |h|^2\, d\lambda \le \int\limits_{W_{\delta_2}} |h|^2\, d\lambda \le Ce^{L(1-\frac{\delta_1}{\delta_2})}.$$

On the other hand, from (5) and (7) we obtain

$$\int\limits_{U_2} |g|^2\, d\lambda = \int\limits_{W_+} |f_+|^2\, d\lambda + \int\limits_{W_-\setminus W_+} |f+f_-|^2\, d\lambda$$

$$\leq Ce^{\frac{L}{\delta_2}(\alpha-\delta_1)} + (1+C^{1/2})^2 e^{\frac{L}{\delta_2}(\alpha-\delta_1)}$$

$$= [(1+C^{1/2})^2 + C]e^{\frac{L}{\delta_2}(\alpha-\delta_1)}.$$

Therefore

$$V_q \subset Ce^{L(1-\frac{\delta_1}{\delta_2})}V_k + [(1+C^{1/2})^2 + C]e^{\frac{L}{\delta_2}(\alpha-\delta_1)}V_2, \quad \forall\, L > 0.$$

By setting $s = e^{-L(1-\frac{\delta_1}{\delta_2})}$, the above inclusion can be rewritten as

$$V_q \subset \frac{C}{s}V_k + [(1+C^{1/2}) + C]s^d V_2, \quad \forall\, s > 0,$$

where $d = \frac{\alpha-\delta_1}{\delta_1-\delta_2}$.

Thus (1) is proved, the proof is thereby finished. □

It remains to prove Lemma 3.2. A proof of this result has been given by Zaharjuta in [**Za**]. For the convenience of readers, we offer below a full proof.

Proof of Lemma 3.2: For the sake of simplicity, we only consider the case where $\hat{D}$ is schlicht, i.e. $\hat{D}$ is a pseudoconvex domain in $\mathbf{C}^n$. The general case, i.e. $\hat{D}$ is a Stein Riemann domain over $\mathbf{C}^n$ can be treated in the same way. Let $\hat{K}_{\hat{D}}$ denote the holomorphic hull of K in $\hat{D}$. We will show that the conclusion of the lemma is satisfied for $\tilde{K} = \hat{K}_{\hat{D}}$.

Let f be a holomorphic function on a neighbourhood U of K. Then by our hypothesis, there exist a neighbourhood U' of K, a sequence $\{f_j\}_{j\geq 1}$ of holomorphic functions on D converges uniformly to f on compact sets of U'. We still denote by f_j the holomorphic extension of f_j to $\hat{D}$. Since K is compact in U', we can find $r > 0$ sufficiently small such that $K + r\Delta^n$ is relatively contained in U' where Δ^n is the unit polydisk in $\mathbf{C}^n$.

Notice that, by shrinking U' we may assume that there exists $M > 0$ such that

$$||f_j||_{U'} < M, \quad \forall\, j \geq 1.$$

By applying the Cauchy inequalities we obtain

$$\left|\frac{f_j^{(\alpha)}(\xi)}{\alpha!}\right| \leq \frac{M}{r^\alpha}, \quad \forall\, \alpha \in \mathbf{N}^n, \quad \forall\, \xi \in K.$$

It follows that

$$\left|\frac{f_j^{(\alpha)}(\xi)}{\alpha!}\right| \leq \frac{M}{r^\alpha}, \quad \forall\, \alpha \in \mathbf{N}^n, \quad \forall\, \xi \in \tilde{K}.$$

By the Cartan-Thullen theorem (see [**Hö**, Theorem 2.5.4]), we have $\tilde{K} + \frac{r}{2}\Delta^n \subset \hat{D}$. Moreover, for any $\lambda \in \frac{r}{2}\Delta^n$ and $\xi \in \tilde{K}$ we have

$$|f_j(\xi+\lambda)| = \left|\sum_{\alpha\in\mathbf{N}^n} \frac{f_j^{(\alpha)}(\xi)\lambda^\alpha}{\alpha!}\right| \leq \sum_{\alpha\in\mathbf{N}^n} \frac{M}{r^\alpha}\left(\frac{r}{2}\right)^\alpha < M' < \infty.$$

Thus the sequence $\{f_j\}_{j\geq 1}$ is uniformly bounded on a *fixed* neighbourhood of $\tilde{K}$. By passing to a subsequence, we may assume that $\{f_j\}_{j\geq 1}$ converges to f^* which is holomorphic on a neighbourhood of $\tilde{K}$. Clearly f^* is an extension of f to a neighbourhood of $\tilde{K}$. $\square$

We derive below some consequences of the preceding theorem.

Corollary 3.4. *If $[H(K)]' \in (\underline{DN})$ then $[\mathcal{H}(\Omega_\varphi(K))]' \in (\underline{DN})$, for every continuous function φ on K.*

Proof: By Theorem 3.1 there exists a Runge neighbourhood D of K. Now let $\tilde{\varphi}$ be a continuous extension of φ to $\mathbf{C}^n$. We show that $\Omega_{\tilde{\varphi}-1}(D)$ is a Runge neighbourhood of $\Omega_\varphi(K)$. Indeed, let f be a holomorphic function on a neighbourhood W of $\Omega_\varphi(K)$ in $\Omega_{\tilde{\varphi}-1}(D)$. Since $\tilde{\varphi}$ is continuous we can find a neighbourhood U of K in D and $0 < \varepsilon < 1$ such that

$$\Omega_\varphi(K) \subset \Omega_{\tilde{\varphi}-\varepsilon}(U) \subset W.$$

We expand f in the Hartogs series

$$f(z,\lambda) = \sum_{k=0}^{\infty} a_k(z)\lambda^k, \quad (z,\lambda) \in \Omega_{\tilde{\varphi}-\varepsilon}(U),$$

where a_k are holomorphic functions on U defined by

$$a_k(z) = \frac{1}{2\pi i} \int\limits_{|\lambda|=e^{-\tilde{\varphi}(z)+\delta}} \frac{f(z,\lambda)}{\lambda^{k+1}}\, d\lambda, \quad z \in U,\ 0 < \delta < \varepsilon.$$

As K is Runge in D we deduce that f can be uniformly approximated on neighbourhoods of $\Omega_\varphi(K)$ by holomorphic functions on $\Omega_{\tilde{\varphi}-1}(D)$. By applying Theorem 3.1 we conclude the proof. $\square$

Corollary 3.5. *Let K be a star shaped compact set in $\mathbf{C}^n$ containing 0, that is for every $z \in K$ the segment $\{tz : 0 \leq t \leq 1\} \subset K$. Then $[\mathcal{H}(K)]' \in (\underline{DN})$.*

Proof: Since K is star shaped, K has a neighbourhood basis of star shaped domains. The existence of a Runge neighbourhood for K now follows from Proposition 1 in [**Ka**]. $\square$

Corollary 3.6. *Let K be a compact in $\mathbf{C}$. Then the following assertions are equivalent.*

(i) $[H(K)]' \in (\underline{DN})$.
(ii) $\mathbf{C}\backslash K$ *has a finite number of connected components.*
(iii) *There exists a neighbourhood D of K such that $K = \hat{K}_D$.*

Proof: (i) $\Rightarrow$ (ii) follows from the following duality relationship

$$[\mathcal{H}(K)]' \cong \mathcal{H}_0(\mathbf{C}\backslash K),$$

where $\mathcal{H}_0(\mathbf{C}\backslash K)$ is the space of holomorphic functions on the complement of K that vanish at ∞.

(ii) $\Rightarrow$ (iii). Let $U_1, \ldots, U_k$ be bounded connected components of $\mathbf{C}\backslash K$. For each $1 \leq j \leq k$ we choose a point $a_j \in U_j$. Let $D = \mathbf{C}\backslash\{a_1, \ldots, a_k\}$. It is easy to check that $D\backslash K$ has no relatively bounded component in D. Thus the classical Runge theorem [**Hö**, Theorem 1.3.1] implies (iii).

(iii) $\Rightarrow$ (i) follows from Theorem 1.3.1 in [**Hö**] and Theorem 3.1. $\square$

4. The property $(\overline{\Omega})$

The first result in this section is the following

Theorem 4.1. *Let K be a compact Stein in $\mathbf{C}^n$ i.e. K has a Stein neighbourhoods basis in $\mathbf{C}^n$. Then $[\mathcal{H}(K)]' \in (\overline{\Omega})$ if and only if K is pluriregular in every neighbourhood W of K.*

Proof: Assume that $[\mathcal{H}(K)]' \in (\overline{\Omega})$. Let W be an arbitrary Stein neighbourhood of K in $\mathbf{C}^n$. Since $[\mathcal{H}(K)]' \in (\overline{\Omega})$, we can find a neighbourhood U of K in W such that for any neighbourhood V of K in U, for any $\varepsilon > 0$ the following inequalities hold

$$||f||_U \leq C||f||_V^{1-\varepsilon}||f||_W^{\varepsilon}, \quad \forall\, f \in \mathcal{H}^\infty(W), \tag{8}$$

where $C > 0$ is independent of f. By applying (8) to f^n we obtain

$$||f||_U^n \leq C||f||_V^{(1-\varepsilon)n}||f||_W^{\varepsilon n}, \quad \forall\, f \in \mathcal{H}^\infty(W), \quad \forall\, n \geq 1.$$

Taking n-th root and let n tend to ∞ we get

$$||f||_U \leq ||f||_V^{1-\varepsilon}||f||_W^{\varepsilon}, \quad \forall\, f \in \mathcal{H}^\infty(W).$$

It follows that

$$||f||_U \leq ||f||_K^{1-\varepsilon}||f||_W^{\varepsilon}, \quad \forall\, f \in \mathcal{H}^\infty(W).$$

Therefore
$$\frac{\log|f(z)| - \log||f||_K}{\log||f||_W - \log||f||_K} \leq \varepsilon, \quad \forall\, f \in \mathcal{H}^\infty(W), \quad \forall\, z \in U.$$
Now it follows from a result of Zaharjuta (see [**Sa**, Proposition 10.2]) that
$$\omega^*(W, K, z) \leq \varepsilon, \quad \forall\, z \in U.$$
Since $\varepsilon > 0$ can be arbitrary small, we conclude that K is pluriregular in W.

Conversely, assume that K is pluriregular in every neighbourhood of K. Let W be an arbitrary neighbourhood of K. Then for every $\varepsilon > 0$, there exists a neighbourhood U of K in W such that
$$\omega^*(W, K, z) < \varepsilon, \quad \forall\, z \in U.$$
It implies that
$$\frac{\log|f(z)| - \log||f||_K}{\log||f||_W - \log||f||_K} \leq \varepsilon, \quad \forall\, z \in U, \quad \forall\, f \in H^\infty(W).$$
Hence
$$||f||_U \leq ||f||_K^{1-\varepsilon}||f||_W^{\varepsilon}, \quad \forall\, f \in \mathcal{H}^\infty(W).$$
So $[\mathcal{H}(K)]' \in (\overline{\Omega})$. □

Combining the above theorem with Proposition 6.1 in [**Sic**] (or Lemma 12.3 in [**Sa**]) we easily deduce the following

Corollary 4.2. *Let K be a polynomially convex compact set in $\mathbf{C}^n$. Then $[\mathcal{H}(K)]' \in (\overline{\Omega})$ if and only if V_K, the Siciak extremal function of K, is continuous on $\mathbf{C}^n$, or equivalently $V_K^* \equiv 0$ on K.*

We next apply Theorem 4.1 to the case of compact Hartogs sets.

Corollary 4.3. *Let K be a compact Stein set in $\mathbf{C}^n$, which is locally pluriregular. Assume that φ is a plurisubharmonic function on a neighbourhood of K and continuous on K. Then $[\mathcal{H}(\Omega_\varphi(K))]' \in (\overline{\Omega})$.*

Proof: We first notice that $\Omega_\varphi(K)$ has a Stein neighbourhood basis. Indeed, it suffices to observe that if U is a Stein neighbourhood of K then the Hartogs domain
$$\Omega_\varphi(U) = \{(z, \lambda) : |\lambda| < e^{-\varphi(z)}\}$$
is a Stein neighbourhood of $\Omega_\varphi(K)$.

It remains, in view of Theorem 4.1, to check that $\Omega_\varphi(K)$ is locally pluriregular. For this, we fix an arbitrary point $p = (z_0, w_0) \in \Omega_\varphi(K)$. By [**Sa**, Proposition 12.5], it is enough to show that $V^*_{\overline{U} \cap \Omega_\varphi(K)}(p) = 0$

for every neighbourhood U of p in $\mathbf{C}^{n+1}$. Then we can find a neighbourhood $\tilde{U}$ of z_0 in $\mathbf{C}^n$ so small that

$$(\overline{\tilde{U}} \cap K) \times \{w : |w| \leq e^{-M}\} \subset \overline{U} \cap \Omega_\varphi(K),$$

where $M = \sup_{z \in \tilde{U} \cap K} \varphi(z)$. It follows that

$$\begin{aligned} V^*_{\overline{U} \cap \Omega_\varphi(K)}(p) &\leq V^*_{(\overline{\tilde{U}} \cap K) \times \{|w| \leq e^{-M}\}}(p) \\ &= \max\Big(V^*_{\overline{\tilde{U}} \cap K}(z_0), V^*_{\{|w| \leq e^{-M}\}}(w_0)\Big) \\ &= \log^+\left(\frac{|w_0|}{e^{-M}}\right) \leq M - \varphi(z_0), \end{aligned}$$

here we have used the product property of Siciak extremal functions in the second line (see [**Sic**, Proposition 5.9]). Since $\varphi \in \mathcal{C}(K)$ we deduce that $M - \varphi(z_0) \to 0$ when $\tilde{U}$ shrinks to z_0. Thus for every neighbourhood U of p we have

$$V^*_{\overline{U} \cap \Omega_\varphi(K)}(p) = 0.$$

This means that p is a locally pluriregular point of $\Omega_\varphi(K)$. We finish the proof. $\square$

The next result should be compared to Corollary 3.4.

Proposition 4.4. *There exists a polynomially convex compact set K in $\mathbf{C}^2$ and a continuous real valued function φ on K such that*

(i) $[\mathcal{H}(K)]' \in (\overline{\Omega})$.
(ii) $\Omega_\varphi(K)$ *is polynomially convex.*
(iii) $[\mathcal{H}(\Omega_\varphi(K))]' \notin (\overline{\Omega})$.

Proof: We let K be the compact set in [**Sa**, Proposition 8.1], i.e., $K = K_1 \cup K_2$ where

$$\begin{cases} K_1 = \{(z,w) \in \mathbf{C}^2 : |z| \leq 1,\ w = 0\} \\ K_2 = \{(z,w) \in \mathbf{C}^2 : z = e^{i\theta},\ \operatorname{Re} w = 0,\ 0 \leq \operatorname{Im} w \leq e^{\frac{1}{\cos\theta - 1}},\ \theta \in [-\pi, \pi]\}. \end{cases}$$

It is proved there that K is polynomially convex and $V_K^* \equiv 0$ on K. Thus Corollary 4.2 implies that $[\mathcal{H}(K)]' \in (\overline{\Omega})$. It remains to construct a continuous function φ satisfying (ii) and (iii). Let v be an arbitrary continuous subharmonic function in $\mathbf{C}$ satisfying

$$v(z) = 0, \quad \forall\, z \in \partial\Delta, \quad v \not\equiv 0 \text{ on } \Delta,$$

where Δ is the open unit disk in $\mathbf{C}$. We set $\varphi(z,w) = v(z) + |w|$.

We have

$$\Omega_\varphi(K) = \{(z,w,u) \in K \times \mathbf{C}^2 : \log|u| + \varphi(z,w) \leq 0\}.$$

It follows from Theorem 4.3.4 in [**Hö**] that $\Omega_\varphi(K)$ is polynomially convex in $\mathbf{C}^3$. Assume for the sake of contradiction that $[\mathcal{H}(\Omega_\varphi(K))]' \in (\overline{\Omega})$. Then by Corollary 4.2 we would have

$$V^*_{\Omega_\varphi(K)}(z) = 0, \quad \forall\, z \in \Omega_\varphi(K). \tag{9}$$

Next we write

$$\Omega_\varphi(K) = K_1' \cup K_2',$$

where $K_i' = \Omega_\varphi(K_i)$, $i = 1,2$. Since K_2 is locally pluriregular (as was shown in [**Sa**, p. 73]), by the proof of Corollary 4.3 we get K_2' is pluriregular. Now we have

$$V^*_{K_1'\cup K_2'} \equiv V^*_{(K_1'\cup K_2')\widehat{}} \leq V^*_{\widehat{K_2'}\cup K_1'} \leq V^*_{K_1'\cup K_2'}.$$

Notice that K_1' is pluripolar in $\mathbf{C}^3$, so we obtain

$$V^*_{K_1'\cup K_2'} \equiv V^*_{\widehat{K_2'}\cup K_1'} = V^*_{\widehat{K_2'}}.$$

Thus

$$V^*_{K_1'\cup K_2'} \equiv V^*_{\widehat{K_2'}}.$$

Combining this with (9) we get $K_1' \subset \widehat{K_2'}$. On the other hand, by the choice of v we have

$$K_1' = \{(z,w,u) : (z,w) \in K_1,\ |u| \leq e^{-v(z)}\}$$

$$K_2' = \{(z,w,u) : (z,w) \in K_2,\ |u| \leq e^{-|w|}\} \subset K_2 \times \overline{\Delta}.$$

It follows that $K_1' \subset \mathbf{C}^2 \times \overline{\Delta}$. Thus $v \geq 0$ on Δ. By the maximum principle we have $v \equiv 0$ on Δ, a contradiction to the choice of v. □

5. The case K is closure of a Reinhardt domain

We devote this section to study the structure of $\mathcal{H}(K)$ in the case where K is the closure $\overline{D}$ of a bounded Reinhardt domain D. First of all, we recall some basic facts about Reinhardt domains that will be used in the sequel.

Let D be a domain in $\mathbf{C}^n$. D is said to be *Reinhardt* if for every $(\theta_1,\ldots,\theta_n) \in \mathbf{R}^n$ we have

$$(z_1,\ldots,z_n) \in D \Rightarrow (e^{i\theta_1}z_1,\ldots,e^{i\theta_n}z_n) \in D.$$

For each Reinhardt domain D in $\mathbf{C}^n$ we denote by $\log D_*$ its *logarithmic image*, more precisely

$$\log D_* = \{(\log|z_1|, \dots, \log|z_n|) : (z_1, \dots, z_n) \in D_*\},$$

where $D_* = \{(z_1, \dots, z_n) \in D : z_1 \dots z_n \neq 0\}$. We also write $\hat{D}$ for the envelope of holomorphy of D. It is well known that $\hat{D}$ is a pseudoconvex Reinhardt in $\mathbf{C}^n$. Next we let for $1 \leq j \leq n$

$$V_j = \{z \in \mathbf{C}^n : z_j = 0\}, \quad \text{and} \quad V = \bigcup_{1 \leq j \leq n} V_j.$$

The following useful criterion for pseudoconvexity of a Reinhardt domain can be found in [**Zw1**].

Lemma 5.1. *Let D be a Reinhardt domain in $\mathbf{C}^n$. Then the following assertions are equivalent.*

(i) *D is pseudoconvex.*

(ii) *$\log D_*$ is convex and if $D \cap V_j \neq \emptyset$ for some $1 \leq j \leq n$ then*

$$(z_1, \dots, z_{j-1}, z_j, \dots, z_n) \in D \Rightarrow (z_1, \dots, z_{j-1}, \lambda z_j, \dots, z_n) \in D \quad \forall\, |\lambda| < 1.$$

We next recall the concept of hyperconvexity. A bounded domain D in $\mathbf{C}^n$ is said to be *hyperconvex* if there is a negative exhaustive continuous plurisubharmonic function for D. It is a remarkable fact that for bounded hyperconvex domains, it is enough to have a weak plurisubharmonic barrier at every boundary point. This fact is perhaps most clearly explained in [**Bl**]. More precisely, we have

Theorem 5.2. *Let D be a bounded domain in $\mathbf{C}^n$. Then D is hyperconvex if and only if every boundary point ξ has a weak plurisubharmonic barrier, i.e., there exists a non constant negative plurisubharmonic function ψ on D such that*

$$\lim_{z \to \xi} \psi(z) = 0.$$

If the domain in question is pseudoconvex Reinhardt then we have a simpler criterion.

Lemma 5.3. *Let D be a bounded pseudoconvex Reinhardt domain in $\mathbf{C}^n$. Then*

(i) *There exists a weak plurisubharmonic barrier at every point $\xi \in \partial D \backslash V$, which extends to a plurisubharmonic function in a neighbourhood of ξ in $\mathbf{C}^n$.*

(ii) *D is hyperconvex if and only if there exists a weak plurisubharmonic barrier at every point $\xi \in (\partial D) \cap V$.*

Proof: (i) The proof is implicitly contained in that of Theorem 2.14 in [**CCW**], we omit the details.

(ii) follows immediately from Theorem 5.2 and Lemma 5.3(i). □

For pseudoconvex Reinhardt domains we mention the following beautiful result in [**Zw2**].

Lemma 5.4. *A bounded pseudoconvex Reinhardt domain D in $\mathbf{C}^n$ is hyperconvex if and only if $D \cap V_j \neq \emptyset$ for any $j \in \{1, \ldots, n\}$ satisfying $\overline{D} \cap V_j \neq \emptyset$.*

Now we are able to formulate the main result in this section

Theorem 5.5. *Let D be a bounded Reinhardt domain in $\mathbf{C}^n$. Then $\mathcal{H}(\overline{D}) \cong \mathcal{H}(\overline{\Delta}^n)$.*

The key element in the proof is the following lemma, which is of independent interest.

Lemma 5.6. *Let $\{U_n\}_{n\geq 1}$ be a decreasing sequence of bounded pseudoconvex Reinhardt domains satisfying $\overline{U}_{n+1} \subset U_n$. Let*

$$G = \bigcap_{n\geq 1} U_n$$

and $\Omega = \operatorname{Int} G$. Then Ω is a bounded hyperconvex Reinhardt domain.

Proof: We first show that Ω is connected. For this, it suffices to check that so is $\Omega_* = \Omega \setminus V$. For this, we take two arbitrary points $a, b \in \Omega_*$ and set

$$a' = (\log|a_1|, \ldots, \log|a_n|) \in \log \Omega_*$$

$$b' = (\log|b_1|, \ldots, \log|b_n|) \in \log \Omega_*.$$

There exist two small balls S_1 and S_2 centered at a', b' such that $S_1 \cup S_2 \subset \log \Omega_*$.

Then

$$S_1 \cup S_2 \subset \log(U_n)_*, \quad n \geq 1.$$

Since $\log(U_n)_*$ is a convex domain in $\mathbf{R}^n$ we infer that

$$\operatorname{conv}(S_1 \cup S_2) \subset \log(U_n)_*, \quad \forall\, n \geq 1.$$

Thus $\operatorname{conv}(S_1 \cup S_2) \subset \log G_*$ and consequently

$$\operatorname{conv}(S_1 \cup S_2) \subset \log \Omega_*.$$

It implies that Ω_* is connected. This also shows that Ω is a pseudoconvex Reinhardt domain.

It remains to check that Ω is hyperconvex. By Lemma 5.3 it is enough to show the existence of a weak plurisubharmonic barrier at every point a of $\partial\Omega \cap V$. There are two cases to consider

Case 1: $a = 0$. Let $p = (p_1, \ldots, p_n)$ be some point in Ω_*. Fix $n \geq 1$. Since $\overline{\Omega} \subset \overline{U}_{n+1} \subset U_n$, we can find $\varepsilon > 0$ such that

$$\{z \in \mathbf{C}^n : |z_1| < \varepsilon, \ldots, |z_n| < \varepsilon\} \cup \{p\} \subset U_n.$$

Hence

$$\begin{aligned} \log(U_n)_* &\supset \operatorname{conv}\Big\{(\log|p_1|, \ldots, \log|p_n|) \cup \{(x_1, \ldots, x_n) : x_j < \log\varepsilon\}\Big\} \\ &\supset \{(x_1, \ldots, x_n) : x_1 < \log|p_1|, \ldots, x_n < \log|p_n|\}. \end{aligned}$$

It follows that

$$\{z \in \mathbf{C}^n : |z_i| < |p_i|, 1 \leq i \leq n\} \subset U_n, \quad \forall\, n \geq 1.$$

Therefore, $0 \in \operatorname{Int}(G) = \Omega$, a contradiction!

Case 2: If $a = (0, 0, \ldots, 0, a_{k+1}, \ldots, a_n)$, where $a_j \neq 0$ for $k+1 \leq j \leq n$, $1 \leq k < n$. Let π denote the projection $(z_1, \ldots, z_n) \mapsto (z_{k+1}, \ldots, z_n)$. By Lemma 5.1 $\pi(\Omega)$ is a pseudoconvex Reinhardt domain in $\mathbf{C}^{n-k}$. We claim that $\pi(a) \notin \pi(\Omega)$. Indeed, otherwise we can find $\alpha = (c_1, c_2, \ldots, c_k, a_{k+1}, \ldots, a_n) \in \Omega_*$ with $\pi(\alpha) = \pi(a)$. We denote

$$\tilde{\alpha} = (\log|c_1|, \ldots, \log|c_k|, \log|a_{k+1}|, \ldots, \log|a_n|) \in \log\Omega_*.$$

Then there exists a small ball $\tilde{S}$ centered at $\tilde{\alpha}$ such that $\tilde{S} \subset \log\Omega_*$. It implies that there exists a Reinhardt neighbourhood (in $\mathbf{C}^n_*$) S of α such that

$$\log S = \tilde{S} \subset \log\Omega_*.$$

We choose δ so small that

$$0 < \delta < \min(|z_1|, \ldots, |z_n|), \quad \forall\, (z_1, \ldots, z_n) \in S.$$

Notice also that $U_n \cap V_j \neq \emptyset$ for $j = \overline{1, k}$, thus by Lemma 5.1 we get

$$(z_1, \ldots, z_n) \in S \Rightarrow (0, \ldots, 0, z_{k+1}, \ldots, z_n) \in U_n. \tag{10}$$

Next we let $\tilde{\pi}$ be the projection $(x_1, \ldots, x_n) \mapsto (x_{k+1}, \ldots, x_n)$ and fix $(z_1^0, \ldots, z_n^0) \in S$. According to (10) we can find $\varepsilon > 0$ small enough such that

$$\{(z_1, \ldots, z_k, z_{k+1}^0, \ldots, z_n^0) : |z_j| < \varepsilon,\ j = \overline{1, k}\} \subset U_n.$$

It implies that

$$\log(U_n)_* \supset (\log|z_1^0|, \ldots, \log|z_n^0|) \cup \{\{(x_1, \ldots, x_k) : x_j < \log\varepsilon,\ j = \overline{1,k}\} \times (\log|z_{k+1}^0|, \ldots, \log|z_n^0|)\}.$$

Thus

$$\begin{aligned}\log(U_n)_* &\supset \operatorname{conv}\Big\{(\log|z_1^0|, \ldots, \log|z_n^0|) \\ &\quad \cup \{(x_1, \ldots, x_k, \log|z_{k+1}^0|, \ldots, \log|z_n^0|) : x_j < \log\varepsilon,\ j = \overline{1,k}\}\Big\} \\ &\supset \Big\{(x_1, \ldots, x_k, \log|z_{k+1}^0|, \ldots, \log|z_n^0|) : x_j < \log|z_j^0|,\ j = \overline{1,k}\Big\}.\end{aligned}$$

It follows that

$$\log(U_n)_* \supset \{(x_1, \ldots, x_k) : x_j < \log\delta\} \times \tilde{\pi}(\log S).$$

This implies that G contains a neighbourhood of a. In other words, $a \in \mathrm{Int}(G) = \Omega$. This is absurd. Thus $\pi(a) \notin \pi(D)$ and therefore $\pi(a) \in \partial\pi(D)$. Hence, we can find a weak plurisubharmonic u barrier at $\pi(a)$ in $\pi(D)$. It follows that $u \circ \pi$ is a weak plurisubharmonic barrier for a in D. Therefore Ω is hyperconvex. $\square$

We now proceed to the

Proof of Theorem 5.5: We divide the proof into four steps.

Step 1: We will prove that there exists a bounded hyperconvex Reinhardt domain $\tilde{D}$ in $\mathbf{C}^n$ such that $D \subset \tilde{D}$ and every holomorphic function on a neighbourhood of $\overline{D}$ extends to a neighbourhood of $\overline{\tilde{D}}$. For this, we let D_n be a decreasing sequence of Reinhardt domains such that

$$\bigcap_{n=1}^{\infty} D_n = \overline{D}, \quad \overline{D_{n+1}} \subset D_n.$$

Let

$$U_n = \hat{D}_n, \quad \tilde{D} = \mathrm{Int}\left(\bigcap_{n=1}^{\infty} U_n\right).$$

It is clear that $\overline{U_{n+1}} \subset U_n$ and $D \subset \tilde{D}$. By Lemma 5.6 we conclude that $\tilde{D}$ is the domain we seek for.

Step 2: $\overline{\tilde{D}}$ has a Runge neighbourhood. Indeed, it follows from Lemma 5.4 and the remark in [**Fu**, p. 171] that $\overline{\tilde{D}}$ is *rationally convex*, i.e. $\mathbf{C}^n \setminus \overline{\tilde{D}}$ is union of algebraic hypersurfaces. Thus $\overline{\tilde{D}}$ has a Runge neighbourhood basis.

Step 3: $\overline{\tilde{D}}$ is pluriregular in every neighbourhood of it. Let U be a neighbourhood of $\overline{\tilde{D}}$. It is clear that

$$\omega^*(U, \overline{\tilde{D}}, z) = 0, \quad \forall\, z \in \tilde{D}. \tag{11}$$

It remains to check (11) at every point $z = (z_1, \ldots, z_n) \in \partial\tilde{D}$. Since $z \neq 0$ we may assume that $z = (0, \ldots, z_{k+1}, \ldots, z_n)$, where $z_j \neq 0$, $\forall\, 1 \leq j \leq k$. Since $\overline{\tilde{D}} \cap V_j \neq \emptyset$, $1 \leq j \leq k$ we deduce that $\tilde{D} \cap V_j \neq \emptyset$, $\forall\, 1 \leq j \leq k$. By Lemma 5.1 we can find a point $(0, \ldots, 0, a_{k+1}, \ldots, a_n)$, where $a_j \neq 0$, $\forall\, k+1 \leq j \leq n$. Let π be the projection $(z_1, \ldots, z_n) \mapsto (z_{j+1}, \ldots, z_n)$ and we denote

$$\tilde{z} = (\log|z_{k+1}|, \ldots, \log|z_n|),$$

$$\tilde{a} = (\log|a_{k+1}|, \ldots, \log|a_n|) \in \log(\tilde{D})_*.$$

We have $\log(\pi(\tilde{D}))_*$ is a convex domain in $\mathbf{R}^{n-k}$, and $\tilde{z} \in \partial(\log(\pi(\tilde{D}))_*)$, $\tilde{a} \in \log(\pi(\tilde{D}))_*$. It follows that the segment having $\tilde{z}$ and $\tilde{a}$ as endpoints lies inside this convex domain. Thus by Lemma 5.1 we can find a real analytic curve $\gamma \subset \tilde{D}$ with endpoints z and a. It is also easy to see that γ is contained in a larger real analytic curve through z. Thus z is not a pluri-thin point of γ, therefore $\omega^*(U, \overline{\tilde{D}}, z) = 0$.

Step 4: $\mathcal{H}(\overline{D}) \cong H(\overline{\Delta}^n)$. It follows from Step 1 that $\mathcal{H}(\overline{D}) \cong \mathcal{H}(\overline{\tilde{D}})$. Notice that $\overline{\tilde{D}}$ has a Runge neighbourhood and $\overline{\tilde{D}}$ is pluriregular in every neighbourhood of it, in particular $\overline{\tilde{D}}$ is $\mathbf{C}^n$ regular (in the sense of Zaharjuta, see [**Za**, p. 141]). Under these conditions Theorem 4.2 in [**Za**] tells us that $H(\overline{\tilde{D}}) \cong H(\overline{\Delta}^n)$. □

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