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ON SOME MEAN OSCILLATION INEQUALITIES FOR MARTINGALES

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Abstract

Let $(X, \|\cdot\|_X)$ be a Banach function space over a nonatomic probability space $(\Omega, \Sigma, \mathbb{P})$. If $f = (f_n)_{n\in\mathbb{Z}_+}$ is a martingale with respect to a filtration $\mathcal{F} = (\mathcal{F}_n)_{n\in\mathbb{Z}_+}$, then we define

$$\theta_{\mathcal{F}} f = \sup_{0\le n\le m<\infty} \mathbb{E}\big[\,|f_m - f_{n-1}|\,\big|\,\mathcal{F}_n\big],$$

where $f_{-1} \equiv 0$. In this paper, we give a necessary and sufficient condition for the existence of constants c and C such that for any martingale $f = (f_n)_{n\in\mathbb{Z}_+}$,

$$c \varlimsup_{n\to\infty} \|f_n\|_X \le \|\theta_{\mathcal{F}} f\|_X \le C \varliminf_{n\to\infty} \|f_n\|_X\,.$$

1. Introduction

Let $(\Omega, \Sigma, \mathbb{P})$ be a nonatomic probability space and let $f = (f_n)_{n\in\mathbb{Z}_+}$ be a martingale with respect to a filtration $\mathcal{F} = (\mathcal{F}_n)_{n\in\mathbb{Z}_+}$ (where by a *filtration*, we mean a nondecreasing sequence of sub-σ-algebras of Σ). We set $f_{-1} \equiv 0$ and define

$$\theta_{\mathcal{F}} f = \sup_{0\le n\le m<\infty} \mathbb{E}\big[\,|f_m - f_{n-1}|\,\big|\,\mathcal{F}_n\big].$$

Let $(X, \|\cdot\|_X)$ be a Banach function space over Ω (see Definition 1 below). In this paper, we consider the inequalities of the form

$$c \varlimsup_{n\to\infty} \|f_n\|_X \le \|\theta_{\mathcal{F}} f\|_X \le C \varliminf_{n\to\infty} \|f_n\|_X\,. \tag{1}$$

If $X = L_p$ for some $1 < p < \infty$, then (1) holds for any martingale $f = (f_n)$. Indeed, by using Theorem 7 of [**7**] and the estimate

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$\theta_{\mathcal{F}} f \le 2\sup_{0\le n\le m<\infty} \mathbb{E}\big[\,|f_m|\,\big|\,\mathcal{F}_n\big]$, we have that

$$c_p \Big\| \sup_n |f_n| \Big\|_p \le \|\theta_{\mathcal{F}} f\|_p \le 2 \Big\| \sup_{0\le n\le m<\infty} \mathbb{E}\big[\,|f_m|\,\big|\,\mathcal{F}_n\big] \Big\|_p ,$$

which, together with the Doob inequality, implies that

$$\begin{aligned} c_p \lim_{n\to\infty} \|f_n\|_p = c_p \sup_n \|f_n\|_p \le \|\theta_{\mathcal{F}} f\|_p \\ \le \frac{2p}{p-1} \sup_n \|f_n\|_p = \frac{2p}{p-1} \lim_{n\to\infty} \|f_n\|_p . \end{aligned}$$

We can derive a similar result for Orlicz function spaces: if Φ is an N-function satisfying the Δ_2- and ∇_2-conditions and if $X = L_\Phi$, then (1) holds for any martingale $f = (f_n)$.

The purpose of this paper is to show that (1) holds for any martingale $f = (f_n)$ if and only if X can be renormed so that it is rearrangement-invariant and $0 < \alpha_X \le \beta_X < 1$, where α_X and β_X denote the lower and upper Boyd indices of X, respectively. An analogous problem has been studied in [**4**]. Combining our Main Theorem with the result of [**4**] shows that (1) holds for any martingale $f = (f_n)$ if and only if there exist constants k and K such that for any uniformly integrable martingale $f = (f_n)$,

$$k\,\|f_\infty\|_X \le \|Sf\|_X \le K\,\|f_\infty\|_X ,$$

where Sf denotes the square function of f, and f_∞ denotes the almost sure limit of f.

2. Preliminaries

Let $(\Omega, \Sigma, \mathbb{P})$ be a fixed probability space. In this paper, we will deal with martingales on Ω with respect to various filtrations on $(\Omega, \Sigma, \mathbb{P})$ (and with respect to $\mathbb{P}$).

Assumption. We assume that the probability space $(\Omega, \Sigma, \mathbb{P})$ is *non-atomic*, that is, that $(\Omega, \Sigma, \mathbb{P})$ contains no atom.

This assumption is essential and will be used implicitly throughout the paper. In addition to Ω, we have to deal with the *canonical* probability space $(I, \mathfrak{M}, \mu)$, where I denotes the interval $(0, 1]$, $\mathfrak{M}$ denotes the σ-algebra of Lebesgue measurable subsets of I, and μ denotes Lebesgue measure. We distinguish these two probability spaces. Although the reader may assume that $(\Omega, \Sigma, \mathbb{P})$ is the canonical probability space, our argument will not become so simple by doing so.

Let X and Y be normed linear spaces of random variables. We write $X \hookrightarrow Y$ to mean that X is continuously embedded in Y, that is, that

$X \subset Y$ and the inclusion map is continuous. If $X \hookrightarrow Y \hookrightarrow X$, then we write $X \approx Y$. Thus $X \approx Y$ if and only if $X = Y$ (as a set) and the norms of these spaces are equivalent.

Definition 1. A *Banach function space* is a real Banach space of (equivalence classes of) random variables satisfying the following conditions:

(B1) $L_\infty \hookrightarrow X \hookrightarrow L_1$;

(B2) if $|x| \le |y|$ a.s. and $y \in X$, then $x \in X$ and $\|x\|_X \le \|y\|_X$;

(B3) if $0 \le x_n \uparrow x$ a.s., $x_n \in X$ for all n, and $\sup_n \|x_n\|_X < \infty$, then $x \in X$ and $\|x\|_X = \sup_n \|x_n\|_X$.

For convenience, we adopt the convention that if $x \notin X$, then $\|x\|_X = \infty$. Thus $\|x\|_X < \infty$ if and only if $x \in X$.

Given random variables x and y, we write $x \simeq_d y$ to mean that x and y have the same distribution.

Definition 2. A Banach function space $(X, \|\cdot\|_X)$ is said to be *rearrangement-invariant* (r.i.) provided that

(RI) if $x \simeq_d y$ and $y \in X$, then $x \in X$ and $\|x\|_X = \|y\|_X$.

In this paper, a rearrangement-invariant Banach function space is called a *rearrangement-invariant space* or an *r.i. space*.

For example, Lebesgue, Orlicz, and Lorentz spaces are r.i. spaces. On the other hand, the weighted Lebesgue space $L_{p,w}$, with a suitable weight w, is a Banach function space that is not r.i. in general (see the remark following the Main Theorem).

Definition 3. Let $(X, \|\cdot\|_X)$ be a Banach function space over Ω. The *associate space* of X is the Banach function space $(X', \|\cdot\|_{X'})$ consisting of those random variables y for which

$$\|y\|_{X'} := \sup\{E[xy] \mid x \in X,\ \|x\|_X \le 1\} < \infty.$$

The associate space of a Banach function space over $I = (0, 1]$ is defined in the same way.

For example, the associate space of L_1 is L_∞, and the associate space of L_∞ is L_1. For any Banach function space X, we have $(X')' = X$; however, the associate space X' is not the dual space of X in general. See [**2**, Chapter 1] for more details.

By definition, we have that for any $x \in X$ and $y \in X'$,

$$\mathbb{E}\big[|xy|\big] \le \|x\|_X \, \|y\|_{X'} \,,$$

which we call *Hölder's inequality.*

It is known that a Banach function space $(X, \|\cdot\|_X)$ is r.i. if and only if so is the associate space of X (see [**2**, p. 60]).

Suppose that $(X, \|\cdot\|_X)$ is an r.i. space over Ω. If $A \in \Sigma$ and 1_A denotes the indicator function of A, then the norm of 1_A in X depends only on the probability of A. Thus we may define a function φ_X on I by setting

$$\varphi_X(t) = \|1_A\|_X\,, \quad \text{where } A \in \Sigma \text{ and } \mathbb{P}(A) = t.$$

We call φ_X the *fundamental function* of X. It is clear that if $1 \le p \le \infty$, then $\varphi_{L_p}(t) = t^{1/p}$ $(t \in I)$. Hence if we denote by p' the conjugate exponent of p, then $\varphi_{L_p}(t)\,\varphi_{L_{p'}}(t) = t$ for all $t \in I$. The same is true for any r.i. space X and its associate space X' (see [**2**, p. 66]); that is,

$$\varphi_X(t)\,\varphi_{X'}(t) = t \quad \text{for all } t \in I. \tag{2}$$

Now let x be a random variable on Ω. The *nonincreasing rearrangement* of x, denoted by x^*, is a (unique) nonincreasing right-continuous function on I such that

$$\mathbb{P}(|x| > \lambda) = \mu(x^* > \lambda) \quad (\lambda > 0).$$

Note that x^* is represented as

$$x^*(t) = \inf\bigl\{\lambda > 0 \,\big|\, \mathbb{P}(|x| > \lambda) \le t\bigr\} \quad (t \in I),$$

with the convention that $\inf \emptyset = \infty$. If ϕ is a measurable function on I, then the nonincreasing rearrangement ϕ^* is defined by regarding ϕ as a random variable on the canonical probability space.

If ϕ and ψ are integrable functions on I, we write $\phi \prec \psi$ to mean that

$$\int_0^t \phi^*(s)\,ds \le \int_0^t \psi^*(s)\,ds \quad \text{for all } t \in I.$$

Furthermore if x and y are integrable random variables on Ω and if $x^* \prec y^*$, then we write $x \prec y$. It is obvious that $x \simeq_d y$ if and only if $x \prec y \prec x$.

In the literature of this subject, a Banach function space $(X, \|\cdot\|_X)$ is said to be universally rearrangement-invariant (u.r.i.) provided that

(URI) if $x \prec y$ and if $y \in X$, then $x \in X$ and $\|x\|_X \le \|y\|_X$.

In our setting, however, there is no need to distinguish between r.i. spaces and u.r.i. spaces; *a Banach function space is r.i. if and only if it is u.r.i., provided that the underlying probability space is nonatomic* (see [**2**, Exercise 16, p. 90]).

Now let X be an r.i. space over Ω. Then there exists a unique r.i. space $\widehat{X}$ over I, equipped with the norm $\|\cdot\|_{\widehat{X}}$, such that:

- $x \in X$ if and only if $x^* \in \widehat{X}$;
- $\|x\|_X = \|x^*\|_{\widehat{X}}$ for all $x \in X$.

In fact $\widehat{X}$ consists of those functions φ for which

$$\|\varphi\|_{\widehat{X}} := \sup\left\{ \int_0^1 \varphi^*(s)\, y^*(s)\, ds \;\middle|\; y \in X',\ \|y\|_{X'} \le 1 \right\} < \infty.$$

We call $(\widehat{X}, \|\cdot\|_{\widehat{X}})$ the *Luxemburg representation* of $(X, \|\cdot\|_X)$. For instance, the Luxemburg representation of $L_p(\Omega)$ is $L_p(I)$. See [**2**, pp. 62–64] for details.

In order to describe our results, we have to recall the notion of *Boyd indices*. Given any positive number s and any measurable function ϕ on I, we define

$$(D_s\phi)(t) = \begin{cases} \phi(st) & \text{if } st \in I, \\ 0 & \text{if } st \notin I, \end{cases} \qquad (t \in I).$$

If Z is an r.i. space over I, then each D_s (restricted to Z) is a bounded linear operator from Z into itself and $\|D_s\|_{B(Z)} \le (1/s) \vee 1$, where $\|D_s\|_{B(Z)}$ stands for the operator norm of $D_s \colon Z \to Z$. The *lower* and *upper Boyd indices* of Z are defined by

$$\alpha_Z = \sup_{0<s<1} \frac{\log \|D_{s^{-1}}\|_{B(Z)}}{\log s} = \lim_{s \downarrow 0} \frac{\log \|D_{s^{-1}}\|_{B(Z)}}{\log s}$$

and

$$\beta_Z = \inf_{1<s<\infty} \frac{\log \|D_{s^{-1}}\|_{B(Z)}}{\log s} = \lim_{s \uparrow \infty} \frac{\log \|D_{s^{-1}}\|_{B(Z)}}{\log s},$$

respectively. If X is an r.i. space over Ω, then the Boyd indices of X are defined by $\alpha_X = \alpha_{\widehat{X}}$ and $\beta_X = \beta_{\widehat{X}}$, where $\widehat{X}$ is the Luxemburg representation of X.

For instance, $\alpha_{L_p} = \beta_{L_p} = 1/p$ $(1 \le p \le \infty)$. Note that $0 \le \alpha_X \le \beta_X \le 1$ for any r.i. space X. See [**2**, p. 149] for details.

Now, let Z_1 and Z_2 be r.i. spaces over I, and let T be a linear operator on $L_1(I)$. We write $T \in B(Z_1, Z_2)$ to mean that the restriction of T to Z_1 is a bounded operator from Z_1 into Z_2. If $Z_1 = Z_2 = Z$, then we also write $T \in B(Z)$ for $T \in B(Z, Z)$.

As shown in [**4**] and [**5**], there are deep connections between some martingale inequalities and the boundedness of some linear operators on $L_1(I)$. We will establish another connection between the inequalities of the form (1) and the boundedness of the operators $\mathcal{P}$ and $\mathcal{Q}$ defined for $\phi \in L_1(I)$ by

$$(\mathcal{P}\phi)(t) = \frac{1}{t}\int_0^t \phi(s)\,ds \quad (t \in I);$$

$$(\mathcal{Q}\phi)(t) = \int_t^1 \frac{\phi(s)}{s}\,ds \qquad (t \in I).$$

Note that $\mathcal{Q}$ is the (formal) adjoint of $\mathcal{P}$. It is well known that $\mathcal{P} \in B(Z)$ (resp. $\mathcal{Q} \in B(Z)$) if and only if $\beta_Z < 1$ (resp. $\alpha_Z > 0$). For a proof, (see [**2**, p. 150]) (*cf.* [**8**]).

Now let $(X, \|\cdot\|_X)$ be an r.i. space over Ω. For each random variable x, we let

$$\|x\|_{H(X)} = \|\mathcal{P}x^*\|_{\widehat{X}} \quad \text{and} \quad \|x\|_{K(X)} = \|\mathcal{Q}x^*\|_{\widehat{X}}\,.$$

Define $H(X)$ (resp. $K(X)$) to be the set of all random variables x for which $\|x\|_{H(X)}$ (resp. $\|x\|_{K(X)}$) is finite. Then $H(X)$ is an r.i. space equipped with the norm $\|\cdot\|_{H(X)}$. Moreover, $K(X)$ is an r.i. spaces *if the function* $t \mapsto -\log t$ *is in* $\widehat{X}$; otherwise $K(X)$ consists of the zero function only. Therefore we will assume that *the function* $t \mapsto -\log t$ *is in* $\widehat{X}$ *whenever we consider the space* $K(X)$. See [**5**] for details.

3. Results

Let $f = (f_n)_{n\in\mathbb{Z}_+}$ be a martingale with respect to a filtration $\mathcal{F} = (\mathcal{F}_n)_{n\in\mathbb{Z}_+}$. If f is uniformly integrable, then $f_n = \mathbb{E}[f_\infty \,|\, \mathcal{F}_n]$ a.s. for each $n \in \mathbb{Z}_+$, and moreover

$$\theta_{\mathcal{F}} f = \sup_{n\in\mathbb{Z}_+} \mathbb{E}\big[|f_\infty - f_{n-1}| \,\big|\, \mathcal{F}_n\big] \quad \text{a.s.}$$

Here and in what follows f_∞ denotes the almost sure limit of f.

Main Theorem. *Let* $(X, \|\cdot\|_X)$ *be a Banach function space over* Ω. *Then the following are equivalent:*

(i) *there are positive constants* c *and* C *such that if* $f = (f_n)_{n\in\mathbb{Z}_+}$ *is a martingale with respect to a filtration* $\mathcal{F} = (\mathcal{F}_n)_{n\in\mathbb{Z}_+}$, *then*

$$c\,\varlimsup_{n\to\infty} \|f_n\|_X \le \|\theta_{\mathcal{F}} f\|_X \le C \varliminf_{n\to\infty} \|f_n\|_X\,;$$

(ii) *there are positive constants* c *and* C *such that if* $f = (f_n)_{n\in\mathbb{Z}_+}$ *is a martingale with respect to a filtration* $\mathcal{F} = (\mathcal{F}_n)_{n\in\mathbb{Z}_+}$, *then*

$$c\,\varliminf_{n\to\infty} \|f_n\|_X \le \|\theta_{\mathcal{F}} f\|_X \le C \varlimsup_{n\to\infty} \|f_n\|_X\,; \tag{3}$$

(iii) *there are positive constants c and C such that if $f = (f_n)_{n\in\mathbb{Z}_+}$ is a uniformly integrable martingale with respect to a filtration $\mathcal{F} = (\mathcal{F}_n)_{n\in\mathbb{Z}_+}$, then*

$$c\,\|f_\infty\|_X \le \|\theta_{\mathcal{F}} f\|_X \le C\,\|f_\infty\|_X\,; \tag{4}$$

(iv) *there exists a norm $|||\cdot|||_X$ on X which is equivalent to $\|\cdot\|_X$ and with respect to which X is a rearrangement-invariant space such that $0 < \alpha_X \le \beta_X < 1$.*

Remarks. (a) Recall from [**5**, Remark 4.3] that if $f = (f_n)$ is a martingale, then $f_n \prec f_{n+1}$ for all $n \in \mathbb{Z}_+$. Hence if (iv) of the Main Theorem holds, then

$$|||f_n|||_X \le |||f_{n+1}|||_X \quad (n \in \mathbb{Z}_+).$$

Thus if the equivalent conditions of the Main Theorem hold, then

$$c \sup_{n\in\mathbb{Z}_+} |||f_n|||_X \le |||\theta_{\mathcal{F}} f|||_X \le C \sup_{n\in\mathbb{Z}_+} |||f_n|||_X$$

for any martingale $f = (f_n)$.

(b) Let $1 < p < \infty$ and let w be a strictly positive random variable. The weighted Lebesgue space $L_{p,w}$ consists of those random variables x for which $x^p w$ is integrable with respect to $\mathbb{P}$. If $w^{-1/(p-1)} \in L_1$, then $L_\infty \hookrightarrow L_{p,w} \hookrightarrow L_1$ and $L_{p,w}$ is a Banach function space (with respect to $\mathbb{P}$). In the case where $X = L_{p,w}$, the equivalent conditions of the Main Theorem hold if and only if there are strictly positive constants a and b such that $a \le w \le b$ a.s. There is a similar result for weighted Orlicz spaces (see [**4**, Section 4]).

As shown in the final section, the Main Theorem is a consequence of Propositions 1, 2, and 3 below.

Proposition 1. *Let $(X, \|\cdot\|_X)$ be a Banach function space over Ω. Suppose that there is a positive constant C such that for any $x \in X$ and for any sub-σ-algebra $\mathcal{G}$ of Σ,*

$$\big\|\mathbb{E}[x \,|\, \mathcal{G}]\big\|_X \le C\,\|x\|_X\,. \tag{5}$$

Then there exists a norm $|||\cdot|||_X$ on X which is equivalent to $\|\cdot\|_X$ and with respect to which X is a rearrangement-invariant space.

If the second inequality of (3) *holds for any martingale $f = (f_n)_{n\in\mathbb{Z}_+}$, then* (5) *holds for any $x \in X$ and any sub-σ-algebra $\mathcal{G}$ of Σ.*

Given a martingale $f = (f_n)_{n\in\mathbb{Z}_+}$, we denote by Mf the *maximal function* of f; $Mf = \sup_{n\in\mathbb{Z}_+} |f_n|$.

Proposition 2. *Let $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ be rearrangement-invariant spaces over Ω. Then the following are equivalent:*

(i) $\mathcal{P} \in B(\widehat{Y}, \widehat{X})$;

(ii) *there is a positive constant C such that if $f = (f_n)_{n\in\mathbb{Z}_+}$ is a martingale with respect to a filtration $\mathcal{F} = (\mathcal{F}_n)_{n\in\mathbb{Z}_+}$, then*

$$\|\theta_{\mathcal{F}} f\|_X \le C \sup_{n\in\mathbb{Z}_+} \|f_n\|_Y\,; \tag{6}$$

(iii) *there is a constant C such that if $f = (f_n)_{n\in\mathbb{Z}_+}$ is a martingale, then*

$$\|Mf\|_X \le C \sup_{n\in\mathbb{Z}_+} \|f_n\|_Y\,. \tag{7}$$

Moreover, if these equivalent conditions hold, then $Y \hookrightarrow H(X)$.

Corollary 1. *Let $(X, \|\cdot\|_X)$ be a rearrangement-invariant space over Ω. Then, for any martingale $f = (f_n)_{n\in\mathbb{Z}_+}$ with respect to a filtration $\mathcal{F} = (\mathcal{F}_n)_{n\in\mathbb{Z}_+}$,*

$$\|\theta_{\mathcal{F}} f\|_X \le 2 \sup_{n\in\mathbb{Z}_+} \|f_n\|_{H(X)} \quad \text{and} \quad \|Mf\|_X \le \sup_{n\in\mathbb{Z}_+} \|f_n\|_{H(X)}\,.$$

From Proposition 2 and Corollary 1, it follows that $H(X)$ is maximal among all r.i. spaces Y which satisfies the inequality of the form (6).

Recall that the Zygmund space $L(\log L)$ is an r.i. space equipped with the norm defined by

$$\|x\|_{L(\log L)} := \int_0^1 (1 - \log s) x^*(s)\, ds,$$

and recall also that $x \in L(\log L)$ if and only if $|x| \log\bigl(1 + |x|\bigr) \in L_1$. If $X = L_1$, then $H(X)$ coincides with $L(\log L)$ (see [**5**, Section 5]) and hence

$$\|\theta_{\mathcal{F}} f\|_1 \le 2 \sup_{n\in\mathbb{Z}_+} \|f_n\|_{L(\log L)}$$

for any martingale $f = (f_n)$ with respect to $\mathcal{F} = (\mathcal{F}_n)$.

From Proposition 2, we can derive an extension of the results of Antipa [**1**].

Corollary 2 (*cf.* [**1**])**.** *Let $(X, \|\cdot\|_X)$ be a Banach function space over Ω. The following are equivalent:*

(i) *there is a positive constant C such that if $f = (f_n)_{n\in\mathbb{Z}_+}$ is a martingale with respect to a filtration $\mathcal{F} = (\mathcal{F}_n)_{n\in\mathbb{Z}_+}$, then*

$$\|Mf\|_X \le C \varliminf_{n\to\infty} \|f_n\|_X\,; \tag{8}$$

(ii) *there exists a norm* $|\!|\!|\cdot|\!|\!|_X$ *on* X *which is equivalent to* $\|\cdot\|_X$ *and with respect to which* X *is a rearrangement-invariant space such that* $\beta_X < 1$.

Suppose that these equivalent conditions hold. Then for any martingale $f = (f_n)_{n\in\mathbb{Z}_+}$,

$$\|Mf\|_X \le C \sup_{n\in\mathbb{Z}_+} \|f_n\|_X \,. \tag{9}$$

Proposition 3. *Let* $(X, \|\cdot\|_X)$ *and* $(Y, \|\cdot\|_Y)$ *be as in Proposition 2.*

(i) *Assume that* $\mathcal{Q} \in B(\widehat{Y}, \widehat{X})$. *Then there is a positive constant* C *such that if* $f = (f_n)_{n\in\mathbb{Z}_+}$ *is a martingale with respect to a filtration* $\mathcal{F} = (\mathcal{F}_n)_{n\in\mathbb{Z}_+}$, *then*

$$\sup_{n\in\mathbb{Z}_+} \|f_n\|_X \le \|Mf\|_X \le C\,\|\theta_{\mathcal{F}} f\|_Y \,. \tag{10}$$

(ii) *Assume that the following two conditions are satisfied:*

(a) *there is a positive constant* C *such that if* $f = (f_n)_{n\in\mathbb{Z}_+}$ *is a martingale with respect to a filtration* $\mathcal{F} = (\mathcal{F}_n)_{n\in\mathbb{Z}_+}$, *then*

$$\sup_{n\in\mathbb{Z}_+} \|f_n\|_X \le C\,\|\theta_{\mathcal{F}} f\|_Y \,; \tag{11}$$

(b) $\beta_Y < 1$, *or equivalently* $\mathcal{P} \in B(\widehat{Y})$.

Then $\mathcal{Q} \in B(\widehat{Y}, \widehat{X})$ *and* $Y \hookrightarrow K(X)$.

Thus (11) *holds for any martingale* $f = (f_n)_{n\in\mathbb{Z}_+}$ *if and only if* $\mathcal{Q} \in B(\widehat{Y}, \widehat{X})$, *provided that* $\beta_Y < 1$.

Corollary 3. *Let* $(X, \|\cdot\|_X)$ *be a rearrangement-invariant space over* Ω. *Then, for any martingale* $f = (f_n)_{n\in\mathbb{Z}_+}$ *with respect to a filtration* $\mathcal{F} = (\mathcal{F}_n)_{n\in\mathbb{Z}_+}$,

$$\|Mf\|_X \le 16\,\|\theta_{\mathcal{F}} f\|_{K(X)} \,. \tag{12}$$

Given $a \in (0,\infty)$, we denote by $L_{\exp:a}$ the r.i. space consisting of those random variables x for which

$$\|x\|_{\exp:a} := \sup_{t\in I} \frac{1}{t(1-\log t)^{1/a}} \int_0^t x^*(s)\,ds < \infty.$$

Then $x \in L_{\exp:a}$ if and only if $\exp\bigl(\lambda|x|^a\bigr) \in L_1$ for some $\lambda > 0$. It is not difficult to verify that $K(L_{\exp:1}) \approx L_\infty$ and that $K(L_{\exp:a}) \approx L_{\exp:\frac{a}{1-a}}$ for $a \in (0,1)$ (see [**5**, Section 5]). Hence it follows from (12) that

$$\|Mf\|_{\exp:a} \le C_a\,\|\theta_{\mathcal{F}} f\|_{\exp:\frac{a}{1-a}} \qquad (0 < a < 1);$$
$$\|Mf\|_{\exp:1} \le C_1\,\|\theta_{\mathcal{F}} f\|_\infty \,.$$

4. Proof of Proposition 1

In order to prove Proposition 1, we need the following lemmas.

Lemma 1. *Let $(X, \|\cdot\|_X)$ be a Banach function space over Ω, and let S_+ be the collection of all nonnegative simple random variables on Ω. The following are equivalent:*

(i) *there is a constant $c > 0$ such that if $x, y \in S_+$, $x \simeq_d y$, and $x \wedge y = 0$ a.s., then $\|y\|_X \leq c\,\|x\|_X$;*

(ii) *there is a constant $c > 0$ such that if $x, y \in X$, $x \simeq_d y$, and $|x| \wedge |y| = 0$ a.s., then $\|y\|_X \leq c\,\|x\|_X$;*

(iii) *there is a constant $c > 0$ such that if $x, y \in X$ and $x \simeq_d y$, then $\|y\|_X \leq c\,\|x\|_X$;*

(iv) *there exists a norm $|||\cdot|||_X$ on X which is equivalent to $\|\cdot\|_X$ and with respect to which X is an r.i. space.*

Proof: A complete proof can be found in [**6**]: for convenience, we sketch the proof.

(iii) $\Rightarrow$ (i). Obvious.

(i) $\Rightarrow$ (ii). Suppose that x and y are random variables in X such that $x \simeq_d y$ and $|x| \wedge |y| = 0$ a.s. Then there exist sequences $\{x_n\}$ and $\{y_n\}$ in S_+ such that $x_n \simeq_d y_n$ for all n, and such that $x_n \uparrow |x|$ and $y_n \uparrow |y|$. If (i) holds, then $\|y_n\|_X \leq c\,\|x_n\|_X$ for all n. This together with (B3) implies that $\|y\|_X \leq c\,\|x\|_X$.

(ii) $\Rightarrow$ (iii). Suppose that (ii) holds. We want to show that

$$\sup\bigl\{\|y\|_X \bigm| x, y \in X,\ x \simeq_d y,\ \|x\|_X \leq 1\bigr\} < \infty.$$

Assume to the contrary that this supremum is infinite. Let d be a constant such that $\|z\|_1 \leq d\,\|z\|_X$ for all $z \in X$, let $m = (2c^2) \vee (3d)$, and let $\alpha = \|\mathbf{1}\|_X$, where $\mathbf{1}$ denotes the constant function (on Ω) taking the value one. By assumption, we can find random variables x and y in X such that

$$x \simeq_d y, \quad \|x\|_X \leq 1, \quad \text{and} \quad (\alpha+1)m \leq \|y\|_X\,.$$

Let $x_1 = |x|1_{\{|x|>m\}}$ and $y_1 = |y|1_{\{|y|>m\}}$. Then $x_1 \simeq_d y_1$ and

$$\mathbb{P}(x_1 \neq 0) = \mathbb{P}(y_1 \neq 0) \leq \frac{1}{3}.$$

Hence there is a random variable z such that $x_1 \simeq_d z$ and $\{z \neq 0\} \subset \{x_1 = 0,\, y_1 = 0\}$ (*cf.* [**3**, p. 44]). Using (ii), we have that $\|z\|_X \leq c\,\|x_1\|_X \leq c$ and that $\|y_1\|_X \leq c\,\|z\|_X$. Thus $\|y_1\|_X \leq c^2$. On the other hand, since $(\alpha+1)m \le \|y\|_X \le \|y_1 + m\|_X \le \|y_1\|_X + \alpha m$, we see that $m \le \|y_1\|_X$. As a result, $m \le c^2$, which contradicts the definition of m.

Thus (i), (ii), and (iii) are equivalent. Moreover it is clear that (iv) implies (iii), and hence it only remains to show that (iii) implies (iv).

Suppose that (iii) holds. If we set

$$|||x|||_X = \sup\left\{ \int_0^1 x^*(s)\, z^*(s)\, ds \;\middle|\; z \in X',\ \|z\|_{X'} \le 1 \right\}$$

for each $x \in X$, then the functional $|||\cdot|||_X$ is a norm on X satisfying the conditions described in (iv). □

Proof of Proposition 1: Suppose that (5) holds for any $x \in X$ and for any sub-σ-algebra $\mathcal{G}$ of Σ. It suffices to show that (i) of Lemma 1 holds. Suppose that $x, y \in S_+$, $x \simeq_d y$, and $x \wedge y = 0$ a.s. Then we can write

$$x = \sum_{j=1}^{\ell} \alpha_j 1_{A_j} \quad \text{and} \quad y = \sum_{j=1}^{\ell} \alpha_j 1_{B_j},$$

where $\alpha_j > 0$, $j \in \{1, 2, \ldots, \ell\}$, and where $\{A_j\}_{j=1}^{\ell}$ and $\{B_j\}_{j=1}^{\ell}$ are pairwise disjoint sequences of sets in Σ such that

$$\left(\bigcup_{j=1}^{\ell} A_j\right) \cap \left(\bigcup_{j=1}^{\ell} B_j\right) = \emptyset$$

and

$$\mathbb{P}(A_j) = \mathbb{P}(B_j) > 0, \quad j \in \{1, 2, \ldots, \ell\}.$$

Let $\Lambda_j = A_j \cup B_j$ for each $j \in \{1, 2, \ldots, \ell\}$, let $\Gamma = \bigcup_{j=1}^{\ell} \Lambda_j$, and let

$$\mathcal{G} = \sigma\big(\{\Lambda_1, \Lambda_2, \ldots, \Lambda_\ell\} \cup \{\Lambda \setminus \Gamma \mid \Lambda \in \Sigma\}\big).$$

Then

$$\mathbb{E}[x \mid \mathcal{G}] = \sum_{j=1}^{\ell} \frac{1_{\Lambda_j}}{\mathbb{P}(\Lambda_j)} \int_{\Lambda_j} x\, d\mathbb{P} = \frac{1}{2}(x+y) \ge \frac{y}{2}.$$

Hence by (5) we have that

$$\|y\|_X \le 2\big\|\mathbb{E}[x \mid \mathcal{G}]\big\|_X \le 2C\, \|x\|_X,$$

as was to be shown.

To prove the last statement, let $x \in X$ and let $\mathcal{G}$ be a sub-σ-algebra of Σ. Consider the martingale $f = (f_n)$ given by

$$f_n = \mathbb{E}[x \mid \mathcal{F}_n] \quad \text{a.s.,} \quad \text{where } \mathcal{F}_n = \begin{cases} \mathcal{G} & \text{if } n = 0, \\ \Sigma & \text{if } n \ge 1, \end{cases} \quad (n \in \mathbb{Z}_+).$$

Since $\left|\mathbb{E}[x \,|\, \mathcal{G}]\right| \le \mathbb{E}\big[|x| \,\big|\, \mathcal{G}\big] \le \theta_{\mathcal{F}} f$ a.s., we may apply the second inequality of (3) to obtain (5). □

5. Proof of Proposition 2

We begin with some preliminary lemmas.

Lemma 2. *Let $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ be as in Proposition 2. If* (6) *holds for any martingale $f = (f_n)$ with respect to $\mathcal{F} = (\mathcal{F}_n)$, then $Y \hookrightarrow X$, or equivalently $\widehat{Y} \hookrightarrow \widehat{X}$. The same conclusion holds if* (11) *holds for any martingale $f = (f_n)$ with respect to $\mathcal{F} = (\mathcal{F}_n)$.*

Proof: Let $\mathcal{F} = (\mathcal{F}_n)_{n \in \mathbb{Z}_+}$ be the filtration defined by

$$\mathcal{F}_n = \begin{cases} \{\emptyset, \Omega\} & \text{if } n = 0, \\ \Sigma & \text{if } n \ge 1. \end{cases}$$

Given $x \in Y$, define a martingale $f = (f_n)$ by $f_n = \mathbb{E}[x \,|\, \mathcal{F}_n]$ a.s. for each $n \in \mathbb{Z}_+$ Then, since $f_0 \prec f_n = x$ for all $n \ge 1$, we have $\sup_n \|f_n\|_X = \|x\|_X$ and $\sup_n \|f_n\|_Y = \|x\|_Y$. Moreover,

$$(13) \qquad |x| \le \left|x - \mathbb{E}[x]\right| + \|x\|_1 \le \theta_{\mathcal{F}} f + \|x\|_1 \le \theta_{\mathcal{F}} f + d\,\|x\|_Y \quad \text{a.s.},$$

and

$$(14) \qquad \theta_{\mathcal{F}} f = \left|x - \mathbb{E}[x]\right| \vee \mathbb{E}\big[|x|\big] \le |x| + \|x\|_1 \le |x| + d\,\|x\|_Y \qquad \text{a.s.},$$

where d is a positive constant which is independent of x. If $f = (f_n)$ satisfies (6), then by (13)

$$\begin{aligned} \|x\|_X &\le \|\theta_{\mathcal{F}} f\|_X + d\,\|x\|_Y\,\|\mathbf{1}\|_X \\ &\le C \sup_n \|f_n\|_Y + d\,\|x\|_Y\,\|\mathbf{1}\|_X = (C + d\,\|\mathbf{1}\|_X)\,\|x\|_Y\,. \end{aligned}$$

Furthermore, if $f = (f_n)$ satisfies (11), then by (14)

$$\|x\|_X = \sup_n \|f_n\|_X \le C\,\|\theta_{\mathcal{F}} f\|_Y \le C(1 + d\,\|\mathbf{1}\|_Y)\,\|x\|_Y\,.$$

In any case, we conclude that $Y \hookrightarrow X$.

In order to prove that $\widehat{Y} \hookrightarrow \widehat{X}$, let $\phi \in \widehat{Y}$. Then there is an $x \in Y$ such that $x^* = \phi^*$ (see [**3**, p. 44]). Since $Y \hookrightarrow X$,

$$\|\phi\|_{\widehat{X}} = \|x\|_X \le c\,\|x\|_Y = c\,\|\phi\|_{\widehat{Y}}\,.$$

Thus $\widehat{Y} \hookrightarrow \widehat{X}$, as desired. □

Before stating the next lemma, we introduce the following notation.

Notation. If Z is a Banach function space over I, then $\mathcal{D}(Z)$ denotes the collection of functions in Z that are *nonnegative, nonincreasing*, and *right-continuous.*

Lemma 3. *Let $(Z_1, \|\cdot\|_{Z_1})$ and $(Z_2, \|\cdot\|_{Z_2})$ be r.i. spaces over I.*

(i) *If there is a positive constant c such that $\|\mathcal{P}\phi\|_{Z_2} \le c\,\|\phi\|_{Z_1}$ for all $\phi \in \mathcal{D}(Z_1)$, then $\mathcal{P} \in B(Z_1, Z_2)$.*
(ii) *If there is a positive constant c such that $\|\mathcal{Q}\phi\|_{Z_2} \le c\,\|\phi\|_{Z_1}$ for all $\phi \in \mathcal{D}(Z_1)$, then $\mathcal{Q} \in B(Z_1, Z_2)$.*

Proof: (i) Suppose that $\|\mathcal{P}\phi\|_{Z_2} \le c\,\|\phi\|_{Z_1}$ for all $\phi \in \mathcal{D}(Z_1)$. We know that $|\mathcal{P}\psi| \le \mathcal{P}\psi^*$ on I for any $\psi \in L_1(I)$ (see [**2**, Lemma 2.1, p. 44]). Therefore, if $\psi \in Z_1$, then $\psi^* \in \mathcal{D}(Z_1)$ and

$$\|\mathcal{P}\psi\|_{Z_2} \le \|\mathcal{P}\psi^*\|_{Z_2} \le c\,\|\psi^*\|_{Z_1} = c\,\|\psi\|_{Z_1}\,.$$

Thus $\mathcal{P} \in B(Z_1, Z_2)$.

(ii) Suppose that $\|\mathcal{Q}\phi\|_{Z_2} \le c\,\|\phi\|_{Z_1}$ for all $\phi \in \mathcal{D}(Z_1)$. We now use the fact that $|\mathcal{Q}\psi| \le \mathcal{Q}|\psi| \prec \mathcal{Q}\psi^*$ for any $\psi \in L_1(I)$ (see the proof of Lemma 3 of [**4**]). Therefore if $\psi \in Z_1$, then

$$\|\mathcal{Q}\psi\|_{Z_2} \le \|\mathcal{Q}\psi^*\|_{Z_2} \le c\,\|\psi^*\|_{Z_1} = c\,\|\psi\|_{Z_1}\,.$$

Thus $\mathcal{Q} \in B(Z_1, Z_2)$. □

In what follows ξ stands for a random variable such that

$$\xi^*(t) = 1 - t \quad \text{for all } t \in I. \tag{15}$$

Such a random variable surely exists, since Ω is nonatomic (see [**3**, p. 44]). It is easy to prove the following lemma.

Lemma 4. *Let $\phi \in L_1(I)$ and let $x = \phi(1-\xi)$, where ξ is a random variable satisfying* (15)*. Define a family of sets $\{A(t) \in \Sigma \,|\, t \in [0,1]\}$ by setting*

$$A(t) = \{\omega \in \Omega \,|\, \xi(\omega) > 1 - t\} \quad \textit{for each } t \in [0,1].$$

Then:

(i) *$x^*(t) = \phi^*(t)$ for all $t \in I$;*
(ii) *$A(s) \subset A(t)$ whenever $0 \le s \le t \le 1$;*
(iii) *$\mathbb{P}(A(t)) = t$ for all $t \in [0,1]$;*
(iv) *$\int_{A(t)} x\,d\mathbb{P} = \int_0^t \phi(s)\,ds$ for all $t \in [0,1]$.*

Proof of Proposition 2: We prove the following chain of implications: (iii) $\Rightarrow$ (ii) $\Rightarrow$ (i) $\Rightarrow$ (iii).

Assume that (iii) holds, and that $f = (f_n)_{n\in\mathbb{Z}_+}$ is a martingale with respect to $\mathcal{F} = (\mathcal{F}_n)_{n\in\mathbb{Z}_+}$. For fixed $k \in \mathbb{Z}_+$, we define a martingale $g = (g_n)$ by $g_n = \mathbb{E}\big[|f_k| \,\big|\, \mathcal{F}_n\big]$ a.s. for each $n \in \mathbb{Z}_+$. If we denote by $f^{(k)}$ the stopped martingale $(f_{n\wedge k})_{n\in\mathbb{Z}_+}$, then

$$\theta_{\mathcal{F}} f^{(k)} = \sup_{0\le n\le k} \mathbb{E}\big[|f_k - f_{n-1}| \,\big|\, \mathcal{F}_n\big] \le 2Mg \quad \text{a.s.} \tag{16}$$

Because the maximal inequality (7) applies to $g = (g_n)$, we see that

$$\big\|\theta_{\mathcal{F}} f^{(k)}\big\|_X \le 2\,\|Mg\|_X \le 2C \sup_{n\in\mathbb{Z}_+} \|g_n\|_Y = 2C\,\|f_k\|_Y\,.$$

Since $\theta_{\mathcal{F}} f^{(k)} \uparrow \theta_{\mathcal{F}} f$ as $k \uparrow \infty$, we conclude that $f = (f_n)$ satisfies (6) with C replaced by $2C$. Thus (iii) implies (ii).

Assume now that (ii) holds. To prove (i), it suffices by Lemma 3 to prove that $\|\mathcal{P}\phi\|_{\widehat{X}} \le k\,\|\phi\|_{\widehat{Y}}$ for all $\phi \in \mathcal{D}(\widehat{Y})$ with a constant $k > 0$ independent of ϕ.

Suppose that $0 \not\equiv \phi \in \mathcal{D}(\widehat{Y})$. For convenience, we set $(\mathcal{P}\phi)(0) = \lim_{t\downarrow 0}(\mathcal{P}\phi)(t)$; thus $(\mathcal{P}\phi)(0) = \|\phi\|_\infty$ if $\phi \in L_\infty$, and $(\mathcal{P}\phi)(0) = \infty$ otherwise. Bearing this in mind, we define a nonincreasing sequence $\{t_n\}_{n\in\mathbb{Z}_+}$ in I by setting

$$t_0 = 1 \quad\text{and}\quad t_n = \inf\big\{s \in [\,0,\,1\,] \,\big|\, (\mathcal{P}\phi)(s) \le 2(\mathcal{P}\phi)(t_{n-1})\big\} \quad (n \ge 1).$$

Then, for each $n \ge 1$,

$$(\mathcal{P}\phi)(t_n) \le 2(\mathcal{P}\phi)(t_{n-1}). \tag{17}$$

More precisely, equality holds in (17) whenever $2^n(\mathcal{P}\phi)(1) \le (\mathcal{P}\phi)(0)$. In particular, if $\phi \notin L_\infty$, then equality holds for all $n \ge 1$. In any case $t_n \downarrow 0$ as $n \uparrow \infty$.

Define a random variable x and a family of sets $\big\{A(t) \in \Sigma \,\big|\, t \in [0,1]\big\}$ as in Lemma 4, and define a filtration $\mathcal{F} = (\mathcal{F}_n)_{n\in\mathbb{Z}_+}$ by setting

$$\mathcal{F}_n = \sigma\big\{\Lambda \setminus A(t_n) \,\big|\, \Lambda \in \Sigma\big\} \quad (n \in \mathbb{Z}_+).$$

We consider the martingale $f = (f_n)$ defined by $f_n = \mathbb{E}[x \,|\, \mathcal{F}_n]$ a.s. for each $n \in \mathbb{Z}_+$. If $t_n > 0$, then

$$f_n = \frac{1_{A(t_n)}}{\mathbb{P}\big(A(t_n)\big)} \int_{A(t_n)} x\,d\mathbb{P} + x 1_{\Omega\setminus A(t_n)} \quad \text{a.s.},$$

and hence by Lemma 4

$$f_n = (\mathcal{P}\phi)(t_n) 1_{A(t_n)} + x\,1_{\Omega\setminus A(t_n)} \quad \text{a.s.} \tag{18}$$

If $t_n = 0$, then (18) can be written as $f_n = x$ a.s. Since $f_\infty := \lim_n f_n = x$,

$$f_\infty - f_{n-1} = \big\{x - (\mathcal{P}\phi)(t_{n-1})\big\} 1_{A(t_{n-1})} \quad \text{a.s.} \quad (n \geq 1).$$

Hence by (17)

$$\begin{aligned} \mathbb{E}\big[|f_\infty - f_{n-1}| \,\big|\, \mathcal{F}_n\big] 1_{A(t_{n-1})\setminus A(t_n)} & \\ &= \big|x - (\mathcal{P}\phi)(t_{n-1})\big|\, 1_{A(t_{n-1})\setminus A(t_n)} \\ &\geq (\mathcal{P}\phi)(t_{n-1})\, 1_{A(t_{n-1})\setminus A(t_n)} - |x|\, 1_{A(t_{n-1})\setminus A(t_n)} \\ &\geq \frac{1}{2}(\mathcal{P}\phi)(t_n)\, 1_{A(t_{n-1})\setminus A(t_n)} - |x|\, 1_{A(t_{n-1})\setminus A(t_n)} \quad \text{a.s.,} \end{aligned}$$

from which it follows that

$$\sum_{n=1}^{\infty} (\mathcal{P}\phi)(t_n)\, 1_{A(t_n)\setminus A(t_{n-1})} \leq 2\big(\theta_{\mathcal{F}} f + |x|\big) \quad \text{a.s.}$$

Let us write η for the sum on the left-hand side. (Notice that if $\phi \in L_\infty$, then η is a finite sum.) It is easy to see that

$$\eta^*(t) = \sum_{n=1}^{\infty} (\mathcal{P}\phi)(t_n) 1_{[t_n,\, t_{n-1})}(t) \quad (t \in I).$$

(Here the indicator function $1_{[t_n,\, t_{n-1})}$ should be replaced by $1_{(t_n,\, t_{n-1})}$ if $0 = t_n < t_{n-1}$.) It then follows that

$$(\mathcal{P}\phi)(t) \leq \eta^*(t) \leq 2\big(\theta_{\mathcal{F}} f + |x|\big)^*(t) \quad (t \in I).$$

Therefore

$$\begin{aligned} \|\mathcal{P}\phi\|_{\widehat{X}} &\leq 2\,\big\|\big(\theta_{\mathcal{F}} f + |x|\big)^*\big\|_{\widehat{X}} \\ &= 2\,\big\|\theta_{\mathcal{F}} f + |x|\big\|_X \\ &\leq 2\,\|\theta_{\mathcal{F}} f\|_X + 2\,\|x\|_X\,. \end{aligned} \tag{19}$$

According to Lemma 2, there is a constant $d > 0$ such that $\|\cdot\|_X \leq d\,\|\cdot\|_Y$. Hence by (6) and (19)

$$\begin{aligned} \|\mathcal{P}\phi\|_{\widehat{X}} &\leq 2C \sup_{n\in\mathbb{Z}_+} \|f_n\|_Y + 2d\,\|x\|_Y \\ &= 2C \sup_{n\in\mathbb{Z}_+} \|f_n\|_Y + 2d\,\|\phi\|_{\widehat{Y}}\,, \end{aligned} \tag{20}$$

where the last equality follows from the fact that $x^* = \phi$.

Now let us estimate the norm of f_n in Y. If $t_n = 0$, then $f_n = x$ a.s. and hence $\|f_n\|_Y = \|x\|_Y = \|\phi\|_{\widehat{Y}}$. So let us assume that $t_n > 0$. Using Hölder's inequality and (2), we see that

$$(\mathcal{P}\phi)(t_n) = \frac{1}{t_n}\int_{A(t_n)} x\, d\mathbb{P} \le \frac{\|x\|_Y\, \varphi_{Y'}(t_n)}{t_n} = \frac{\|x\|_Y}{\varphi_Y(t_n)}.$$

Therefore it follows from (18) that

$$\|f_n\|_Y \le \frac{\|x\|_Y}{\varphi_Y(t_n)} \cdot \left\|1_{A(t_n)}\right\|_Y + \left\|x\, 1_{\Omega\setminus A(t_n)}\right\|_Y \le 2\,\|x\|_Y\,.$$

Thus

$$\sup_{n\in\mathbb{Z}_+} \|f_n\|_Y \le 2\,\|x\|_Y = 2\,\|\phi\|_{\widehat{Y}}\,.$$

Combining this with (20), we have the estimate

$$\|\mathcal{P}\phi\|_{\widehat{X}} \le 2(2C+d)\,\|\phi\|_{\widehat{Y}}\,,$$

and thus $\mathcal{P} \in B(\widehat{Y}, \widehat{X})$.

We now show that (i) implies (iii). Let $f = (f_n)_{n\in\mathbb{Z}_+}$ be a martingale. Then, for each $k \in \mathbb{Z}_+$

$$(M_k f)^*(t) \le \big(\mathcal{P} f_k^*\big)(t) \quad (t \in I),$$

where $M_k f = \sup_{0\le n\le k} |f_n|$ (see the proof of Proposition 3 of [**4**] or the proof of Theorem 4.1 of [**5**]). Since $\mathcal{P} \in B(\widehat{Y}, \widehat{X})$ by assumption,

$$\begin{aligned}\|M_k f\|_X &= \|(M_k f)^*\|_{\widehat{X}} \le \|\mathcal{P} f_k^*\|_{\widehat{X}} \\ &\le C\,\|f_k^*\|_{\widehat{Y}} = C\,\|f_k\|_Y \le C \sup_{n\in\mathbb{Z}_+} \|f_n\|_Y\,,\end{aligned}$$

where $C = \|\mathcal{P}\|_{B(\widehat{Y},\widehat{X})}$. By letting $k \to \infty$ we obtain (7), as desired.

Finally, we need to prove the last statement of Proposition 2. Let $\mathcal{P} \in B(\widehat{Y}, \widehat{X})$ and $C = \|\mathcal{P}\|_{B(\widehat{Y},\widehat{X})}$. Then, for any $x \in Y$,

$$\|x\|_{H(X)} = \|\mathcal{P}x^*\|_{\widehat{X}} \le C\,\|x^*\|_{\widehat{Y}} = C\,\|x\|_Y\,,$$

which shows that $Y \hookrightarrow H(X)$. □

Proof of Corollary 1: From the proof of Proposition 2, we already know that if $\mathcal{P} \in B(\widehat{Y}, \widehat{X})$, then (7) holds with $C = \|\mathcal{P}\|_{B(\widehat{Y},\widehat{X})}$. We also know that if (7) holds with a constant C, then (6) holds with C replaced by $2C$. Therefore, to prove the corollary, it suffices to show that $\mathcal{P} \in B\big(\widehat{H(X)}, \widehat{X}\big)$ and $\|\mathcal{P}\|_{B(\widehat{H(X)},\widehat{X})} \le 1$. Here $\widehat{H(X)}$ denotes the Luxemburg representation of $H(X)$. Suppose $\phi \in \widehat{H(X)}$. Then there is a random variable x such that $x^* = \phi^*$ (see [**3**, p. 44]). Since $|\mathcal{P}\phi| \le \mathcal{P}\phi^* = \mathcal{P}x^*$ on I,

$$\|\mathcal{P}\phi\|_{\widehat{X}} \le \|\mathcal{P}x^*\|_{\widehat{X}} = \|x\|_{H(X)} = \|\phi\|_{\widehat{H(X)}}.$$

This completes the proof. □

Proof of Corollary 2: (i) ⇒ (ii). Let $f = (f_n)_{n\in\mathbb{Z}_+}$ be a martingale with respect to $\mathcal{F} = (\mathcal{F}_n)_{n\in\mathbb{Z}_+}$. For fixed $k \in \mathbb{Z}_+$, we define a martingale $g = (g_n)_{n\in\mathbb{Z}_+}$ as in the proof of Proposition 2. Suppose that (i) holds. Then,

$$\left\|\theta_{\mathcal{F}} f^{(k)}\right\|_X \le 2\,\|Mg\|_X \le 2C \varliminf_{n\to\infty} \|g_n\|_X = 2C\,\|f_k\|_X\,,$$

where the first inequality follows from (16) and the second inequality follows from (8) applied to $g = (g_n)$. Letting $k \to \infty$, we see that

$$\|\theta_{\mathcal{F}} f\|_X \le 2C \varlimsup_{k\to\infty} \|f_k\|_X\,. \tag{21}$$

Hence, by Proposition 1, there is a norm $|||\cdot|||_X$ on X which is equivalent to the original norm of X and with respect to which X is an r.i. space. Since $f_n \prec f_{n+1}$ for all $n \in \mathbb{Z}_+$, inequality (21) can be rewritten as

$$|||\theta_{\mathcal{F}} f|||_X \le C' \sup_n |||f_n|||_X\,.$$

It then follows from Proposition 2 that $\mathcal{P} \in B(\widehat{X})$, or equivalently that $\beta_X < 1$. Thus (i) implies (ii).

(ii) ⇒ (i). Assume that (ii) holds. Then $\mathcal{P} \in B(\widehat{X})$, since $\beta_X < 1$. Hence Proposition 2 implies that, for any martingale $f = (f_n)$

$$|||Mf|||_X \le C' \sup_n |||f_n|||_X = C' \lim_{n\to\infty} |||f_n|||_X \tag{22}$$

with a positive constant C', independent of f. Since the norms $\|\cdot\|_X$ and $|||\cdot|||_X$ are equivalent, (22) can be rewritten as (8). Thus (ii) implies (i). Moreover, since (22) can also be rewritten as (9), the last statement follows. □

6. Proof of Proposition 3

In order to prove Proposition 3, we will use the fact that if $f = (f_n)$ is a martingale with respect to $\mathcal{F} = (\mathcal{F}_n)$, then

$$\mathbb{E}[Mf] \le 16\,\mathbb{E}[\theta_{\mathcal{F}} f]. \tag{23}$$

A more general inequality was established by R. L. Long in [**7**]. For an elementary proof of (23) (for uniformly integrable martingales), (see [**5**, Appendix A]).

Proof of Proposition 3: (i) Let $f = (f_n)_{n\in\mathbb{Z}_+}$ be a martingale with respect to $\mathcal{F} = (\mathcal{F}_n)_{n\in\mathbb{Z}_+}$ such that $\theta_{\mathcal{F}} f \in Y$. To prove the second inequality of (10), it suffices to show that

$$(Mf)^* \prec 16\,\mathcal{Q}(\theta_{\mathcal{F}} f)^*. \tag{24}$$

Indeed, since $\mathcal{Q} \in B(\widehat{Y}, \widehat{X})$ by assumption, estimate (24) implies that

$$\begin{aligned}\|Mf\|_X = \|(Mf)^*\|_{\widehat{X}} &\le 16\,\|\mathcal{Q}(\theta_{\mathcal{F}} f)^*\|_{\widehat{X}} \\ &\le 16\,\|\mathcal{Q}\|_{B(\widehat{Y},\widehat{X})}\,\|(\theta_{\mathcal{F}} f)^*\|_{\widehat{Y}} \\ &= 16\,\|\mathcal{Q}\|_{B(\widehat{Y},\widehat{X})}\,\|\theta_{\mathcal{F}} f\|_Y\,.\end{aligned}$$

If we can show that

$$\mathbb{E}[Mf - M_{k-1}f \,|\, \mathcal{F}_k] \le 16\,\mathbb{E}[\theta_{\mathcal{F}} f \,|\, \mathcal{F}_k] \quad \text{a.s.} \quad (k \in \mathbb{Z}_+), \tag{25}$$

then (24) will follow from Theorem 3.3 of [**5**] (or Lemma 4 of [**4**]). Notice that (25) is the conditional form of (23). To prove (25), we can use a standard way to derive the conditional form from a inequality for processes. Fix $k \in \mathbb{Z}_+$ and let $A \in \mathcal{F}_k$. We define a filtration $\mathcal{F}' = (\mathcal{F}'_n)$ and a process $f' = (f'_n)$ by setting

$$\mathcal{F}'_n = \mathcal{F}_{k+n} \quad \text{and} \quad f'_n = (f_{k+n} - f_{k-1})\,1_A \quad (n \in \mathbb{Z}_+).$$

Then $f' = (f'_n)$ is a martingale with respect to $\mathcal{F}' = (\mathcal{F}'_n)$. Since

$$Mf \le (M_{k-1}f) + \sup_{n\in\mathbb{Z}_+} |f_{k+n} - f_{k-1}|,$$

we see that

$$(Mf - M_{k-1}f)\,1_A \le \sup_{n\in\mathbb{Z}_+} |f_{k+n} - f_{k-1}|\,1_A = Mf'.$$

On the other hand,

$$\theta_{\mathcal{F}'} f' = \sup_{k\le n\le m<\infty} \mathbb{E}\big[|f_m - f_{n-1}| \,\big|\, \mathcal{F}_n\big]\,1_A \le (\theta_{\mathcal{F}} f)\,1_A.$$

Applying (23) to the martingale f' (with respect to $\mathcal{F}'$), we conclude that

$$\mathbb{E}\big[(Mf - M_{k-1}f)\,1_A\big] \le 16\,\mathbb{E}\big[(\theta_{\mathcal{F}} f)\,1_A\big] \quad (A \in \mathcal{F}_k),$$

which implies (25).

(ii) Suppose that (a) and (b) in (ii) hold. We want to show that $\|\mathcal{Q}\psi\|_{\widehat{X}} \le K\,\|\psi\|_{\widehat{Y}}$ for all $\psi \in \widehat{Y}$, with a positive constant K independent of ψ. According to Lemma 3, we may assume that $\psi \in \mathcal{D}(\widehat{Y})$ and $\psi \not\equiv 0$. Given $\varepsilon > 0$, we can find a sequence $\{t_n\}_{n\in\mathbb{Z}_+}$ in I such that

$$t_0 = 1 \quad \text{and} \quad (\mathcal{Q}\psi)(t_n) = (\mathcal{Q}\psi)(t_{n-1}) + \varepsilon \quad (n = 1, 2, \ldots), \tag{26}$$

since $\lim_{t\downarrow 0}(\mathcal{Q}\psi)(t) = \infty$. It is clear that $t_n > 0$ for all n and $t_n \downarrow 0$.

Let $\phi = \mathcal{Q}\psi - \psi$, and define x and $\{A(t) \,|\, t \in I\}$ as in Lemma 4. Define a filtration $\mathcal{F} = (\mathcal{F}_n)$ and a martingale $f = (f_n)$ by setting

$$\mathcal{F}_n = \sigma\big\{\Lambda \setminus A(t_n) \,\big|\, \Lambda \in \Sigma\big\} \quad \text{and} \quad f_n = \mathbb{E}[x \,|\, \mathcal{F}_n] \quad \text{a.s.} \quad (n \in \mathbb{Z}_+).$$

Because $\mathcal{P}\phi = \mathcal{P}(\mathcal{Q}\psi) - \mathcal{P}\psi = \mathcal{Q}\psi$,

$$f_n = (\mathcal{Q}\psi)(t_n) 1_{A(t_n)} + x\, 1_{\Omega \setminus A(t_n)} \quad \text{a.s.} \quad (n \in \mathbb{Z}_+).$$

Hence for each $n \geq 1$,

$$f_\infty - f_{n-1} = \big\{x - (\mathcal{Q}\psi)(t_{n-1})\big\} 1_{A(t_{n-1})} \quad \text{a.s.,}$$

and thus

$$\begin{aligned} &\mathbb{E}\big[|f_\infty - f_{n-1}| \,\big|\, \mathcal{F}_n\big] \\ &= \frac{1_{A(t_n)}}{t_n} \int_{A(t_n)} \big|x - (\mathcal{Q}\psi)(t_{n-1})\big| \, d\mathbb{P} + \big|x - (\mathcal{Q}\psi)(t_{n-1})\big| \, 1_{A(t_{n-1}) \setminus A(t_n)} \\ &\equiv E_1^{(n)} + E_2^{(n)} \quad \text{a.s.} \end{aligned}$$

To estimate $E_1^{(n)}$, note that $(\mathcal{Q}\psi)(1-\xi) \geq (\mathcal{Q}\psi)(t_n) \geq (\mathcal{Q}\psi)(t_{n-1})$ on the set $A(t_n) = \{1 - \xi < t_n\}$. Then we see that

$$\begin{aligned} &\int_{A(t_n)} \big|x - (\mathcal{Q}\psi)(t_{n-1})\big| \, d\mathbb{P} \\ &\qquad = \int_{A(t_n)} \big|(\mathcal{Q}\psi)(1-\xi) - \psi(1-\xi) - (\mathcal{Q}\psi)(t_{n-1})\big| \, d\mathbb{P} \\ &\qquad \leq \int_{\{1-\xi<t_n\}} \Big\{(\mathcal{Q}\psi)(1-\xi) - (\mathcal{Q}\psi)(t_{n-1})\Big\} \, d\mathbb{P} \\ &\qquad\qquad\qquad\qquad + \int_{\{1-\xi<t_n\}} \psi(1-\xi) \, d\mathbb{P} \\ &\qquad = \int_0^{t_n} (\mathcal{Q}\psi)(s) \, ds + \int_0^{t_n} \psi(s) \, ds - t_n (\mathcal{Q}\psi)(t_{n-1}). \end{aligned}$$

Hence by (26)

$$E_1^{(n)} \le \frac{1_{A(t_n)}}{t_n}\left\{\int_0^{t_n}(\mathcal{Q}\psi)(s)\,ds + \int_0^{t_n}\psi(s)\,ds - t_n(\mathcal{Q}\psi)(t_{n-1})\right\}$$
$$= \Big\{\big(\mathcal{P}(\mathcal{Q}\psi)\big)(t_n) + (\mathcal{P}\psi)(t_n) - (\mathcal{Q}\psi)(t_{n-1})\Big\}1_{A(t_n)}$$
$$= \big\{2(\mathcal{P}\psi)(t_n) + (\mathcal{Q}\psi)(t_n) - (\mathcal{Q}\psi)(t_{n-1})\big\}1_{A(t_n)}$$
$$= \big\{2(\mathcal{P}\psi)(t_n) + \varepsilon\big\}1_{A(t_n)}.$$

To estimate $E_2^{(n)}$, observe that on the set $A(t_{n-1}) \setminus A(t_n)$,

$$x - (\mathcal{Q}\psi)(t_{n-1}) = (\mathcal{Q}\psi)(1-\xi) - \psi(1-\xi) - (\mathcal{Q}\psi)(t_{n-1}) \ge -\psi(1-\xi)$$

and

$$x - (\mathcal{Q}\psi)(t_{n-1}) \le (\mathcal{Q}\psi)(t_n) - (\mathcal{Q}\psi)(t_{n-1}) = \varepsilon.$$

Then we have that

$$E_2^{(n)} = \big|x - (\mathcal{Q}\psi)(t_{n-1})\big|\,1_{A(t_{n-1})\setminus A(t_n)}$$
$$\le \big\{\psi(1-\xi) + \varepsilon\big\}1_{A(t_{n-1})\setminus A(t_n)}.$$

As a result, for each $n \ge 1$,

$$\mathbb{E}\big[\,|f_\infty - f_{n-1}|\,\big|\,\mathcal{F}_n\big]$$
$$\le \big\{2(\mathcal{P}\psi)(t_n) + \varepsilon\big\}1_{A(t_n)} + \big\{\psi(1-\xi) + \varepsilon\big\}1_{A(t_{n-1})\setminus A(t_n)}$$
$$\le 2(\mathcal{P}\psi)(t_n)1_{A(t_n)} + \psi(1-\xi)1_{A(t_{n-1})\setminus A(t_n)} + \varepsilon \quad \text{a.s.}$$

Moreover, if $n = 0$, then

$$\mathbb{E}\big[|f_\infty - f_{n-1}|\,\big|\,\mathcal{F}_n\big]$$
$$= \|x\|_1 \le \|\mathcal{Q}\psi\|_1 + \|\psi\|_1 = 2\,\|\psi\|_1 = 2(\mathcal{P}\psi)(t_0) \quad \text{a.s.}$$

It then follows that

$$\theta_{\mathcal{F}} f \le 2 \sup_{n\in\mathbb{Z}_+}(\mathcal{P}\psi)(t_n)1_{A(t_n)} + \psi(1-\xi) + \varepsilon$$
$$= 2\sum_{k=1}^{\infty}(\mathcal{P}\psi)(t_{k-1})1_{A(t_{k-1})\setminus A(t_k)} + \psi(1-\xi) + \varepsilon \quad \text{a.s.}$$

Thus

$$\|\theta_{\mathcal{F}} f\|_Y \le 2\left\|\sum_{k=1}^{\infty}(\mathcal{P}\psi)(t_{k-1})1_{A(t_{k-1})\setminus A(t_k)}\right\|_Y + \|\psi(1-\xi)\|_Y + \varepsilon\,\|\mathbf{1}\|_{\widehat{Y}}\,. \tag{27}$$

Since $\big(\psi(1-\xi)\big)^*(t) = \psi(t)$ and since

$$\Big(\sum_{k=1}^{\infty}(\mathcal{P}\psi)(t_{k-1})1_{A(t_{k-1})\setminus A(t_k)}\Big)^*(t) = \sum_{k=1}^{\infty}(\mathcal{P}\psi)(t_{k-1})1_{[t_k,t_{k-1})}(t) \le (\mathcal{P}\psi)(t) \quad (t\in I),$$

we see from (27) that

$$\|\theta_{\mathcal{F}} f\|_Y \le 2\,\|\mathcal{P}\psi\|_{\widehat{Y}} + \|\psi\|_{\widehat{Y}} + \varepsilon\,\|\mathbf{1}\|_{\widehat{Y}}\,. \tag{28}$$

On the other hand,

$$\|\mathcal{Q}\psi\|_{\widehat{X}} - \|\psi\|_{\widehat{X}} \le \big\|(\mathcal{Q}\psi)-\psi\big\|_{\widehat{X}} = \|\phi\|_{\widehat{X}} = \|x\|_X \le \sup_n \|f_n\|_X\,, \tag{29}$$

where the last inequality follows from (B3) and the fact that x is the almost sure limit of $f=(f_n)$. Combining (11), (28) and (29), we have that

$$\|\mathcal{Q}\psi\|_{\widehat{X}} \le C\big(2\,\|\mathcal{P}\psi\|_{\widehat{Y}} + \|\psi\|_{\widehat{Y}} + \varepsilon\,\|\mathbf{1}\|_{\widehat{Y}}\big) + \|\psi\|_{\widehat{X}}\,.$$

Since $\widehat{Y}\hookrightarrow\widehat{X}$ by Lemma 2, the norm $\|\psi\|_{\widehat{X}}$ on the right-hand side may be replaced by a constant multiple of $\|\psi\|_{\widehat{Y}}$. It follows that

$$\|\mathcal{Q}\psi\|_{\widehat{X}} \le k\big(\|\psi\|_{\widehat{Y}} + \|\mathcal{P}\psi\|_{\widehat{Y}} + \varepsilon\big)$$

with a positive constant k, independent of ψ. Letting $\varepsilon\downarrow 0$, we have the estimate

$$\|\mathcal{Q}\psi\|_{\widehat{X}} \le k\big(\|\psi\|_{\widehat{Y}} + \|\mathcal{P}\psi\|_{\widehat{Y}}\big).$$

Since $\mathcal{P}\in B(\widehat{Y})$, we conclude that $\|\mathcal{Q}\psi\|_{\widehat{X}} \le K\,\|\psi\|_{\widehat{Y}}$, as desired.

To complete the proof, it only remains to show that $Y\hookrightarrow K(X)$. Suppose $x\in Y$. Then

$$\|x\|_{K(X)} = \|\mathcal{Q}x^*\|_{\widehat{X}} \le \|\mathcal{Q}\|_{B(\widehat{Y},\widehat{X})}\,\|x^*\|_{\widehat{Y}} = \|\mathcal{Q}\|_{B(\widehat{Y},\widehat{X})}\,\|x\|_Y\,.$$

Thus $Y\hookrightarrow K(X)$, as desired. □

Proof of Corollary 3: From (24) we see that

$$\|Mf\|_X = \|(Mf)^*\|_{\widehat{X}} \le 16\,\|\mathcal{Q}(\theta_{\mathcal{F}} f)^*\|_{\widehat{X}} = 16\,\|\theta_{\mathcal{F}} f\|_{K(X)}\,,$$

as desired. □

7. Proof of the Main Theorem

This final section is devoted to the proof of our Main Theorem.

Proof of the Main Theorem: (i) $\Rightarrow$ (ii). Obvious.

(ii) $\Rightarrow$ (iv). Suppose that (ii) holds. Then, by Proposition 1, there exists a norm $|||\cdot|||_X$ on X which is equivalent to $\|\cdot\|_X$ and with respect to which X is r.i. To show that $0<\alpha_X\le\beta_X<1$, it suffices to show that $\mathcal{P},\mathcal{Q}\in B(\widehat{X})$. Since $|||f_n|||_X\le|||f_{n+1}|||_X$ for each $n\in\mathbb{Z}_+$, (3) can be rewritten as

$$c\sup_{n\in\mathbb{Z}_+}|||f_n|||_X\le|||\theta_{\mathcal{F}}f|||_X\le C\sup_{n\in\mathbb{Z}_+}|||f_n|||_X\,. \tag{30}$$

From Proposition 2 and (30), it follows that $\mathcal{P}\in B(\widehat{X})$, and hence $\beta_X<1$, where $\widehat{X}$ is the Luxemburg representation of the r.i. space $(X,|||\cdot|||_X)$. It also follows from Proposition 3 that $\mathcal{Q}\in B(\widehat{X})$. Thus $0<\alpha_X\le\beta_X<1$, as desired.

(iv) $\Rightarrow$ (iii). Suppose that (iv) holds. Then $\mathcal{P},\mathcal{Q}\in B(\widehat{X})$. Hence we obtain (30) by using Propositions 2 and 3. If $f=(f_n)$ is a uniformly integrable martingale, then $\sup_n|||f_n|||_X\le|||f_\infty|||_X$ since $f_n\prec f_\infty$ for all $n\in\mathbb{Z}_+$. On the other hand, by (B3), $|||f_\infty|||_X\le\underline{\lim}_n|||f_n|||_X=\sup_n|||f_n|||_X$. Thus $|||f_\infty|||_X=\sup_n|||f_n|||_X$ and (30) can be rewritten as (4).

(iii) $\Rightarrow$ (i). Suppose that (iii) holds, and let $f=(f_n)_{n\in\mathbb{Z}_+}$ be a martingale. Applying (4) to the stopped martingale $f^{(k)}=(f_{n\wedge k})_{n\in\mathbb{Z}_+}$, we see that

$$c\,\|f_k\|_X\le\left\|\theta_{\mathcal{F}}f^{(k)}\right\|_X\le C\,\|f_k\|_X\quad(k\in\mathbb{Z}_+).$$

Since $\theta_{\mathcal{F}}f^{(k)}\uparrow\theta_{\mathcal{F}}f$ as $k\uparrow\infty$, we conclude from (B3) that

$$c\,\overline{\lim_{k\to\infty}}\,\|f_k\|_X\le\|\theta_{\mathcal{F}}f\|_X\le C\,\underline{\lim}_{k\to\infty}\,\|f_k\|_X\,,$$

as desired. □

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