

## COMPACT HYPERBOLIC TETRAHEDRA WITH NON-OBTUSE DIHEDRAL ANGLES

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*Abstract*

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Given a combinatorial description  $C$  of a polyhedron having  $E$  edges, the space of dihedral angles of all compact hyperbolic polyhedra that realize  $C$  is generally not a convex subset of  $\mathbb{R}^E$  [9]. If  $C$  has five or more faces, Andreev's Theorem states that the corresponding space of dihedral angles  $A_C$  obtained by restricting to *non-obtuse* angles is a convex polytope. In this paper we explain why Andreev did not consider tetrahedra, the only polyhedra having fewer than five faces, by demonstrating that the space of dihedral angles of compact hyperbolic tetrahedra, after restricting to non-obtuse angles, is non-convex. Our proof provides a simple example of the “method of continuity”, the technique used in classification theorems on polyhedra by Alexandrov [4], Andreev [5], and Rivin-Hodgson [18].

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Given a combinatorial description  $C$  of a polyhedron having  $E$  edges, the space of dihedral angles of all compact hyperbolic polyhedra that realize  $C$  is generally not a convex subset of  $\mathbb{R}^E$ . This is proved in a nice paper by Díaz [9]. However, Andreev's Theorem [5], [13], [19], [20] shows that by restricting to compact hyperbolic polyhedra with non-obtuse dihedral angles, the space of dihedral angles is a convex polytope, which we label  $A_C \subset \mathbb{R}^E$ . It is interesting to note that the statement of Andreev's Theorem requires that  $C$  have five or more faces, ruling out the tetrahedron which is the only polyhedron having fewer than five faces.

In this paper, we explain why hyperbolic tetrahedra are a special case that is not covered by Andreev's Theorem. We provide an explicit description of the space of dihedral angles,  $A_\Delta$ , corresponding to compact hyperbolic tetrahedra with non-obtuse dihedral angles, finding that  $A_\Delta$  is a non-convex, path-connected subset of  $\mathbb{R}^6$ .

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2000 *Mathematics Subject Classification.* 52B10, 52A55, 51M09.

*Key words.* Hyperbolic geometry, polyhedra, tetrahedra.

A description of the space of Gram matrices (and hence indirectly of the space of dihedral angles) corresponding to compact hyperbolic tetrahedra having arbitrary dihedral angles is available in Milnor's collected works [16]. Our description of the space of dihedral angles  $A_\Delta$  can be derived from the result in [16], using the assumption that the dihedral angles are non-obtuse. However, we use the "method of continuity," providing the reader with a simple example of a method that plays an important role in the classification theorems on polyhedra by Alexandrow [4], Andreev [5], and Rivin-Hodgson [18].

Let  $E^{3,1}$  be  $\mathbb{R}^4$  with the indefinite metric  $\|\mathbf{x}\|^2 = -x_0^2 + x_1^2 + x_2^2 + x_3^2$ . In this paper, we work in the hyperbolic space  $\mathbb{H}^3$  given by the component of the subset of  $E^{3,1}$  given by

$$\|\mathbf{x}\|^2 = -x_0^2 + x_1^2 + x_2^2 + x_3^2 = -1$$

having  $x_0 > 0$ , with the Riemannian metric induced by the indefinite metric

$$-dx_0^2 + dx_1^2 + dx_2^2 + dx_3^2.$$

There is a natural compactification of the hyperbolic space obtained by adding the set of rays asymptotic to the hyperboloid. We refer to these points as the *points at infinity*. There is no natural extension of the Riemannian structure of  $\mathbb{H}^3$  to these points at infinity, however, there is a natural way to extend the conformal structure on  $\mathbb{H}^3$  to these points at infinity.

One can check that the hyper-plane orthogonal to a vector  $\mathbf{v} \in E^{3,1}$  intersects  $\mathbb{H}^3$  if and only if  $\langle \mathbf{v}, \mathbf{v} \rangle > 0$ . Let  $\mathbf{v} \in E^{3,1}$  be a vector with  $\langle \mathbf{v}, \mathbf{v} \rangle > 0$ , and define

$$P_{\mathbf{v}} = \{\mathbf{w} \in \mathbb{H}^3 \mid \langle \mathbf{w}, \mathbf{v} \rangle = 0\}$$

to be the hyperbolic plane orthogonal to  $\mathbf{v}$ ; and the corresponding closed half space:

$$H_{\mathbf{v}}^+ = \{\mathbf{w} \in \mathbb{H}^3 \mid \langle \mathbf{w}, \mathbf{v} \rangle \geq 0\}.$$

Notice that given two planes  $P_{\mathbf{v}}$  and  $P_{\mathbf{w}}$  in  $\mathbb{H}^3$  with  $\langle \mathbf{v}, \mathbf{v} \rangle = 1$  and  $\langle \mathbf{w}, \mathbf{w} \rangle = 1$ , they:

- Intersect in a line if and only if  $\langle \mathbf{v}, \mathbf{w} \rangle^2 < 1$ , in which case their dihedral angle is  $\arccos(-\langle \mathbf{v}, \mathbf{w} \rangle)$ .
- Intersect in a single point at infinity if and only if  $\langle \mathbf{v}, \mathbf{w} \rangle^2 = 1$ , in this case their dihedral angle is 0.

A *hyperbolic polyhedron* is an intersection

$$P = \bigcap_{i=0}^n H_{\mathbf{v}_i}^+$$

having non-empty interior. There are many papers on hyperbolic polyhedra, including [5], [6], [9], [13], [16], [18], [19], [20], [21], [22], [24], [26], and particularly on the groups of reflections generated by them [3], [24], [25], [27]. A *hyperbolic tetrahedron* is therefore a hyperbolic polyhedron having the combinatorial type of a tetrahedron. There are also many papers on hyperbolic tetrahedra including [8], [11], [12], [15], [17], [23], many of these studying volume and symmetries.

If we normalize the vectors  $v_i$  that are orthogonal to the faces of a polyhedron  $P$ , the *Gram matrix* of  $P$  is the matrix with terms  $M_{ij} = \langle \mathbf{v}_i, \mathbf{v}_j \rangle$ . By construction, a Gram matrix is symmetric and unidiagonal (i.e. has 1s on the diagonal). The following theorem appears in [16]:

**Theorem 1.** *A symmetric unidiagonal matrix  $M$  is the Gram matrix of a compact hyperbolic tetrahedron if and only if  $\det(M) < 0$  and each principal minor is positive definite.*

Although the hyperboloid model of hyperbolic space is very natural, it is not easy to visualize, since the ambient space is four-dimensional. We will often use the Poincaré ball model of hyperbolic space, given by the unit ball in  $\mathbb{R}^3$  with the metric

$$4 \frac{dx_1^2 + dx_2^2 + dx_3^2}{(1 - \|\mathbf{x}\|^2)^2}$$

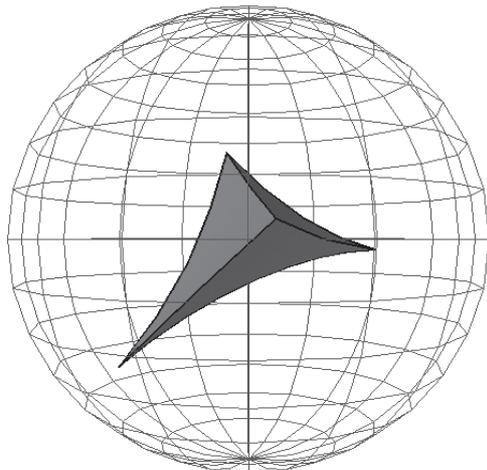
and the upper half-space model of hyperbolic space, given by the subset of  $\mathbb{R}^3$  with  $x_3 > 0$  equipped with the metric

$$\frac{dx_1^2 + dx_2^2 + dx_3^2}{x_3^2}.$$

Both of these models are isometric to  $\mathbb{H}^3$ . The points at infinity in the Poincaré Ball model correspond to points on the unit sphere, and the points at infinity in the upper half-space model correspond to the points in the plane  $x_3 = 0$ . More background is available on hyperbolic geometry in [7].

Hyperbolic planes in these models correspond to portions of Euclidean spheres and Euclidean planes that intersect the boundary perpendicularly. Furthermore, these models are conformally correct, that is, the hyperbolic angle between a pair of such intersecting hyperbolic planes is exactly the Euclidean angle between the corresponding spheres or planes.

See below for an image of a compact hyperbolic tetrahedron depicted in the Poincaré ball model using Geomview [2]. The sphere at infinity is shown for reference.



The following two lemmas will be necessary when discussing compact hyperbolic polyhedra having non-obtuse dihedral angles. They are well known results and appear in many of the works on hyperbolic polyhedra mentioned above, including [5].

**Lemma 2.** *Suppose that three planes  $P_{\mathbf{v}_1}$ ,  $P_{\mathbf{v}_2}$ ,  $P_{\mathbf{v}_3}$  intersect pairwise in  $\mathbb{H}^3$  with non-obtuse dihedral angles  $\alpha$ ,  $\beta$ , and  $\gamma$ . Then,  $P_{\mathbf{v}_1}$ ,  $P_{\mathbf{v}_2}$ ,  $P_{\mathbf{v}_3}$  intersect at a vertex in  $\overline{\mathbb{H}^3}$  if and only if  $\alpha + \beta + \gamma \geq \pi$ . The planes intersect in  $\mathbb{H}^3$  if and only if the inequality is strict.*

*Proof:* The planes intersect in a point of  $\overline{\mathbb{H}^3}$  if and only if the subspace spanned by  $\mathbf{v}_1$ ,  $\mathbf{v}_2$ ,  $\mathbf{v}_3$  is positive semi-definite, so that the orthogonal is a negative semi-definite line of  $E^{3,1}$ . If the inner product on this line is negative, the line defines a point of intersection with the hyperboloid model. Otherwise, the inner product on the line is zero and this line corresponds to a point in  $\partial\mathbb{H}^3$ , since the line will then lie in the cone to which the hyperboloid is asymptotic.

The symmetric matrix defining the inner product on the span of  $\mathbf{v}_1$ ,  $\mathbf{v}_2$ , and  $\mathbf{v}_3$  is

$$\begin{bmatrix} 1 & \langle \mathbf{v}_1, \mathbf{v}_2 \rangle & \langle \mathbf{v}_1, \mathbf{v}_3 \rangle \\ \langle \mathbf{v}_1, \mathbf{v}_2 \rangle & 1 & \langle \mathbf{v}_2, \mathbf{v}_3 \rangle \\ \langle \mathbf{v}_1, \mathbf{v}_3 \rangle & \langle \mathbf{v}_2, \mathbf{v}_3 \rangle & 1 \end{bmatrix} = \begin{bmatrix} 1 & -\cos \alpha & -\cos \beta \\ -\cos \alpha & 1 & -\cos \gamma \\ -\cos \beta & -\cos \gamma & 1 \end{bmatrix}$$

where  $\alpha$ ,  $\beta$ , and  $\gamma$  are the dihedral angles between the pairs of faces  $(P_{\mathbf{v}_1}, P_{\mathbf{v}_2})$ ,  $(P_{\mathbf{v}_1}, P_{\mathbf{v}_3})$ , and  $(P_{\mathbf{v}_2}, P_{\mathbf{v}_3})$ , respectively.

Since the principal minor is positive definite for  $0 < \alpha \leq \pi/2$ , it is enough to find out when the determinant

$$1 - 2 \cos \alpha \cos \beta \cos \gamma - \cos^2 \alpha - \cos^2 \beta - \cos^2 \gamma$$

is non-negative.

A bit of trigonometric trickery (we used complex exponentials) shows that the expression above is equal to

$$(1) \quad -4 \cos\left(\frac{\alpha+\beta+\gamma}{2}\right) \cos\left(\frac{\alpha-\beta+\gamma}{2}\right) \cos\left(\frac{\alpha+\beta-\gamma}{2}\right) \cos\left(\frac{-\alpha+\beta+\gamma}{2}\right).$$

Let  $\delta = \alpha + \beta + \gamma$ . When  $\delta < \pi$ , (1) is strictly negative; when  $\delta = \pi$ , (1) is clearly zero; and when  $\delta > \pi$ , (1) is strictly positive. Hence the inner product on the space spanned by  $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$  is positive semidefinite if and only if  $\delta \geq \pi$ . It is positive definite if and only if  $\delta > \pi$ .

Therefore, the three planes  $P_{\mathbf{v}_1}, P_{\mathbf{v}_2}, P_{\mathbf{v}_3} \subset \mathbb{H}^3$  intersect at a point in  $\overline{\mathbb{H}^3}$  if and only if they intersect pairwise in  $\mathbb{H}^3$  and the sum of the dihedral angles  $\delta \geq \pi$ . It is also clear that they intersect at a finite point if and only if the inequality is strict.  $\square$

**Lemma 3.** *Given a trivalent vertex of a hyperbolic polyhedron, we can compute the angles of the faces in terms of the dihedral angles. If the dihedral angles are non-obtuse, these angles are also  $\leq \pi/2$ .*

*Proof:* Let  $v$  be a finite trivalent vertex of  $P$ . After an appropriate isometry, we can assume that  $v$  is the origin in the Poincaré ball model, so that the faces at  $v$  are subsets of Euclidean planes through the origin. A small sphere centered at the origin will intersect  $P$  in a spherical triangle  $Q$  whose angles are the dihedral angles between faces. Call these angles  $\alpha_1, \alpha_2, \alpha_3$ .

The edge lengths of  $Q$  are precisely the angles in the faces at the origin. Supposing that  $Q$  has edge lengths  $(\beta_1, \beta_2, \beta_3)$  with the edge  $\beta_i$  opposite of angle  $\alpha_i$  for each  $i = 1, 2, 3$ , The law of cosines in spherical geometry states that:

$$(2) \quad \cos(\beta_i) = \frac{\cos(\alpha_i) + \cos(\alpha_j) \cos(\alpha_k)}{\sin(\alpha_j) \sin(\alpha_k)}.$$

Hence, the face angles are calculable from the dihedral angles. They are non-obtuse, since the right-hand side of the equation is positive for  $\alpha_i, \alpha_j, \alpha_k$  non-obtuse.  $\square$

We can now state our classification of compact hyperbolic tetrahedra:

**Theorem 4.** *Let  $\alpha_1, \dots, \alpha_6$  be a set of proposed non-obtuse dihedral angles and let  $\beta_1(\alpha_1, \dots, \alpha_6), \dots, \beta_{12}(\alpha_1, \dots, \alpha_6)$  be the face angles given by equation (2), corresponding to these proposed dihedral angles.*

*There is a compact hyperbolic tetrahedron with dihedral angles  $\alpha_1, \dots, \alpha_6$  if and only if:*

- (1) *For each edge  $e_i$ ,  $0 < \alpha_i \leq \pi/2$ .*
- (2) *Whenever 3 distinct edges  $e_i, e_j, e_k$  meet at a vertex,  $\alpha_i + \alpha_j + \alpha_k > \pi$ .*
- (3) *For each face the sum of the face angles satisfies  $\beta_i + \beta_j + \beta_k < \pi$ .*

*Furthermore this tetrahedron is unique.*

Recall from Lemma 3 that the face angles  $\beta_i$  are calculable from the dihedral angles  $\alpha_i$  and are themselves non-obtuse so that condition (3) is a highly non-linear condition on the dihedral angles. We will denote the subset of  $\mathbb{R}^6$  of dihedral angles satisfying conditions (1)–(3) by  $A_\Delta$ .

We present a proof of Theorem 4 using the “method of continuity”, the classical method used by Alexandrow [4], Andreev [5], later by Rivin and Hodgson [18], and in this author’s more recent proof of Andreev’s Theorem [19], [20]. The idea of this method is to establish a bijection between two manifolds of the same dimension: one,  $X$ , consisting of the geometric objects that you want to construct, and the other,  $Y$ , a subset of  $\mathbb{R}^n$  consisting of various angles, lengths, etc. The space  $X$  should be viewed as unknown and the space  $Y$  as known.

You then consider the mapping  $f: X \rightarrow Y$  which takes your geometric object, in  $X$ , and reads off its appropriate measurements, in  $Y$ . Of course, you need to show that the image is actually in  $Y$ , namely, that the constraints that you put on the coordinates of  $Y$  (typically something like the triangle inequality for the edges of a triangle) are indeed satisfied for each geometric object of  $X$ .

This map  $f$  will always be obviously continuous, and it is not too hard to show that it is proper and injective. (Recall that a mapping is said to be proper if the pullback of a compact set is compact.) Then, the following lemma can be used to show that the image of  $f$  is a union of connected components of  $Y$ .

**Lemma 5.** *Let  $X$  and  $Y$  metric spaces, and let  $f: X \rightarrow Y$  be a proper local homeomorphism. Then the image of  $f$  is a union of connected components of  $Y$ .*

*Proof of Lemma 5:* It is sufficient to show that  $f(X)$  is both open and closed in  $Y$ . Because  $f$  is a local homeomorphism, it is an open mapping, so  $f(X)$  is open in  $Y$ ; and since  $f: X \rightarrow Y$  is proper, it immediately

follows that the limit of any sequence in the image of  $f$  which converges in  $Y$  must lie in the image of  $f$ , so  $f(X)$  is closed in  $Y$ .  $\square$

In fact, a stronger result is true: any local homeomorphism between metric spaces which is also proper will be a finite-sheeted covering map [10, p. 23] and [14, p. 127]. This gives an alternative route to proving Lemma 5.

Therefore, this lemma reduces the problem to showing that  $X$  is nonempty and that  $Y$  is connected, which are usually the hardest parts!

The result of the “method of continuity” is that you have established a bijection between your geometric objects, set  $X$ , and the measurements  $Y$ .

Let  $C$  be a cell complex on  $\mathbb{S}^2$  that describes the combinatorics of a convex polyhedron. We say that a hyperbolic polyhedron  $P \subset \mathbb{H}^3$  realizes  $C$  if there is a cellular homeomorphism from  $C$  to  $\partial P$  (i.e., a homeomorphism mapping faces of  $C$  to faces of  $P$ , edges of  $C$  to edges of  $P$ , and vertices of  $C$  to vertices of  $P$ ). We will call each isotopy class of cellular homeomorphisms  $\phi: C \rightarrow \partial P$  a *marking* on  $P$ .

Let  $\Delta$  be the cell complex on  $\mathbb{S}^2$  describing the combinatorics of the tetrahedron. Throughout this paper we will call hyperbolic polyhedra realizing  $\Delta$  hyperbolic tetrahedra.

We will define  $\mathcal{P}_\Delta$  to be the set of pairs  $(P, \phi)$  so that  $P$  is a hyperbolic tetrahedron and  $\phi$  is a marking on  $P$  with the equivalence relation that  $(P, \phi) \sim (P', \phi')$  if there exists an automorphism  $\rho: \mathbb{H}^3 \rightarrow \mathbb{H}^3$  such that  $\rho(P) = P'$  and both  $\phi'$  and  $\rho \circ \phi$  represent the same marking on  $P'$ .

**Proposition 6.** *The space  $\mathcal{P}_\Delta$  is a manifold of dimension 6.*

*Proof:* Let  $\mathcal{H}$  be the space of closed half-spaces of  $\mathbb{H}^3$ ; clearly  $\mathcal{H}$  is a 3-dimensional manifold. Let  $\mathcal{O}_\Delta$  be the set of marked hyperbolic polyhedra realizing  $\Delta$ . By forgetting this marking, an element of  $\mathcal{O}_\Delta$  is a 4-tuple of half-spaces that intersect in a polyhedron realizing  $\Delta$ . This induces a mapping from  $\mathcal{O}_\Delta$  to  $\mathcal{H}^4$  whose image is an open set. We give  $\mathcal{O}_\Delta$  the topology that makes this mapping from  $\mathcal{O}_\Delta$  into  $\mathcal{H}^4$  a local homeomorphism. Since  $\mathcal{H}^4$  is a 12-dimensional manifold,  $\mathcal{O}_\Delta$  must be a 12-dimensional manifold as well.

If  $\rho(P, \phi) = (P', \phi')$ , we have that  $\rho \circ \phi$  is isotopic to  $\phi'$  through cellular homeomorphisms. Hence, the automorphism  $\rho$  must fix all vertices of  $P$ , and consequently restricts to the identity on all edges and faces. However, an automorphism of  $\mathbb{H}^3$  which fixes four non-coplanar points must be the identity. Therefore  $\text{Aut}(\mathbb{H}^3)$  acts freely on  $\mathcal{O}_\Delta$ .

This quotient is  $\mathcal{P}_\Delta$ , hence  $\mathcal{P}_\Delta$  is a manifold with dimension equal to  $\dim(\mathcal{O}_\Delta) - \dim(\text{Aut}(\mathbb{H}^3)) = 3 \cdot 4 - 6 = 6$ .  $\square$

In fact, we will restrict to the subset  $\mathcal{P}_\Delta^0$  of tetrahedra with dihedral angles in  $(0, \pi/2]$ . Notice that  $\mathcal{P}_\Delta^0$  is not, *a priori*, a manifold or even a manifold with boundary. All that we will need for the proof of Theorem 4 is that  $\mathcal{P}_\Delta$  is a manifold and that the subspace  $\mathcal{P}_\Delta^0$  is a metric space.

Consider the map  $\alpha: \mathcal{P}_\Delta \rightarrow \mathbb{R}^6$  which is obtained by measuring the dihedral angles (ordered by the marking) of an element of  $\mathcal{P}_\Delta$ . Using the topology on  $\mathcal{P}_\Delta$  that is described in the proof of Proposition 6, it is clear that  $\alpha$  is continuous. Therefore, we will use the method of continuity to show that  $\alpha$  restricted to  $\mathcal{P}_\Delta^0$  is a homeomorphism onto  $A_\Delta$ , in order to prove Theorem 4.

At this point it is necessary to clarify the statement of uniqueness in Theorem 4. We will show that the map  $\alpha$  is injective, which shows that for each set of proposed dihedral angles  $\alpha_1, \dots, \alpha_6$  there is a unique *marked tetrahedron* with the dihedral angles  $\alpha_1, \dots, \alpha_6$ , as ordered by this marking. This is what we mean by uniqueness in Theorem 4 and in the later Theorem 10.

*Proof of Theorem 4:* The first step is to make sure that the dihedral angles of a compact tetrahedron satisfy conditions (1)–(3). For condition (1), notice that if two adjacent faces intersect along a line segment with dihedral angle 0, they would coincide. In addition, the dihedral angle between adjacent faces is  $\leq \pi/2$  by hypothesis. For condition (2), let  $x$  be a vertex of  $P$ . The compactness of  $P$  implies that  $x \in \mathbb{H}^3$ , and by Lemma 2, the sum of the dihedral angles between the three planes intersecting at  $x$  must be  $> \pi$ . Furthermore, each face of a hyperbolic tetrahedron is a hyperbolic triangle of non-zero area so the Gauss-Bonnet formula gives condition (3). Therefore conditions (1)–(3) are necessary.

There is an elementary proof that  $\alpha: \mathcal{P}_\Delta \rightarrow \mathbb{R}^6$  is injective: Since the face angles are uniquely determined by the dihedral angles and each face is a hyperbolic triangle, one can calculate the length of each edge using the hyperbolic law of cosines.

Before proving that  $\alpha: \mathcal{P}_\Delta^0 \rightarrow A_\Delta$  is proper, we will need the following lemma:

**Lemma 7.** *Given three points  $v_1, v_2, v_3$  that form a non-obtuse, non-degenerate triangle in the Poincaré model of  $\mathbb{H}^3$ , there is a unique orientation preserving isometry taking  $v_1$  to a positive point on the  $x$ -axis,  $v_2$  to a positive point on the  $y$ -axis, and  $v_3$  to a positive point on the  $z$ -axis.*

*Proof of Lemma 7:* The points  $v_1, v_2,$  and  $v_3$  form a triangle  $T$  in a plane  $P_T$ . It is sufficient to show that there is a plane  $Q_T$  in the Poincaré ball model that intersects the positive octant in a triangle isomorphic to  $T$ . The isomorphism taking  $v_1, v_2,$  and  $v_3$  to the  $x, y,$  and  $z$ -axes will then be the one that takes the plane  $P_T$  to the plane  $Q_T$  and the triangle  $T$  to the intersection of  $Q_T$  with the positive octant.

Let  $s_1, s_2,$  and  $s_3$  be the side lengths of  $T$ . The plane  $Q_T$  must intersect the  $x, y,$  and  $z$ -axes at distances  $a_1, a_2,$  and  $a_3$  satisfying the hyperbolic Pythagorean theorem:

$$\begin{aligned} \cosh(s_1) &= \cosh(a_2) \cosh(a_3), \\ \cosh(s_2) &= \cosh(a_3) \cosh(a_1), \\ \cosh(s_3) &= \cosh(a_1) \cosh(a_2). \end{aligned}$$

These equations can be solved for  $(\cosh^2(a_1), \cosh^2(a_2), \cosh^2(a_3))$ , obtaining

$$\left( \frac{\cosh(s_2) \cosh(s_3)}{\cosh(s_1)}, \frac{\cosh(s_3) \cosh(s_1)}{\cosh(s_2)}, \frac{\cosh(s_1) \cosh(s_2)}{\cosh(s_3)} \right).$$

The only concern in solving for  $a_i$  is that each of these terms is  $\geq 1$ . However, this follows from the triangle  $T$  being non-obtuse.  $\square$

**Lemma 8.** *The mapping  $\alpha: \mathcal{P}_\Delta^0 \rightarrow A_\Delta$  is proper.*

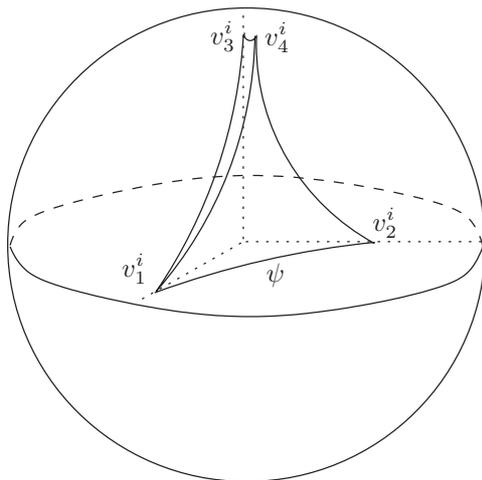
*Proof:* To see that  $\alpha: \mathcal{P}_\Delta^0 \rightarrow A_\Delta$  is a proper mapping, suppose that there is a sequence of polyhedra  $P_i$  realizing  $\Delta$ , with  $\alpha(P_i) = \mathbf{a}_i \in A_\Delta$ . We must show that if  $\mathbf{a}_i$  converges to  $\mathbf{a} \in A_\Delta$ , then a subsequence of the  $P_i$  converges to some  $P_\infty$  in  $\mathcal{P}_\Delta^0$ .

Throughout this part of the proof, we consider each  $P_i$  to be in the Poincaré ball. Denote the vertices of  $P_i$  by  $v_1^i, v_2^i, v_3^i,$  and  $v_4^i$ . According to Lemma 7, we can normalize each  $P_i$  so that  $v_1^i$  is on the  $x$ -axis,  $v_2^i$  is on the  $y$ -axis, and  $v_3^i$  is on the  $z$ -axis.

Because  $\overline{\mathbb{H}^3}$  is a compact space (in the Euclidean metric), we can take a subsequence of the  $P_i$  so that the vertices  $v_1^i, \dots, v_4^i$  converge to some points  $v_1, \dots, v_4$  in  $\overline{\mathbb{H}^3}$ . We must use that  $\mathbf{a}$  satisfies conditions (1)–(3) to show that  $v_1, \dots, v_4$  are actually at distinct finite points in  $\mathbb{H}^3$  whose span is a tetrahedron.

**The vertices  $v_1^i, v_2^i, v_3^i,$  and  $v_4^i$  converge to distinct points in  $\overline{\mathbb{H}^3}$ .**

Notice that at most two of the vertices could converge to the same point in  $\partial\mathbb{H}^3$ , since  $v_1^i$  is on the  $x$ -axis,  $v_2^i$  is on the  $y$ -axis, and  $v_3^i$  is on the  $z$ -axis. We suppose, without loss of generality, that  $v_4^i$  converges to the same point in  $\partial\mathbb{H}^3$  as  $v_3^i$ , that is, both  $v_4^i$  and  $v_3^i$  converge to the north pole of the Poincaré ball. Then, however, the dihedral angle,  $\psi$ , between the face spanned by  $(v_1^i, v_2^i, v_3^i)$  and the face spanned by  $(v_1^i, v_2^i, v_4^i)$  must limit to 0, contrary to condition (1). This configuration is depicted in the diagram below.



Hence, we conclude that any of the vertices  $v_j^i$  that converge to points in  $\partial\mathbb{H}^3$ , must converge to distinct points.

Any face of  $P_i$  that degenerates to a point or a line segment has (hyperbolic) area that limits to zero, since the vertices of  $P_i$  that converge to points in  $\partial\mathbb{H}^3$  converge to distinct points. Hence, by the Gauss Bonnet formula, the sum of the face angles for such a degenerating face would limit to  $\pi$ , contrary to condition (3). This is enough to show that  $v_1^i, \dots, v_4^i$  converge to distinct points  $v_1, \dots, v_4$  in  $\overline{\mathbb{H}^3}$ .

**The limit points  $v_1, v_2, v_3,$  and  $v_4$  are finite points whose span is a tetrahedron.**

The sum of the dihedral angles at the edges leading to each  $v_j^i$  converges to a value  $> \pi$ . Therefore, according to Lemma 2, we conclude that the limit points of vertices  $v_1, \dots, v_4$  are actually at finite points.

Since each face is non-degenerate, and the dihedral angles are non-obtuse, the  $P_i$  cannot degenerate to a single triangle. So, their span realizes a tetrahedron, with dihedral angles  $\mathbf{a}$ .

This is enough to conclude that  $\alpha: \mathcal{P}_\Delta^0 \rightarrow A_\Delta$  is proper.  $\square$

Invariance of domain gives that  $\alpha: \mathcal{P}_\Delta \rightarrow \mathbb{R}^6$  is a local homeomorphism because it is a continuous and injective mapping between manifolds of the same dimension. Therefore, the restriction  $\alpha: \mathcal{P}_\Delta^0 \rightarrow A_\Delta$  is also a local homeomorphism. Because  $\alpha: \mathcal{P}_\Delta^0 \rightarrow A_\Delta$  is also a proper mapping, by Lemma 5,  $\alpha(\mathcal{P}_\Delta^0)$  is a union of connected components of  $A_\Delta$ . We will show that  $\mathcal{P}_\Delta^0$  is nonempty and that  $A_\Delta$  is connected, thus proving that  $\alpha: \mathcal{P}_\Delta^0 \rightarrow A_\Delta$  is surjective.

The easiest way to see that  $\mathcal{P}_\Delta^0 \neq \emptyset$  is by explicit construction. Let  $\mathbf{v}_1 = (0, 1, 0, 0)$ ,  $\mathbf{v}_2 = (0, 0, 1, 0)$ ,  $\mathbf{v}_3 = (0, 0, 0, 1)$ , and  $\mathbf{v}_4 = \frac{1}{\sqrt{2}}(-1, -1, -1, -1)$ . Then the intersection of the half-spaces  $H_{\mathbf{v}_1} \cap H_{\mathbf{v}_2} \cap H_{\mathbf{v}_3} \cap H_{\mathbf{v}_4}$  is a hyperbolic tetrahedron with dihedral angles  $\alpha_{1,2} = \pi/2$ ,  $\alpha_{1,3} = \pi/2$ ,  $\alpha_{2,3} = \pi/2$ ,  $\alpha_{1,4} = \alpha_{2,4} = \alpha_{3,4} = \arccos(1/\sqrt{2}) = \pi/4$ . Hence, we conclude that  $\mathcal{P}_\Delta^0 \neq \emptyset$ .

To see that  $A_\Delta$  is connected is significantly harder than for  $A_C$  with  $C$  not the tetrahedron because the inequalities specifying  $A_\Delta$  are not linear. We will have to do detailed analysis of the equation that expresses a face's angles in terms of the dihedral angles.

**Lemma 9.**  *$A_\Delta$  is path connected.*

*Proof:* Recall from Lemma 3 that the face angle  $\beta_i$  at a vertex  $(e_i, e_j, e_k)$  in the face containing  $e_j$  and  $e_k$  is

$$\cos(\beta_i) = \frac{\cos(\alpha_i) + \cos(\alpha_j)\cos(\alpha_k)}{\sin(\alpha_j)\sin(\alpha_k)}.$$

Let  $A_i \subset \partial A_\Delta$  be the subset obtained by restricting the dihedral angle sum at each of the vertices, except  $v_i$ , to equal  $\pi$ . Using the formula for the  $\beta_j$ , one can check that at each vertex with dihedral angle sum exactly  $\pi$ , each of the face angles is 0. One can also check that each of the face angles at  $v_i$  is non-obtuse, since each of the dihedral angles is non-obtuse. Therefore, for any point in  $A_i$ , for each  $i = 1, \dots, 4$ , each of the face angle sums is  $\leq \pi/2$ . Therefore, since the formula for face angles in terms of dihedral angles is continuous, there exists a neighborhood  $NA_i$  of each  $A_i$  in  $A_\Delta$ . If necessary, we can restrict  $NA_i$  to a smaller set which is connected, since  $A_i$  is convex.

For  $i = 1, \dots, 4$ , each  $A_i$  contains  $(\pi/3, \dots, \pi/3)$ , which are the dihedral angles of the regular ideal tetrahedron, hence  $NA_1 \cap NA_2 \cap NA_3 \cap NA_4 \neq \emptyset$ . Therefore  $NA_1 \cup NA_2 \cup NA_3 \cup NA_4$  is path connected. Denote this set by  $\mathcal{N}$ .

Given any  $\mathbf{a} \in A_\Delta$ , we will create a path from  $\mathbf{a}$  to a point in  $\mathcal{N}$ . This will be sufficient to prove that  $A_\Delta$  is connected. First, notice that for any  $\mathbf{a} \in A_\Delta$ , decreasing any of the components of  $\mathbf{a}$  does not increase any of the  $\beta_i$ . One can check that if:

$$F(x, y, z) = \frac{\cos(x) + \cos(y) \cos(z)}{\sin(y) \sin(z)}.$$

Then we have:

$$\frac{\partial F}{\partial x} = -\frac{\sin(x)}{\sin(y) \sin(z)},$$

$$\frac{\partial F}{\partial y} = \frac{-\sin(y) \sin(z) \sin(y) \cos(z) - \cos(y) \cos(z) \cos(y) \sin(z)}{\sin^2(y) \sin^2(z)},$$

$$\frac{\partial F}{\partial z} = \frac{-\sin(y) \sin(z) \cos(y) \sin(z) - \cos(y) \cos(z) \sin(y) \cos(z)}{\sin^2(y) \sin^2(z)}.$$

These have the nice property that for all  $(x, y, z) \in (0, \pi/2]^3$  we have  $\frac{\partial F}{\partial x} < 0$ ,  $\frac{\partial F}{\partial y} < 0$ , and  $\frac{\partial F}{\partial z} < 0$ . Because  $\arccos$  is a decreasing function, this gives that  $\beta(\gamma_i, \gamma_j, \gamma_k) \leq \beta(\mathbf{a}_i, \mathbf{a}_j, \mathbf{a}_k)$  when  $\gamma_i \leq \mathbf{a}_i$ ,  $\gamma_j \leq \mathbf{a}_j$ , and  $\gamma_k \leq \mathbf{a}_k$ .

Therefore, given  $\mathbf{a} \in A_\Delta$ , decreasing the angles of  $\mathbf{a}$  cannot result in a violation of condition (3).

Consider  $t \cdot \mathbf{a}$  decreasing  $t$  from 1 to 0. For some first value of  $t$ , the sum of dihedral angles at one of the vertices, say  $v_1$ , will be  $\pi$ . Next, decrease only the dihedral angles of edges not entering  $v_1$  in the same uniform way until the sum of the dihedral angles at another of the vertices, say  $v_2$  equals  $\pi$ . Finally, decrease the dihedral angle on the edge that does not enter  $v_1$  or  $v_2$  until one the two remaining vertices has dihedral angle sum  $\pi$ , call this vertex  $v_3$ .

Since we have decreased the dihedral angles during the duration of this path, condition (3) was satisfied throughout. Condition (1) was satisfied throughout because we decreased the dihedral angles, so none exceeded  $\pi/2$  and since we decreased them by scaling, so that none reached 0.

Therefore, we have constructed a path from  $\mathbf{a}$  to  $A_1$ . This path must have entered  $\mathcal{N}$  because it connected the point  $\mathbf{a} \in A$  to  $A_1$ .  $\square$

Therefore, since  $\alpha_\Delta: \mathcal{P}_\Delta^0 \rightarrow A_\Delta$  is an injective covering map with  $\mathcal{P}_\Delta^0 \neq \emptyset$  and  $A_\Delta$  path connected, we conclude that  $\alpha_\Delta$  is a homeomorphism. This proves Theorem 4.  $\square$

Using equation (2), we can re-express Theorem 4 entirely in terms of the dihedral angles.

**Corollary 10.** *There is a compact hyperbolic tetrahedron with non-obtuse dihedral angles  $\alpha_1, \dots, \alpha_6$  if and only if:*

- (1) *For each edge  $e_i$ ,  $0 < \alpha_i \leq \pi/2$ .*
- (2) *Whenever 3 (distinct) edges  $e_i, e_j, e_k$  meet at a vertex,  $\alpha_i + \alpha_j + \alpha_k > \pi$ .*
- (3) *For each face  $F$  bounded by edges  $e_i, e_j, e_k$  with edges  $e_{i,j}, e_{j,k}, e_{k,i}$  emanating from the vertices, we have:*

$$\begin{aligned} & \arccos\left(\frac{\cos(\alpha_{i,j}) + \cos(\alpha_i)\cos(\alpha_j)}{\sin(\alpha_i)\sin(\alpha_j)}\right) \\ & + \arccos\left(\frac{\cos(\alpha_{j,k}) + \cos(\alpha_j)\cos(\alpha_k)}{\sin(\alpha_j)\sin(\alpha_k)}\right) \\ & + \arccos\left(\frac{\cos(\alpha_{k,i}) + \cos(\alpha_k)\cos(\alpha_i)}{\sin(\alpha_k)\sin(\alpha_i)}\right) < \pi. \end{aligned}$$

Furthermore, this hyperbolic polyhedron is unique.

The proof is evidently a direct consequence of Theorem 4 and the formula for the face angles.

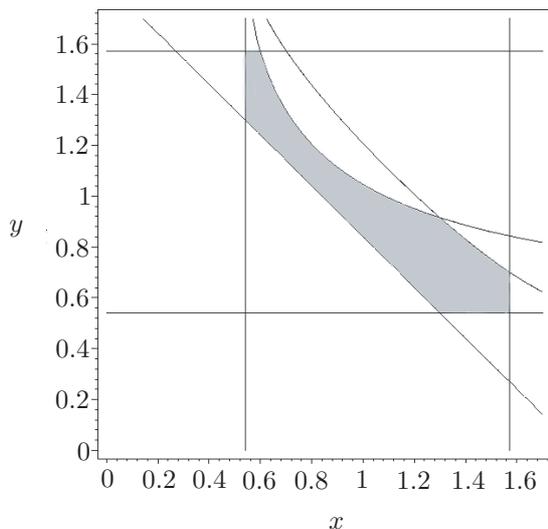
The reader should notice that Theorem 4 can also be proved directly from Theorem 1, the characterization of hyperbolic tetrahedra in terms of the Gram matrix  $M$ . In Lemma 2 we checked that if we restrict to non-obtuse dihedral angles, condition (2) from Theorem 4 is equivalent to the condition that every principal minor of  $M$  is positive definite.

Similar trigonometric tricks can be used to show that conditions (1) and (3) from Theorem 4 are equivalent to  $\det(M) < 0$ . If a face  $F$  contains face angles  $\beta_i, \beta_j, \beta_k$  and the edges surrounding  $F$  have dihedral angles  $\alpha_l, \alpha_m, \alpha_n$ , then:

$$\begin{aligned} \det(M) &= -4(1 - \cos^2 \alpha_l)(1 - \cos^2 \alpha_m)(1 - \cos^2 \alpha_n) \\ &\quad \times \cos\left(\frac{\beta_i + \beta_j + \beta_k}{2}\right) \cos\left(\frac{\beta_i - \beta_j + \beta_k}{2}\right) \\ &\quad \times \cos\left(\frac{\beta_i + \beta_j - \beta_k}{2}\right) \cos\left(\frac{-\beta_i + \beta_j + \beta_k}{2}\right). \end{aligned}$$

Condition (1) from Theorem 4 requires that the dihedral angles are positive and non-obtuse, so the second line of this equation is negative. The third line is positive if and only if  $\beta_i + \beta_j + \beta_k < \pi$ , since the face angles are non-obtuse (because the dihedral angles are non-obtuse). Hence, Theorem 4 does follow from Theorem 1. However, the author feels that the proof using the method of continuity is more intuitive.

In terms of the dihedral angles, condition (3) is reasonably nasty. In fact, it results in  $A_\Delta$  being non-convex! Consider the hyperbolic tetrahedron with dihedral angles  $x$  and  $y$  assigned to two edges that meet at a vertex and dihedral angles  $\alpha$  assigned to the remaining edges. The following figure was computed in Maple [1] and shows the cross section of  $A_\Delta$  when  $\alpha = 1.3$ .



This classification of hyperbolic tetrahedra in terms of their dihedral angles gives us some understanding of how a generalization of Andreev's Theorem [5], [13], [19], [20] to include obtuse dihedral angles would be significantly more complicated than Andreev's Theorem.

One obvious difficulty in considering arbitrary dihedral angles is that one cannot restrict to studying hyperbolic polyhedra realizing trivalent abstract polyhedra, a restriction that was essential in the proof of Andreev's Theorem.

However, a further difficulty that arises even for trivalent hyperbolic polyhedra is that for each  $n$ -sided face  $\mathcal{F}$ , there is the necessary condition

that the sum of the face angles of  $\mathcal{F}$  must be  $< (n - 2)\pi$ , resulting from the Gauss-Bonnet Theorem. As in conditions (3) in Theorems 4 and 10 from this paper, this results in highly non-linear necessary conditions on the dihedral angles.

The restriction to non-obtuse dihedral angles results in non-obtuse face angles via equation (2). Therefore, when studying hyperbolic polyhedra with non-obtuse dihedral angles, this condition on face angles is irrelevant for faces with more than 5 edges. Part of the proof of Andreev's Theorem is to show that, as long as the polyhedron has more than 4 faces, this condition on face angles for 3 and 4-sided faces is automatically a consequence of two other *linear* necessary conditions on the dihedral angles. In the statement of Andreev's Theorem as written in [19], [20], these are conditions (3)–(5).

However, once the dihedral angles can be obtuse, these conditions on face angles for 3 and 4-sided faces are no longer a consequence of conditions (3)–(5) of Andreev's Theorem. Furthermore, this condition on face angles becomes relevant for faces with 5 and more edges because the face angles are no longer restricted to be non-obtuse.

Of course, one can also expect that other conditions may be necessary to prevent more exotic types of degeneracies.

**Acknowledgements.** The author thanks the anonymous referee for his or her detailed comments, and especially for informing the author of the characterization of hyperbolic tetrahedra in terms of the Gram matrix. The author also thanks Joshua Bowman, William D. Dunbar, John H. Hubbard, and Rodrigo Pérez for their mathematical suggestions.

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Primera versió rebuda el 3 de maig de 2005,  
darrera versió rebuda el 19 d'octubre de 2005.