Publ. Mat. **50** (2006), 57–70

TOPOLOGICAL LINEAR COMPACTNESS FOR GROTHENDIECK CATEGORIES. THEOREM OF TYCHONOFF. APPLICATIONS TO COALGEBRAS

P. Enache, C. Năstăsescu and B. Torrecillas

Abstract

We show the Tychonoff's theorem for a Grothendieck category with a set of small projective generators. Strictly quasi-finite objects for semiartinian Grothendieck categories are characterized. We apply these results to the study of the Morita duality of dual algebra of a coalgebra.

Introduction

The classical Morita theory duality of module categories (cf. [**17**] and [**24**]) was extended to Grothendieck categories by Colby and Fuller [**6**] and a weaker form by Ánh and Wiegandt [**4**]. Many authors have considered this theory in general Grothendieck categories (cf. [**12**] and [**20**]) and also in particular Grothendieck categories (e.g. graded modules aver graded rings [**16**], modules over rings with enough idempotents [**3**], comodules over a coalgebra [**13**]). Linear compact objects with respect to a topology play a crucial role in the analysis of Morita dualities in Grothendieck categories (see [**4**], [**12**]).

In this paper we study linear compact objects in Grothendieck categories. The first section is notational and moreover it contains some preliminary results. Section 2 is devoted to characterizing linear compact objects in semiartinian Grothendieck categories (Theorem 2.3).

In Section 3 we consider the Tychonoff's theorem. This theorem is a classical result for linear compact modules over unitary rings, which does not remain true for general Grothendieck categories. We obtain a

2000 *Mathematics Subject Classification.* 16W30, 18E15, 16D90.
Key words. Tychonoff's theorem, coalgebras, quasi-finite objects, semiartinian Grothendieck categories, Morita duality.

positive answer for Grothendieck categories having a set of small projective generators. This result applies for right semiperfect coalgebras, graded rings and rings with local units.

In Section 4 we study the linear compactness of the dual algebra of a coalgebra. For this algebras the discrete linear compactness is equivalent to the noetherian property, and when the coalgebra is almost connected, the topological linear compactness is equivalent to the almost noetherian property (Theorem 4.8). Furthermore, we show that if this algebra has a right Morita duality it has a self-duality.

1. Notation and preliminary results for categories

Let $\mathcal{C}$ be an abelian category with direct products. Recall that a topology τ on an object $X \in \mathcal{C}$ is a filter base $(X_i)_{i\in I}$ of subobjects of X. For a subobject Y of X the τ-closure of Y is given by $\overline{Y} = \cap_I (Y + X_i)$, thus Y is τ-closed if $Y = \cap_I (Y + X_i)$. Y is open if it belongs to the filter generated by $(X_i)_{i\in I}$, i.e. $X_i \subseteq Y$ for some i (see [**4**]).

Let $\mathcal{A}$ and $\mathcal{B}$ be two abelian categories with direct products and let $F\colon \mathcal{A} \to \mathcal{B}$ be an exact functor which commutes with direct products, then

i) If (X, τ) is a topological object with (X_i) the filter base, then $(F(X), \tau')$ is a topological object with filter base $(F(X_i))$. This is clear because F commutes with intersections.

ii) If $Y \subseteq X$ is a τ-closed, then $F(Y) \subseteq F(X)$ is τ'-closed. Indeed, we note that, since F is exact, F commutes with finite sums and, moreover, since F preserve direct products F commutes with arbitrary intersections. Thus

$$F(Y) = F(\cap_{i\in I}(Y + X_i)) = \cap_{i\in I} F(Y + X_i) = \overline{F(Y)}.$$

iii) Let (X_i, τ_i) be a family of objects of $\mathcal{A}$. We can construct the topological product object $(\prod_{i\in I} X_i, \tau)$, where τ is the product (Tychonoff) topology. Throughout this paper, we shall always assume that $\prod_i X_i$ is endowed with the product topology.

Definition 1.1. A topological object X is *linearly compact* if for every filter base (X_i) of closed subobjects of X, the canonical morphism

$$X \longrightarrow \varprojlim X/X_i$$

is an epimorphism.

X is *discrete linearly compact* if X is linearly compact with respect to the topology defined by the filter base $\{0\}$.

The next result shows the behavior of linearly compact objects under localization.

Lemma 1.2. *Let $\mathcal{T}$ be any localizing subcategory of the Grothendieck category $\mathcal{A}$, and let $T\colon \mathcal{A} \to \mathcal{A}/\mathcal{T}$ be the localizing functor. Assume that T commutes with direct products. If (X,τ) is linearly compact then $(T(X),\tau')$ is linearly compact, where τ' is the linear topology induced by τ via functor T.*

Proof: If τ is the linear topology given by the filter base $(X_i)_{i\in I}$, then τ' is given by the filter base $(T(X_i))_{i\in I}$. If $Y \subseteq T(X)$ is a closed subobject relative to τ', we consider $Z = \psi_X^{-1}(S(Y))$ where S is the right adjoint to T and $\psi_X\colon X \to ST(X)$ is the unit of the adjoint situation. Thus, $T(Z) = T(S(Y)) = Y$. Moreover,

$$T(\overline{Z}) = \cap_{i\in I}(T(Z) + T(X_i)) = \cap_{i\in I}(Y + T(X_i)) = Y$$

since Y is closed. Hence if we have a filter base of closed subobjects $(Y_j)_j$ of $T(X)$, there exists a filter base $(Z_j)_j$ of closed subobjects of X such that $T(Z_j) = Y_j$. Now

$$X \longrightarrow \varprojlim X/Z_j \longrightarrow 0$$

is exact. Since T is exact and it commutes with direct products then T commutes also with direct projective limits so the following sequence

$$T(X) \longrightarrow T(\varprojlim X/Z_j) \cong \varprojlim T(X)/Y_j \longrightarrow 0$$

is exact. Thus $T(X)$ is linearly compact in $\mathcal{A}/\mathcal{T}$. $\square$

Example 1.3. Let $\mathcal{A}$ be a Grothendieck category and $P \in \mathcal{A}$ be a projective object. We consider the class

$$\mathcal{C}_P = \{A \in \mathcal{A} \mid \mathrm{Hom}_{\mathcal{A}}(P,A) = 0\}.$$

It is easy to see that $\mathcal{C}_P$ is a TTF class. Following [**10**], we can consider the quotient category $\mathcal{A}/\mathcal{C}_P$ and the canonical functor

$$\mathcal{A} \underset{S_P}{\overset{T_P}{\rightleftarrows}} \mathcal{A}/\mathcal{C}_P.$$

Since $\mathcal{C}_P$ is a TTF class it is well-known that T_P commutes with direct products. Hence, by Lemma 1.2, if M is linearly compact then $T_P(M)$ is also linearly compact.

Moreover, we recall that if P is small, then, by [**5**, Theorem 1.2],

$$\mathcal{A}/\mathcal{C}_P \cong R\text{-}\mathrm{Mod}$$

where $R = \mathrm{End}_{\mathcal{A}}(P)$.

Following [**23**] the ring R is said to be a ring with local units if for every finite subset X of R there exists an idempotent $e \in R$ such that X is contained in the ring eRe. For a ring R with local units we denote by R-MOD the category of all left R-modules M with the property that $RM = M$. Clearly the category R-MOD is a Grothendieck category. Recently, the interest for this class of rings has been motivated by the fact that for certain H-comodule algebras (e.g. H a Hopf algebra with a nonzero integral) its associated Doi-Hopf module category is isomorphic to the category of R-MOD for certain ring R with local units (see [**8**]). It is well-knonw (see [**23**]) that $\{Re \mid e^2 = e\}$ is a generating set of finitely generated projective modules in R-MOD. The following result connects rings with local units with Grothendieck categories.

Proposition 1.4. *Let $\mathcal{A}$ be a Grothendieck category with a set $(P_i)_{i\in I}$ of small projective generators. Then there exists a ring R with local units such that $\mathcal{A}$ is equivalent to the category R-MOD.*

Proof: See [**5**, Theorem 3.1] and [**2**]. □

2. Quasi-finite objects

Recall that an object X satisfies $AB-5^*$ if for any subobjet Y and an inverse family of subobjects $\{X_i\}_{i\in I}$ of X, then

$$Y + \cap_{i\in I} X_i = \cap_{i\in I}(Y + X_i).$$

As in the case of R-modules, it is easy to show that any discrete linearly compact object satisfies $AB-5^*$ and that the class of linearly compact objects is a Serre subcategory. A semiartinian object is called quasi-finite if its socle has finite generated homogeneous component. A semiartinian object is strictly quasi-finite if any quotient is quasi-finite.

The next lemma can be shown using the same argument as [**13**, Lemma 2.7 and Proposition 2.8].

Lemma 2.1. *Let $\mathcal{T}$ be any localizing subcategory of the semiartinian Grothendieck category $\mathcal{A}$, and let $T\colon \mathcal{A} \to \mathcal{A}/\mathcal{T}$ be the localizing functor.*

- i) *If X is a simple object of $\mathcal{A}/\mathcal{T}$, then there exists a simple object $Y \in \mathcal{A}$ such that $T(Y) \cong X$.*
- ii) *If M is a strictly quasi-finite object in $\mathcal{A}$, then $T(M)$ is strictly quasi-finite.*

Let $\{S_i \mid i \in I\}$ be a complete set of representatives of the isomorphism types of simple objects of $\mathcal{A}$. For every $i \in I$, let $T_i\colon \mathcal{A} \to \mathcal{A}/\mathcal{T}_{E(S_i)}$ denote the localization functor, where

$$\mathcal{T}_{E(S_i)} = \{X \in \mathcal{A} \mid \operatorname{Hom}(X, E(S_i)) = 0\}.$$

Proposition 2.2. *Let $\mathcal{A}$ be a semiartinian Grothendieck category. An object A is strictly quasi-finite if and only if $T_i(A)$ is artinian for any i.*

Proof: Let A be strictly quasi-finite, then by Lemma 2.1 $T_i(A)$ is strictly quasi-finite in the quotient category $\mathcal{A}/\mathcal{T}_{(E(S_i))}$. This category has an unique isomorphic type of simple module, hence strictly quasi-finite of $T_i(A)$ implies that $T_i(A)/B$ is quasi-finite for any subobject B of $T_i(A)$. Therefore $T_i(A)/B$ is finitely cogenerated and $T_i(A)$ is artinian.

Conversely, assume that $T_i(A)$ is artinian for every $i \in I$. Since the functors T_i are exact, it is enough to prove that A is quasi-finite. Let S be a simple object and consider the unique $i \in I$ such that $S \cong S_i$. If A_i denotes the largest subobject of A such that $A_i \in \mathcal{T}_{E(S_i)}$, then we have an exact sequence

$$0 \longrightarrow A_i \longrightarrow A \longrightarrow A/A_i \longrightarrow 0.$$

Then $\operatorname{Hom}(S_i, A_i) = 0$ and $\operatorname{Hom}(S_i, A) \leq \operatorname{Hom}(S_i, A/A_i)$. Thus, we can assume that $A_i = 0$, that is, A is $\mathcal{T}_{E(S_i)}$ torsionfree. The socle $\operatorname{soc}(A)$ is then a direct sum of copies of S_i and, since $T_i(A)$ is artinian, we have that $T_i(\operatorname{soc}(A)) \leq T_i(A)$ is artinian as well. Therefore, $\operatorname{soc}(A)$ consists of a direct sum of finitely many copies of $S_i \cong S$. □

By [**19**, Proposition 2.5], if A is a semiartinian object in $\mathcal{A}$ satisfying $AB-5^*$, then A is strictly quasi-finite.

Theorem 2.3. *Let $\mathcal{A}$ be a semiartinian Grothendieck category. Assume that every simple object of $\mathcal{A}$ has a projective cover. The following properties of an object M are equivalent:*

1) *M is discrete linearly compact;*
2) *M is $AB-5^*$;*
3) *M is strictly quasi-finite.*

Proof: It remains to show that 3) ⇒ 1). Let $\{S_i : i \in I\}$ be a complete set of representatives of the isomorphism types of simple objects of $\mathcal{A}$. For every $i \in I$, we may consider the localizing subcategory

$$\mathcal{T}_{E(S_i)} = \{X \in \mathcal{A} \mid \operatorname{Hom}(X, E(S_i) = 0\}.$$

Let $T_i\colon \mathcal{A} \to \mathcal{A}/\mathcal{T}_{E(S_i)}$ denote the localization functor. By [**5**, Theorem 3.3] every $\mathcal{T}_{E(S_i)}$ is a TTF-class. Since the projective cover of the simple modules form a family of finitely generated projective generators by [**5**, Theorem 3.2], T_i has a left adjoint functor H_i. Hence T_i commutes with projective limits. Let

$$M \xrightarrow{p_\lambda} M_i \longrightarrow 0$$

be an inverse system of epimorphisms. Then

$$T_i(p_\lambda)\colon T_i(M) \longrightarrow T_i(M_\lambda)$$

are epimorphisms. Since M is strictly quasi-finite $T_i(M)$ is an artinian object. Hence

$$\varprojlim T_i(p_\lambda)\colon M \longrightarrow \varprojlim T_i(M_\lambda)$$

is an epimorphism. Thus

$$T_i(\varprojlim p_\lambda)$$

is an epimorphism for any i. Since T_i is exact and $\cap \operatorname{Ker} T_i = 0$, then

$$\varprojlim p_\lambda\colon M \longrightarrow \varprojlim M_\lambda$$

is an epimorphism. □

3. The theorem of Tychonoff for Grothendieck categories

Theorem 3.1. *Let $\mathcal{A}$ be a Grothendieck category with a set of small projective generators and let $(X_k, \tau_k)_{k\in\Lambda}$ be a family of topological objects. If (X_k, τ_k) is linearly compact for every $k \in \Lambda$, then $(\prod_{k\in\Lambda} X_k, \tau)$ is linearly compact.*

Proof: By Proposition 1.4 we have that $\mathcal{A} \cong R$-MOD where R is a ring with local units. Now we observe that if $(M_i)_{i\in I}$ is a family of objects in R-MOD, then the direct product of this family in R-MOD is equal to $R\prod_{i\in I} M_i = \cup_e \prod_{i\in I} M_i$, where $\prod_{i\in I} M_i$ is the usual cartesian product and e runs over all the idempotents of R. Moreover, the same statement holds for inverse limits. These two observations join with the definition of linearly compact object reduce the problem to the case of unitary rings of the form eRe, where $e \in R$ is idempotent. But for unitary rings the result is true (see [**22**, Theorem 28.7]. □

Remark 3.2. An alternative proof of the preceding result and the classical result for linearly compact modules over unitary rings can be given using Lemma 1.2, Proposition 1.4 and Example 1.3.

Examples.

1. Right semiperfect coalgebras

If C is a right semiperfect coalgebra then $\mathbf{M}^C$ is a semiartinian Grothendieck category and simple objects have projective covers. The next result shows that Tychonoff's theorem holds in $\mathbf{M}^C$ when C is right semiperfect.

Corollary 3.3. *Let C be a semiperfect coalgebra and let $(M_i)_{i\in I}$ be a family of linearly compact right comodules, then $\prod_{i\in I} M_i$ is linearly compact in $\mathbf{M}^C$ (here we consider the direct product in the category $\mathbf{M}^C$ with the product topology).*

In general the result is not true for the category of comodules, as we can show in the following

Counterexample 3.4. Let $C = k[x]$ be the Hopf algebra of the polynomials in one indeterminate x over a field k of characteristic zero; its structure of coalgebra is given by $\Delta(x) = x \otimes 1 + 1 \otimes x$ and $\epsilon(x) = 0$. In this case $C^* = K[[x]]$ and $\mathbf{M}^C$ is isomorphic to the subcategory of torsion $K[[x]]$-modules. Take $U_i = K[[x]]/(x^i)$ for $i = 1, 2, \dots$. Then the U_i are artinian objects, so U_i are (discrete) linearly compact in $\mathbf{M}^C$ and C^*-Mod. Consider

$$M' = t\left(\prod_{i=1}^{\infty} U_i\right) \subseteq M = \prod_{i=1}^{\infty} U_i,$$

where t is the classical torsion radical on $K[[x]]$.

We will see that M', the product of U_i in $\mathbf{M}^C$, is not linearly compact with respect to the product topology. Let

$$H_i = ((x)/(x^i) \times K[[x]]/(x^{i+1}) \times \cdots) \cap M'.$$

Clearly each H_i is a closed submodule of M'. Consider the set of coset $a_i = (0, \dots, 1 + x^i, 0, \dots) + H_i$, then $(x)a_i \subseteq H_i$. This set of cosets has the finite intersection property but not the intersection property.

2. Graded rings

Let $R = \oplus_{\sigma\in G} R_\sigma$ be a graded ring. Since $\{R(\sigma), \sigma \in G\}$ is a family of small projective generators in R-gr, then the products of linearly compact graded modules are linearly compact (see [**16**] for details about linearly compact graded modules).

3. Rings with local units

Linearly compact modules for this category has been considered in [**3**]. From the proof of Theorem 3.1 it follows that the Tychonoff's theorem is valid for modules over ring with local units.

4. Linear compactness of the dual algebra of a coalgebra

Let C be a coalgebra and let $C^* = \mathrm{Hom}_k(C, k)$ be its dual convolution algebra. Denote by ${}^C\mathbf{M}$ the category of left C-comodules.

A right Morita duality is an additive contravariant category-equivalence between two categories of Mod-R and S-Mod which are both closed under submodules and factor modules and contains all finitely generated modules. This is equivalent to say that there exists a (S, R)-bimodule such that (a) E_R and ${}_SE$ are injective cogenerators and (b) the right and left multiplication induces isomorphisms $\mathrm{End}_R(E) \cong S$ and $\mathrm{End}_S(E) \cong R$. If $S = R$, then we say that R has a right Morita self-duality. It is well-known the connection between Morita duality theory and the notion of linear compactness (see [**18**]). First, we examine the Morita duality theory for C^*.

Proposition 4.1. *Let C be a coalgebra. If C^* possesses a right Morita duality then C^* is left noetherian.*

Proof: If C^* has a right Morita duality, then C^* is right linearly compact [**18**, Theorem 1]. Hence C^*/J is right linearly compact, therefore it has finite Goldie dimension, then it is semisimple, where J denotes the Jacobson radical of C^*. Thus C^* has only a finite number of isomorphic types of simple modules. If follows that C_{C^*} has finite generated socle. Since the injective envelope of each simple right C^*-module is reflexive, then C_{C^*}, as finite cogenerated, is reflexive. For any right C^*-submodule X of C, we have that $(C/X)_{C^*}$ is reflexive. Hence C/X has a finite generated socle and it is finitely cogenerated. Therefore C as right C^*-module is artinian. As C_{C^*} is quasi-injective, we can apply [**1**, Corollary 4.4] and C^* is left noetherian. □

Theorem 4.2. *Let C be a coalgebra. C^* has a Morita self-duality if and only if C^* is both sides noetherian.*

Proof: Assume that C^* is left noetherian, then by [**7**, Theorem 3.2] the class of rational left C^*-modules is closed under injective envelopes. Hence the left C^*-module C is injective. By [**11**, Theorem 2], C as right C^*-module is linearly compact. Hence C_{C^*} has a finitely generated socle, then C is an almost connected coalgebra. Hence C is an injective cogenerator of C^*-Mod. By a similar argument, C is also an injective cogenerator of Mod-C^*. Since C satisfies that $\mathrm{End}_{C*}(C_{C^*}) = \mathrm{End}_{C^*}({}_{C^*}C) = C^*$, then ${}_{C^*}C_{C^*}$ gives a sef-duality of C^*.

The other implication follows from Proposition 4.1 and its left side version. □

Proposition 4.3. *Let M be a left C-comodule (then M is also a right C^*-module). The following conditions are equivalent:*

i) *${}_{C^*}M^*$ is discrete linearly compact;*
ii) *M^* is a left noetherian C^*-module;*
iii) *M is artinian as a left C^*-comodule.*

Proof: i) $\Rightarrow$ iii) Let X be a right C^*-submodule of M and consider $Y = \mathrm{soc}(M/X)$. We have the following commutative diagram of right C^*-modules

$$\begin{tikzcd}
 & & & 0 \arrow[d] & \\
 & & & Y \arrow[d] & \\
0 \arrow[r] & X \arrow[r] & M \arrow[r] & M/X \arrow[r] & 0
\end{tikzcd}$$

By taking dual we obtain a commutative diagram of left C^*-modules

$$\begin{tikzcd}
0 \arrow[r] & (M/X)^* \arrow[r] \arrow[d] & M^* \arrow[r] & X^* \arrow[r] & 0 \\
 & Y^* \arrow[d] & & & \\
 & 0 & & &
\end{tikzcd}$$

Hence Y^* is left linearly compact. If $Y = \oplus_I S_i$, then $Y^* = \prod_I S_i^*$, but Y^* linearly compact implies that I is finite. Hence M/X is finitely cogenerated and therefore M is artinian as right C^*-comodule.

iii) $\Leftrightarrow$ ii) We can apply [**1**, Corollary 4.3]. We note that C is quasi-injective as right C^*-module, $\mathrm{Hom}_{C^*}(M,C) \cong \mathrm{Hom}_k(M,k) = M^*$ and any submodule of M is closed.

iii) $\Rightarrow$ i) Let X_i be an inverse family of left C^*-submodules of M^*. Since M^* is left noetherian, then the X_i's are finitely generated. Now by [**1**, Proposition 4.1] and $X_i = \mathrm{Hom}_{C^*}(M/Y_i, C) = (M/Y_i)^*$ for some $Y_i \subseteq M_{C^*}$, and the Y_i will form a direct system. From the exact sequence

$$0 \longrightarrow Y_i \longrightarrow M \longrightarrow M/Y_i \longrightarrow 0$$

we obtain

$$0 \longrightarrow (M/Y_I)^* \longrightarrow M^* \longrightarrow Y_i^* \longrightarrow 0$$

so $M/X_i \cong Y_i^*$. Now

$$\varprojlim M^*/X_i = \varprojlim Y_i^* \cong (\varinjlim Y_i)^* = (\cup_i Y_i)^*.$$

Since $\cup_i Y_i \subseteq M$, it follows

$$M^* \longrightarrow (\cup_i Y_i)^* \longrightarrow 0.$$

□

Corollary 4.4. *C^* is left linearly compact for the discrete topology if and only if C^* is left noetherian.*

B. J. Müller [**18**] showed that for commutative rings the existence of Morita dualities implies the existence of self dualities.

Corollary 4.5. *Let C be a coalgebra. If C^* has a right Morita dualiy then C^* has a self-Morita duality.*

Proof: If C^* has a right Morita duality, then C^* is left noetherian by Proposition 4.1. Moreover, by [**18**, Theorem 1] $C^*_{C^*}$ is discrete linearly compact, then Corollary 4.4 implies that C^* is right noetherian. By Theorem 4.2, C^* has self-Morita duality. □

Let R be a k-algebra. A left R-module M is called *almost noetherian* if every cofinite submodule of M is finitely generated. Submodules, quotients, and extensions of almost noetherian modules are almost noetherian. The algebra R is left almost noetherian if its almost noetherian as a left module.We call R almost noetherian if it is left and right almost noetherian. See [**7**, Section 1.1] for further details on almost noetherian modules and algebra.

The coradical filtration on C induces a filtration on C^* ,

$$C^* \supseteq C_0^{\perp} \supseteq C_1^{\perp} \cdots . \tag{1}$$

The two-sided ideal $J = C_0^{\perp}$ is the Jacobson radical of C.

Proposition 4.6. *Let C be a coalgebra and R the graded dual ring of* $\mathrm{gr}(C)$. *Denote by* $\mathrm{gr}(C^*)$ *the graded ring associated to the filtration* (1). *Then*

i) $\mathrm{gr}(C^*) \cong R$ *as graded rings.*
ii) C^* *is complete with respect to* (1).

Lemma 4.7. C^* *is linearly compact with respect to* (1) *if and only if* C_i^* *is linearly compact for the discrete topology for all* i.

Proof: It follows from [**22**, Theorem 28.15] and the fact that $C^*/C_i^{\perp} \cong C_i^*$ and C^* is complete with respect to (1).

It is well-known that the category of right C-comodules $\mathbf{M}^C$ is isomorphic to $\mathrm{Rat}(C^*\text{-}\mathrm{Mod})$, the subcategory of $C^*\text{-}\mathrm{Mod}$ consisting of all rational left C^*-modules. $\mathrm{Rat}(C^*\text{-}\mathrm{Mod})$ is closed under submodules, quotients and arbitrary direct sums. In the sense of [**21**], it is a hereditary pretorsion class in $C^*\text{-}\mathrm{Mod}$. See loc.cit. for further detail on torsion classes. The left exact preradical associated to $\mathrm{Rat}(C^*\text{-}\mathrm{Mod})$ is the rational functor $\mathrm{Rat}_C(-)\colon C^*\text{-}\mathrm{Mod} \to C^*\text{-}\mathrm{Mod}$. Given $M \in C^*\text{-}\mathrm{Mod}$, $\mathrm{Rat}_C(M)$ is the sum of all rational modules contained in M. The left linear topology $\mathcal{F}_C$ on C corresponding to $\mathrm{Rat}(C^*\text{-}\mathrm{Mod})$ is the family of all closed (in the weak-* topology) and cofinite left ideals of C^*. If $I \in \mathcal{F}_C$ there is a finite-dimensional left coideal W of C such that $I = W^{\perp}$. Recall that a coalgebra C is called almost connected if its coradical C_0 is finite dimensional. ☐

Theorem 4.8. *If* C^* *is linearly compact for the topology* (1), *then* C^* *is almost noetherian. Moreover, if* C *is almost connected both conditions are equivalent.*

Proof: Assume that C^* is linearly compact for the topology (1). Since $C^*/C_0^{\perp} \cong C_0^*$ is discrete linearly compact it follows that C is almost connected. Then since $C_i^{\perp}$ is open, it follows that $C^*/C_i^{\perp}$ is linearly compact for the discrete topology. But $C^*/C_i^{\perp} \cong C_i^*$, hence C_i^* is linearly compact. Now the exact sequence

$$0 \longrightarrow C_1 \longrightarrow C_0 \longrightarrow C_0/C_1 \longrightarrow 0$$

induces

$$0 \longrightarrow (C_1/C_0)^* \longrightarrow C_1^* \longrightarrow C_0^* \longrightarrow 0.$$

Thus $(C_1/C_0)^*$ is linearly compact. Assume that $C_1/C_0 = \oplus_I S_i$, then $(C_1/C_0)^* \cong \prod_I S_i^*$. It follows that I is finite because $(C_1/C_0)^*$ is linearly compact. Hence C_1/C_0 is finite dimensional, and C_1 is finite dimensional. Using induction we obtain that C_i is finite dimensional for all i. By [**7**, Theorem 2.8], C^* is almost noetherian.

Suppose that C^* is almost connected, then J is in $\mathcal{F}_C$. If C^* is almost noetherian, by [**7**, Theorem 2.8], we have that Rat is a torsion radical and hence $J^n \in \mathcal{F}_C$ for all n. Now $C_i^{\perp} = \overline{J^{i+1}} = J^{i+1}$. Thus the J-adic topology and the topology defined by (1) coincide. Therefore C^* is complete for the J-adic topology and $C^*/J^i \cong C_i^*$ is finite dimensional for all i. Now the result follows from Lemma 4.7. □

Acknowledgements. The authors thank the referee for suggesting a different proof of Theorem 3.1.

The first author has a Ramón y Cajal's contract with the University of Almería.

The second author has been partially supported by grant BFM2005-03227 from Ministerio de Ciencia y Tecnología of Spain.

References

[1] T. Albu and C. Năstăsescu, *"Relative finiteness in module theory"*, Monographs and Textbooks in Pure and Applied Mathematics **84**, Marcel Dekker, Inc., New York, 1984.

[2] P. N. Ánh, A representation of some Grothendieck categories, *Rad. Mat.* **3(1)** (1987), 159–164.

[3] P. N. Ánh and C. Menini, Morita duality for rings with local units, *J. Algebra* **164(3)** (1994), 632–641.

[4] P. N. Ánh and R. Wiegandt, Morita duality for Grothendieck categories, *J. Algebra* **168(1)** (1994), 273–293.

[5] F. Castaño Iglesias, N. Chifan and C. Năstăsescu, Localization on certain Grothendieck categories, preprint.

[6] R. R. Colby and K. R. Fuller, Exactness of the double dual and Morita duality for Grothendieck categories, *J. Algebra* **82(2)** (1983), 546–558.

[7] J. Cuadra, C. Năstăsescu and F. Van Oystaeyen, Graded almost Noetherian rings and applications to coalgebras, *J. Algebra* **256(1)** (2002), 97–110.

[8] S. Dăscălescu, C. Năstăsescu and Ş. Raianu, *"Hopf algebras. An introduction"*, Monographs and Textbooks in Pure and Applied Mathematics **235**, Marcel Dekker, Inc., New York, 2001.

[9] C. Faith, *"Algebra. II. Ring theory"*, Grundlehren der Mathematischen Wissenschaften **191**, Springer-Verlag, Berlin-New York, 1976.

[10] P. GABRIEL, Des catégories abéliennes, *Bull. Soc. Math. France* **90** (1962), 323–448.

[11] J. L. GÓMEZ PARDO, Counterinjective modules and duality, *J. Pure Appl. Algebra* **61(2)** (1989), 165–179.

[12] J. L. GÓMEZ PARDO AND P. A. GUIL ASENSIO, Linear compactness and Morita duality for Grothendieck categories, *J. Algebra* **148(1)** (1992), 53–67.

[13] J. GÓMEZ-TORRECILLAS, C. NĂSTĂSESCU AND B. TORRECILLAS, Localization in coalgebras. Applications to finiteness condition, submited.

[14] R. G. HEYNEMAN AND D. E. RADFORD, Reflexivity and coalgebras of finite type, *J. Algebra* **28** (1974), 215–246.

[15] A. V. JATEGAONKAR, Morita duality and Noetherian rings, *J. Algebra* **69(2)** (1981), 358–371.

[16] C. MENINI AND Á. DEL RÍO, Morita duality and graded rings, *Comm. Algebra* **19(6)** (1991), 1765–1794.

[17] K. MORITA, Duality for modules and its applications to the theory of rings with minimum condition, *Sci. Rep. Tokyo Kyoiku Daigaku Sect. A* **6** (1958), 83–142.

[18] B. J. MÜLLER, Linear compactness and Morita duality, *J. Algebra* **16** (1970), 60–66.

[19] C. NĂSTĂSESCU AND B. TORRECILLAS, Morita duality for Grothendieck categories with applications to coalgebras, *Comm. Algebra* **33(11)** (2005), 4083–4096.

[20] K. OHTAKE, Morita duality for Grothendieck categories and its application, *J. Algebra* **174(3)** (1995), 801–822.

[21] B. STENSTRÖM, *"Rings of quotients"*, Die Grundlehren der Mathematischen Wissenschaften **217**. An introduction to methods of ring theory, Springer-Verlag, New York-Heidelberg, 1975.

[22] S. WARNER, *"Topological rings"*, North-Holland Mathematics Studies **178**, North-Holland Publishing Co., Amsterdam, 1993.

[23] R. WISBAUER, *"Foundations of module and ring theory"*, A handbook for study and research, Revised and translated from the 1988 German edition, Algebra, Logic and Applications **3**, Gordon and Breach Science Publishers, Philadelphia, PA, 1991.

[24] W. XUE, *"Rings with Morita duality"*, Lecture Notes in Mathematics **1523**, Springer-Verlag, Berlin, 1992.

P. Enache:
Faculty of Mathematics
University of Bucharest
Str. Academiei 14
RO-70109, Bucharest 1
Romania
E-mail address: `ecipi@yahoo.com`

C. Năstăsescu:
Faculty of Mathematics
University of Bucharest
Str. Academiei 14
RO-70109, Bucharest 1
Romania
E-mail address: `cnastase@al.math.unibuc.ro`

B. Torrecillas:
Departamento de Álgebra y Análisis
Universidad de Almería
04071 Almería
Spain
E-mail address: `btorreci@ual.es`

Primera versió rebuda el 2 de desembre de 2004,
darrera versió rebuda el 26 de setembre de 2005.