Publ. Mat. **50** (2006), 301–313

HARMONIC MAPS AND THE TOPOLOGY OF MANIFOLDS WITH POSITIVE SPECTRUM AND STABLE MINIMAL HYPERSURFACES

Qiaoling Wang*

Abstract

In this paper, we prove two Liouville theorems for harmonic maps and apply them to study the topology of manifolds with positive spectrum and stable minimal hypersurfaces in Riemannian manifolds with non-negative bi-Ricci curvature.

1. Introduction

Harmonic maps are natural generalizations of harmonic functions and are critical points of the energy functional defined on the space of maps between two Riemannian manifolds. The Liouville type properties for harmonic maps have been studied extensively in the past years (Cf. [**Ch**], [**C**], [**EL1**], [**EL2**], [**ES**], [**H**], [**HJW**], [**J**], [**SY**], [**S**], [**Y1**], etc.). In 1975, Yau [**Y1**] proved that any harmonic function bounded from one side on a complete Riemannian manifold with non-negative Ricci curvature must be a constant. Schoen and Yau [**SY**] have shown that a harmonic map of finite energy from a complete Riemannian manifold with non-negative Ricci curvature to a complete manifold with non-positive sectional curvature is constant. This Liouville theorem of Schoen-Yau was used [**SY**] to show the important result which states that any smooth map of finite energy from a complete Riemannian manifold with non-negative Ricci curvature to a compact manifold with non-positive sectional curvature is homotopic to constant on each compact set. In this paper, we use the same idea of Schoen-Yau to study complete non-compact manifolds with Ricci curvature bounded from below and stable minimal hypersurfaces

2000 *Mathematics Subject Classification.* 53C20, 53C42.
Key words. Liouville theorems, harmonic maps, topology, manifolds, spectrum, minimal hypersurfaces.
*Partially supported by CNPq.

in manifolds with non-negative bi-Ricci curvature. Our first result can be stated as follows.

Theorem 1.1. *Let M^n $(n \geq 2)$ and N^s $(s \geq 1)$ be two complete Rimannian manifolds. Suppose that M^n is non-compact and that N^s has non-positive sectional curvature. Assume that the first eigenvalue $\lambda_1(M^n)$ of M^n is positive and that the Ricci curvature of M^n satisfies*

$$\mathrm{Ric}_{M^n} \geq -\left(1 + \frac{1}{2ns}\right)\lambda_1(M^n) + \delta \tag{1.1}$$

for some $\delta > 0$. Then any harmonic map from M^n to N^s with finite energy is a constant.

It has been shown by Schoen-Yau (Cf. [**SY**]) that for any smooth map f of finite energy from a complete Riemannian manifold M to a compact manifold N with non-positive sectional curvature, there is a harmonic map $h\colon M \to N$ with finite energy such that f is homotopic to h on each compact set of M. Thus Theorem 1.1 implies immediately the following

Corollary 1.2. *Let s be a positive integer and let M^n $(n \geq 2)$ be a complete non-compact Rimannian manifold with $\lambda_1(M^n) > 0$ and*

$$\mathrm{Ric}_{M^n} \geq -\left(1 + \frac{1}{2ns}\right)\lambda_1(M^n) + \delta$$

for some $\delta > 0$. Let N^s be a compact manifold with non-positive sectional curvature. If $f\colon M^n \to N^s$ is a smooth map with finite energy, then f is homotopic to constant on each compact set.

As an application of this corollary, one has the following result.

Corollary 1.3. *Let M^n $(n \geq 2)$ be as in Theorem 1.1 and let D be a compact domain in M^n with smooth simply connected boundary. Then there exists no non-trivial homomorphism from $\pi_1(D)$ into the fundamental group of a compact manifold N with non-positive sectional curvature.*

Proof: We use the arguments in [**SY**]. Let $h\colon \pi_1(D) \to \pi_1(N)$ be a homomorphism. Since N is $K(\pi, 1)$, there is a smooth map $f\colon D \to N$ such that $f_* = h$. Observe that f is a homotopic to a constant map on ∂D because ∂D is simply connected. Thus, f can be extended to be a smooth map $\tilde{f}\colon M \to N$ such that outside a compact set, $\tilde{f}$ is constant. As $\tilde{f}$ has finite energy, we conclude from Corollary 1.2 that f is homotopic to a contant and that h is trivial. □

Before stating our next result, we fix some notation. Let M be a complete oriented minimal hypersurface immersed in an oriented Riemannian manifold $\overline{M}$. Let ∇ be the gradient operator of M and denote by $|A|^2$ the squared norm of the second fundamental form A of M in $\overline{M}$. M is said to be *stable* if

$$\int_M \{|\nabla f|^2 - (|A|^2 + \overline{\mathrm{Ric}}(\mu,\mu))f^2\} \geq 0 \tag{1.2}$$

for all $f\colon M \to R$ with compact support, where $\overline{\mathrm{Ric}}(\mu,\mu)$ is the Ricci curvature of $\overline{M}$ in the unit normal direction μ to M.

Definition 1.4 ([**ShY**], [**T**])**.** Let $\overline{M}$ be an m-dimensional complete Riemannian manifold, and u, v be orthonormal tangent vectors. We set

$$\text{b-Ric}(u,v) = \overline{\mathrm{Ric}}(u,u) + \overline{\mathrm{Ric}}(v,v) - \overline{K}(u,v),$$

and call it the bi-Ricci curvature in the directions u, v. Here $\overline{\mathrm{Ric}}$ and $\overline{K}$ denote the Ricci curvature and sectional curvature of $\overline{M}$, respectively.

If $m = 3$, then b-Ric $= s/2$, where s denotes the scalar curvature of $\overline{M}$. It is clear from the definition that the non-negativity of the sectional curvature implies the non-negativity of the bi-Ricci curvature of $\overline{M}$.

We then prove the following theorem which generalizes a main result in [**SY**].

Theorem 1.5. *Let M^n be an n-dimensional complete oriented non-compact stable minimal hypersurface in a complete $(n+1)$-dimensional Riemannian manifold $\overline{M}^{n+1}$ with non-negative bi-Ricci curvature. Assume that N is a complete Riemannian manifold with non-positive sectional curvature. If $f\colon M^n \to N$ is a harmonic map with finite energy, then f is constant.*

As in Corollaries 1.2 and 1.3, we have

Corollary 1.6. *Let M be an n-dimensional complete oriented non-compact stable minimal hypersurface in a complete $(n+1)$-dimensional Riemannian manifold $\overline{M}$ with non-negative bi-Ricci curvature. Assume that N is a compact Riemannian manifold with non-positive sectional curvature. If $f\colon M \to N$ is a smooth map with finite energy, then f is homotopic to constant on each compact set.*

Corollary 1.7. *Let M be as in Corollary 1.6 and let D be a compact domain in M with smooth simply connected boundary. Then there exists no non-trivial homomorphism from $\pi_1(D)$ into the fundamental group of a compact manifold with non-positive sectional curvature.*

2. Preliminaries

Let M^n and N^s be complete Riemannian manifolds. Let $f\colon M^n \to N^s$ be a harmonic map. Let $\{e_i\}_{i=1}^n$ and $\{\overline{e}_\alpha\}_{\alpha=1}^s$ be local orthonormal frames of M^n and N^s, respectively. Suppose $\{\omega_i\}_{i=1}^n$ and $\{\theta_\alpha\}_{\alpha=1}^s$ are the dual coframes of $\{e_i\}_{i=1}^n$ and $\{\overline{e}_\alpha\}_{\alpha=1}^s$, respectively, and $\{\omega_{ij}\}_{i,j=1}^n$ and $\{\theta_{\alpha\beta}\}_{\alpha,\beta=1}^s$ are the corresponding connection forms. Denote by R_{ijkl} and $K_{\alpha\beta\gamma\delta}$ the curvature tensors of M^n and N^s, respectively. Then we have the structure equations:

$$\begin{cases} d\omega_i = \sum_j \omega_{ij} \wedge \omega_j \\ \omega_{ij} + \omega_{ji} = 0 \\ d\omega_{ij} = \sum_k \omega_{ik} \wedge \omega_{kj} - \frac{1}{2} \sum_{k,l} R_{ijkl} \omega_k \wedge \omega_l. \end{cases}$$

$$\begin{cases} d\theta_\alpha = \sum_\beta \theta_{\alpha\beta} \wedge \theta_\beta \\ \theta_{\alpha\beta} + \theta_{\beta\alpha} = 0 \\ d\theta_{\alpha\beta} = \sum_\gamma \theta_{\alpha\gamma} \wedge \theta_{\gamma\beta} - \frac{1}{2} \sum_{\gamma,\delta} K_{\alpha\beta\gamma\delta} \theta_\gamma \theta_\delta. \end{cases}$$

Define $f_{\alpha i}$, $1 \le \alpha \le s$, $1 \le i \le n$ by

$$f^*(\theta_\alpha) = \sum_i f_{\alpha i} \omega_i. \tag{2.1}$$

Then the energy density $e(f)$ is given by

$$e(f) = \sum_{\alpha,i} f_{\alpha i}^2.$$

Taking the exterior differentiation of (2.1), we get

$$f^*(d\theta_\alpha) = \sum_i \left(df_{\alpha i} \wedge \omega_i + f_{\alpha i}\, d\omega_i \right),$$

which gives

$$\sum_i \left(df_{\alpha i} - \sum_j f_{\alpha j} \omega_{ij} - f^*(\theta_{\alpha\beta}) f_{\beta i} \right) \wedge \omega_i = 0. \tag{2.2}$$

Define $f_{\alpha ij}$ by

$$df_{\alpha i} + \sum_\beta f_{\beta i} f^*(\theta_{\beta\alpha}) + \sum_j f_{\alpha j} \omega_{ji} = \sum_j f_{\alpha ij} \omega_j. \tag{2.3}$$

Then (2.2) and (2.3) imply that $f_{\alpha ij} = f_{\alpha ji}$ and f is harmonic means

$$\sum_i f_{\alpha ii} = 0, \quad \forall\, \alpha = 1, \ldots, s.$$

Exterior differentiating (2.3), we have

$$(2.4)\quad \sum_l \left(df_{\alpha il} + \sum_j (f_{\alpha ij}\omega_{jl} + f_{\alpha jl}\omega_{ji}) + \sum_\beta f_{\beta il} f^*(\theta_{\beta\alpha}) \right) \wedge \omega_l$$
$$= \frac{1}{2}\sum_{j,k,l} R_{ijkl} f_{\alpha j}\omega_k \wedge \omega_l + \frac{1}{2}\sum_{\beta,\delta,\gamma,k,l} K_{\alpha\beta\gamma\delta} f_{\beta i} f_{\gamma k} f_{\delta l}\omega_k \wedge \omega_l.$$

Define

$$\sum_k f_{\alpha ijk}\omega_k = df_{\alpha ij} + \sum_k (f_{\alpha ik}\omega_{kj} + f_{\alpha kj}\omega_{ki}) + \sum_\beta f_{\beta ij} f^*(\theta_{\alpha\beta});$$

then (2.4) implies that

$$f_{\alpha ikl} - f_{\alpha ilk} = \sum_j R_{ijlk} f_{\alpha j} + \sum_{\beta,\gamma,\delta} K_{\alpha\beta\gamma\delta} f_{\beta i} f_{\gamma l} f_{\delta k}.$$

Set $e = e(f)$ and let Δ be the Laplacian operator acting on functions on M^n. From the above formula, one can easily get the following Bochner type formula for harmonic maps which was first derived by Eells-Sampson [**ES**].

$$(2.5)\quad \frac{1}{2}\Delta e = \sum_{\alpha,i,j} f_{\alpha ij}^2 + \sum_{\alpha,i,j} R_{ij} f_{\alpha i} f_{\alpha j} - \sum_{\alpha,\beta,\gamma,\delta,i,j} K_{\alpha\beta\gamma\delta} f_{\alpha i} f_{\beta j} f_{\gamma i} f_{\delta j},$$

where R_{ij} is the Ricci tensor of M^n. Since

$$|\nabla\sqrt{e}|^2 = \frac{1}{e}\sum_j \left(\sum_{i,\alpha} f_{\alpha i} f_{\alpha ij} \right)^2,$$

we have

$$\sum_{\alpha,i,j} f_{\alpha ij}^2 - |\nabla\sqrt{e}|^2 = \frac{1}{2e}\sum_{i,j,k,\alpha,\beta} (f_{\alpha i} f_{\beta kj} - f_{\beta k} f_{\alpha ij})^2$$
$$\geq \frac{1}{2e}\sum_{i,j,\alpha} (f_{\alpha i} f_{\alpha jj} - f_{\alpha j} f_{\alpha ij})^2.$$

By Schwartz inequality,

$$\sum_{i,j,\alpha}(f_{\alpha i}f_{\alpha jj}-f_{\alpha j}f_{\alpha ij})^2 \geq \frac{1}{ns}\sum_i\left(\sum_{j,\alpha}(f_{\alpha i}f_{\alpha jj}-f_{\alpha j}f_{\alpha ij})\right)^2$$
$$= \frac{1}{ns}\sum_i\left(\sum_{\alpha j}f_{\alpha j}f_{\alpha ji}\right)^2$$
$$= \frac{1}{ns}|\nabla\sqrt{e}|^2.$$

Therefore, it holds [**SY**]

$$\sum_{\alpha,i,j} f_{\alpha ij}^2 \geq \left(1+\frac{1}{2ns}\right)|\nabla\sqrt{e}|^2. \tag{2.6}$$

3. Proofs of the Theorems

Proof of Theorem 1.1: Let $f\colon M^n \to N^s$ be a harmonic map with finite energy and set $e = e(f)$. Let $\lambda_1 = \lambda_1(M^n)$; then by definition,

$$\lambda_1 \int_{M^n}\psi^2 \leq \int_{M^n}|\nabla\psi|^2$$

for any compactly supported function $\psi \in H_{1,2}(M^n)$.

Replacing ψ by $\phi\sqrt{e}$ with $\phi \in C_0^\infty(M^n)$, we get

$$\lambda_1\int_{M^n}e\phi^2 \leq \int_{M^n}e|\nabla\phi|^2 + \int_{M^n}\phi^2|\nabla\sqrt{e}|^2 + 2\int_{M^n}\sqrt{e}\phi\nabla\sqrt{e}\nabla\phi. \tag{3.1}$$

Since N^s has non-positive sectional curvature, we conclude from (1.1), (2.5) and (2.6) that

$$\frac{1}{2}\Delta e \geq \left(1+\frac{1}{2ns}\right)|\nabla\sqrt{e}|^2 + \left(-\left(1+\frac{1}{2ns}\right)\lambda_1+\delta\right)e.$$

Thus one gets from the divergence theorem that

$$
\begin{aligned}
2\int_{M^n}\sqrt{e}\phi\nabla\sqrt{e}\nabla\phi &= \frac{1}{2}\int_{M^n}\nabla e\nabla\phi^2 \\
&= -\frac{1}{2}\int_{M^n}\phi^2\Delta e \\
&\leq -\left(1+\frac{1}{2ns}\right)\int_{M^n}\phi^2|\nabla\sqrt{e}|^2 \\
&\quad + \left(\left(1+\frac{1}{2ns}\right)\lambda_1-\delta\right)\int_{M^n}e\phi^2.
\end{aligned}
\tag{3.2}
$$

On the other hand, for any $l > 0$, we also have

$$
2\int_{M^n}\sqrt{e}\phi\nabla\sqrt{e}\nabla\phi \leq \frac{1}{l}\int_{M^n}|\nabla\sqrt{e}|^2\phi^2 + l\int_{M^n}e|\nabla\phi|^2. \tag{3.3}
$$

Take a sufficiently large l so that $2ns(1+l)\delta > \lambda_1$. We get from (3.2) and (3.3) that

$$
\begin{aligned}
&2\int_{M^n}\sqrt{e}\phi\nabla\sqrt{e}\nabla\phi \\
&\quad= \left(\frac{4ns(1+l)}{2ns(1+l)+l}+\frac{2l}{2ns(1+l)+l}\right)\int_{M^n}\sqrt{e}\phi\nabla\sqrt{e}\nabla\phi \\
&\quad\leq \frac{2ns(1+l)}{2ns(1+l)+l}\left\{-\left(1+\frac{1}{2ns}\right)\int_{M^n}\phi^2|\nabla\sqrt{e}|^2\right. \\
&\qquad\qquad \left.+\left(\left(1+\frac{1}{2ns}\right)\lambda_1-\delta\right)\int_{M^n}e\phi^2\right\} \\
&\qquad + \frac{l}{2ns(1+l)+l}\left\{\frac{1}{l}\int_{M^n}|\nabla\sqrt{e}|^2\phi^2 + l\int_{M^n}e|\nabla\phi|^2\right\} \\
&\quad= \frac{2ns(1+l)}{2ns(1+l)+l}\left(\left(1+\frac{1}{2ns}\right)\lambda_1-\delta\right)\int_{M^n}e\phi^2 - \int_{M^n}\phi^2|\nabla\sqrt{e}|^2 \\
&\qquad + \frac{l^2}{2ns(1+l)+l}\int_{M^n}e|\nabla\phi|^2,
\end{aligned}
$$

which, combining with (3.1), gives

$$\lambda_1 \int_{M^n} e\phi^2 \leq \frac{2ns(1+l)}{2ns(1+l)+l}\left((1+\frac{1}{2ns})\lambda_1 - \delta\right)\int_{M^n} e\phi^2 + \left(1+\frac{l^2}{2ns(1+l)+l}\right)\int_{M^n} e|\nabla\phi|^2. \tag{3.4}$$

Fix a point $p \in M$. For $r > 0$, we choose ϕ to satisfy the properties that

$$\phi = \begin{cases} 1 & \text{on } B(p,r) \\ 0 & \text{on } M^n \setminus B(p,2r) \end{cases}$$

and

$$|\nabla\phi| \leq Cr^{-1}$$

for some constant $C > 0$, where $B(p,r)$ is the geodesic ball of radius r and center p. Thus, (3.4) becomes

$$(2ns(1+l)\delta - \lambda_1)\int_{B(p,r)} e \leq C^2 r^{-2}((1+l)(2ns+l))\int_{B(p,2r)\setminus B(p,r)} e.$$

Letting $r \to \infty$, the right hand side tends to 0 since f has finite energy. Since $2ns(1+l)\delta > \lambda_1$, we conclude that $e \equiv 0$ and consequently f is constant. This completes the proof of Theorem 1.1. □

Proof of Theorem 1.5: Let $e_1, \dots, e_n, \mu$ be a local orthonormal frame on $\overline{M}$ such that $e_1, \dots, e_n$ when restricted to M form a local orthonormal frame in a neighborhood of a point $x_0 \in M$. Let $\omega_1, \dots, \omega_{n+1}$ be the dual coframe. Let ω_{ij} be the connection 1-forms of $\overline{M}$ and $\omega_{n+1i} = \sum_{j=1}^n h_{ij}\omega_j$ when restricted to M defines the second fundamental form of M. The squared norm of the second fundamental form of M is then given by

$$|A|^2 = \sum_{i,j} h_{ij}^2.$$

The condition that M is stable is characterized by the following inequality

$$\int_{M^n}\left(\sum_{i,j} h_{ij}^2\right)\psi^2 + \int_{M^n}\psi^2\overline{\mathrm{Ric}}(\mu,\mu) \leq \int_{M^n}|\nabla\psi|^2 \tag{3.5}$$

for any compactly supported function $\psi \in H_{1,2}(M^n)$.

Set $e = e(f)$. Replacing ψ in (3.5) by $\sqrt{e}\phi$ with $\phi \in C_0^\infty(M^n)$, we get

$$\begin{aligned}(3.6)\qquad &\int_{M^n} e\phi^2 \left(\sum_{i,j} h_{ij}^2 + \overline{\mathrm{Ric}}(\mu,\mu)\right) \\ &\leq \int_{M^n} e|\nabla\phi|^2 + 2\int_{M^n} \sqrt{e}\phi\nabla\sqrt{e}\nabla\phi + \int_{M^n} \phi^2|\nabla\sqrt{e}|^2 \\ &= \int_{M^n} e|\nabla\phi|^2 - \frac{1}{2}\int_{M^n} \phi^2\Delta e + \int_{M^n} \phi^2|\nabla\sqrt{e}|^2.\end{aligned}$$

Since M is minimal, we conclude from the Gauss equation that the Ricci curvature tensor (R_{ij}) of M is given by

$$(3.7)\qquad R_{ij} = \sum_{k=1}^{n} \overline{R}(e_i, e_k, e_j, e_k) - \sum_{k=1}^{n} h_{ik}h_{kj},$$

where $\overline{R}$ is the curvature tensor of $\overline{M}$. Set $u_\alpha = \sum_{k=1}^n f_{\alpha k}e_k$; then we have from (2.5), (2.6), (3.7) and the non-positivity of the sectional curvature of N that

$$\begin{aligned}&\frac{1}{2}\Delta e \geq \sum_{\alpha,i,j} f_{\alpha ij}^2 + \sum_{\alpha,i,j} R_{ij}f_{\alpha i}f_{\alpha j} \\ (3.8)\qquad &= \sum_{\alpha,i,j} f_{\alpha ij}^2 + \sum_{\alpha}\left(\overline{\mathrm{Ric}}(u_\alpha,u_\alpha) - |u_\alpha|^2\overline{K}(u_\alpha,\mu)\right) - \sum_{\alpha,i}\left(\sum_j h_{ij}f_{\alpha j}\right)^2 \\ &\geq \sum_{\alpha,i,j} f_{\alpha ij}^2 + \sum_{\alpha}\left(\overline{\mathrm{Ric}}(u_\alpha,u_\alpha) - |u_\alpha|^2\overline{K}(u_\alpha,\mu)\right) - \left(\sum_{i,j} h_{ij}^2\right)e.\end{aligned}$$

Substituting the above inequality into (3.6), we obtain

$$\begin{aligned}(3.9)\quad \int_M e|\nabla\phi|^2 \geq &\int_M \phi^2\left(e\overline{\mathrm{Ric}}(\mu,\mu) + \sum_\alpha\left(\overline{\mathrm{Ric}}(u_\alpha,u_\alpha) - |u_\alpha|^2\overline{K}(u_\alpha,\mu)\right)\right) \\ &+ \int_M \phi^2\left(\sum_{\alpha,i,j} f_{\alpha ij}^2 - |\nabla\sqrt{e}|^2\right).\end{aligned}$$

Observe from the non-negativity of the bi-Ricci curvature of $\overline{M}$ that

$$\begin{aligned} & e\overline{\mathrm{Ric}}(\mu,\mu)+\sum_{\alpha}\left(\overline{\mathrm{Ric}}(u_\alpha,u_\alpha)-|u_\alpha|^2\overline{K}(u_\alpha,\mu)\right) \\ & \quad =\sum_{\alpha}\left\{|u_\alpha|^2\left(\overline{\mathrm{Ric}}(\mu,\mu)+\overline{\mathrm{Ric}}(u'_\alpha,u'_\alpha)-\overline{K}(u'_\alpha,\mu)\right)\right\} \\ & \quad \geq 0, \end{aligned} \tag{3.10}$$

where u'_α is the unit vector in the direction u_α. It then follows that

$$\int_M e|\nabla\phi|^2 \geq \int_M \phi^2\left(\sum_{\alpha,i,j} f_{\alpha ij}^2-|\nabla\sqrt{e}|^2\right), \tag{3.11}$$

which, combining with (2.6), gives

$$\frac{1}{2ns}\int_M \phi^2|\nabla\sqrt{e}|^2 \leq \int_M e|\nabla\phi|^2. \tag{3.12}$$

Fix a $p\in M$ and choose ϕ to be a cut-off function with the properties

$$\phi=\begin{cases}1 & \text{on } B(p,r)\\ 0 & \text{on } M^n\setminus B(p,3r)\end{cases}$$

and

$$|\nabla\phi|\leq\frac{1}{r}.$$

We then obtain from (3.12) that

$$\int_{B(p,r)}|\nabla\sqrt{e}|^2\leq\frac{2ns}{r^2}E(f).$$

Letting $r\to\infty$, one knows that e is a constant.

Substituting the above ϕ into (3.11), taking $r\to\infty$ and noticing that e is constant, we conclude that

$$f_{\alpha ij}\equiv 0,\quad \forall\,\alpha,i,j,$$

which in turn implies from (3.8) that

$$\sum_{\alpha}\left(\overline{\mathrm{Ric}}(u_\alpha,u_\alpha)-\overline{K}(u_\alpha,u)\right)-\left(\sum_{i,j}h_{ij}^2\right)e\leq 0.$$

By introducing the above ϕ into (3.9), taking $r\to\infty$ and using (3.10), we get

$$e\overline{\mathrm{Ric}}(\mu,\mu)+\sum_{\alpha}\left(\overline{\mathrm{Ric}}(u_\alpha,u_\alpha)-\overline{K}(u_\alpha,\mu)\right)=0.$$

Therefore

$$e\left(\overline{\mathrm{Ric}}(\mu,\mu)+\sum_{i,j}h_{ij}^2\right)\geq 0.$$

On the other hand, it follows by introducing the above ϕ into (3.6) and taking $r\to\infty$ that

$$\int_M e\left(\sum_{i,j}h_{ij}^2+\overline{\mathrm{Ric}}(\mu,\mu)\right)\leq 0.$$

Hence

$$e\left(\sum_{i,j}h_{ij}^2+\overline{\mathrm{Ric}}(\mu,\mu)\right)=0 \tag{3.13}$$

holds on M.

Let v be an arbitrary unit tangent vector to M. From the non-negativity of the bi-Ricci curvature of $\overline{M}$ for the orthonormal pair $\{v,\mu\}$, it follows that

$$\overline{\mathrm{Ric}}(v,v)+\overline{\mathrm{Ric}}(\mu,\mu)-\overline{K}(v,\mu)\geq 0. \tag{3.14}$$

Set

$$v=\sum_i a_ie_i,\quad \sum_i a_i^2=1;$$

then (3.7) implies that the Ricci curvature of M in the direction v satisfies

$$\begin{aligned}\mathrm{Ric}(v,v)&=\sum_{i,j}a_ia_jR_{ij}\\&=\sum_{i,j,k}a_ia_j\overline{R}(e_i,e_k,e_j,e_k)-\sum_k\left(\sum_i a_ih_{ik}\right)^2\\&=\sum_k\overline{R}(v,e_k,v,e_k)-\sum_k\left(\sum_i a_ih_{ik}\right)^2 \qquad (3.15)\\&=\overline{\mathrm{Ric}}(v,v)-\overline{K}(v,\mu)-\sum_k\left(\sum_i a_ih_{ik}\right)^2\\&\geq\overline{\mathrm{Ric}}(v,v)-\overline{K}(v,\mu)-\sum_{i,j}h_{ij}^2.\end{aligned}$$

Assume now that $e \neq 0$. We get from (3.13)–(3.15) that

$$\begin{aligned}\mathrm{Ric}(v,v) &\geq -\overline{\mathrm{Ric}}(\mu,\mu) - \sum_{i,j} h_{ij}^2 \\ &= 0.\end{aligned}$$

That is, M has non-negative Ricci curvature and so it has infinite volume since it is non-compact [**Y2**]. But this contradicts to the fact that $E(f)$ is finite. Consequently, we conclude that $e \equiv 0$ and f is constant. This completes the proof of Theorem 1.5. □

Acknowledgement. The author would like to thank the referee for the encouragements and the valuable suggestions.

References

[Ch] S. Y. CHENG, Liouville theorem for harmonic maps, in: *"Geometry of the Laplace operator"* (Proc. Sympos. Pure Math., Univ. Hawaii, Honolulu, Hawaii, 1979), Proc. Sympos. Pure Math. **XXXVI**, Amer. Math. Soc., Providence, R.I., 1980, pp. 147–151.

[C] H. I. CHOI, On the Liouville theorem for harmonic maps, *Proc. Amer. Math. Soc.* **85(1)** (1982), 91–94.

[EL1] J. EELLS AND L. A. LEMAIRE, A report on harmonic maps, *Bull. London Math. Soc.* **10(1)** (1978), 1–68.

[EL2] J. EELLS AND L. A. LEMAIRE, Another report on harmonic maps, *Bull. London Math. Soc.* **20(5)** (1988), 385–524.

[ES] J. EELLS, JR. AND J. H. SAMPSON, Harmonic mappings of Riemannian manifolds, *Amer. J. Math.* **86** (1964), 109–160.

[H] S. HILDEBRANDT, Liouville theorems for harmonic mappings, and an approach to Bernstein theorems, in: *"Seminar on Differential Geometry"*, Ann. of Math. Stud. **102**, Princeton Univ. Press, Princeton, N.J., 1982, pp. 107–131.

[HJW] S. HILDEBRANDT, J. JOST AND K.-O. WIDMAN, Harmonic mappings and minimal submanifolds, *Invent. Math.* **62(2)** (1980/81), 269–298.

[J] Z. R. JIN, Liouville theorems for harmonic maps, *Invent. Math.* **108(1)** (1992), 1–10.

[S] Y. SHEN, A Liouville theorem for harmonic maps, *Amer. J. Math.* **117(3)** (1995), 773–785.

[ShY] Y. SHEN AND R. YE, On stable minimal surfaces in manifolds of positive bi-Ricci curvatures, *Duke Math. J.* **85(1)** (1996), 109–116.

[SY] R. Schoen and S. T. Yau, Harmonic maps and the topology of stable hypersurfaces and manifolds with non-negative Ricci curvature, *Comment. Math. Helv.* **51(3)** (1976), 333–341.

[T] S. Tanno, L^2 harmonic forms and stability of minimal hypersurfaces, *J. Math. Soc. Japan* **48(4)** (1996), 761–768.

[Y1] S. T. Yau, Harmonic functions on complete Riemannian manifolds, *Comm. Pure Appl. Math.* **28** (1975), 201–228.

[Y2] S. T. Yau, Some function-theoretic properties of complete Riemannian manifold and their applications to geometry, *Indiana Univ. Math. J.* **25(7)** (1976), 659–670.

Departamento de Matemática-IE
Universidade de Brasília
70910-900 Brasília-DF
Brazil
E-mail address: wang@mat.unb.br

Primera versió rebuda el 29 de març de 2005,
darrera versió rebuda el 2 de desembre de 2005.