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CONVERGENCE OF THE 'RELATIVISTIC' HEAT EQUATION TO THE HEAT EQUATION AS $c \to \infty$

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Abstract

We prove that the entropy solutions of the so-called relativistic heat equation converge to solutions of the heat equation as the speed of light c tends to ∞ for any initial condition $u_0 \geq 0$ in $L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$.

1. Introduction

To limit the speed of propagation of different types of waves which are solutions of nonlinear degenerate parabolic equations some mechanisms of saturation of the flux as the gradient becomes unbounded have been proposed by different authors [**16**], [**11**], [**17**].

The speed of light c is the highest admissible velocity for transport of radiation in transparent media, and, to ensure it, J. R. Wilson (in an unpublished work, see [**16**]) proposed to use a flux limiter. The flux limiter merely enforces the physical restriction that the flux cannot exceed energy density times the speed of light, that is, the flux cannot violate causality. The basic idea is to modify the diffusion-theory formula for the flux in a way that gives the standard result in the high opacity limit, while simulating free streaming (at light speed) in transparent regions. As an example, one of the expressions suggested for the flux of the (positive) energy density u is

$$F = -\nu u \frac{\nabla u}{u + \nu c^{-1}|\nabla u|} \tag{1.1}$$

(where ν is a constant representing a kinematic viscosity and c the speed of light) which yields in the limit $\nu \to \infty$ the flux $F = -cu\frac{\nabla u}{|\nabla u|}$. Observe also that when $c \to \infty$, the flux tends to $F = -\nu \nabla u$, and the

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corresponding diffusion equation becomes the heat equation, which has an infinite speed of propagation.

The diffusion equation corresponding to (1.1) is

$$u_t = \nu \operatorname{div}\left(\frac{u\nabla u}{u + \frac{\nu}{c}|\nabla u|}\right) \tag{1.2}$$

and is one among the various *flux limited diffusion equations* used in the theory of radiation hydrodynamics [**16**]. Indeed, the same effects can be guaranteed for a similar equation [**8**]

$$u_t = \nu \operatorname{div}\left(\frac{u\nabla u}{\sqrt{u^2 + \frac{\nu^2}{c^2}|\nabla u|^2}}\right), \tag{1.3}$$

which was introduced by Y. Brenier [**8**]. He was able to derive (1.3) from Monge-Kantorovich's mass transport theory and described it as a *relativistic heat equation*. Both equations, (1.2) and (1.3), interpolate (see [**8**]) between the usual heat equation (when $c \to \infty$) and the diffusion equation in transparent media (when $\nu \to \infty$) with constant speed of propagation c

$$u_t = c \operatorname{div}\left(u\frac{\nabla u}{|\nabla u|}\right). \tag{1.4}$$

Let us mention that many other models of nonlinear degenerate parabolic equations with flux saturation as the gradient becomes unbounded have been proposed by Rosenau and his coworkers [**11**], [**17**], and Bertsch and Dal Passo [**7**], [**12**].

We consider the solution of (1.3) with initial condition $u(0,x) = u_0(x) \in L^\infty(\mathbb{R}^N) \cap L^1(\mathbb{R}^N)$, $u_0 \geq 0$. In a series of papers [**3**], [**4**], [**5**] we developed a theory of existence and uniqueness of entropy solutions for (1.3) and we studied the propagation of discontinuity fronts at the speed of light c. Moreover, in [**6**] we proved the convergence of (1.3) to equation (1.4) when $\nu \to \infty$. Our purpose in this paper is to study the asymptotic limit of equation (1.3) as $c \to \infty$. If $u_c(t,x)$ denotes the entropy solution of (1.3) we shall prove that u_c converges as $c \to \infty$ to the solution of the classical heat equation

$$u_t = \nu \Delta u \tag{1.5}$$

with initial condition $u(0,x) = u_0(x)$.

Let us explain the plan of the paper. In Section 2 we recall the basic existence and uniqueness result of entropy solutions of (1.3) and state the main result of convergence of solutions of (1.3) to solutions of (1.5) as $c \to \infty$. Section 3 is devoted to the proof of this convergence result.

2. Preliminaries

In [**3**], [**4**] we studied the well-posedness of (1.3) for initial conditions in $(L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N))^+$. We proved that the underlying elliptic operator to (1.3) defines a nonlinear contraction semigroup in $L^1(\mathbb{R}^N)^+$, we defined the concept of entropy solution proving its existence and uniqueness and we proved that entropy solutions coincide with the semigroup ones.

For the concept of entropy solution of (1.3) we refer to [**4**]. Let us recall the basic existence and uniqueness result proved in [**4**].

Theorem 2.1. *For any initial datum $0 \le u_0 \in L^\infty(\mathbb{R}^N) \cap L^1(\mathbb{R}^N)$ there exists a unique entropy solution u of* (1.3) *in $Q_T = (0,T) \times \mathbb{R}^N$ for every $T > 0$ such that $u(0) = u_0$. Moreover, if $u(t)$, $\overline{u}(t)$ are the entropy solutions corresponding to initial data u_0, $\overline{u}_0 \in \left(L^\infty(\mathbb{R}^N) \cap L^1(\mathbb{R}^N)\right)^+$, respectively, then*

$$\|(u(t) - \overline{u}(t))^+\|_1 \le \|(u_0 - \overline{u}_0)^+\|_1 \quad \text{for all } t \ge 0. \tag{2.1}$$

Moreover, the map $T(t)u_0 = u(t)$, $t \ge 0$, defines a nonlinear contraction semigroup in $L^1(\mathbb{R}^N)^+$.

Our main purpose is to prove the following result.

Theorem 2.2. *Let u_c be the entropy solution of* (1.3) *with $u(0,x) = u_0(x) \in (L^\infty(\mathbb{R}^N) \cap L^1(\mathbb{R}^N))^+$. As $c \to \infty$, u_c converges in $C([0,T], L^1(\mathbb{R}^N))$ to the solution U of the heat equation* (1.5) *with $U(0,x) = u_0(x)$.*

We observe that the same result is true for equation (1.2). Since the proof is similar, we skip the details.

Observe that $v(t,x)$ is an entropy solution of (1.3) if and only if $u(t,x) = v(\nu t, \nu x)$ is an entropy solution of

$$u_t = \operatorname{div}\left(\frac{u \nabla u}{\sqrt{u^2 + \frac{1}{c^2}|\nabla u|^2}}\right). \tag{2.2}$$

Thus, without loss of generality we may assume that $\nu = 1$, and, for simplicity, we shall assume it in the sequel.

Since, by Theorem 2.1, $T(t)u_0 = u(t)$ determines a nonlinear contraction semigroup in $L^1(\mathbb{R}^N)^+$, we may reduce the proof of Theorem 2.2 to a dense set of functions in $(L^\infty(\mathbb{R}^N) \cap L^1(\mathbb{R}^N))^+$ with respect to the norm in $L^1(\mathbb{R}^N)$. We shall consider the set of functions

$$\mathcal{A} := \left\{ u \in \mathcal{S}(\mathbb{R}^N) : u_0(x) > 0 \text{ for all } x \in \mathbb{R}^N, \left\| \frac{\nabla u_0}{u_0} \right\|_\infty < \infty \right\},$$

$$\mathcal{A}^e := \{ u \in \mathcal{A} : u_0(x) \geq \lambda_0 e^{-\beta \frac{|x|^2}{2}} \text{ for some } \lambda_0,\, \beta > 0 \},$$

where $\mathcal{S}(\mathbb{R}^N)$ denotes the space of rapidly decreasing functions in $\mathbb{R}^N$. Functions in $\mathcal{S}(\mathbb{R}^N)$ which are positive and behave like $\lambda e^{-\beta|x|}$, for some $\lambda, \beta > 0$ and for $|x|$ large enough, belong to $\mathcal{A}^e$ and, thus, both $\mathcal{A}^e$ and $\mathcal{A}$ are dense in $L^1(\mathbb{R}^N)^+$.

We shall concentrate our efforts in proving Theorem 2.2 when $u_0 \in \mathcal{A}^e$. In that case, we shall prove that solutions satisfy a Lipschitz bound which enables to pass to the limit as $c \to \infty$ in (2.2).

3. Convergence of solutions of (1.3) to solutions of heat equation

Our first purpose is to prove the following result.

Proposition 3.1. *Assume that $u_0 \in \mathcal{A}^e$. Let u be the entropy solution of (2.2). Then for any $t > 0$, $u(t) \in \mathcal{A}^e$. Moreover, we have*

$$\sup_{[0,T]\times\mathbb{R}^N} \frac{|\nabla u(t,x)|}{|u(t,x)|} \leq \left\| \frac{\nabla u_0}{u_0} \right\|_\infty . \tag{3.1}$$

Observe first that, according to Proposition 3 in [**5**], $u(t,x) > 0$ for any $t > 0$ and any $x \in \mathbb{R}^N$. To prove the gradient bound (3.1) we reduce it to the case where $c = 1$. For that, we observe that $u(t,x)$ is the entropy solution of (2.2) with $u(0,x) = u_0(x)$ if and only if $\tilde{u}(t,x) = u\left(\frac{t}{c^2}, \frac{x}{c}\right)$ is the entropy solution of

$$u_t = \operatorname{div}\left(\frac{u\nabla u}{\sqrt{u^2 + |\nabla u|^2}} \right) \tag{3.2}$$

with $\tilde{u}(0,x) = u_0\left(\frac{x}{c}\right)$. Now, assume that we have proved that

$$\sup_{[0,T]\times\mathbb{R}^N} \frac{|\nabla \tilde{u}(t,x)|}{|\tilde{u}(t,x)|} \leq \left\| \frac{\nabla \tilde{u}(0)}{\tilde{u}(0)} \right\|_\infty .$$

Writing this inequality in terms of u, we have

$$c^{-1} \frac{\left|\nabla u\left(\frac{t}{c^2}, \frac{x}{c}\right)\right|}{\left|u\left(\frac{t}{c^2}, \frac{x}{c}\right)\right|} \leq \sup_{x\in\mathbb{R}^N} c^{-1} \frac{\left|\nabla u_0\left(\frac{x}{c}\right)\right|}{u_0\left(\frac{x}{c}\right)}.$$

This implies (3.1). Thus, without loss of generality we assume that $c = 1$.

For any $R > 0$, let $\Omega_R := [-R, R]^N$. In order to prove the Lipschitz bound (3.1) for the entropy solutions of (3.2) we need to approximate (3.2) by

$$\begin{cases} \dfrac{\partial u}{\partial t} = \operatorname{div}\left(\dfrac{u\nabla u}{\sqrt{u^2+|\nabla u|^2}}\right) & \text{in } Q_T^R = (0,T)\times\Omega_R \\ \dfrac{u\nabla u}{\sqrt{u^2+|\nabla u|^2}}\cdot\nu^{\Omega_R} = 0 & \text{on } S_T^R = (0,T)\times\partial\Omega_R \\ u(0,x) = u_0(x) & \text{in } x\in\Omega_R, \end{cases} \tag{3.3}$$

with ν^{Ω_R} the unit outward normal on $\partial\Omega_R$.

Proposition 3.2. *Assume that $u_0 \in \mathcal{A}$. Then there exists an entropy solution $u^R \in C([0,T], L^1(\Omega_R))$ of (3.3). Moreover $u(t,x) > 0$ for any $t > 0$, $x \in \mathbb{R}^N$ and*

$$\sup_{[0,T]\times\Omega_R} \frac{|\nabla u^R(t,x)|}{|u^R(t,x)|} \le \sup_{\Omega_R} \frac{|\nabla u_0(x)|}{|u_0(x)|}. \tag{3.4}$$

To prove Proposition 3.2 we consider the following approximation. First we define $\tilde{u}_0^R(x) = u_0(x - R(1,\dots,1))$ in $[0,2R]^N$, then we extend it to $\Omega_{2R} := [-2R, 2R]^N$ by symmetry so that

$$\tilde{u}_0^R(x_1,\dots,x_i,\dots,x_N) = \tilde{u}_0^R(x_1,\dots,-x_i,\dots,x_N) \tag{3.5}$$

for any $x = (x_1,\dots,x_N) \in [-2R,2R]^N$ and any $i = 1,\dots,N$, and then we extend it by periodicity so that the extended function, call it $\tilde{u}_0^R$, is periodic of fundamental period $[-2R,2R]^N$ and has no discontinuities in $\mathbb{R}^N$. Then we smooth it by defining

$$\tilde{u}_{0n}^R = \rho_n * \tilde{u}_0^R,$$

where $\rho_n(x) = n^{-N}\rho(\frac{x}{n})$, $\rho \in C_0^\infty(\mathbb{R}^N)$, $\rho \ge 0$, $\operatorname{supp}\rho \subseteq B(0,1)$, $\int_{\mathbb{R}^N}\rho(x)\,dx = 1$. Then the functions $\tilde{u}_{0n}^R \in C^\infty(\mathbb{R}^N)$ have also the symmetries (3.5) and are periodic with fundamental period $[-2R,2R]^N$.

We consider the following approximating problem

$$\begin{cases} \dfrac{\partial u}{\partial t} = \operatorname{div}\left(\dfrac{u\nabla u}{\sqrt{u^2+|\nabla u|^2}}\right) + \eta\Delta u & \text{in } Q_T = (0,T)\times\Omega_{2R} \\ \left(\dfrac{u\nabla u}{\sqrt{u^2+|\nabla u|^2}} + \eta\nabla u\right)\cdot\nu^{\Omega_{2R}} = 0 & \text{on } S_T^R = (0,T)\times\partial\Omega_{2R} \\ u(0,x) = \tilde{u}_{0n}^R & \text{in } x\in\Omega_{2R}. \end{cases} \tag{3.6}$$

Lemma 3.3. *Assume that* $u_0 \in \mathcal{S}(\mathbb{R}^N)$, $u_0(x) > 0$ *for all* $x \in \mathbb{R}^N$. *Then there is a solution* $u^R_{\eta,n}$ *of* (3.6) *which is even,* $\frac{\partial u^R_{\eta,n}}{\partial t}$, $D^2 u^R_{\eta,n} \in L^2([0,T]\times\Omega_{2R})$, *and* $\nabla u^R_{\eta,n} \in C^{1+\beta/2,2+\beta}([0,T]\times\Omega_{2R})$ *for any* $T>0$. *Moreover, we have the estimates*

$$u^R_{\eta,n} \geq \inf_{x\in[0,2R]^N} \tilde{u}^R_{0n}, \tag{3.7}$$

$$\|u^R_{\eta,n}(t)\|_{L^q([0,2R]^N)} \leq \|\tilde{u}^R_{0n}\|_{L^q([0,2R]^N)} \quad \text{for any } q\in[1,\infty] \text{ and any } t\geq 0, \tag{3.8}$$

and

$$\int_0^T \int_{[0,2R]^N} |\nabla T_{a,b}(u^R_{\eta,n})|\,dx\,dt \leq C \tag{3.9}$$

for any $T_{a,b}(r) := \max(\min(b,r),a)$ $(0<a<b)$ *where the constant* $C>0$ *only depends on* a, b *and* u_0 *and does not depend on* η, n, R.

We denote $\partial_t = \frac{\partial}{\partial t}$, $\partial_i := \frac{\partial}{\partial x_i}$, $\partial_{ij} := \frac{\partial^2}{\partial x_i \partial x_j}$.

As usual, if $Q_{a,b} = [a,b]\times Q$, $0\leq a\leq b$, and Q is a compact subset of $\mathbb{R}^N$, then $C^{1+\beta/2,2+\beta}(Q_{a,b})$ denotes the parabolic Hölder space of functions in $Q_{a,b}$ [**13**], [**14**]. If V is an open subset of $\mathbb{R}^N$, by $C^{1+\beta/2,2+\beta}_{\text{loc}}([0,T]\times V)$ we denote the set of functions u in $C^{1+\beta/2,2+\beta}(Q_{a,b})$ for any set $[0,T]\times Q$ where Q a compact subset of V.

Proof: The present lemma follows from Theorem 3.1 and Theorem 8.1 in Chapter 5 of [**14**]. Let us sketch the proof. Let us approximate (3.6) by the following PDE:

$$\begin{aligned} &u_t = \operatorname{div}\left(\frac{u\nabla u}{\sqrt{\epsilon^2+u^2+|\nabla u|^2}}\right) + \eta\Delta u \quad \text{on} \quad (0,T)\times\mathbb{R}^N \\ &u(0,x) = \tilde{u}^R_{0n} \qquad\qquad x\in\mathbb{R}^N. \end{aligned} \tag{3.10}$$

The equation (3.10) can be written as

$$u_t = \operatorname{div} A^{\epsilon,\eta}(u,\nabla u)$$

where

$$A^{\epsilon,\eta}(w,p) = \frac{wp}{\sqrt{\epsilon^2+w^2+|p|^2}} + \eta p \quad (w,p)\in[0,\infty)\times\mathbb{R}^N.$$

Then $A^{\epsilon,\eta}(w,p)$ is continuously differentiable in (w,p). We denote $A^{\epsilon,\eta i}$ the coordinates of $A^{\epsilon,\eta}$, $A_w^{\epsilon,\eta i} := \partial_w A^{\epsilon,\eta i}$, $A_{p_j}^{\epsilon,\eta i} := \partial_{p_j} A^{\epsilon,\eta i}$ and similarly for higher order derivatives. We require some computations:

$$A_w^{\epsilon,\eta i} = \frac{p_i(\epsilon^2+|p|^2)}{(\epsilon^2+w^2+|p|^2)^{3/2}},$$

$$A_{p_j}^{\epsilon,\eta i} = \frac{w\delta_{ij}}{(\epsilon^2+w^2+|p|^2)^{1/2}} - \frac{wp_ip_j}{(\epsilon^2+w^2+|p|^2)^{3/2}} + \eta\delta_{ij},$$

$$A_{ww}^{\epsilon,\eta i} = \frac{-3wp_i(\epsilon^2+|p|^2)}{(\epsilon^2+w^2+|p|^2)^{5/2}},$$

$$A_{wp_j}^{\epsilon,\eta i} = \frac{\delta_{ij}(\epsilon^2+|p|^2)}{(\epsilon^2+w^2+|p|^2)^{3/2}} + \frac{p_ip_j(2w^2-\epsilon^2-|p|^2)}{(\epsilon^2+w^2+|p|^2)^{5/2}},$$

$$A_{p_jp_k}^{\epsilon,\eta i} = -\frac{w(\delta_{ij}p_k+\delta_{ik}p_j+\delta_{jk}p_i)}{(\epsilon^2+w^2+|p|^2)^{3/2}} + \frac{3wp_ip_jp_k}{(\epsilon^2+w^2+|p|^2)^{5/2}}.$$

Then we have

$$\eta|\xi|^2 \le A_{p_j}^{\epsilon,\eta i}\xi_i\xi_j \le (1+\eta)|\xi|^2, \tag{3.11}$$

$$\sum_{i=1}^{N}(|A^{\epsilon,\eta i}|+|A_w^{\epsilon,\eta i}|) \le \sqrt{N}(4+\eta|p|), \tag{3.12}$$

the functions $A^{\epsilon,\eta i}$ are Lipschitz in (w,p) with bounds independent of ϵ and η, as soon as η remains bounded, and the functions $A_w^{\epsilon,\eta i}$, $A_{p_j}^{\epsilon,\eta i}$ are also Lipschitz with a bound depending on $\frac{1}{\epsilon}$. Then by Theorem 8.1 on Chapter 5 of [**14**], we have that there exists a solution $u_{\eta,n,\epsilon}^R \in C^{1+\beta/2,2+\beta}(\overline{Q_T})$ for some $\beta > 0$ (indeed any $\beta < 1$) where Q_T is any bounded cylinder in $[0,T]\times\mathbb{R}^N$. The solution satisfies $\|u_{\eta,n,\epsilon}^R(t)\|_\infty \le \|u_0\|_\infty$. Observe that since $A_w^{\epsilon,\eta i}$, $A_{p_j}^{\epsilon,\eta i}$ are Lipschitz (with a bound depending on $\frac{1}{\epsilon}$) and $A_{wp_j}^{\epsilon,\eta i}p_i$, $A_w^{\epsilon,\eta i}$ and $A_{ww}^{\epsilon,\eta i}p_i$ are bounded, then the uniqueness conditions of Theorem 8.1 on Chapter 5 of [**14**] hold, and we have a unique solution in the class of functions which are bounded together with its derivatives of first and second order. In particular since $u_{\eta,n,\epsilon}^R(\cdot+2R(1,\ldots,1))$ and $u_{\eta,n,\epsilon}^R(x_1,\ldots,-x_i,\ldots,x_N)$, $i=1,\ldots,N$, are also solutions of (3.10) in this class, we deduce that $u_{\eta,n,\epsilon}^R$ has the same symmetries and is periodic with fundamental period $[-2R,2R]^N$. In particular, this implies that

$$\nabla u_{\eta,n,\epsilon}^R \cdot \nu^{[0,2R]^N} = 0 \quad t>0,\ x\in\partial[0,2R]^N. \tag{3.13}$$

Using (3.13), the estimates (3.7) and (3.8) and

$$\int_0^T \int_{[0,2R]^N} |\nabla u^R_{\eta,n,\epsilon}|^2 \, dx \, dt \leq \frac{1}{2\eta} \int_{[0,2R]^N} (\tilde{u}^R_{0n}(x))^2 \, dx \tag{3.14}$$

follow easily by multiplication by suitable test functions and integration by parts in $[0,2R]^N$. The estimate (3.9) can be proved as in [**3**].

The regularity of $u^R_{\eta,n,\epsilon}$ permits now to use Theorem 3.1 in Chapter 5 of [**14**]. Indeed, observe that the estimates in (3.11), (3.12) are independent of ϵ, and the terms on the right hand side of (3.11), (3.12) which do not depend on $|p|$ are constants. Then Theorem 3.1 in Chapter 5 of [**14**] proves that $\nabla u^R_{\eta,n,\epsilon}$ is Hölder-continuous in $[0,T] \times Q$ and $\frac{\partial u^R_{\eta,n,\epsilon}}{\partial t}$, $D^2 u^R_{\eta,n,\epsilon} \in L^2([0,T] \times Q)$ for any compact set $Q \in \mathbb{R}^N$ with bounds which depend on N, $\|u_0\|_\infty$, η and the constants in the right hand side of (3.11), (3.12) (which are universal). Letting $\epsilon \to 0$, we obtain that there is a weak solution of (3.10) $u^R_{\eta,n}$ whose restriction to $[0,2R]^N$ is in $L^1((0,T) \times [0,2R]^N) \cap L^2((0,T), W^{1,2}([0,2R]^N))$, it satisfies the estimates in (3.7), (3.8), (3.9) and (3.14) and is such that $\frac{\partial u^R_{\eta,n}}{\partial t}$, $D^2 u^R_{\eta,n} \in L^2([0,T] \times Q)$ and $\nabla u^R_{\eta,n}$ is Hölder-continuous in $[0,T] \times Q$ for any compact set $Q \subseteq \mathbb{R}^N$.

To prove the last assertion, let $A^\eta(w,p) = A^{0,\eta}(w,p)$ and let us observe the previous computations of $A^{\eta i}_{ww}$, $A^{\eta i}_{wp_j}$ and $A^{\eta i}_{p_j p_k}$ permit to check that $A^{\eta i}$, $A^{\eta i}_w$, $A^{\eta i}_{p_j}$ are Lipschitz continuous in any set of the form $\{(w,p) : w \geq \alpha > 0, \, p \in \mathbb{R}^N\}$. Since, by (3.7), $u^R_{\eta,n}$ is bounded away from zero in $[0,T] \times \mathbb{R}^N$, then using Theorem 8.1 of Chapter 5 in [**14**] we obtain that $u^R_{\eta,n} \in C^{1+\beta/2, 2+\beta}([0,T] \times \mathbb{R}^N)$ for some $\beta > 0$. In particular $u^R_{\eta,n}$, $\nabla u^R_{\eta,n}$, $D^2 u^R_{\eta,n}$ are locally Hölder continuous in (t,x) and the functions

$$\mathcal{A}_{ij}(t,x) = A^i_{p_j}(u^R_{\eta,n}(t,x), \nabla u^R_{\eta,n}(t,x)),$$

$$\mathcal{B}_i(t,x) = A^i_w(u^R_{\eta,n}(t,x), \nabla u^R_{\eta,n}(t,x))$$

and its derivatives are Hölder continuous in (t,x). Observe that $\partial_k u^R_{\eta,n}$ is a weak solution of the partial differential equation

$$\partial_t \omega = \partial_i \left(\mathcal{A}_{ij} \partial_j \omega \right) + \partial_i \left(\mathcal{B}_i \omega \right). \tag{3.15}$$

We consider this PDE in $[0,T] \times B(0,R')$ (where $B(0,R')$ denotes the ball centered at 0 of radius $R' > 0$) with boundary conditions

$$\omega(0,x) = \partial_k \tilde{u}_{0n}(x) \qquad x \in B(0,R') \tag{3.16}$$

$$\omega(t,x) = \partial_k u^R_{\eta,n}(t,x) \quad t \in [0,T], \, x \in \partial B(0,R'). \tag{3.17}$$

Now, we consider the PDE obtained by expanding (3.15)

$$\partial_t \omega = \mathcal{A}_{ij}\partial_{ij}\omega + \partial_i \mathcal{A}_{ij}\partial_j \omega + \mathcal{B}_i \partial_i \omega + \partial_i \mathcal{B}_i \omega \tag{3.18}$$

with boundary conditions (3.16), (3.17). Since its coefficients of (3.18) are Hölder continuous in (t,x) and the initial condition is in $C^{2+\beta}(\overline{B(0,R')})$, there is a solution $\omega \in C^{1+\beta/2,2+\beta}_{\mathrm{loc}}([0,T]\times B(0,R'))$ of (3.18) which is also a weak solution of (3.15) [**13**], [**14**]. By uniqueness of weak solutions of (3.15) which satisfy the boundary conditions (3.16), (3.17) classically, we have that $\nabla u^R_{\eta,n} \in C^{1+\beta/2,2+\beta}_{\mathrm{loc}}([0,T]\times \mathbb{R}^N)$. This implies our statement. □

Lemma 3.4. *Assume that $u_0 \in \mathcal{A}$. Then we have*

$$\sup_{[0,T]\times[0,2R]^N} \frac{|\nabla u^R_{\eta,n}(t,x)|}{|u^R_{\eta,n}(t,x)|} \le \sup_{x\in[0,2R]^N} \frac{|\nabla \tilde{u}^R_{0n}(x)|}{|\tilde{u}^R_{0n}(x)|}. \tag{3.19}$$

Proof: We consider here $u^R_{\eta,n}$ as a function in $[0,T]\times\mathbb{R}^N$ with the symmetries stated after Proposition 3.2. By the regularity of $u^R_{\eta,n}$ stated in Lemma 3.3 we can proceed to the following change of variables and the subsequent computations. Define $u^R_{\eta,n} = e^v$. Since $u^R_{\eta,n}(t,x) > 0$, then $v(t,x)\in\mathbb{R}$. Then $v(t,x)$ is a solution of

$$v_t = \operatorname{div}\left(\frac{\nabla v}{\sqrt{1+|\nabla v|^2}}\right) + \frac{|\nabla v|^2}{\sqrt{1+|\nabla v|^2}} + \eta\Delta v + \eta|\nabla v|^2 \tag{3.20}$$

which is even and periodic with fundamental period $[-2R,2R]^N$. We shall use Bernstein method to obtain an estimate on $\|Dv\|_\infty$. Let us write (3.20) as

$$v_t = a_{ij}(\nabla v)\partial_{ij} v + F(|\nabla v|^2), \tag{3.21}$$

where we use the Einstein convention, $a_{ij}(p) = \frac{\delta_{ij}}{\sqrt{1+|p|^2}} - \frac{p_i p_j}{(1+|p|^2)^{3/2}} + \eta\delta_{ij}$, $F(|p|^2) = \frac{|p|^2}{\sqrt{1+|p|^2}} + \eta|p|^2$. Let $v_k = \partial_k v$. Let $\omega = |\nabla v|^2$. Differentiating (3.21) with respect to x_k, and multiplying the resulting equation by v_k, we obtain

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\omega = v_k v_{kt} &= \frac{1}{2}a_{ij}\partial_{ij}\omega + \frac{1}{2}\frac{\partial a_{ij}}{\partial p_l}\omega_l \partial_{ij} v - a_{ij} v_{ki} v_{kj} + F'(|\nabla v|^2)\omega_k \cdot v_k \\ &\le \frac{1}{2}a_{ij}\partial_{ij}\omega + \frac{1}{2}\frac{\partial a_{ij}}{\partial p_l}\omega_l \partial_{ij} v + F'(|\nabla v|^2)\omega_k \cdot v_k.\end{aligned}$$

As an application of the maximum principle, we obtain

$$\|\nabla v(t)\|_\infty \le \|\nabla v_0\|_\infty. \tag{3.22}$$

Then (3.19) is implied by (3.22). □

We denote by $C^1_{ct}((0,T)\times\overline{\Omega})$ the set of functions obtained as restrictions to $(0,T)\times\overline{\Omega}$ of functions $\phi\in C^1_0((0,T)\times\mathbb{R}^N)$.

Definition 3.5. Let Ω be an open bounded set in $\mathbb{R}^N$ with Lispchitz boundary. Let $u\in L^1((0,T)\times\Omega)$ and $z\in L^1((0,T)\times\Omega,\mathbb{R}^N)$. We say that

$$u_t = \operatorname{div} z \quad \text{in } (0,T)\times\Omega$$
$$z\cdot\nu^\Omega = 0 \quad \text{on } (0,T)\times\partial\Omega$$

with test functions in $C^1_{ct}((0,T)\times\overline{\Omega})$ if

$$\int_0^T\int_\Omega u\phi_t\,dx\,dt = \int_0^T\int_\Omega z\cdot\nabla\phi\,dx\,dt \quad \forall\,\phi\in C^1_{ct}((0,T)\times\overline{\Omega}).$$

Next lemma is obvious and we state it for convenience.

Lemma 3.6. *Let $u_n,u\in L^1((0,T)\times\Omega)$, $z_n,z\in L^1((0,T)\times\Omega,\mathbb{R}^N)$ be such that $u_n\to u$ weakly in $L^1((0,T)\times\Omega)$ and $z_n\to z$ weakly in $L^1((0,T)\times\Omega,\mathbb{R}^N)$. If*

$$u_{nt} = \operatorname{div} z_n \quad \textit{in } (0,T)\times\Omega$$
$$z_n\cdot\nu^\Omega = 0 \quad \textit{on } (0,T)\times\partial\Omega$$

with test functions in $C^1_{ct}((0,T)\times\overline{\Omega})$, then

$$u_t = \operatorname{div} z \quad \textit{in } (0,T)\times\Omega$$
$$z\cdot\nu^\Omega = 0 \quad \textit{on } (0,T)\times\partial\Omega$$

with test functions in $C^1_{ct}((0,T)\times\overline{\Omega})$.

Lemma 3.7. *Let Ω be an open bounded set in $\mathbb{R}^N$ with Lispchitz boundary. Let $u\in L^1((0,T)\times\Omega)$ be such that $\nabla u\in L^1((0,T)\times\Omega,\mathbb{R}^N)$ and let $z\in L^\infty((0,T)\times\Omega)$ be such that*

$$\begin{aligned} u_t &= \operatorname{div} z \quad \textit{in } (0,T)\times\Omega \\ z\cdot\nu^\Omega &= 0 \quad \textit{on } (0,T)\times\partial\Omega \end{aligned} \tag{3.23}$$

with test functions in $C^1_{ct}((0,T)\times\overline{\Omega})$. Let $j:\mathbb{R}\to\mathbb{R}$ be a convex function with $p=j'\in W^{1,\infty}(\mathbb{R})$. Then

$$-\int_0^T\int_\Omega j(u)\phi_t\,dx\,dt+\int_0^T\int_\Omega z\cdot\nabla p(u)\phi\,dx\,dt+\int_0^T\int_\Omega z\cdot\nabla\phi p(u)\,dx\,dt=0 \tag{3.24}$$

holds for any $\phi\in C^1_{ct}((0,T)\times\overline{\Omega})$.

Proof: For any function $v(t,x)$, let us denote $\Delta_\tau^+ v(t) = \frac{v(t+\tau)-v(t)}{\tau}$ and $\Delta_\tau^- v(t) = \frac{v(t)-v(t-\tau)}{\tau}$, and $v_\tau^-(t,x) := \frac{1}{\tau}\int_{t-\tau}^t v(s,x)\,ds$, $v_\tau^+(t,x) := \frac{1}{\tau}\int_t^{t+\tau} v(s,x)\,ds$. Let $\phi \in C_{ct}^1((0,T)\times\mathbb{R}^N)$, $\phi \geq 0$ with $\phi(t) = 0$ when $0 \leq t \leq \tau_0$, $\tau_0 > 0$. Using Definition 3.5 (which formally amounts to multiply (3.23) by ϕ_τ^+ and integrate by parts), we deduce that

$$\Delta_\tau^- u = \operatorname{div} z_\tau^- \quad \text{in } (\tau,T)\times\Omega \tag{3.25}$$

and

$$z_\tau^- \cdot \nu^\Omega = 0 \quad \text{on } (\tau,T)\times\partial\Omega, \tag{3.26}$$

again using test functions in $C_{ct}^1((\tau,T)\times\overline{\Omega})$. From the convexity of j we have that

$$\Delta_\tau^- j(u) \leq p(u)\Delta_\tau^- u \quad \text{for } t \geq \tau.$$

Hence, denoting by $\tilde{\rho}_n$ an approximation of the identity in (t,x) with compact support, if we choose $0 < \tau < \tau_0$, we have

$$\begin{aligned}
-\int_0^T\int_\Omega j(u)\Delta_\tau^+\phi\,dx\,dt &= \int_0^T\int_\Omega \Delta_\tau^- j(u)\phi\,dx\,dt \\
&\leq \int_0^T\int_\Omega p(u)\Delta_\tau^- u\phi\,dx\,dt \\
&= \lim_n \int_0^T\int_\Omega (\tilde{\rho}_n * p(u))\Delta_\tau^- u\phi\,dx\,dt \\
&= -\lim_n \int_0^T\int_\Omega z_\tau^-\cdot\nabla(\tilde{\rho}_n * p(u))\phi\,dx\,dt \\
&\quad -\lim_n \int_0^T\int_\Omega z_\tau^-\cdot\nabla\phi(\tilde{\rho}_n * p(u))\,dx\,dt \\
&= -\int_0^T\int_\Omega z_\tau^-\cdot\nabla p(u)\phi\,dx\,dt \\
&\quad -\int_0^T\int_\Omega z_\tau^-\cdot\nabla\phi p(u)\,dx\,dt.
\end{aligned}$$

Letting $\tau \to 0+$, we obtain that

$$-\int_0^T\int_\Omega j(u)\phi_t\,dx\,dt + \int_0^T\int_\Omega z\cdot\nabla p(u)\phi\,dx\,dt + \int_0^T\int_\Omega z\cdot\nabla\phi p(u)\,dx\,dt \leq 0 \tag{3.27}$$

and this holds for any $\phi \in C_{ct}^1((0,T)\times\overline{\Omega})$, $\phi \geq 0$.

Similarly, we deduce that

$$\Delta_\tau^+ u = \operatorname{div} z_\tau^+ \quad \text{in } (0, T-\tau)\times\Omega \tag{3.28}$$

and

$$z_\tau^+ \cdot \nu^\Omega = 0 \quad \text{on } (0, T-\tau)\times\partial\Omega, \tag{3.29}$$

using test functions in $C^1_{ct}((0,T-\tau)\times\overline{\Omega})$. From the convexity of j we have that

$$\Delta_\tau^+ j(u) \geq p(u)\Delta_\tau^+ u \quad \text{for } t \leq T-\tau.$$

With a similar argument as above we deduce that

$$-\int_0^T\int_\Omega j(u)\phi_t\,dx\,dt + \int_0^T\int_\Omega z\cdot\nabla p(u)\phi\,dx\,dt + \int_0^T\int_\Omega z\cdot\nabla\phi p(u)\,dx\,dt \geq 0 \tag{3.30}$$

holds for any $\phi \in C^1_{ct}((0,T)\times\overline{\Omega})$, $\phi\geq 0$. Thus, (3.24) holds for such ϕ. Since any function $\phi\in C^1_{ct}((0,T)\times\overline{\Omega})$ can be written as $\phi = \max(\phi,0) - \max(-\phi,0) = \lim_n \tilde{\rho}_n * \max(\phi,0) - \lim_n \tilde{\rho}_n * \max(-\phi,0)$, the integrability properties of u and ∇u permit to prove that (3.24) holds for any $\phi\in C^1_{ct}((0,T)\times\overline{\Omega})$. □

We will use the entropy inequalities for (3.3) in the proof of Proposition 3.2. For that, we consider the truncature functions of the form $T^l_{a,b}(r) := T_{a,b}(r) - l$ ($l\in\mathbb{R}$) and we denote

$$\mathcal{T}^+ := \{T^l_{a,b} : 0 < a < b,\ l\in\mathbb{R},\ T^l_{a,b}\geq 0\}.$$

Proof of Proposition 3.2:

Step 1. Assume that $\eta\in(0,1]$. Let us prove that there is some $\gamma > 0$ such that

$$|u^R_{\eta,n}(t,x) - u^R_{\eta,n}(s,x)| \leq C|t-s|^\gamma \quad \forall\, 0\leq s,t\leq T,\ \forall\, x\in\mathbb{R}^N, \tag{3.31}$$

where $C>0$ is a positive constant which does not depend on η. Using its symmetry properties, we consider $u^R_{\eta,n}$ as a solution of (3.6) in $[0,T]\times\mathbb{R}^N$. Now, observe that we may write (3.6) as

$$u_t = \operatorname{div}(a^R_{\eta,n}(t,x)\nabla u),$$

where

$$a^R_{\eta,n}(t,x) = \frac{u^R_{\eta,n}}{\sqrt{(u^R_{\eta,n})^2 + |\nabla u^R_{\eta,n}|^2}} + \eta.$$

Since

$$a^R_{\eta,n}(t,x) = \frac{1}{\sqrt{1+\left(\frac{|\nabla u^R_{\eta,n}|}{u^R_{\eta,n}}\right)^2}} + \eta$$

we have that

$$\frac{1}{\sqrt{1+M_n^2}} + \eta \le a_{\eta,n}^R(t,x) \le 1+\eta,$$

where M_n is the constant in the right hand side of (3.19). Hence the assumptions of Theorem 10.1 of Chapter 3 in [**14**] are satisfied and (3.31) holds.

Step 2. We let $\eta \to 0^+$. The estimates (3.8), (3.19) and (3.31) imply the pre-compactness of $\{u_{\eta,n}^R\}_\eta$ in $C([0,T]\times[0,2R]^N)$. Any limit u_n^R satisfies the bounds (3.8), (3.19) and (3.31). Moreover, by extraction of a subsequence, we may assume that $u_{\eta,n}^R \to u_n^R$ in $C([0,T]\times[0,2R]^N)$ and

$$A^\eta(u_{\eta,n}^R, \nabla u_{\eta,n}^R) \to z_n^R(t,x) \quad \text{weakly}^* \text{ in } L^\infty([0,T]\times[0,2R]). \tag{3.32}$$

Using Lemma 3.6 and the techniques in [**2**] one can prove the following facts:

$$\begin{aligned} \frac{\partial u_n^R}{\partial t} &= \operatorname{div} z_n^R \qquad \text{in } (0,T)\times(0,2R)^N \\ z_n^R \cdot \nu^{[0,2R]^N} &= 0 \quad \text{on } (0,T)\times\partial[0,2R]^N, \end{aligned} \tag{3.33}$$

with test functions in $C_{ct}^1((0,T)\times[0,2R]^N)$ and

$$z_n^R = \frac{u_n^R \nabla u_n^R}{\sqrt{(u_n^R)^2 + |\nabla u_n^R|^2}}. \tag{3.34}$$

By Lemma 3.7, the entropy inequalities are satisfied (see [**2**]). Indeed, let $S,T \in \mathcal{T}^+$ and J_{ST} be the primitive of ST, i.e., $J_{ST}(r) = \int_0^r T(s)S(s)\,ds$, let

$$A(w,p) := \frac{wp}{\sqrt{w^2+|p|^2}} \qquad w\in[0,\infty),\ p\in\mathbb{R}^N,$$

$$h(w,p) = A(w,p)\cdot p = \frac{w|p|^2}{\sqrt{w^2+|p|^2}} \quad w\in[0,\infty),\ p\in\mathbb{R}^N,$$

and

$$h_S(u,\nabla T(u)) = S(u)h(u,\nabla T(u)).$$

Replacing j by J_{ST}, u by u_n^R and z by z_n^R in (3.24), using only the inequality $\leq$, we obtain the entropy inequalities [**2**], [**4**]

$$\begin{aligned}
&\int_0^T \int_{[0,2R]^N} \phi h_S(u_n^R, \nabla T(u_n^R))\,dx\,dt \\
&+ \int_0^T \int_{[0,2R]^N} \phi h_T(u_n^R, \nabla S(u_n^R))\,dx\,dt \\
&\leq \int_0^T \int_{[0,2R]^N} J_{TS}(u_n^R(t))\phi'(t)\,dx\,dt \\
&- \int_0^T \int_{[0,2R]^N} A(u_n^R(t), \nabla u_n^R(t)) \cdot \nabla\phi T(u_n^R(t))S(u_n^R(t))\,dx\,dt
\end{aligned}$$

for any truncatures $S, T \in \mathcal{T}^+$ and any smooth function $\phi \in C_{ct}^1((0,T)\times[0,2R])^N$, $\phi \geq 0$.

Let us mention that the concept of entropy solution for the problem (3.3) defined in [**2**], [**4**] requires that the equation (3.33) holds with more general test functions and using the techniques in [**2**], [**4**] it can be proved that u_n^R is an entropy solution of (3.3). On the other hand, the concept of entropy solution for (3.3) implies that the equation in (3.3) holds with test functions in $C_{ct}^1((0,T)\times[0,2R])$, that truncatures of the solution are functions of bounded variation in x, and the entropy conditions are satisfied. The last two conditions are enough to prove uniqueness using the method in [**2**], [**4**]. In the present context, this amounts to say that using the uniqueness proof in [**2**], [**4**] we can prove: a) any entropy solution of (3.3) coincides with u_n^R and b) any two functions with the regularity of u_n^R satisfying the entropy conditions above must coincide. In particular, this implies that the whole sequence $u_{\eta,n}^R \to u_n^R$ as $\eta \to 0$ in $C([0,T]\times[0,2R]^N)$. Finally observe that, letting $\eta \to 0^+$ in (3.19) we obtain

$$\sup_{[0,T]\times[0,2R]^N} \frac{|\nabla u_n^R(t,x)|}{|u_n^R(t,x)|} \leq \sup_{x\in[0,2R]^N} \frac{|\nabla \tilde{u}_{0n}^R(x)|}{|\tilde{u}_{0n}^R(x)|}. \tag{3.35}$$

Step 3. Let us prove that

$$\sup_{x\in[0,2R]^N} \frac{|\nabla \tilde{u}_{0n}^R(x)|}{|\tilde{u}_{0n}^R(x)|} \to \sup_{x\in[0,2R]^N} \frac{|\nabla \tilde{u}_{0}^R(x)|}{|\tilde{u}_{0}^R(x)|} \quad \text{as } n\to\infty. \tag{3.36}$$

Since $\nabla \tilde{u}_{0n}^R \to \nabla \tilde{u}_0^R$ weakly* in $L^\infty([0,2R]^N, \mathbb{R}^N)$ and $\tilde{u}_{0n}^R \to \tilde{u}_0^R$ uniformly in $[0,2R]^N$ we deduce that

$$\sup_{x\in[0,2R]^N} \frac{|\nabla \tilde{u}_{0n}^R(x)|}{|\tilde{u}_{0n}^R(x)|} \leq \liminf_n \sup_{x\in[0,2R]^N} \frac{|\nabla \tilde{u}_0^R(x)|}{|\tilde{u}_0^R(x)|}. \tag{3.37}$$

On the other hand, we observe that

$$\begin{aligned} |\nabla \tilde{u}_{0n}^R(x)| &\leq \int_{\mathbb{R}^N} \rho_n(x-y)|\nabla \tilde{u}_0^R(y)|\,dy \\ &\leq \int_{\mathbb{R}^N} \rho_n(x-y) \sup_{y'\in B(x,\frac{1}{n})\cap[0,2R]^N} |\nabla \tilde{u}_0^R(y')|\,dy \\ &= \sup_{y'\in B(x,\frac{1}{n})\cap[0,2R]^N} |\nabla \tilde{u}_0^R(y')| \leq |\nabla \tilde{u}_0^R(x)| + \frac{\|D^2\tilde{u}_0^R\|_{L^\infty([0,2R]^N)}}{n}. \end{aligned}$$

Together with (3.37), this inequality and the uniform convergence of $\tilde{u}_{0n}^R$ to $\tilde{u}_0^R$ as $n\to\infty$ imply (3.36).

Step 4. We let $n\to\infty$. Indeed, since $\tilde{u}_{0n}^R \to \tilde{u}_0^R$ in $L^1([0,2R]^N)$ and the entropy solution of (3.3) gives a contractive semigroup in $L^1([0,2R]^N)$ [**2**], then $u_n^R(t,x)$ converges as $n\to\infty$ to the entropy solution $\overline{u}^R(t,x)$ of (3.3) with initial datum $\tilde{u}_0^R$. Let us observe that the estimates we have permit to obtain easily the same conclusion. The estimate analogous to (3.8) for u_n^R, the estimate (3.35) and the boundedness of its right hand side prove that $\sup_{[0,T]\times[0,2R]^N} \frac{|\nabla u_n^R(t,x)|}{|u_n^R(t,x)|}$, hence also, $\|\nabla u_n^R\|_\infty$, is bounded independently of n. Now, the argument in Step 1 proves that the Hölder bound (3.31) holds for u_n^R with a bound independent of n. Hence, after extraction of a subsequence if necessary, we have that $u_n^R \to \overline{u}^R$ uniformly in $[0,T]\times[0,2R]^N$ as $n\to\infty$ for some $\overline{u}^R \in C([0,T]\times[0,2R]^N)$. Now, by Step 3, we may pass to the limit in (3.35) and we obtain

$$\sup_{[0,T]\times[0,2R]^N} \frac{|\nabla \overline{u}^R(t,x)|}{|\overline{u}^R(t,x)|} \leq \sup_{x\in[0,2R]^N} \frac{|\nabla \tilde{u}_0^R(x)|}{|\tilde{u}_0^R(x)|}. \tag{3.38}$$

Moreover, by extraction of a subsequence, we may assume that

$$A(u_n^R,\nabla u_n^R) \to \overline{z}^R(t,x) \quad \text{weakly* in } L^\infty([0,T]\times[0,2R]) \text{ as } n\to\infty. \tag{3.39}$$

Using Lemma 3.6 and the techniques in [**2**] one can prove the following facts:

$$\begin{aligned} &\frac{\partial \overline{u}^R}{\partial t} = \operatorname{div} \overline{z}^R \qquad \text{in } (0,T)\times(0,2R)^N \\ &\overline{z}^R\cdot\nu^{[0,2R]^N} = 0 \quad \text{in } (0,T)\times\partial(0,2R)^N, \end{aligned} \tag{3.40}$$

with test functions $\phi \in C^1_{ct}([0,T]\times\mathbb{R}^N)$ and

$$\overline{z}^R = \frac{\overline{u}^R\nabla\overline{u}^R}{\sqrt{(\overline{u}^R)^2+|\nabla\overline{u}^R|^2}}.$$

As in Step 2, by Lemma 3.7, the entropy inequalities are satisfied (see [**2**]). The same observations made in Step 2 can be done in this case. Let us only mention that it can be proved that $\overline{u}^R$ is an entropy solution of (3.3) in the sense of [**4**] (see also [**2**]). But we only need to observe that solutions of (3.3) satisfying the same regularity conditions as $\overline{u}^R$ and the entropy inequalities are unique. This implies that the whole sequence $u_n^R \to \overline{u}^R$ as $n\to\infty$ in $C([0,T]\times[0,2R]^N)$. The entropy solution of (3.3) is obtained by a suitable translation of $\overline{u}^R$, i.e., $u^R(x) = \overline{u}^R(x+R(1,\ldots,1))$. □

Remark 3.8. Observe that the regularity of $u^R_{\eta,n}$ permits to obtain the estimate

$$\|u^R_{\eta,n}(t)-u^R_{\eta,m}(t)\|_{L^1([0,2R]^N)} \le \|\tilde{u}^R_{0n}-\tilde{u}^R_{0m}\|_{L^1([0,2R]^N)} \qquad \forall\, t\ge 0, \quad \forall\, n,m\ge 1,$$

in the standard way. Letting $\eta\to 0^+$ we obtain

$$\|u^R_{n}(t)-u^R_{m}(t)\|_{L^1([0,2R]^N)} \le \|\tilde{u}^R_{0n}-\tilde{u}^R_{0m}\|_{L^1([0,2R]^N)} \qquad \forall\, t\ge 0, \quad \forall\, n,m\ge 1.$$

This implies the convergence of $u_n^R(t)$ to some $u^R(t,x)$ in $C([0,T],L^1([0,2R]^N))$.

Lemma 3.9. *Assume that $u_0\in\mathcal{S}(\mathbb{R}^N)$ and $u_0(x)\ge\lambda_0 e^{-\beta\frac{|x|^2}{2}}$ for some λ_0, $\beta>0$. Let u be the entropy solution of (3.2) with $u(0,x)=u_0(x)$. Then $u(t,x)\ge\lambda_0 e^{-\alpha t-\beta\frac{|x|^2}{2}}$ for any $\alpha\ge\beta(2N-1)$.*

Proof: Let us prove that if $\alpha \geq 2N-1$, then $U(t,x) = e^{-\alpha t - \beta\frac{|x|^2}{2}}$ is a subsolution of (3.2). This follows from $U_t = -\alpha U$ and the inequality

$$\operatorname{div}\left(\frac{U\nabla U}{\sqrt{U^2+|\nabla U|^2}}\right) = \left[-\frac{N\beta + (N-1)\beta^3|x|^2}{(1+\beta^2|x|^2)^{3/2}} + \frac{\beta^2|x|^2}{(1+\beta^2|x|^2)^{1/2}}\right] U$$
$$\geq (-2N+1)\beta U.$$

Since the equation (3.2) is homogeneous of degree 1, then $\lambda e^{-\alpha t - \beta\frac{|x|^2}{2}}$ is also a subsolution of (3.2) for any $\lambda \geq 0$. By comparison with this type of subsolution, we deduce that $u(t,x) \geq \lambda_0 e^{-\alpha t - \beta\frac{|x|^2}{2}}$ when $\alpha \geq \beta(2N-1)$. $\square$

Remark 3.10. If u is the entropy solution of (2.2) then $u(t,x) \geq \lambda_0 e^{-c^2\alpha t - \beta\frac{|x|^2}{2}}$ for any $\alpha \geq \frac{\beta}{c^2}(2N-1)$.

Proof of Proposition 3.1:
Step 1. Estimates on u^R. Fix a compact set $Q \subseteq \mathbb{R}^N$ and let $R > 0$ be such that $Q \subseteq (-R,R)^N$. Since the right hand side of estimate (3.38) is uniformly bounded, as in Step 2 of the proof of Proposition 3.2, the estimate (3.38) implies that there is some $\gamma > 0$ such that

$$|u^R(t,x) - u^R(s,x)| \leq C|t-s|^\gamma \quad \forall\, 0 \leq s \leq t \leq T,\ \forall\, x \in Q$$

where $C > 0$ depends on Q but it does not depend on R. Now, we observe that letting $\eta \to 0^+$, $n \to \infty$ in this order in the estimate (3.8) we obtain that

$$\|u^R(t,x)\|_{L^q([-R,R]^N)} \leq \|u_0\|_{L^q(\mathbb{R}^N)} \quad \forall\, q \in [1,\infty]. \tag{3.41}$$

Combining (3.41) with (3.38) we have that

$$\|\nabla u^R(t)\|_{L^\infty([-R,R]^N)} \leq \left\|\frac{\nabla u_0(x)}{u_0(x)}\right\|_{L^\infty([-R,R]^N)} \|u^R(t)\|_{L^\infty([-R,R]^N)}$$
$$\leq \left\|\frac{\nabla u_0(x)}{u_0(x)}\right\|_{L^\infty(\mathbb{R}^N)} \|u_0\|_{L^\infty(\mathbb{R}^N)}.$$

These estimates imply that, by extracting a subsequence, if necessary, we may assume that $u^R(t,x) \to u(t,x)$ locally uniformly in $[0,T] \times \mathbb{R}^N$ for some function $u(t,x) \in C([0,T] \times \mathbb{R}^N)$. Letting $R \to \infty$ we obtain

$$\|u(t,x)\|_{L^q(\mathbb{R}^N)} \leq \|u_0\|_{L^q(\mathbb{R}^N)} \quad \forall\, q \in [1,\infty], \tag{3.42}$$

and

$$\|\nabla u(t)\|_{L^\infty(\mathbb{R}^N)} \leq \left\|\frac{\nabla u_0(x)}{u_0(x)}\right\|_{L^\infty(\mathbb{R}^N)} \|u_0\|_{L^\infty(\mathbb{R}^N)}.$$

We may also assume that

$$A(u^R, \nabla u^R) \to z \quad \text{weakly}^* \text{ in } L^\infty([0,T] \times \mathbb{R}^N, \mathbb{R}^N)$$

for some vector field $z \in L^\infty([0,T] \times \mathbb{R}^N, \mathbb{R}^N)$. Letting $R \to \infty$ in the PDE satisfied by u^R (see (3.40) and the last lines of Step 4 of the proof of Proposition 3.2) we have that

$$\frac{\partial u}{\partial t} = \operatorname{div} z \quad \text{in } (0,T) \times \mathbb{R}^N \tag{3.43}$$

with test functions $\phi \in C^1_{ct}((0,T) \times \mathbb{R}^N)$.

Step 2. Identification of z. Let us prove that

$$z = A(u, \nabla u) \quad \text{a.e. in } (0,T) \times \mathbb{R}^N. \tag{3.44}$$

For that, following [**2**], [**4**], we prove that

$$\int_0^T \int_{\mathbb{R}^N} \phi(z - A(u, \nabla g)) \cdot (\nabla u - \nabla g)\, dx\, dt \geq 0, \tag{3.45}$$

for any $\phi(t,x) \in C_0^\infty((0,T) \times \mathbb{R}^N)$, $\phi \geq 0$, and any $g \in C^1([0,T] \times \mathbb{R}^N)$. Let us fix a function ϕ and a function g in the previous classes. For each $R > 0$ such that $\operatorname{supp} \phi \subseteq (0,T) \times (-R,R)^N$, we write the inequalities

$$\int_0^T \int_{\mathbb{R}^N} \phi(A(u^R, \nabla u^R) - A(u^R, \nabla g)) \cdot (\nabla u^R - \nabla g)\, dx\, dt \geq 0$$

where the domain of integration is considered to be $\mathbb{R}^N$ by our choice of $R > 0$.

Observe that, since $u^R \to u$ as $R \to \infty$ locally uniformly in $[0,T] \times \mathbb{R}^N$ and $\nabla u^R \to \nabla u$ weakly* in $L^\infty([0,T] \times \mathbb{R}^N, \mathbb{R}^N)$, we have

$$\int_0^T \int_{\mathbb{R}^N} \phi A(u^R, \nabla g) \cdot \nabla u^R\, dx\, dt \to \int_0^T \int_{\mathbb{R}^N} \phi A(u, \nabla g) \cdot \nabla u\, dx\, dt, \tag{3.46}$$

$$\int_0^T \int_{\mathbb{R}^N} \phi A(u^R, \nabla g) \cdot \nabla g\, dxdt \to \int_0^T \int_{\mathbb{R}^N} \phi A(u, \nabla g) \cdot \nabla g\, dxdt \tag{3.47}$$

and

$$\int_0^T \int_{\mathbb{R}^N} \phi A(u^R, \nabla u^R) \cdot \nabla g\, dx\, dt \to \int_0^T \int_{\mathbb{R}^N} \phi z \cdot \nabla g\, dx\, dt. \tag{3.48}$$

Finally, using the PDE satisfied by u^R (see (3.40)), (3.43) and Lemma 3.7 we have

$$\int_0^T \int_{\mathbb{R}^N} \phi A(u^R, \nabla u^R) \cdot \nabla u^R \, dx \, dt$$

$$= - \int_0^T \int_{\mathbb{R}^N} \nabla \phi \cdot A(u^R, \nabla u^R) u^R \, dx \, dt + \frac{1}{2} \int_0^T \int_{\mathbb{R}^N} \phi_t (u^R)^2 \, dx \, dt$$

$$\to - \int_0^T \int_{\mathbb{R}^N} \nabla \phi \cdot z u \, dx \, dt + \frac{1}{2} \int_0^T \int_{\mathbb{R}^N} \phi_t u^2 \, dx \, dt = \int_0^T \int_{\mathbb{R}^N} \phi z \cdot \nabla u \, dx \, dt.$$

Collecting all the above inequalities, we obtain (3.45).

In particular, the inequalities (3.45) imply that

$$[z(t,x) - A(u(t,x), \nabla g(t,x))] \cdot (\nabla u(t,x) - \nabla g(t,x)) \geq 0 \quad \text{a.e. in } (0,T) \times \mathbb{R}^N.$$

Since we may take a countable set of functions in $C^1([0,T] \times \mathbb{R}^N)$ dense in $C^1([0,T] \times \overline{B_k})$ for any ball B_k centered at 0 of radius $k \in \mathbb{N}$, we have that the above inequality holds for all $(t,x) \in S$ where $S \subseteq (0,T) \times \mathbb{R}^N$ is such that $\mathcal{L}^N((0,T) \times \mathbb{R}^N \setminus S) = 0$, and all $g \in \cup_k C^1([0,T] \times \overline{B_k})$. Now, fixed $(t,x) \in S$, and given $y \in \mathbb{R}^N$, there is $g \in \cup_k C^1([0,T] \times \overline{B_k})$ such that $\nabla g(t,x) = y$. Then

$$(z(t,x) - A(u(t,x), y)) \cdot (\nabla u(t,x) - y) \geq 0 \quad \forall\, y \in \mathbb{R}^N \text{ and } \forall\, (t,x) \in S,$$

and, by an application of the Minty-Browder method in $\mathbb{R}^N$, it follows that

$$z(t,x) = A(u(t,x), \nabla u(t,x)) \quad \text{a.e. } (t,x) \in Q_T,$$

and (3.44) follows.

Step 3. Concluding the proof. By Steps 1 and 2 we have that

$$\frac{\partial u}{\partial t} = \operatorname{div} A(u, \nabla u) \quad \text{in } (0,T) \times \mathbb{R}^N \tag{3.49}$$

with test functions $\phi \in C^1_{ct}((0,T) \times \mathbb{R}^N)$. Since $\nabla u \in L^1_{\text{loc}}((0,T) \times \mathbb{R}^N, \mathbb{R}^N)$, with the same proof of Lemma 3.7, we have that the entropy inequalities hold for u. Moreover, letting $\eta \to 0$, $n \to \infty$ and $R \to \infty$ in this order, we obtain that

$$\int_0^T \int_{\mathbb{R}^N} |\nabla T_{a,b}(u(t))| \, dx \, dt \leq C \tag{3.50}$$

for some constant $C > 0$. This estimate, together with the entropy inequalities, permit to prove that if $\overline{u}$ is the entropy solution of (3.2) with initial datum $\overline{u}(0,x) = u_0$, then $\overline{u} = u$. In other words, u is the entropy solution of (3.2). Since $u_0 \in \mathcal{A}^e$, then $u_0(x) \geq \lambda_0 e^{-\beta\frac{|x|^2}{2}}$ for some λ_0, $\beta > 0$ and, by Lemma 3.9 we have that $u(t,x) \geq \lambda_0 e^{-\alpha t-\beta\frac{|x|^2}{2}}$ for any $\alpha \geq \beta(2N-1)$. Thus, given R_0, u_R is bounded away from zero in $[0,T]\times[-R_0,R_0]^N$ for R large enough, and $\frac{1}{u^R} \to \frac{1}{u}$ locally uniformly in $[0,T]\times\mathbb{R}^N$. This implies that $\frac{\nabla u^R(t,x)}{u^R(t,x)}$ converges weakly* in L^∞ to $\frac{\nabla u(t,x)}{u(t,x)}$ and, hence

$$\left\|\frac{\nabla u(t)}{u(t)}\right\|_\infty \leq \left\|\frac{\nabla u_0}{u_0}\right\|_\infty. \qquad \square$$

We conclude the proof of Theorem 2.2 with the following lemma.

Lemma 3.11. *Assume that $u_0 \in \mathcal{A}^e$. Let u_c be the solution of* (2.2). *As $c \to \infty$, u_c converges to the solution $U \in C([0,T], L^1(\mathbb{R}^N))$ of the heat equation $u_t = \Delta u$ with $U(0,x) = u_0(x)$.*

Proof: As in Step 2 of the proof of Proposition 3.2, let us prove that there is some $\gamma > 0$ such that for any compact subset $Q \subseteq \mathbb{R}^N$ we have

$$|u_c(t,x) - u_c(s,x)| \leq C|t-s|^\gamma \quad \forall\, 0 \leq s,t \leq T, \quad \forall\, x \in Q, \tag{3.51}$$

where $C > 0$ is a positive constant depending on Q which does not depend on c. For that, observe that we may write (2.2) as

$$u_t = \operatorname{div}(a_c(t,x)\nabla u),$$

where

$$a_c(t,x) = \frac{1}{\sqrt{1+\frac{1}{c^2}\left(\frac{|\nabla u|}{u}\right)^2}}.$$

Observe that

$$\frac{1}{\sqrt{1+\frac{M^2}{c^2}}} \leq a_c(t,x) \leq 1$$

where M is the constant in the right hand side of (3.1). Hence the assumptions of Theorem 10.1 of Chapter 3 in [**14**] are satisfied and (3.51) holds. The estimates (3.42), (3.1), (3.51) for u_c and the convergence $a_c(t,x) \to 1$ uniformly as $c \to \infty$, imply that u_c converges as $c \to \infty$ to the solution $U \in C([0,T], L^1(\mathbb{R}^N))$ of the heat equation with $U(0,x) = u_0(x)$. $\square$

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References

[1] F. Andreu, V. Caselles and J. M. Mazón, A strongly degenerate quasilinear equation: the elliptic case, *Ann. Sc. Norm. Super. Pisa Cl. Sci. (5)* **3(3)** (2004), 555–587.

[2] F. Andreu, V. Caselles and J. M. Mazón, A strongly degenerate quasilinear equation: the parabolic case, *Arch. Ration. Mech. Anal.* **176(3)** (2005), 415–453.

[3] F. Andreu, V. Caselles and J. M. Mazón, A strongly degenerate quasilinear elliptic equation, *Nonlinear Anal.* **61(4)** (2005), 637–669.

[4] F. Andreu, V. Caselles and J. M. Mazón, The Cauchy problem for a strongly degenerate quasilinear equation, *J. Eur. Math. Soc. (JEMS)* **7(3)** (2005), 361–393.

[5] F. Andreu, V. Caselles, J. M. Mazón and S. Moll, Finite propagation speed for limited flux diffusion equations, *Arch. Ration. Mech. Anal.* **182(2)** (2006), 269–297.

[6] F. Andreu, V. Caselles, J. M. Mazón and S. Moll, A diffusion equation in transparent media, Preprint (2005).

[7] M. Bertsch and R. Dal Passo, Hyperbolic phenomena in a strongly degenerate parabolic equation, *Arch. Rational Mech. Anal.* **117(4)** (1992), 349–387.

[8] Y. Brenier, Extended Monge-Kantorovich theory, in: *“Optimal transportation and applications”* (Martina Franca, 2001), Lecture Notes in Math. **1813**, Springer, Berlin, 2003, pp. 91–121.

[9] F. E. Browder, Pseudo-monotone operators and nonlinear elliptic boundary value problems on unbounded domains, *Proc. Nat. Acad. Sci. U.S.A.* **74(7)** (1977), 2659–2661.

[10] J. Carrillo and P. Wittbold, Uniqueness of renormalized solutions of degenerate elliptic-parabolic problems, *J. Differential Equations* **156(1)** (1999), 93–121.

[11] A. Chertock, A. Kurganov and P. Rosenau, Formation of discontinuities in flux-saturated degenerate parabolic equations, *Nonlinearity* **16(6)** (2003), 1875–1898.

[12] R. Dal Passo, Uniqueness of the entropy solution of a strongly degenerate parabolic equation, *Comm. Partial Differential Equations* **18(1–2)** (1993), 265–279.
[13] A. Friedman, *"Partial differential equations of parabolic type"*, Prentice-Hall, Inc., Englewood Cliffs, N.J., 1964.
[14] O. A. Ladyzhenskaya and N. N. Ural'tseva, *"Linear and quasilinear elliptic equations"*, Translated from the Russian by Scripta Technica, Inc. Translation editor: Leon Ehrenpreis, Academic Press, New York-London, 1968.
[15] J.-L. Lions, *"Quelques méthodes de résolution des problèmes aux limites non linéaires"*, Dunod; Gauthier-Villars, Paris, 1969.
[16] D. Mihalas and B. W. Mihalas, *"Foundations of radiation hydrodynamics"*, Oxford University Press, New York, 1984.
[17] P. Rosenau, Tempered diffusion: A transport process with propagating fronts and inertial delay, *Phys. Rev. A* **46(12)** (1992), R7371–R7374

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