Publ. Mat. **51** (2007), 17–44

OPTIMAL SOBOLEV EMBEDDINGS ON $\mathbb{R}^n$

Jan Vybíral

Abstract

We study Sobolev-type embeddings involving rearrangement-invariant norms. In particular, we focus on the question when such embeddings are optimal. We concentrate on the case when the functions involved are defined on $\mathbb{R}^n$. This subject has been studied before, but only on bounded domains. We first establish the equivalence of the Sobolev embedding to a new type of inequality involving two integral operators. Next, we show this inequality to be equivalent to the boundedness of a certain Hardy operator on a specific new type of cone of positive functions. This Hardy operator is then used to provide optimal domain and range rearrangement-invariant norm in the embedding inequality. Finally, the limiting case of the Sobolev embedding on $\mathbb{R}^n$ is studied in detail.

1. Introduction

Embeddings of spaces of smooth functions into other spaces of integrable functions form an important field of study in the theory of function spaces. Consider, for example, the classical Sobolev inequality [**13**] on bounded domains Ω in $\mathbb{R}^n$, $n \geq 2$. This states that, given $1 < p < n$ and setting $q = np/(n-p)$,

$$\overset{\circ}{W}{}_p^1(\Omega) \hookrightarrow L^q(\Omega) \quad \text{for} \quad 1 < p < n. \tag{1.1}$$

(Here $L^q(\Omega)$ is the classical Lebesgue space, $W_p^1(\Omega)$ denotes the usual Sobolev space, $\overset{\circ}{W}{}_p^1(\Omega)$ the closure of $C_0^\infty(\Omega)$ in $W_p^1(\Omega)$ and $\hookrightarrow$ denotes a continuous embedding.)

Now, (1.1) is the so-called sublimiting case of the Sobolev embedding (since p is strictly less than the dimension of Ω). The limiting case $p = n$ is of crucial importance and great interest. Standard examples show that

2000 *Mathematics Subject Classification.* 46E35, 46E30.
Key words. Sobolev embeddings, rearrangement-invariant norms, Hardy operators, cones of positive functions.

although $np/(n-p)$ tends to infinity as p approaches n from the left, we may not replace the L^q-norm on the right side of (1.1) by the L^∞-norm.

It has been proved in many situations that the scale of Lebesgue spaces, although of primary interest, is not rich enough to describe all the important situations. Especially in limiting situations things can be very delicate and we have to consider finer scales of function spaces. It turns out to be very rewarding to study Sobolev-type embeddings in a broader context of general rearrangement-invariant spaces. These involve Lebesgue spaces, but also Lorentz and Orlicz spaces together with their numerous mutations, and more.

On bounded domains, a comprehensive study of Sobolev-type inequalities involving rearrangement-invariant function spaces has been carried out in [**5**].

In this paper, we study (1.1) with Ω replaced by the entire $\mathbb{R}^n$. In such situation, the techniques which have been successfully used for bounded domains do not work. We develop a new method suitable to deal with such problems.

Let us now briefly outline our approach. Let ϱ_R and ϱ_D be rearrangement-invariant Banach function norms on $(0,\infty)$ (precise definitions will be given in Section 2). Our aim is to study the embedding

$$W^1_{\varrho_D}(\mathbb{R}^n) \hookrightarrow L^{\varrho_R}(\mathbb{R}^n), \tag{1.2}$$

with

$$L^{\varrho_R}(\mathbb{R}^n) = \left\{u \in L^1_{\mathrm{loc}}(\mathbb{R}^n) : ||u|L^{\varrho_R}(\mathbb{R}^n)|| = \varrho_R(u^*) < \infty\right\} \tag{1.3}$$

and

$$W^1_{\varrho_D}(\mathbb{R}^n) = \left\{u \in L^1_{\mathrm{loc}}(\mathbb{R}^n) : ||u|W^1_{\varrho_D}(\mathbb{R}^n)|| = \varrho_D(u^*) + \varrho_D(|\nabla u|^*) < \infty\right\}, \tag{1.4}$$

where u^* is the non-increasing rearrangement of u.

The embedding (1.2) is then equivalent to

$$\varrho_R(u^*) \le c[\varrho_D(u^*) + \varrho_D(|\nabla u|^*)], \quad u \in W^1_{\varrho_D}(\mathbb{R}^n). \tag{1.5}$$

The inequality (1.5) is the main subject of our study. Let us mention that a similar question in the frame of Bessel potential spaces was studied recently in [**9**].

We are interested in two main questions:

1. Suppose that the 'range' norm ϱ_R is given. We want to find the optimal (that is, essentially smallest) norm ϱ_D for which (1.5) holds. The optimality means that if (1.5) holds with ϱ_D replaced by some other rearrangement-invariant norm σ, then there exists a constant $C > 0$ such that $\varrho_D(u^*) \le C\sigma(u^*)$ for all functions $u \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$.

2. Suppose that the 'domain' norm ϱ_D is given. We would like to construct the corresponding optimal 'range' norm ϱ_R. This means that the ϱ_R will be the essentially largest rearrangement-invariant norm for which (1.5) holds.

In Section 3, we reduce (1.5) to a certain new type of inequality involving two different Hardy-type operators. Similar inequalities appeared recently in [**3**], but in a completely different context. In Section 4 we prove another equivalent version of (1.5), namely inequality (4.4), which connects certain specific Hardy operator with an interesting cone of positive functions. The delicate interplay between this operator on the one side and the cone on the other side plays a crucial role in the subsequent sections, and is of independent interest. Especially, we emphasise that knowledge of both of these notions is indispensable in most of the results yet to come. We refer to Lemma 5.1 and Lemma 6.1 for details. The action of Hardy operators on cones of positive functions was very recently studied in [**11**] and [**12**] in a different context. It seems to be a very promising subject of study which opens interesting new directions of research and which might provide new ways how to approach to various difficult problems.

In Sections 5 and 6 we find optimal domain and optimal range spaces for (1.2) under two rather restrictive conditions (5.2) and (6.10). In Section 7 we show that these conditions are satisfied in sub-limiting cases and give a complete answer in these situations.

In order to be able to give definitive answer in the limiting case as well, we have to develop a yet finer method. This is done in Section 8, where the limiting case is investigated in detail.

A crucial step is provided by Lemmas 5.1 and 6.1. The rather technical proofs of these results are given in the Appendix. These lemmas play a substantial role in our approach as they describe the wonderful interplay between the Hardy operator (4.6) and the convex cone (4.5).

In [**14**] we studied the inequality

$$\varrho_R(u^*) \le c\varrho_D(|\nabla u|^*), \quad u \in W^1_{\varrho_D}(\mathbb{R}^n),$$

which corresponds to one part of (1.5). As we shall see, the study of (1.5) requires several new techniques to be developed.

Throughout the paper, c stands for a positive constant, not necessarily the same at each occurrence. Sometimes we abbreviate the inequality $A \le c\,B$ to $A \lesssim B$. The same applies to symbols "$\gtrsim$" and "$\approx$".

2. Rearrangement-invariant norms

We denote by $\mathfrak{M}(\mathbb{R}^n)$ the set of real-valued Lebesgue-measurable functions on $\mathbb{R}^n$ finite almost everywhere and by $\mathfrak{M}_+(\mathbb{R}^n)$ the class of non-negative functions in $\mathfrak{M}(\mathbb{R}^n)$. Finally, $\mathfrak{M}_+(0,\infty,\downarrow)$ denotes the set of all non-increasing functions from $\mathfrak{M}_+(0,\infty)$. Given $f \in \mathfrak{M}(\mathbb{R}^n)$ we define its non-increasing rearrangement by

$$f^*(t) = \inf\{\lambda > 0 : |\{|f(x)| > \lambda\}| \le t\}, \quad 0 < t < \infty. \tag{2.1}$$

For a set $A \subset \mathbb{R}^n$ we denote by $|A|$ its Lebesgue measure. A detailed treatment of rearrangements may be found in [**1**]. Furthermore, we set

$$f^{**}(t) = \frac{1}{t}\int_0^t f^*(s)\,\mathrm{d}s, \quad 0 < t < \infty. \tag{2.2}$$

We point out two important properties, namely

$$(f+g)^*(t) \le f^*\left(\frac{t}{2}\right) + g^*\left(\frac{t}{2}\right), \quad 0 < t < \infty, \tag{2.3}$$

and

$$(f+g)^{**}(t) \le f^{**}(t) + g^{**}(t), \quad 0 < t < \infty, \quad f, g \in \mathfrak{M}(\mathbb{R}^n). \tag{2.4}$$

We briefly recall some basic aspects of the theory of Banach function norms. For details, see [**1**].

Definition 2.1. A functional $\varrho\colon \mathfrak{M}_+(0,\infty) \to [0,\infty]$ is called a *Banach function norm* on $(0,\infty)$ if, for all f, g, f_n $(n = 1, 2, \dots)$ in $\mathfrak{M}_+(0,\infty)$, for all constants $a \ge 0$ and for all measurable subsets E of $(0,\infty)$, it satisfies the following axioms

(A_1) $\varrho(f) = 0$ if and only if $f = 0$ a.e.;
$\varrho(af) = a\varrho(f)$;
$\varrho(f+g) \le \varrho(f) + \varrho(g)$;
(A_2) if $0 \le g \le f$ a.e. then $\varrho(g) \le \varrho(f)$;
(A_3) if $0 \le f_n \uparrow f$ a.e. then $\varrho(f_n) \uparrow \varrho(f)$;
(A_4) if $|E| < \infty$ then $\varrho(\chi_E) < \infty$;
(A_5) if $|E| < \infty$ then $\int_E f \le C_E\,\varrho(f)$

with some constant $0 < C_E < \infty$, depending on ϱ and E but independent of f.

If, in addition, $\varrho(f) = \varrho(f^*)$, we say that ϱ is *rearrangement-invariant (r.i.) Banach function norm.* We often use the notions *norm* and *r.i. norm* to shorten the notation.

Definition 2.2. The *dilation operator* E_s, $0 < s < \infty$, is defined by

$$(E_s f)(t) = f(st), \quad 0 < t < \infty, \quad f \in \mathfrak{M}(0,\infty). \tag{2.5}$$

The *dual* of a norm ϱ is the functional

$$\varrho'(g) = \sup_{h:\varrho(h)=1} \int_0^\infty g(t)h(t)\,\mathrm{d}t, \quad g,h \in \mathfrak{M}_+(0,\infty). \tag{2.6}$$

Theorem 2.3 (G. H. Hardy, J. E. Littlewood)**.** *If $f, g \in \mathfrak{M}(\mathbb{R}^n)$, then*

$$\int_{\mathbb{R}^n} |f(x)g(x)|\,\mathrm{d}x \le \int_0^\infty f^*(s)g^*(s)\,\mathrm{d}s. \tag{2.7}$$

Theorem 2.4 (G. G. Lorentz, W. A. J. Luxemburg)**.** *Let ϱ be a Banach function norm. Then*

$$\varrho'' = \varrho. \tag{2.8}$$

Theorem 2.5 (G. H. Hardy, J. E. Littlewood, G. Pólya)**.** *Let ϱ be an r.i. norm on $(0,\infty)$ and let $f_1, f_2 \in \mathfrak{M}(\mathbb{R}^n)$ satisfy*

$$\int_0^t f_1^*(s)\,\mathrm{d}s \le \int_0^t f_2^*(s)\,\mathrm{d}s, \quad s > 0.$$

Then

$$\varrho(f_1^*) \le \varrho(f_2^*).$$

Lemma 2.6 (Hardy's Lemma)**.** *Let f_1 and f_2 be non-negative measurable functions on $(0,\infty)$ and suppose*

$$\int_0^t f_1(s)\,\mathrm{d}s \le \int_0^t f_2(s)\,\mathrm{d}s$$

for all $t > 0$. Let $h \in \mathfrak{M}_+(0,\infty,\downarrow)$. Then

$$\int_0^\infty f_1(s)h(s)\,\mathrm{d}s \le \int_0^\infty f_2(s)h(s)\,\mathrm{d}s.$$

If $1 \le p \le \infty$, we define

$$\varrho_p(g) = ||g||_p := \begin{cases} \left(\int_{\mathbb{R}^n} |g(x)|^p\,\mathrm{d}x\right)^{1/p} & \text{if } 1 \le p < \infty, \\ \operatorname{ess\,sup}_{x\in\mathbb{R}^n} |g(x)| & \text{if } p = \infty. \end{cases}$$

3. Reduction to Hardy operators

In this section we present the first step in the study of (1.5), namely a reduction of (1.5) to the boundedness of certain Hardy operators.

Theorem 3.1. *Let ϱ_D, ϱ_R be two r.i. Banach function norms on* $(0,\infty)$. *Then the inequality*

$$\varrho_R(u^*) \le c[\varrho_D(u^*) + \varrho_D(|\nabla u|^*)], \quad u \in W^1_{\varrho_D}(\mathbb{R}^n), \tag{3.1}$$

holds if and only if there is a constant $K>0$ such that

$$\varrho_R\left(\int_t^\infty f(s)s^{1/n-1}\,\mathrm{d}s\right) \le K\varrho_D\left(f(t) + \int_t^\infty f(s)s^{1/n-1}\,\mathrm{d}s\right) \tag{3.2}$$

for all $f \in \mathfrak{M}_+(0,\infty)$.

Proof: Step 1. Let us suppose that (3.1) holds and that a function $f \in \mathfrak{M}_+(0,\infty)$ is given. We define a new function u by

$$u(x) = \int_{\omega_n|x|^n}^\infty f(t)t^{1/n-1}\,\mathrm{d}t, \quad x \in \mathbb{R}^n,$$

where ω_n is the volume of unit ball in $\mathbb{R}^n$. We may assume, that $u(x)$ is finite a.e. (otherwise both sides of (3.2) are identically infinite and there is nothing to prove). Considering level sets of u we obtain

$$u^*(t) = \int_t^\infty f(s)s^{1/n-1}\,\mathrm{d}s,$$

$$|(\nabla u)(x)| = n\omega_n^{1/n} f(\omega_n|x|^n),$$

$$|(\nabla u)|^*(t) = n\omega_n^{1/n} f^*(t).$$

We point out, that if $u \notin W^1_{\varrho_D}(\mathbb{R}^n)$, then (3.1) holds trivially. Therefore we may apply (3.1) and obtain

$$\varrho_R\left(\int_t^\infty f(s)s^{1/n-1}\,\mathrm{d}s\right) = \varrho_R(u^*(t)) \le c\left[\varrho_D(f) + \varrho_D\left(\int_t^\infty f(s)s^{1/n-1}\,\mathrm{d}s\right)\right],$$

which is equivalent to (3.2).

Step 2. Let us now assume that (3.2) is true and $u \in W^1_{\varrho_D}(\mathbb{R}^n)$ with compact support is given. First note that

$$u^*(t) = -\int_t^\infty \frac{\mathrm{d}u^*(s)}{\mathrm{d}s}\,\mathrm{d}s. \tag{3.3}$$

Next, we recall the following generalization of the Pólya-Szegö principle from [**4**, (4.3)]:

$$\int_0^t \left[-s^{1-1/n}\frac{\mathrm{d}u^*}{\mathrm{d}s}\right]^*(s)\,\mathrm{d}s \le c\int_0^t |\nabla u|^*(s)\,\mathrm{d}s, \tag{3.4}$$

which holds for every $t > 0$ and every weakly differentiable function u such that $(\nabla u) \in L^1(\mathbb{R}^n) + L^\infty(\mathbb{R}^n)$ and

$$|\{x \in \mathbb{R}^n : |u(x)| > s\}| < \infty \quad \text{for all} \quad s > 0.$$

As $\nabla u \in L_{\varrho_D}(\mathbb{R}^n) \subset L^1(\mathbb{R}^n) + L^\infty(\mathbb{R}^n)$ and u has compact support, these assumptions are satisfied and (3.4) applies to u.

Using Theorem of Hardy, Littlewood and Pólya (Theorem 2.5) on (3.4) we obtain

$$\varrho_D\left(-s^{1-1/n}\frac{\mathrm{d}u^*(s)}{\mathrm{d}s}\right) \le \varrho_D(|(\nabla u)|^*(t)). \tag{3.5}$$

We combine our assumption with these observations and use (3.3), (3.2) with $f = s^{1-1/n}\frac{\mathrm{d}u^*(s)}{\mathrm{d}s}$ and (3.5) to obtain

$$\begin{aligned}
\varrho_R(u^*(t)) &= \varrho_R\left(-\int_t^\infty \frac{\mathrm{d}u^*(s)}{\mathrm{d}s}\,\mathrm{d}s\right)\\
&\le c\left[\varrho_D\left(-\int_t^\infty \frac{\mathrm{d}u^*(s)}{\mathrm{d}s}\,\mathrm{d}s\right) + \varrho_D\left(-s^{1-1/n}\frac{\mathrm{d}u^*(s)}{\mathrm{d}s}\right)\right]\\
&\le c\left[\varrho_D(u^*(t)) + \varrho_D(|\nabla u|^*(t))\right].
\end{aligned}$$

Hence, (3.1) holds for every $u \in W^1_{\varrho_D}(\mathbb{R}^n)$ with compact support. For a general $u \in W^1_{\varrho_D}(\mathbb{R}^n)$ we define

$$u_n = u\varphi_n, \quad \varphi_n(x) = \begin{cases} 1 & \text{if } |x| < n,\\ n+1-|x| & \text{if } n \le |x| \le n+1,\\ 0 & \text{if } |x| > n+1.\end{cases}$$

We apply (3.1) to u_n and use

$$|u_n(x)| \le |u(x)|, \quad |(\nabla u_n)(x)| \le c[|(\nabla u)(x)| + |u(x)|], \quad x\in\mathbb{R}^n, \quad n\in\mathbb{N}.$$

This leads to

$$\varrho_R(u_n^*) \le c[\varrho_D(u_n^*) + \varrho_D(|\nabla u_n|^*)] \le c[\varrho_D(u^*) + \varrho_D(|\nabla u|^*)]. \tag{3.6}$$

The monotone convergence of $|u_n|$ to $|u|$ and axiom (A_3) show that the left side of (3.6) tends to $\varrho_R(u)$ as n tends to infinity. □

4. Another equivalent version of (1.5)

The inequality (3.2) obtained in Theorem 3.1 is still not suitable for further investigation. Therefore we will derive another equivalent version of (3.1). In (3.2) we substitute

$$g(t) = f(t) + \int_t^\infty f(s)s^{1/n-1}\,\mathrm{d}s, \quad f \in \mathfrak{M}_+(0,\infty), \quad t > 0. \tag{4.1}$$

We shall need also the inverse substitution. Namely, if g is defined by (4.1), then

$$f(t) = g(t) - e^{nt^{1/n}} \int_t^\infty g(s)s^{1/n-1}e^{-ns^{1/n}}\,\mathrm{d}s \text{ for a.e. } t > 0. \tag{4.2}$$

If f is differentiable, then it may be proven by differentiation of (4.1). For a general f we observe, that the equation (4.1) has only one solution f for a fixed $g \in \mathfrak{M}_+(0,\infty)$. And a direct computation shows that it is given by (4.2).

Finally, we sum up (4.1) and (4.2) and obtain

$$\int_t^\infty f(s)s^{1/n-1}\,\mathrm{d}s = e^{nt^{1/n}}\int_t^\infty g(u)u^{1/n-1}e^{-nu^{1/n}}\,\mathrm{d}u \text{ for a.e. } t>0. \tag{4.3}$$

This substitution can now be used to reformulate (3.1).

Theorem 4.1. *Let ϱ_D, ϱ_R be two r.i. Banach function norms on $(0,\infty)$. Then, (3.1) is equivalent to*

$$\varrho_R\left(e^{nt^{1/n}} \int_t^\infty g(u)u^{1/n-1}e^{-nu^{1/n}}\,\mathrm{d}u\right) \le c\varrho_D(g) \textit{ for all } g \in \mathbf{G}, \tag{4.4}$$

where $\mathbf{G}$ is the new class of functions, defined by

(4.5)

$$\mathbf{G} = \Big\{g \in \mathfrak{M}_+(0,\infty) : \textit{there is a function } f \in \mathfrak{M}_+(0,\infty) \textit{ such that}$$
$$g(t) = f(t) + \int_t^\infty f(s)s^{1/n-1}\,\mathrm{d}s \textit{ for all } t > 0\Big\}$$
$$= \Big\{g \in \mathfrak{M}_+(0,\infty) : g(t) - e^{nt^{1/n}} \int_t^\infty g(s)s^{1/n-1}e^{-ns^{1/n}}\,\mathrm{d}s \ge 0$$
$$\textit{for all } t > 0\Big\}.$$

Proof: The assertion follows immediately from Theorem 3.1, (4.2) and (4.3). ◻

Hence the inequality (3.1) is equivalent to the boundedness of the Hardy-type operator

$$(4.6) \qquad (Gg)(u) = e^{nu^{1/n}} \int_u^\infty g(s)s^{1/n-1}e^{-ns^{1/n}}\,\mathrm{d}s, \quad u > 0$$

on the set $\mathbf{G}$. Using this notation, we may rewrite (4.3). If g is defined by (4.1), we have $Gg(t) = \int_t^\infty f(s)s^{1/n-1}\,\mathrm{d}s$. Furthermore, the set $\mathbf{G}$ is the image of the positive cone $\mathfrak{M}_+(0,\infty)$ under the operator

$$f \to f(t) + \int_t^\infty f(s)s^{1/n-1}\,\mathrm{d}s.$$

Before we proceed any further, we shall derive some basic properties of the class $\mathbf{G}$.

Remark 4.2. (i) $\mathbf{G}$ contains all non-negative non-increasing functions. To see this, note that for all $g \in \mathfrak{M}_+(0,\infty,\downarrow)$

$$(4.7) \quad g(t) - e^{nt^{1/n}} \int_t^\infty g(s)s^{1/n-1}e^{-ns^{1/n}}\,\mathrm{d}s \\ \geq g(t)\left\{1 - e^{nt^{1/n}} \int_t^\infty s^{1/n-1}e^{-ns^{1/n}}\,\mathrm{d}s\right\} = 0.$$

(ii) For every g from $\mathbf{G}$, Gg is non-increasing. Indeed, let $g \in \mathbf{G}$ and let f be defined by (4.2), then

$$(4.8) \quad (Gg)'(t) = \left[e^{nt^{1/n}}\int_t^\infty g(u)u^{1/n-1}e^{-nu^{1/n}}\,\mathrm{d}u\right]' = -t^{1/n-1}f(t) \leq 0.$$

(iii) The set $\mathbf{G}$ is a *convex cone*, that is, for every $\alpha, \beta > 0$ and $g_1, g_2 \in \mathbf{G}$, we have $\alpha g_1 + \beta\alpha_2 \in \mathbf{G}$. The proof of this statement is trivial.

Remark 4.3. (i) To show some applications, we prove that $W^{1,p}(\mathbb{R}^n) \hookrightarrow L^{\frac{np}{n-p},p}(\mathbb{R}^n)$ for $1 \leq p < n$. In this case, we have $\varrho_R(f) = ||f^*(t)t^{-1/n}||_p$ and $\varrho_D(f) = ||f||_p$. Using Remark 4.2 (ii) and the boundedness of classical Hardy operators on L^p, we get for every function $g \in \mathbf{G}$ that

$$\begin{aligned}\varrho_R(Gg) &= ||t^{-1/n}(Gg)^*(t)||_p \\ &= \left\| t^{-1/n} e^{nt^{1/n}} \int_t^\infty g(u) u^{1/n-1} e^{-nu^{1/n}}\,\mathrm{d}u \right\|_p \\ &\le \left\| t^{-1/n} \int_t^\infty g(u) u^{1/n-1}\,\mathrm{d}u \right\|_p \\ &\le c||t^{-1/n} g(t) t^{1/n}||_p = c||g||_p = c\varrho_D(g).\end{aligned}$$

(ii) Another application of the obtained results is the embedding $W^1(L^{n,1})(\mathbb{R}^n) \hookrightarrow L^\infty(\mathbb{R}^n)$. In this case

$$\begin{aligned}\varrho_R(Gg) = \sup_{t>0}(Gg)(t) = (Gg)(0) &= \int_0^\infty g(u) u^{1/n-1} e^{-nu^{1/n}}\,\mathrm{d}u \\ \le \int_0^\infty g(u) u^{1/n-1}\,\mathrm{d}u &\le \int_0^\infty g^*(u) u^{1/n-1}\,\mathrm{d}u = \varrho_D(g)\end{aligned}$$

for every function $g \in \mathbf{G}$. Now we used Remark 4.2 (ii) and Theorem 2.3.

(iii) Both these applications recover well-known results. They demonstrate some important aspects of this method. First, the second basic property of the class $\mathbf{G}$ (c.f. Remark 4.2 (ii)) lies in the roots of every Sobolev embedding. Second, the boundedness of Hardy operators plays a crucial role in this theory.

(iv) We haven't used the property (4.7) yet. It will play a crucial role in the study of optimality of obtained results.

5. Optimal domain space

In this section we are going to solve one of the main problems stated in the Introduction. We shall construct the optimal domain norm ϱ_D to a given range norm ϱ_R.

We start with a crucial lemma describing one important property of the class $\mathbf{G}$ which shall be useful later on. We postpone its proof to Appendix.

Lemma 5.1. *The inequality*

$$\int_t^\infty g(u)u^{1/n-1}e^{-nu^{1/n}}\,\mathrm{d}u \le c\int_t^\infty g^{**}(u)u^{1/n-1}e^{-nu^{1/n}}\,\mathrm{d}u, \quad t \ge 0, \tag{5.1}$$

holds for every $g \in \mathbf{G}$ with c independent of g.

Now we may solve the problem of the optimal domain space.

Theorem 5.2. *Let the norm ϱ_R satisfy*

$$\varrho_R\left(G(g^{**})\right) \le c\varrho_R\left(G(g^{*})\right), \quad g \in \mathfrak{M}_+(0,\infty). \tag{5.2}$$

Then the optimal domain norm ϱ_D corresponding to ϱ_R in the sense described in the Introduction is defined by

$$\varrho_D(g) := \varrho_R\left(G(g^{**})\right), \quad g \in \mathfrak{M}_+(0,\infty). \tag{5.3}$$

Proof: First, we point out that the functional ϱ_D defined by (5.3) is a norm. The axioms (A_1)–(A_3) are trivially satisfied. To prove (A_4) for ϱ_D, we fix a set $E \subset (0,\infty)$ with $|E| < \infty$. Then we get $G\chi_E^*(t) \le \chi_{(0,|E|)}(t)$ for every $t > 0$, and using (5.2) and (A_4) for ϱ_R, we get

$$\varrho_D(\chi_E) = \varrho_R(G\chi_E^{**}) \le c\varrho_R(G\chi_E^{*}) \le c\varrho_R(\chi_{(0,|E|)}) < \infty.$$

To verify (A_5) for ϱ_D, we fix also a set $E \subset (0,\infty)$ with $|E| = a < \infty$ and use (A_5) for ϱ_R. Consequently,

$$\begin{aligned}
\varrho_D(g) = \varrho_R(Gg^{**}) &\ge c\int_0^{a/2}(Gg^{**})(t)\,\mathrm{d}t \\
&\ge c\int_0^{a/2} e^{nt^{1/n}}\int_{a/2}^{a} g^{**}(s)s^{1/n-1}e^{-ns^{1/n}}\,\mathrm{d}s\,\mathrm{d}t \\
&\ge c\,g^{**}(a)\int_0^{a/2} e^{nt^{1/n}}\,\mathrm{d}t\int_{a/2}^{a} s^{1/n-1}e^{-ns^{1/n}}\,\mathrm{d}s \\
&\ge c_E\int_0^a g^{*}(s)\,\mathrm{d}s \ge c_E\int_E g.
\end{aligned}$$

Now we have to verify that (4.4) really holds. Let us fix a $g \in \mathbf{G}$. Then, by (5.1) and (5.3),

$$\begin{aligned}
&\varrho_R\left(e^{nt^{1/n}}\int_t^\infty g(u)u^{1/n-1}e^{-nu^{1/n}}\,\mathrm{d}u\right) \\
&\le c\varrho_R\left(e^{nt^{1/n}}\int_t^\infty g^{**}(u)u^{1/n-1}e^{-nu^{1/n}}\,\mathrm{d}u\right) \\
&= c\varrho_D(g).
\end{aligned}$$

Finally, we have to show that ϱ_D is optimal. Let us suppose that (4.4) holds with some other r.i. norm σ instead of ϱ_D. We want to show that $\varrho_D(g) \le c\sigma(g)$ for every function $g \in \mathfrak{M}_+(0,\infty)$. Using (5.2) and the first property of the class $\mathbf{G}$ from Remark 4.2, namely that $g^* \in \mathbf{G}$ for every function $g \ge 0$, we get

$$\begin{aligned}\varrho_D(g) &= \varrho_R\left(e^{nt^{1/n}}\int_t^\infty g^{**}(u)u^{1/n-1}e^{-nu^{1/n}}\,\mathrm{d}u\right)\\ &\le c\varrho_R\left(e^{nt^{1/n}}\int_t^\infty g^{*}(u)u^{1/n-1}e^{-nu^{1/n}}\,\mathrm{d}u\right)\\ &\le c\sigma(g^*) = c\sigma(g). \qquad\square\end{aligned}$$

6. Optimal range space

In this section we solve the converse problem. Namely, the norm ϱ_D is now considered to be fixed and we are searching for the optimal ϱ_R. First of all we shall introduce some notation.

We recall (4.6) and define

$$(6.1)\quad (Gg)(t)=e^{nt^{1/n}}\int_t^\infty g(s)s^{1/n-1}e^{-ns^{1/n}}\,\mathrm{d}s,\quad g\in\mathfrak{M}_+(0,\infty),\quad t>0,$$

$$(6.2)\quad (Hh)(t)=t^{1/n-1}e^{-nt^{1/n}}\int_0^t h(s)e^{ns^{1/n}}\,\mathrm{d}s,\quad h\in\mathfrak{M}_+(0,\infty),\quad t>0,$$

$$(6.3)\quad E(s)=e^{-ns^{1/n}}\int_0^s e^{nu^{1/n}}\,\mathrm{d}u,\qquad s>0.$$

The operators G and H are dual in the sense that

$$(6.4)\quad \int_0^\infty h(t)Gg(t)\,\mathrm{d}t = \int_0^\infty g(u)Hh(u)\,\mathrm{d}u \quad\text{for all}\quad g,h\in\mathfrak{M}_+(0,\infty).$$

As in [**5**], we would like to use duality to define ϱ_R. Using the notation introduced above, we can rewrite (4.4) as

$$(6.5)\qquad \sup_{g\in\mathbf{G}}\frac{\varrho_R(Gg)}{\varrho_D(g)}<\infty.$$

We may employ the duality in the following way:

$$\sup_{g\in\mathbf{G}}\frac{\varrho_R(Gg)}{\varrho_D(g)}=\sup_{g\in\mathbf{G},h\in\mathfrak{M}_+(0,\infty,\downarrow)}\frac{\int_0^\infty(Gg)(t)h(t)\,\mathrm{d}t}{\varrho_D(g)\varrho_R'(h)}$$
$$=\sup_{g\in\mathbf{G},h\in\mathfrak{M}_+(0,\infty,\downarrow)}\frac{\int_0^\infty(Hh)(t)g(t)\,\mathrm{d}t}{\varrho_D(g)\varrho_R'(h)}.$$

We have used Remark 4.2 (ii), (6.4) and the so-called *resonance* of the measure space $((0,\infty),\mathrm{d}x)$. We refer to [**1**, Chapter 2, Definition 2.3 and Chapter 2, Theorem 2.7] for details.

Let us now suppose for a moment that extending the supremum over all $g\in\mathfrak{M}_+(0,\infty)$ gives an equivalent quantity. Then we could continue the calculation

$$\sup_{g\in\mathbf{G}}\frac{\varrho_R(Gg)}{\varrho_D(g)}\approx\sup_{g\in\mathfrak{M}_+(0,\infty),h\in\mathfrak{M}_+(0,\infty,\downarrow)}\frac{\int_0^\infty(Hh)(t)g(t)\,\mathrm{d}t}{\varrho_D(g)\varrho_R'(h)}$$
$$=\sup_{h\in\mathfrak{M}_+(0,\infty,\downarrow)}\frac{\varrho_D'(Hh)}{\varrho_R'(h)},\tag{6.6}$$

and the inequality (4.4) would be equivalent to

$$\varrho_D'(Hh)\le c\varrho_R'(h),\quad h\in\mathfrak{M}_+(0,\infty,\downarrow).\tag{6.7}$$

A sufficient condition that would enable us to extend the supremum is given in the following lemma. We postpone its proof to Appendix.

Lemma 6.1. *Assume that the r.i. norm ϱ_D satisfies*

$$\varrho_D\left(\int_s^\infty f(u)\frac{E(u)}{u}u^{1/n-1}\,\mathrm{d}u\right)\le c\varrho_D(f),\quad f\in\mathfrak{M}_+(0,\infty).\tag{6.8}$$

Then

$$\sup_{g\in\mathbf{G}}\frac{\int_0^\infty(Hh)(t)g(t)\,\mathrm{d}t}{\varrho_D(g)}\approx\sup_{g\in\mathfrak{M}_+(0,\infty)}\frac{\int_0^\infty(Hh)(t)g(t)\,\mathrm{d}t}{\varrho_D(g)}\tag{6.9}$$

for all $h\in\mathfrak{M}_+(0,\infty,\downarrow)$.

The constants of equivalence do not depend on the choice of $h\in\mathfrak{M}_+(0,\infty,\downarrow)$.

As we shall see, the condition (6.8) is satisfied in all important examples, including the limiting Sobolev embedding. Equipped with this tool, we can now easily solve our problem.

Theorem 6.2. *Assume that the r.i. norm ϱ_D satisfies* (6.8) *and that its dual norm ϱ'_D satisfies*

$$\varrho'_D(H(h^{**})) \le c\varrho'_D(H(h^*)), \quad h \in \mathfrak{M}_+(0,\infty). \tag{6.10}$$

Then the optimal range norm in (4.4) *associated to ϱ_D is given as a dual norm to $\varrho'_D(H(f^{**}))$. Or, equivalently, the dual of the optimal range norm can be described by $\varrho'_R(f) := \varrho'_D(H(f^{**}))$.*

Proof: According to Lemma 6.1 and the calculation above, (4.4) is equivalent to (6.7). But for our choice of ϱ'_R this inequality is trivially true.

To prove the optimality, suppose, again, that there is another r.i. norm σ, such that (6.7) is true when we substitute its dual norm σ' in place of ϱ'_R. Then,

$$\sigma'(f) = \sigma'(f^*) \ge c\varrho'_D(H(f^*)) \ge c\varrho'_D(H(f^{**})) = c\varrho'_R(f), \quad \text{for all } f \in \mathfrak{M}_+(0,\infty),$$

proving the optimality of ϱ_R.

Finally, we have to prove that the functional $\varrho(f) = \varrho'_D(H(f^{**}))$ is a norm. Again, the axioms (A_1)–(A_3) are trivially satisfied. Using (6.10), Hardy's Lemma 2.6 and axiom (A_4) for ϱ'_D, we get also (A_4) for ϱ. (A_5) follows from the same axiom for ϱ'_D. □

7. The study of (5.2) and (6.10)

In this section we derive sufficient conditions for (6.8) and (6.10). In general, we follow the idea of [**5**, Theorem 4.4]. First of all, for every function $f \in \mathfrak{M}_+(0,\infty)$, we define the dilation operator E by

$$(E_s f)(t) = f(st), \quad t > 0, \quad s > 0.$$

It is well known, [**1**, Chapter 3, Proposition 5.11], that, for every r.i. norm ϱ on $\mathfrak{M}_+(0,\infty)$ and every $s > 0$, the operator E_s satisfies

$$\varrho(E_s f) \le c\varrho(f), \quad f \in \mathfrak{M}_+(0,\infty).$$

The smallest possible constant c in this inequality (which depends of course on s) is denoted by $h_\varrho(s)$. Hence

$$h_\varrho(s) = \sup_{f \ne 0} \frac{\varrho(E_s f)}{\varrho(f)}.$$

Now we are ready to prove the following result.

Theorem 7.1. *If a rearrangement-invariant norm ϱ_R satisfies $\int_0^1 s^{-1/n} h_{\varrho_R}(s)\,\mathrm{d}s < \infty$, then it also satisfies* (5.2).

Proof: Step 1. Let us suppose that the positive real numbers s, t, y satisfy $st < y$ and $0 < s < 1$. Then $t^{1/n} < (y/s)^{1/n}$ and, consequently,

$$e^{nt^{1/n}-n(y/s)^{1/n}} \le \left[e^{nt^{1/n}-n(y/s)^{1/n}}\right]^{s^{1/n}} = e^{n(st)^{1/n}-ny^{1/n}}.$$

So, for every function $f \in \mathfrak{M}_+(0,\infty)$, we obtain

$$e^{nt^{1/n}}\int_{st}^{\infty} f^*(y)y^{1/n-1}e^{-n(y/s)^{1/n}}\,\mathrm{d}y \le e^{n(st)^{1/n}}\int_{st}^{\infty} f^*(y)y^{1/n-1}e^{-ny^{1/n}}\,\mathrm{d}y.$$

Step 2. We may now come to the proof of the theorem. Fix a function $g \in \mathfrak{M}_+(0,\infty)$, with $\varrho'_R(g) = 1$. Then we use several times Fubini's Theorem, a change of variables, and inequality from Step 1 and obtain

$$\begin{aligned}
&\int_0^\infty g^*(t)Gf^{**}(t)\,\mathrm{d}t\\
&= \int_0^\infty g^*(t)e^{nt^{1/n}}\int_t^\infty f^{**}(s)s^{1/n-1}e^{-ns^{1/n}}\,\mathrm{d}s\,\mathrm{d}t\\
&= \int_0^\infty s^{1/n-1}e^{-ns^{1/n}}\int_0^s g^*(u)e^{nu^{1/n}}\,\mathrm{d}u\int_0^1 f^*(st)\,\mathrm{d}t\,\mathrm{d}s\\
&= \int_0^1\int_0^\infty f^*(st)s^{1/n-1}e^{-ns^{1/n}}\int_0^s g^*(u)e^{nu^{1/n}}\,\mathrm{d}u\,\mathrm{d}s\,\mathrm{d}t\\
&= \int_0^1\int_0^\infty g^*(u)e^{nu^{1/n}}\int_u^\infty f^*(st)s^{1/n-1}e^{-ns^{1/n}}\,\mathrm{d}s\,\mathrm{d}u\,\mathrm{d}t\\
&= \int_0^1 t^{-1/n}\int_0^\infty g^*(u)e^{nu^{1/n}}\int_{tu}^\infty f^*(y)y^{1/n-1}e^{-n(y/t)^{1/n}}\,\mathrm{d}y\,\mathrm{d}u\,\mathrm{d}t\\
&= \int_0^1 s^{-1/n}\int_0^\infty g^*(t)e^{nt^{1/n}}\int_{st}^\infty f^*(y)y^{1/n-1}e^{-n(y/s)^{1/n}}\,\mathrm{d}y\,\mathrm{d}t\,\mathrm{d}s\\
&\le \int_0^1 s^{-1/n}\int_0^\infty g^*(t)e^{n(st)^{1/n}}\int_{st}^\infty f^*(y)y^{1/n-1}e^{-ny^{1/n}}\,\mathrm{d}y\,\mathrm{d}t\,\mathrm{d}s\\
&= \int_0^1 s^{-1/n}\int_0^\infty g^*(t)(Gf^*)(st)\,\mathrm{d}t\,\mathrm{d}s.
\end{aligned}$$

Taking a supremum over g, we obtain that the left-hand side of (5.2) can be estimated from above by

$$\begin{aligned}\sup_{g\geq 0:\varrho_R'(g)=1}\int_0^1 s^{-1/n}\int_0^\infty g^*(t)(Gf^*)(st)\,\mathrm{d}t\,\mathrm{d}s\\ &= \int_0^1 s^{-1/n}\varrho_R((Gf^*)(s\cdot))\,\mathrm{d}s\\ &\leq \int_0^1 s^{-1/n}h_{\varrho_R}(s)\varrho_R(Gf^*)\,\mathrm{d}s\\ &= \left(\int_0^1 s^{-1/n}h_{\varrho_R}(s)\,\mathrm{d}s\right)\varrho_R(Gf^*). \qquad \square\end{aligned}$$

An analogous result can be obtained also for (6.10). The proof is omitted as it uses the same ideas as the preceding one.

Theorem 7.2. *If an r.i. norm σ satisfies $\int_0^1 s^{-1/n}h_\sigma(s)\,\mathrm{d}s<\infty$ then it satisfies also* (6.10) *with ϱ_D' replaced by σ.*

We will now present some applications of our results.

Example 7.3. Let

$$\varrho_R(f)=\varrho_\infty(f)=\operatorname*{ess\,sup}_{x\in\mathbb{R}^n}|f(x)|.$$

Then $h_{\varrho_R}(s)=1$ and, according to Theorem 7.1, (5.2) is satisfied and the optimal domain norm is given by

$$\varrho_D(f)\approx\sup_{t>0}(Gf^*)(t)=\int_0^\infty f^*(s)s^{1/n-1}e^{-ns^{1/n}}\,\mathrm{d}s,\quad f\in\mathfrak{M}(\mathbb{R}^n).$$

This norm is essentially smaller than $\varrho_{n,1}(f)=\int_0^\infty t^{1/n-1}f^*(t)\,\mathrm{d}t$, hence this result improves the second example from Remark 4.3. Now, an easy calculation shows that

$$\begin{aligned}\varrho_D(f)&\approx f^*(1)+\int_0^1 f^*(t)t^{1/n-1}\,\mathrm{d}t\\ &\approx \varrho_\infty(f^*\chi_{(1,\infty)})+\varrho_{n,1}(f^*\chi_{(0,1)}),\quad f\in\mathfrak{M}(\mathbb{R}^n).\end{aligned}$$

Example 7.4. Let

$$\varrho_D(f)=\varrho_1(f)=\int_{\mathbb{R}^n}|f(x)|\,\mathrm{d}x.$$

In that case, $\varrho_D' = \varrho_\infty$, whence $h_{\varrho_D'}(s) = 1$. So, by Theorem 7.2, (6.10) is satisfied. It is a simple exercise to verify (6.8). Using Theorem 6.2, the optimal range norm can be described as the dual norm to

$$\sigma(f) = \varrho_\infty(Hf^*) = \varrho_\infty\left(t^{1/n-1}e^{-nt^{1/n}}\int_0^t f^*(s)e^{ns^{1/n}}\,\mathrm{d}s\right).$$

To simplify

$$\varrho_R(g) = \sigma'(g) = \sup_{f:\varrho_\infty(Hf^*)\le 1}\int_0^\infty f^*(t)g^*(t)\,\mathrm{d}t,$$

we take $f(t) = t^{-1/n}\chi_{(0,1)}(t) + \chi_{(1,\infty)}(t)$. Calculation shows that then Hf^* is bounded on $(0,\infty)$. This choice leads to

$$\varrho_R(g) \gtrsim \int_0^1 g^*(t)t^{-1/n}\,\mathrm{d}t + \int_1^\infty g^*(t)\,\mathrm{d}t.$$

To prove the converse estimate, take $f \in \mathfrak{M}_+(0,\infty,\downarrow)$ bounded and $g \in \mathfrak{M}_+(0,\infty,\downarrow)$ bounded, with bounded support and differentiable. Then a direct calculation using only integration by parts and Fubini's Theorem shows that

$$\begin{aligned}\int_0^\infty f^*(t)g^*(t)\,\mathrm{d}t &= \int_0^\infty t^{1/n-1}e^{-nt^{1/n}}\int_0^t f^*(s)e^{ns^{1/n}}\,\mathrm{d}s \\ &\qquad\times\left[g^*(t) - t^{1-1/n}\frac{\mathrm{d}g^*}{\mathrm{d}t}(t)\right]\,\mathrm{d}t \\ &\le \varrho_\infty(Hf^*)\int_0^\infty\left[g^*(t) - t^{1-1/n}\frac{\mathrm{d}g^*}{\mathrm{d}t}(t)\right]\,\mathrm{d}t \\ &\lesssim \varrho_\infty(Hf^*)\left[\int_0^1 g^*(t)t^{-1/n}\,\mathrm{d}t + \int_1^\infty g^*(t)\,\mathrm{d}t\right].\end{aligned}\tag{7.1}$$

If $f \in \mathfrak{M}_+(0,\infty,\downarrow)$ is not bounded, it may be approximated by a monotone sequence of bounded $f_n \nearrow f$, $f_n \in \mathfrak{M}_+(0,\infty,\downarrow)$. This procedure shows that (7.1) holds for every $f \in \mathfrak{M}_+(0,\infty,\downarrow)$ and g as above. Finally, every $g \in \mathfrak{M}_+(0,\infty,\downarrow)$ may also be approximated by bounded differentiable functions $g_n \nearrow g$, $g_n \in \mathfrak{M}_+(0,\infty,\downarrow)$ with bounded supports. This provides (7.1) for all $f, g \in \mathfrak{M}_+(0,\infty,\downarrow)$.

Hence,

$$\varrho_R(g) = \sup_{f:\varrho_\infty(Hf^*)\le 1}\int_0^\infty f^*(t)g^*(t)\,\mathrm{d}t \approx \int_0^1 g^*(t)t^{-1/n}\,\mathrm{d}t + \int_1^\infty g^*(t)\,\mathrm{d}t.$$

8. The limiting embedding

In this section we consider the case of limiting Sobolev embedding, where ϱ_D is set to be $\varrho_D(f) = \varrho_n(f) = \left(\int_{\mathbb{R}^n} |f(x)|^n \, \mathrm{d}x\right)^{1/n}$. In that case, $\varrho'_D(f) = \varrho_{n'}(f)$, where n' is the conjugated exponent to n, namely $\frac{1}{n} + \frac{1}{n'} = 1$. Direct calculation shows that $h_{\varrho'_D}(s) = s^{-1/n'}$ and $\int_0^1 s^{-1/n} h_{\varrho'_D}(s) \, \mathrm{d}s = \infty$. Moreover, standard examples ($h(s) = \frac{1}{s|\log s|^2}\chi_{(0,1/2)}(s)$) show that (6.10) is not satisfied.

To include this important case into the frame of our work, we will develop a finer theory of optimal range space. This is described in the following assertion.

Theorem 8.1. *Let ϱ_D be a given r.i. norm such that* (6.8) *holds and*

$$\varrho'_D(H\chi_{(0,1)}) < \infty. \tag{8.1}$$

Set

$$\sigma(h) = \varrho'_D(Hh^*), \quad h \in \mathfrak{M}_+(0,\infty).$$

Then,

$$\varrho_R := \sigma' \tag{8.2}$$

is an r.i. norm which satisfies (4.4) *and which is optimal for* (4.4).

Proof: Step 1. We will prove that ϱ_R is an r.i. norm. The axioms (A_2) and (A_3) are easy to verify. Let us assume that $\varrho_R(f) = 0$ for some $f \in \mathfrak{M}_+(0,\infty)$. Then

$$0 = \varrho_R(f) = \sup_{\sigma(g)=1} \int_0^\infty f(t)g(t) \, \mathrm{d}t. \tag{8.3}$$

According to (8.1), $\sigma(\chi_E)$ is finite for every measurable set $E \subset (0,\infty)$ with $|E| < \infty$. Together with (8.3), this implies that $\int_E f = 0$ for every such set E and, consequently, $f = 0$ almost everywhere, which proves (A_1).

To verify (A_5), take a set $E \subset (0,\infty)$ with $|E| < \infty$. Then, for every $f \in \mathfrak{M}_+(0,\infty)$,

$$\varrho_R(f) = \sup_{\sigma(h)\neq 0} \frac{\int fh}{\sigma(h)} \geq \frac{\int f\chi_E}{\sigma(\chi_E)} = c_E \int_E f.$$

The axiom (A_4) is an easy consequence of (8.2) and the estimate

$$\sigma(g) \geq c_E \int_0^{|E|} g^*(u) \, \mathrm{d}u, \quad g \in \mathfrak{M}_+(0,\infty). \tag{8.4}$$

To prove (8.4), we use Fubini's Theorem

$$\begin{aligned}\sigma(g) = \varrho_D'(Hg^*) &= \varrho_D'\left(t^{1/n-1}e^{-nt^{1/n}}\int_0^t g^*(u)e^{nu^{1/n}}\,\mathrm{d}u\right)\\ &\geq \frac{\int_0^{2|E|} t^{1/n-1}e^{-nt^{1/n}}\int_0^t g^*(u)e^{nu^{1/n}}\,\mathrm{d}u\,\mathrm{d}t}{\varrho_D(\chi_{(0,2|E|)})}\\ &= c\int_0^{2|E|} g^*(u)e^{nu^{1/n}}\int_u^{2|E|} t^{1/n-1}e^{-nt^{1/n}}\,\mathrm{d}t\,\mathrm{d}u\\ &\geq c_E\int_0^{|E|} g^*(u)\,\mathrm{d}u.\end{aligned}$$

Step 2. We show that ϱ_R and ϱ_D satisfy (4.4). As in Section 6, we obtain

$$\begin{aligned}\sup_{g\in\mathbf{G}}\frac{\varrho_R(Gg)}{\varrho_D(g)} = \sup_{g\in\mathbf{G}}\frac{\sigma'(Gg)}{\varrho_D(g)} &= \sup_{g\in\mathbf{G},h\in\mathfrak{M}_+(0,\infty,\downarrow)}\frac{\int_0^\infty (Gg)(t)h(t)\,\mathrm{d}t}{\varrho_D(g)\sigma(h)}\\ &= \sup_{g\in\mathbf{G},h\in\mathfrak{M}_+(0,\infty,\downarrow)}\frac{\int_0^\infty (Hh)(t)g(t)\,\mathrm{d}t}{\varrho_D(g)\sigma(h)}.\end{aligned} \tag{8.5}$$

Together with Lemma 6.1, this yields

$$\begin{aligned}\sup_{g\in\mathbf{G}}\frac{\varrho_R(Gg)}{\varrho_D(g)} &= \sup_{h\in\mathfrak{M}_+(0,\infty,\downarrow)}\frac{1}{\sigma(h)}\sup_{g\in\mathbf{G}}\frac{\int_0^\infty (Hh)(t)g(t)\,\mathrm{d}t}{\varrho_D(g)}\\ &\approx \sup_{h\in\mathfrak{M}_+(0,\infty,\downarrow)}\frac{1}{\sigma(h)}\sup_{g\in\mathfrak{M}_+(0,\infty)}\frac{\int_0^\infty (Hh)(t)g(t)\,\mathrm{d}t}{\varrho_D(g)}\\ &= \sup_{h\in\mathfrak{M}_+(0,\infty,\downarrow)}\frac{\varrho_D'(Hh^*)}{\sigma(h)} = 1.\end{aligned}$$

Step 3. Finally, we prove the optimality of ϱ_R. Let the r.i. norms ν and ϱ_D satisfy (4.4) with ν instead of ϱ_R, that is,

$$\sup_{g\in\mathbf{G}}\frac{\nu(Gg)}{\varrho_D(g)} < \infty.$$

Then, proceeding as above,

$$\infty > \sup_{g\in\mathbf{G}} \frac{\nu(Gg)}{\varrho_D(g)} = \sup_{g\in\mathbf{G},h\in\mathfrak{M}_+(0,\infty,\downarrow)} \frac{\int_0^\infty (Gg)(t)h(t)\,\mathrm{d}t}{\varrho_D(g)\nu'(h)}$$

$$\approx \sup_{h\in\mathfrak{M}_+(0,\infty,\downarrow)} \frac{\varrho'_D(Hh^*)}{\nu'(h)}.$$

Hence, for every $h \in \mathfrak{M}_+(0,\infty)$,

$$\sigma(h) = \varrho'_D(Hh^*) \le c\,\nu'(h).$$

Consequently,

$$\nu(f) = \nu''(f) \le c\,\sigma'(f) = c\,\varrho_R(f) \quad \text{for all} \quad f \in \mathfrak{M}_+(0,\infty). \qquad \square$$

Let us apply Theorem 8.1 to the limiting Sobolev embeddings with

$$\varrho_D(f) = \varrho_n(f) = \left(\int_0^\infty |f^*(t)|^n\,\mathrm{d}t\right)^{1/n}$$

or

$$\varrho_D(f) = \varrho_{n,1}(f) = \int_0^\infty t^{1/n-1} f^*(t)\,\mathrm{d}t,$$

respectively. Direct calculation shows that (8.1) is satisfied in both these cases.

To verify (6.8), we point out that

$$E(s) \approx \begin{cases} s & \text{for } s \in (0,1], \\ s^{1-1/n} & \text{for } s \in (1,\infty). \end{cases} \tag{8.6}$$

Hence, Fubini's Theorem, (8.6) and Lemma 2.6 imply that

$$\varrho_{n,1}\left(\int_t^\infty f(u)\frac{E(u)}{u}u^{1/n-1}\,\mathrm{d}u\right) = n\int_0^\infty f(u)\frac{E(u)}{u}u^{1/n-1}u^{1/n}\,\mathrm{d}u$$

$$\le c\int_0^\infty t^{1/n-1} f(t)\,\mathrm{d}t \le c\int_0^\infty t^{1/n-1} f^*(t)\,\mathrm{d}t = c\varrho_{n,1}(f).$$

When $\varrho_D = \varrho_n$, (6.8) is a consequence of Hardy's inequality. We refer to [**8**] for details. So, in both the cases, Theorem 8.1 is applicable and gives the optimal range norm. The result is presented in the next theorem.

Theorem 8.2. *Let $\varrho_D = \varrho_n$. Then, the optimal range norm, ϱ_R, satisfies*

$$\varrho_R(f) \approx \varrho_n(f) + \lambda(f^*\chi_{(0,1)}), \tag{8.7}$$

where

$$\lambda(g) := \left(\int_0^1 \left(\frac{g^*(t)}{\log(\frac{e}{t})}\right)^n \frac{\mathrm{d}t}{t}\right)^{\frac{1}{n}}, \quad g \in \mathfrak{M}(0,1).$$

Proof: We first recall that for $\varrho_D = \varrho_n$, both (6.8) and (8.1) are satisfied. Thus, by Theorem 8.1,

$$\begin{aligned}
\varrho_R'(h) &\approx \varrho_{n'}(Hh^*) = \varrho_{n'}\left(t^{1/n-1}e^{-nt^{1/n}}\int_0^t h^*(s)e^{ns^{1/n}}\,\mathrm{d}s\right)\\
&\approx \varrho_{n'}\left(\chi_{(0,1)}(t)t^{1/n-1}e^{-nt^{1/n}}\int_0^t h^*(s)e^{ns^{1/n}}\,\mathrm{d}s\right)\\
&\quad + \varrho_{n'}\left(\chi_{(1,\infty)}(t)t^{1/n-1}e^{-nt^{1/n}}\int_0^t h^*(s)e^{ns^{1/n}}\,\mathrm{d}s\right)\\
&=: \mathrm{I} + \mathrm{II}.
\end{aligned}$$

Since

$$e^{-n} \le e^{n(s^{1/n}-t^{1/n})} \le 1 \quad \text{for} \quad 0 \le s \le t \le 1,$$

we obtain

$$\mathrm{I} \approx \varrho_{n'}\left(\chi_{(0,1)}(t)t^{1/n-1}\int_0^t h^*(s)\,\mathrm{d}s\right) = \left(\int_0^1 \left(\int_0^t h^*(s)\,\mathrm{d}s\right)^{n'} \frac{\mathrm{d}t}{t}\right)^{\frac{1}{n'}}.$$

As for II, we use monotonicity of h^*, (6.3) and (8.6) to get

$$\begin{aligned}
\mathrm{II} &= \left(\int_1^\infty \left(\int_0^t h^*(s)e^{ns^{1/n}}\,\mathrm{d}s\right)^{n'} e^{-nn't^{1/n}}\frac{\mathrm{d}t}{t}\right)^{\frac{1}{n'}}\\
&\ge \left(\int_1^\infty h^*(t)^{n'}\left(e^{-nt^{1/n}}\int_0^t e^{ns^{1/n}}\,\mathrm{d}s\right)^{n'}\frac{\mathrm{d}t}{t}\right)^{\frac{1}{n'}}\\
&\approx \left(\int_1^\infty h^*(t)^{n'}\left(t^{1-1/n}\right)^{n'}\frac{\mathrm{d}t}{t}\right)^{\frac{1}{n'}}\\
&= \left(\int_1^\infty h^*(t)^{n'}\,\mathrm{d}t\right)^{\frac{1}{n'}}.
\end{aligned}$$

Conversely, by the weighted Hardy inequality (cf. [**8**]),

$$\begin{aligned}
\mathrm{II} &\approx \left(\int_1^\infty \left(\int_0^1 h^*(s)e^{ns^{1/n}}\,\mathrm{d}s\right)^{n'} e^{-nn't^{1/n}}\frac{\mathrm{d}t}{t}\right)^{\frac{1}{n'}}\\
&\quad+\left(\int_1^\infty \left(\int_1^t h^*(s)e^{ns^{1/n}}\,\mathrm{d}s\right)^{n'} e^{-nn't^{1/n}}\frac{\mathrm{d}t}{t}\right)^{\frac{1}{n'}}\\
&\le c\left[\int_0^1 h^*(s)\,\mathrm{d}s+\left(\int_1^\infty h^*(t)^{n'}\,\mathrm{d}t\right)^{\frac{1}{n'}}\right]\\
&\le c\left[\left(\int_0^1\left(\int_0^t h^*(s)\,\mathrm{d}s\right)^{n'}\frac{\mathrm{d}t}{t}\right)^{\frac{1}{n'}}+\left(\int_1^\infty h^*(t)^{n'}\,\mathrm{d}t\right)^{\frac{1}{n'}}\right].
\end{aligned}$$

Altogether,

$$\varrho_R'(g)\approx\left(\int_0^1\left(\int_0^t h^*(s)\,\mathrm{d}s\right)^{n'}\frac{\mathrm{d}t}{t}+\int_1^\infty h^*(t)^{n'}\,\mathrm{d}t\right)^{\frac{1}{n'}}.$$

Now, set

$$\nu(g):=\left(\int_0^\infty g^*(t)^n v(t)\,\mathrm{d}t\right)^{\frac{1}{n}},$$

where

$$v(t)=\begin{cases} t^{-1}\left(\log\frac{e}{t}\right)^{-n}, & t\in(0,1),\\ 1, & t\in(1,\infty).\end{cases}$$

Then, by [**10**, Theorem 4], ν is an r.i. norm. More precisely, it is a special case of a classical Lorentz norm whose Köthe dual has been characterised in [**10**, Theorem 1]. Thus,

$$\nu'(f)\approx\left(\int_0^\infty\left(\int_0^t f^*(s)\,\mathrm{d}s\right)^{n'}\frac{v(t)}{\left(\int_0^t v(s)\,\mathrm{d}s\right)^{n'}}\,\mathrm{d}t\right)^{\frac{1}{n'}}\approx\varrho_R'(f),$$

as an easy calculation shows.

Finally, since both ν and ϱ_R are r.i. norms, it follows from the Principle of Duality (2.8) that

$$\varrho_R \approx \nu,$$

as desired. □

Remark 8.3. We note that λ from Theorem 8.2 is the well-known norm discovered in various contexts independently by Maz'ja [**7**], Hansson [**6**] and Brézis-Wainger [**2**].

Appendix A. Proofs of lemmas

As we have promised, we deliver here the proofs of Lemma 5.1 and Lemma 6.1.

Proof of Lemma 5.1: We fix $g \in \mathbf{G}$ and $t \geq 0$. Then, according to (4.5), there is a function $f \geq 0$ such that (4.1) holds. Thus the left-hand side of (5.1) can be rewritten as

$$\begin{aligned} &\int_t^\infty \left(f(u) + \int_u^\infty f(s)s^{1/n-1}\,\mathrm{d}s \right) u^{1/n-1}e^{-nu^{1/n}}\,\mathrm{d}u \\ &= \int_t^\infty f(u)u^{1/n-1}e^{-nu^{1/n}}\,\mathrm{d}u \\ &\quad + \int_t^\infty f(s)s^{1/n-1} \int_t^s u^{1/n-1}e^{-nu^{1/n}}\,\mathrm{d}u\,\mathrm{d}s \\ &= e^{-nt^{1/n}} \int_t^\infty f(s)s^{1/n-1}\,\mathrm{d}s. \end{aligned} \tag{A.1}$$

The right-hand side of (5.1) is more complicated. Using (2.4), (4.1) and Fubini's Theorem, we get

$$\begin{aligned} g^{**}(u) &\approx f^{**}(u) + \left(\int_t^\infty f(s)s^{1/n-1}\,\mathrm{d}s \right)^{**}(u) \\ &= f^{**}(u) + \int_u^\infty f(s)s^{1/n-1}\,\mathrm{d}s + \frac{1}{u}\int_0^u f(s)s^{1/n}\,\mathrm{d}s. \end{aligned} \tag{A.2}$$

We insert the formula (A.2) in (5.1) and use Fubini's Theorem to arrive at

$$\begin{aligned}
&\int_t^\infty g^{**}(u)u^{1/n-1}e^{-nu^{1/n}}\,\mathrm{d}u \\
&\approx \underbrace{\int_t^\infty f^{**}(u)u^{1/n-1}e^{-nu^{1/n}}\,\mathrm{d}u}_{\mathrm{I}} \\
&\quad + \underbrace{\int_t^\infty \left(\int_u^\infty f(s)s^{1/n-1}\,\mathrm{d}s\right)u^{1/n-1}e^{-nu^{1/n}}\,\mathrm{d}u}_{\mathrm{II}} \\
&\quad + \underbrace{\int_t^\infty \left(\int_0^u f(s)s^{1/n-1}\,\mathrm{d}s\right)u^{1/n-2}e^{-nu^{1/n}}\,\mathrm{d}u}_{\mathrm{III}}.
\end{aligned}$$

Each of these three integrals can be further estimated. We start with the second one:

$$\mathrm{II} = e^{-nt^{1/n}}\int_t^\infty f(s)s^{1/n-1}\,\mathrm{d}s - \int_t^\infty f(s)s^{1/n-1}e^{-ns^{1/n}}\,\mathrm{d}s.$$

To deal with integrals I and III, we use the notation $h(s) := \int_s^\infty u^{1/n-2}e^{-nu^{1/n}}\,\mathrm{d}u$. Then, by Fubini's Theorem,

$$\mathrm{I} \geq \int_t^\infty \frac{1}{u}\left(\int_t^u f(s)\,\mathrm{d}s\right)u^{1/n-1}e^{-nu^{1/n}}\,\mathrm{d}u = \int_t^\infty f(s)h(s)\,\mathrm{d}s$$

and

$$\mathrm{III} \geq \int_t^\infty \int_t^u f(s)s^{1/n}\,\mathrm{d}s u^{1/n-2}e^{-nu^{1/n}}\,\mathrm{d}u = \int_t^\infty f(s)s^{1/n}h(s)\,\mathrm{d}s.$$

The last three estimates give us

$$\begin{aligned}
\mathrm{I}+\mathrm{II}+\mathrm{III} \geq \int_t^\infty f(s)h(s)(s^{1/n}+1)\,\mathrm{d}s + e^{-nt^{1/n}}\int_t^\infty f(s)s^{1/n-1}\,\mathrm{d}s \\
- \int_t^\infty f(s)s^{1/n-1}e^{-ns^{1/n}}\,\mathrm{d}s.
\end{aligned}$$

This estimate and (A.1) imply that it is enough to prove that

$$\int_t^\infty f(s)h(s)(s^{1/n}+1)\,\mathrm{d}s \geq \int_t^\infty f(s)s^{1/n-1}e^{-ns^{1/n}}\,\mathrm{d}s.$$

But the last inequality is a trivial consequence of the pointwise estimate

$$h(s)(s^{1/n}+1) \geq s^{1/n-1}e^{-ns^{1/n}}, \quad s > 0,$$

which may be proved by direct calculation. □

Proof of Lemma 6.1: As $\mathbf{G} \subset \mathfrak{M}_+(0,\infty)$, the estimate "$\lesssim$" in (6.9) follows immediately. To prove the reverse one, take a $h \in \mathfrak{M}_+(0,\infty,\downarrow)$. Moreover, if $f \in \mathfrak{M}_+(0,\infty)$, we put $\tilde{f}(s) = f(s)\frac{E(s)}{s}$ for all $s > 0$, where E is defined by (6.3), and $g(t) = \tilde{f}(t) + \int_t^\infty \tilde{f}(s)s^{1/n-1}\,\mathrm{d}s$, $t > 0$. We claim, that the following two conditions are satisfied:

I. $\varrho_D(g) \le c\varrho_D(f)$,

II. $\int_0^\infty (Hh)(t)g(t)\,\mathrm{d}t \ge c\int_0^\infty (Hh)(t)f(t)\,\mathrm{d}t$.

Indeed, to prove I, we use the fact that $s^{-1}E(s) \le 1$ for all $s > 0$. We get (c.f. (8.6) and (6.8))

$$\begin{aligned}\varrho_D(g) &= \varrho_D\left(f(s)\frac{E(s)}{s} + \int_s^\infty f(u)\frac{E(u)}{u}u^{1/n-1}\,\mathrm{d}u\right)\\ &\le \varrho_D\left(f(s)\frac{E(s)}{s}\right) + \varrho_D\left(\int_s^\infty f(u)\frac{E(u)}{u}u^{1/n-1}\,\mathrm{d}u\right)\\ &\le \varrho_D(f) + c\varrho_D(f) = c\varrho_D(f),\end{aligned}$$

where we used (6.8).

The proof of II is more complicated. The left-hand side of the condition II can be simplified by

$$\begin{aligned}\int_0^\infty (Hh)(t)g(t)\,\mathrm{d}t &= \int_0^\infty (Gg)(t)h(t)\,\mathrm{d}t\\ &= \int_0^\infty h(t)\left(\int_t^\infty \tilde{f}(s)s^{1/n-1}\,\mathrm{d}s\right)\,\mathrm{d}t\end{aligned}$$

and the right-hand side by

$$\begin{aligned}\int_0^\infty (Hh)(t)f(t) &= \int_0^\infty f(u)u^{1/n-1}e^{-nu^{1/n}}\left(\int_0^u h(t)e^{nt^{1/n}}\,\mathrm{d}t\right)\,\mathrm{d}u\\ &= \int_0^\infty h(t)\left(e^{nt^{1/n}}\int_t^\infty f(u)u^{1/n-1}e^{-nu^{1/n}}\,\mathrm{d}u\right)\,\mathrm{d}t.\end{aligned}$$

By Hardy's Lemma 2.6, the result will follow if we show that, for all $\xi > 0$ and for all $f \in \mathfrak{M}_+(0,\infty)$,

$$\text{(A.3)}\quad \int_0^\xi\int_t^\infty \tilde{f}(s)s^{1/n-1}\,\mathrm{d}s\,\mathrm{d}t \ge \int_0^\xi e^{nt^{1/n}}\int_t^\infty f(u)u^{1/n-1}e^{-nu^{1/n}}\,\mathrm{d}u\,\mathrm{d}t.$$

Using Fubini's Theorem, we can rewrite the right-hand side of (A.3) as

$$\text{(A.4)}\quad \int_0^\xi f(s)s^{1/n-1}e^{-ns^{1/n}}\left(\int_0^s e^{nt^{1/n}}\,\mathrm{d}t\right)\mathrm{d}s + \int_\xi^\infty f(s)s^{1/n-1}e^{-ns^{1/n}}\,\mathrm{d}s\int_0^\xi e^{nt^{1/n}}\,\mathrm{d}t,$$

and the left-hand side of (A.3) as

$$\text{(A.5)}\quad \int_0^\xi \tilde{f}(s)s^{1/n}\,\mathrm{d}s + \xi\int_\xi^\infty \tilde{f}(s)s^{1/n-1}\,\mathrm{d}s = \int_0^\xi f(s)s^{1/n-1}E(s)\,\mathrm{d}s + \xi\int_\xi^\infty f(s)s^{1/n-2}E(s)\,\mathrm{d}s.$$

The first integral in the last sum in (A.5) is equal to the first integral in (A.4). So, we shall deal with the second integrals. We shall use the following observation

$$\frac{1}{s}\int_0^s e^{nu^{1/n}}\,\mathrm{d}u \geq \frac{1}{\xi}\int_0^\xi e^{nu^{1/n}}\,\mathrm{d}u, \quad s>\xi,$$

and finish the proof by

$$\begin{aligned}\xi\int_\xi^\infty f(s)s^{1/n-2}E(s)\,\mathrm{d}s &= \xi\int_\xi^\infty f(s)s^{1/n-2}e^{-ns^{1/n}}\int_0^s e^{nu^{1/n}}\,\mathrm{d}u\,\mathrm{d}s\\ &\geq \xi\int_\xi^\infty f(s)s^{1/n-1}e^{-ns^{1/n}}\frac{1}{\xi}\int_0^\xi e^{nu^{1/n}}\,\mathrm{d}u\,\mathrm{d}s\\ &= \int_\xi^\infty f(s)s^{1/n-1}e^{-ns^{1/n}}\,\mathrm{d}s\int_0^\xi e^{nu^{1/n}}\,\mathrm{d}u. \quad \square\end{aligned}$$

Acknowledgements. I would like to express my deep gratitude to Luboš Pick for his interest and valuable comments, and to an anonymous referee for many remarks, which helped to improve this paper.

References

[1] C. Bennett and R. Sharpley, *"Interpolation of operators"*, Pure and Applied Mathematics **129**, Academic Press, Inc., Boston, MA, 1988.

[2] H. Brézis and S. Wainger, A note on limiting cases of Sobolev embeddings and convolution inequalities, *Comm. Partial Differential Equations* **5(7)** (1980), 773–789.

[3] M. J. Carro, A. Gogatishvili, J. Martín and L. Pick, Weighted inequalities involving two Hardy operators with applications to embeddings of function spaces, *J. Operator Theory* (to appear).

[4] A. Cianchi and L. Pick, Sobolev embeddings into BMO, VMO, and L_∞, *Ark. Mat.* **36(2)** (1998), 317–340.

[5] D. E. Edmunds, R. Kerman and L. Pick, Optimal Sobolev imbeddings involving rearrangement-invariant quasinorms, *J. Funct. Anal.* **170(2)** (2000), 307–355.

[6] K. Hansson, Imbedding theorems of Sobolev type in potential theory, *Math. Scand.* **45(1)** (1979), 77–102.

[7] V. G. Maz'ja, *"Sobolev spaces"*, Translated from the Russian by T. O. Shaposhnikova, Springer Series in Soviet Mathematics, Springer-Verlag, Berlin, 1985.

[8] B. Muckenhoupt, Hardy's inequality with weights, Collection of articles honoring the completion by Antoni Zygmund of 50 years of scientific activity, I, *Studia Math.* **44** (1972), 31–38.

[9] B. Opic and W. Trebels, Sharp embeddings of Bessel potential spaces with logarithmic smoothness, *Math. Proc. Cambridge Philos. Soc.* **134(2)** (2003), 347–384.

[10] E. Sawyer, Boundedness of classical operators on classical Lorentz spaces, *Studia Math.* **96(2)** (1990), 145–158.

[11] G. Sinnamon, Embeddings of concave functions and duals of Lorentz spaces, *Publ. Mat.* **46(2)** (2002), 489–515.

[12] G. Sinnamon, Transferring monotonicity in weighted norm inequalities, *Collect. Math.* **54(2)** (2003), 181–216.

[13] S. L. Sobolev, *"Applications of functional analysis in mathematical physics"*, Translated from the Russian by F. E. Browder, Translations of Mathematical Monographs **7**, American Mathematical Society, Providence, R.I., 1963.

[14] J. Vybíral, Optimality of function spaces for boundedness of integral operators and Sobolev embeddings, Diploma Thesis, MFF UK, Prague (2002).

Department of Mathematical Analysis
Faculty of Mathematics and Physics
Charles University
Sokolovská 83
186 75 Praha 8
Czech Republic

Current address:
Mathematisches Institut
Friedrich-Schiller-Universität Jena
Ernst-Abbe-Platz 3
07740 Jena
Germany
E-mail address: vybiral@minet.uni-jena.de

Primera versió rebuda el 28 de febrer de 2006,
darrera versió rebuda el 9 d'octubre de 2006.