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LOCALLY NILPOTENT LINEAR GROUPS WITH THE WEAK CHAIN CONDITIONS ON SUBGROUPS OF INFINITE CENTRAL DIMENSION

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Abstract

Let V be a vector space over a field F. If $G \leq GL(V,F)$, *the central dimension of* G is the F-dimension of the vector space $V/C_V(G)$. In [**DEK**] and [**KS**], soluble linear groups in which the set $\mathcal{L}_{\text{icd}}(G)$ of all proper infinite central dimensional subgroups of G satisfies the minimal condition and the maximal condition, respectively, have been described. On the other hand, in [**MOS**], periodic locally radical linear groups in which $\mathcal{L}_{\text{icd}}(G)$ satisfies one of the weak chain conditions (the weak minimal condition or the weak maximal condition) have been characterized. In this paper, we begin the study of the non-periodic case by describing locally nilpotent linear groups in which $\mathcal{L}_{\text{icd}}(G)$ satisfies one of the two weak chain conditions.

1. Introduction

Let V be a vector space over a field F. The subgroups of the group $GL(V,F)$ of all automorphisms of V are called *linear groups*. The Theory of Linear Groups is one of the most developed branches of the Theory of Groups. If V has finite dimension over F, it is well-known that $GL(V,F)$ can be identified with the group of non-singular $n\times n$-matrices with entries in F, where $n = \dim_F V$. Finite dimensional linear groups have played an important role in mathematics due to the identification mentioned above and the interplay between other mathematical ideas and such groups. However the study of the subgroups of $GL(V,F)$,

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when V has infinite dimension over F has received considerably less attention, and requires some additional restrictions.

In the paper [**DEK**], a way of studying infinite dimensional linear groups that are near to finite dimensional groups, in some sense, was begun. If G is a subgroup of $GL(V, F)$, then G really acts on the factor-space $V/C_V(G)$. Following [**DEK**], the dimension of this factor-space will be called *the central dimension* of the subgroup G and will be denoted by $\operatorname{centdim}_F G$. Thus, $\operatorname{centdim}_F G = \dim_F(V/C_V(G))$.

Suppose that G is a linear group of finite central dimension. If $C = C_G(V/C_V(G))$, then C is a normal subgroup of G and G/C is isomorphic to some subgroup of $GL(n, F)$, where $n = \dim_F(V/C_V(G))$. Since C stabilizes the series

$$\langle 0 \rangle \leq C_V(G) \leq V,$$

C is abelian. Even more, C is torsion-free, if $\operatorname{char} F = 0$, and is p-elementary abelian, if $\operatorname{char} F = p > 0$ (see [**KW**, Proposition 1.C.3] and [**FL**, Section 43]). Hence, the structure of G can be determined by the structure of G/C, which is an ordinary finite dimensional linear group.

If $G \leq GL(V, F)$, let $\mathcal{L}_{\mathrm{icd}}(G)$ be the set of all proper subgroups of G of infinite central dimension. In order to study infinite dimensional linear groups G that are close to finite dimensional, it is natural to start making $\mathcal{L}_{\mathrm{icd}}(G)$ *very small* in some sense. That is, we want to impose some restriction to $\mathcal{L}_{\mathrm{icd}}(G)$. Given the success that the study of infinite groups with finiteness conditions has enjoyed, it seemed reasonable to study linear groups with finiteness conditions. Thus, in the paper [**DEK**] linear groups G in which the set $\mathcal{L}_{\mathrm{icd}}(G)$ satisfies the minimal condition (or G satisfies Min-icd) have been described. The dual case, that is linear groups G in which $\mathcal{L}_{\mathrm{icd}}(G)$ satisfies the maximal condition (or G satisfies Max-icd) has been studied in the paper [**KS**]. Since the weak minimal condition and the weak maximal condition are the most natural group-theoretical generalizations of the ordinary minimal and maximal conditions (see [**LR**, 5.1]), it is natural to go on this direction of research with them.

The weak conditions have been introduced by R. Baer [**BR**] and D. I. Zaitsev [**ZD**]. We recall their definitions in the most general way. Let G be a group and let $\mathcal{M}$ be a family of subgroups of G. Then G is said to satisfy *the weak minimal condition for $\mathcal{M}$-subgroups*, if the set $\mathcal{M}$ satisfies the weak minimal condition, that is if given a descending chain of members of $\mathcal{M}$

$$G_0 \geq G_1 \geq \cdots \geq G_n \geq G_{n+1} \geq \cdots$$

there exists some $m \in \mathbb{N}$ such that the index $|G_n : G_{n+1}|$ is finite for every $n \geq m$. The definition of the weak maximal condition is dual: G is said to satisfy *the weak maximal condition for $\mathcal{M}$-subgroups*, if the set $\mathcal{M}$ satisfies the weak maximal condition, that is if given an ascending chain of members of $\mathcal{M}$

$$G_0 \leq G_1 \leq \cdots \leq G_n \leq G_{n+1} \leq \cdots$$

there exists some $m \in \mathbb{N}$ such that the index $|G_{n+1} : G_n|$ is finite for every $n \geq m$. In passing we note that groups satisfying one of these weak conditions for different $\mathcal{M}$ have been studied by many authors (see [**LR**, 5.1] and the surveys [**AK**], [**KK**]).

We say that a group $G \leq GL(V, F)$ *satisfies the weak minimal condition on subgroups of infinite central dimension* (or G satisfies Wmin-icd), if G satisfies the weak minimal condition for $\mathcal{L}_{\text{icd}}(G)$-subgroups. Dually, we say that $G \leq GL(V, F)$ *satisfies the weak maximal condition on subgroups of infinite central dimension* (or G satisfies Wmax-icd), if G satisfies the weak maximal condition for $\mathfrak{L}_{\text{icd}}(G)$-subgroups. Periodic locally radical linear groups satisfying Wmin-icd or Wmax-icd were described in the paper [**MOS**]: they are either Chernikov or nilpotent-by-abelian-by-finite. As a consequence of that description, it was shown that, in the periodic case, the conditions Wmin-icd, Wmax-icd (and even Min-icd) are equivalent. In the current paper, we begin the study of some non-periodic groups satisfying one of these weak chain conditions as we study locally nilpotent linear groups satisfying Wmin-icd or Wmax-icd. Periodic locally nilpotent linear groups satisfying Wmin-icd or Wmax-icd are Chernikov ([**MOS**, Corollary D]), but, as we mentioned above, in the non-periodic case, other types of groups can occur, for example minimax groups. The main results obtained of this paper are the following ones.

Theorem A (Theorem 2.6)**.** *Let G be a subgroup of $GL(V, F)$ of infinite central dimension. Suppose that H is a normal subgroup of G such that G/H is nilpotent. If G satisfies either Wmin-icd or Wmax-icd, then G/H is minimax. In particular, if G is nilpotent, then G is minimax.*

From now on we restrict ourselves to the case of $\operatorname{char} F = p > 0$ since the case of $\operatorname{char} F = 0$ requires a different approach. We recall that, if G is a locally nilpotent group, then the set $t(G)$ of all elements having finite order is a characteristic subgroup of G. The subgroup $t(G)$ is called *the periodic part* or *the torsion subgroup* of the locally nilpotent group G.

Theorem B (Theorem 3.5). *Let G be a locally nilpotent subgroup of $GL(V,F)$ of infinite central dimension. Suppose that* $\operatorname{char} F = p > 0$. *If G satisfies either Wmin-icd or Wmax-icd, then $G/t(G)$ is minimax.*

Let $\mathfrak{F}$ be the class of finite groups. If G is a group, then the the intersection $G^{\mathfrak{F}}$ of all subgroups H of G of finite index is called *the finite residual* of G. The factor-group $G/G^{\mathfrak{F}}$ is said to be *residually finite.* Obviously, G itself is residually finite if $G^{\mathfrak{F}} = \langle 1 \rangle$.

Theorem C (Theorem 3.6). *Let G be a locally nilpotent subgroup of $GL(V,F)$ of infinite central dimension. Suppose that* $\operatorname{char} F = p > 0$. *If G satisfies either Wmin-icd or Wmax-icd, then $G/G^{\mathfrak{F}}$ is minimax and nilpotent.*

Let $\mathfrak{N}$ be the class of nilpotent groups. As above the intersection $G^{\mathfrak{N}}$ of all normal subgroups H of a group G such that $G/H \in \mathfrak{N}$ is called *the nilpotent residual* of G and the factor-group $G/G^{\mathfrak{N}}$ is said to be *residually nilpotent.* G itself is residually nilpotent if $G^{\mathfrak{N}} = \langle 1 \rangle$.

Corollary D (Corollary 3.7). *Let G be a locally nilpotent subgroup of $GL(V,F)$ of infinite central dimension. Suppose that* $\operatorname{char} F = p > 0$. *If G satisfies either Wmin-icd or Wmax-icd, then $G/G^{\mathfrak{N}}$ is minimax.*

2. Nilpotent linear groups satisfying Wmax-icd or Wmin-icd

In this section we establish that nilpotent linear groups (or more precisely, nilpotent factor-groups of linear groups) that satisfy Wmin-icd or Wmax-icd are minimax or have finite central dimension. In order to show this, we need first some preliminary results, the first lemma of which deals with abelian-by-finite linear groups, and it is widely employed throughout this paper.

If G is a group, as usual, we denote by $\Pi(G)$ the set of primes occurring as divisors of the orders of the periodic elements of G.

Lemma 2.1. *Let G be a subgroup of $GL(V,F)$ of infinite central dimension. Let H be a normal subgroup of G such that G/H is abelian-by-finite. If G satisfies either Wmin-icd or Wmax-icd, then G/H is minimax.*

Proof: Suppose that G has infinite central dimension and G/H is not minimax. Let A be a normal subgroup of G such that A/H is abelian and G/A is finite. By our assumption, A/H is not minimax. We note that if U is some G-invariant subgroup of A with $U \leq H$ such that A/U is periodic, then [**MOS**, Lemma 4.4] shows that G/U must be Chernikov. In particular, A/U is likewise Chernikov.

We claim that the 0-rank (or *torsioin-free rank*) $r_0(A/H)$ of A/H is infinite. For, otherwise, we may choose a finite maximal $\mathbb{Z}$-independent subset $\{a_1H, \dots, a_kH\}$ in A/H. Put $B/H = \langle a_1H, \dots, a_kH\rangle^{G/H}$. Since G/A is finite, B/H is a finitely generated abelian subgroup of A/H such that A/B is periodic. Applying the above paragraph, A/B is a Chernikov group. But in this case A/H is minimax, a contradiction. This contradiction shows our claim, that is $r_0(A/H)$ is infinite.

Let c_1H be an element of A/H of infinite order and put $C_1/H = \langle c_1H\rangle^{G/H}$. Since G/A is finite, C_1/H is a finitely generated G-invariant abelian subgroup of A/H. There is a positive integer t such that $D_1/H = (C_1/H)^t$ is free abelian, and, obviously, D_1/H is also G-invariant. Suppose that we have already constructed an ascending series

$$\langle 1\rangle = D_0/H \leq D_1/H \leq \cdots \leq D_\alpha/H$$

of G-invariant subgroups of A/H whose factors are free abelian. Then the subgroup D_α/H is free abelian (see [**KM**, Theorem 7.1.3]).

If A/D_α is not periodic, then it has an element $c_{\alpha+1}D_\alpha$ of infinite order. Put $C_{\alpha+1}/D_\alpha = \langle c_{\alpha+1}D_\alpha\rangle^{G/D_\alpha}$. Since G/A is finite, $C_{\alpha+1}/D_\alpha$ is a finitely generated G-invariant abelian subgroup of A/D_α. There is an $r \geq 0$ such that $D_{\alpha+1}/D_\alpha = (C_{\alpha+1}/D_\alpha)^r$ is free abelian. By construction, $D_{\alpha+1}/D_\alpha$ is also G-invariant. Moreover, there exists an ordinal γ such that A/D_γ is periodic. We also remark that $E/H = D_\gamma/H$ is free abelian. Since A/E is periodic, as we have already noted, A/E is Chernikov. In particular, the set $\Pi(A/E)$ is finite. Suppose that $p \notin \Pi(A/E)$. Since E/H is a free abelian subgroup, $E/H \neq (E/H)^p = L/H$. Furthermore $(E/H)/(L/H)$ is an infinite elementary abelian p-group. It follows that the Sylow p-subgroup of A/L is infinite elementary abelian. If W/L is the Sylow p'-subgroup of A/L, then A/W is an infinite elementary abelian p-group. In particular, it is not Chernikov, which contradicts the fact established above. This final contradiction implies our result. □

Corollary 2.2. *Let G be a subgroup of $GL(V, F)$ of infinite central dimension. If G satisfies Wmin-icd or Wmax-icd, then $G/[G, G]$ is minimax.*

The proof of a similar theorem for nilpotency is the final step of this section. In order to do this, we need some results about the existence of certain subgroups of a group G provided the first central factor-group of G satisfies prescribed properties.

Lemma 2.3. *Let G be a group whose center $\zeta(G)$ has an infinite elementary abelian p-subgroup E such that G/E is abelian minimax. Then G has a normal minimax subgroup L such that $G = LE$. In particular, G/L is an infinite elementary abelian p-group.*

Proof: Clearly G is nilpotent. Hence the torsion subgroup $T = t(G)$ of G is characteristic in G. Let P be the Sylow p-subgroup of G. Then $E \leq P$. Since G/E is minimax, P/E is a Chernikov subgroup. Let D/E be the divisible part of P/E. Then the abelian group G/D has a finite Sylow p-subgroup P/D and therefore

$$G/D = P/D \times R/D,$$

for some subgroup R (see [**FL**, Theorem 27.5]). In particular, G/R is a finite p-group.

We have

$$D/E = K_1/E \times \cdots \times K_r/E,$$

where $K_1/E, \ldots K_r/E$ are Prüfer p-subgroups. Let $1 \leq j \leq r$. Since K_j/E is locally cyclic, the inclusion $E \leq \zeta(G)$ gives that K_j is abelian. By [**KL1**, Lemma 1], the subgroup K_j has a Prüfer p-subgroup V_j, so that $K_j = EV_j$. Clearly V_j is a characteristic subgroup of K_j, in particular, it is normal in D. Then

$$U = V_1 \cdots V_r$$

is a normal divisible Chernikov subgroup of G. Clearly $D = EU$, and the inclusion $E \leq \zeta(G)$ implies that D is abelian. Then EU/U is the Sylow p-subgroup of R/U. Let Q/U be the Sylow p'-subgroup of R/U. We remark that Q is normal in G, Q is minimax and EQ/Q is the periodic part of R/Q. By [**HK**, Proposition 2], R/Q has a normal torsion-free subgroup W/Q such that R/W is a bounded p-group. Proceeding as in the proof of [**HK**, Proposition 2], we see that the subgroup W/Q can be chosen such that W/Q is characteristic in R/Q, and hence W is normal in G. Since W/Q is torsion-free, W is minimax. The finiteness of G/R implies that G/W is bounded. Hence G/W is a central extension of EW/W by a bounded minimax group G/EW. However a bounded abelian minimax group is finite. In other words, G/W is central-by-finite. It follows that G/W has a normal finite subgroup L/W such that $G/W = (L/W)(EW/W)$. Clearly L is likewise minimax. It follows that G/L is an infinite elementary abelian p-group, as required. □

Corollary 2.4. *Let G be a group whose center $\zeta(G)$ has an infinite elementary abelian p-subgroup E such that G/E is nilpotent minimax. Then G has a normal subgroup L such that G/L is an infinite elementary abelian p-group.*

Proof: Let

$$\langle 1\rangle = Z_0/E \leq Z_1/E \leq \cdots \leq Z_n/E = G/E$$

be the upper central series of G/E. We proceed by induction on n. If $n = 1$, then G/E is abelian and the statement follows from Lemma 2.3. Suppose that $n > 1$ and consider the subgroup Z_1. By Lemma 2.3, Z_1 has a normal minimax subgroup U such that $Z_1 = UE$. Put $V = \mathrm{Core}_G(U)$, so that V is minimax and $V \leq Z_1$.

Observe that the second center of G includes Z_1. Therefore for each element $zV \in Z_1/V$ the mapping $gV \mapsto [gV, zV]$ is a homomorphism of G/V into EV/V, whose kernel coincides with $C_{G/V}(zV)$. Thus $[G/V, zV]$ is an elementary abelian subgroup. On the other hand, the obvious inclusion $EV/V \leq C_{G/V}(zV)$ implies that $(G/V)/C_{G/V}(zV)$ is a nilpotent minimax group. The isomorphism

$$[G/V, zV] \cong (G/V)/C_{G/V}(zV)$$

shows that the last factor-group is a bounded minimax group, in particular, it is finite. This means that $Z_1/V \leq FC(G/V)$, where $FC(G/V)$ is the FC-center of G/V. Now let $W/V = (U/V)^{G/V}$, then W/V is finite and hence W is minimax. Therefore $Z_1 = WE$, where W is a minimax G-invariant subgroup. In particular, Z_1/W is an infinite elementary abelian p-group, $Z_1/W \leq \zeta(G/W)$ and $(G/W)/(Z_1/W)$ is a nilpotent minimax group whose nilpotency class is $n - 1$. Consequently we can apply now the inductive hypothesis. □

Lemma 2.5. *Let G be a group and suppose that $\zeta(G)$ has a divisible abelian p-subgroup E such that E is not Chernikov and G/E is nilpotent minimax. Then G has a normal minimax subgroup L such that G/L is a divisible abelian p-group which is not Chernikov.*

Proof: Let

$$\langle 1\rangle = Z_0/E \leq Z_1/E \leq \cdots \leq Z_n/E = G/E$$

be the upper central series of G/E. Clearly, G is nilpotent. Hence the set $T = t(Z_1)$ of all elements of Z_1 having finite order is a characteristic subgroup of Z_1. Let P be the Sylow p-subgroup of Z_1. Then $E \leq P$. Since G/E is minimax, P/E is a Chernikov subgroup. Let D/E be the divisible part of P/E. Therefore D is periodic nilpotent divisible, so that it is abelian (see [**KA**, §65]).

Let Q be the Sylow p'-subgroup of T. Remark that Q is normal in G and Q is Chernikov. For each element $gQ \in Z_1/Q$ the mapping $xQ \mapsto [gQ, xQ]$, $x \in G$, is a homomorphism of G/Q into DQ/Q, whose kernel coincides with $C_{G/Q}(gQ)$. On the other hand, the obvious inclusion $EQ/Q \leq C_{G/Q}(gQ)$ implies that $(G/Q)/C_{G/Q}(gQ)$ is a nilpotent minimax group. The isomorphism

$$[gQ, G/Q] \cong (G/Q)/C_{G/Q}(gQ)$$

shows that the latter is a periodic minimax group and hence Chernikov.

Since Z_1/D is abelian minimax, it has a finitely generated subgroup F/D such that Z_1/F is periodic. Let $F/D = \langle g_1 D, \ldots, g_t D\rangle$. As we have seen above, the subgroups $[g_1 Q, G/Q], \ldots, [g_t Q, G/Q]$ are Chernikov. Put

$$U/Q = [g_1 Q, G/Q] \cdots [g_t Q, G/Q].$$

Then U is a Chernikov subgroup and

$$V/U = \langle g_1, \ldots, g_t\rangle U/U \leq \zeta(G/U).$$

In particular; V is normal in G and V/U is a finitely generated abelian group. Furthermore, V is minimax, so that DV/V is not Chernikov. Since the factor-group Z_1/V is periodic nilpotent,

$$Z_1/V = P_1/V \times Q_1/V,$$

where P_1/V is the Sylow p-subgroup of Z_1/V and Q_1/V is the Sylow p'-subgroup of Z_1/V. Clearly Q_1 is minimax and Z_1/Q_1 is a p-group. Since DQ_1/Q_1 is divisible and not Chernikov and Z_1/DQ_1 is Chernikov, Z_1/Q_1 has a normal divisible subgroup D_1/Q_1 of finite index which is not Chernikov. Recall that in a nilpotent periodic group every divisible subgroup lies in the center (see [**KW**, 1.F.1]), so that Z_1/Q_1 is central-by-finite. As above we see that Z_1/Q_1 has a G-invariant Chernikov subgroup L/Q_1 such that $Z_1/Q_1 = (L/Q_1)(D_1/Q_1)$. Clearly L is likewise minimax. It follows that Z_1/L is a divisible abelian p-group which is not Chernikov. Proceeding in the same way, after finitely many steps we obtain the required result. □

Before stating the last result of the section, we recall some definitions and facts that we use subsequently. Let A be an abelian group of finite special rank, M a maximal $\mathbb{Z}$-independent subset of A and set $B = \langle M\rangle$.

Put

$$\mathrm{Sp}(A) = \{p \mid p \text{ is a prime such that the Sylow } p\text{-subgroup of } A/B \text{ is infinite}\}.$$

The set $\mathrm{Sp}(A)$ is called *the spectrum of the group* A. If V is also a finitely generated subgroup of A such that A/V is periodic, then

$$B/(B \cap V) \cong BV/V \text{ and } V/(B \cap V) \cong BV/B$$

are finite, which shows that the set $\mathrm{Sp}(A)$ is independent of the choice of the finitely generated subgroup B.

Let G be a nilpotent group of finite special rank and let

$$\langle 1 \rangle = \zeta_0 \leq \zeta_1 \leq \cdots \leq \zeta_n = G$$

be the upper central series of G. Put

$$\mathrm{Sp}(G) = \mathrm{Sp}(\zeta_1/\zeta_0) \cup \cdots \cup \mathrm{Sp}(\zeta_n/\zeta_{n-1}).$$

It is not hard to see that the spectrum of G is the union of the spectrum of all factors of every central series of G. We note that, if H is a normal subgroup of G such that G/H is periodic and p is a prime such that $p \notin \mathrm{Sp}(G)$, then the Sylow p-subgroup of G/H is finite.

We are now in a position to establish Theorem A, the main result of this section.

Theorem 2.6. *Let G be a subgroup of $GL(V, F)$ of infinite central dimension. Suppose that H is a normal subgroup of G such that G/H is nilpotent. If G satisfies either Wmin-icd or Wmax-icd, then G/H is minimax.*

Proof: By hypothesis, G has infinite central dimension. Let

$$\langle 1 \rangle = Z_0/H \leq Z_1/H \leq \cdots \leq Z_n/H = G/H$$

be the upper central series of G/H. We proceed by induction on n. If $n = 1$, then G/H is abelian and the statement follows from Corollary 2.2. Suppose that $n > 1$ and we have already proved that G/Z_1 is minimax. We want to prove that Z_1/H is minimax. Suppose the contrary and seek a contradiction. It suffices to show that G has a normal subgroup U such that G/U is periodic abelian and not minimax. Then Corollary 2.2 will give the required contradiction.

Put $L = G/H$, $C_0 = Z_0/H, \ldots, C_n = Z_n/H$. Then $C_1 \leq \zeta(L)$. Choose in C_1 a maximal $\mathbb{Z}$-independent subset $\{b_\lambda \mid \lambda \in \Lambda\}$ and put

$$B = \langle b_\lambda \mid \lambda \in \Lambda \rangle = \mathrm{Dr}_{\lambda \in \Lambda} \langle b_\lambda \rangle,$$

so that C_1/B is a periodic abelian group.

Suppose first that $\Pi(C_1/B)$ is infinite. Since L/C_1 is minimax, $\sigma = \mathrm{Sp}(L/C_1)$ is finite, and hence $\Pi(C_1/B) \setminus \sigma$ is infinite. Let D/B be the Sylow σ-subgroup of C_1/B, so that $\Pi(C_1/D)$ is infinite. For each element $cD \in C_2/D$, the mapping $gD \mapsto [gD, cD]$ defines a homomorphism of L/D into C_1/D whose kernel is exactly $C_{L/D}(cD)$. Since $[G/D, cD] \leq C_1/D$, the isomorphism

$$[L/D, cD] \cong (L/D)/C_{L/D}(cD)$$

shows that the latter is periodic abelian and $\Pi((L/D)/C_{L/D}(cD)) \subseteq \Pi(C_1/D)$. In particular, $\Pi((L/D)/C_{L/D}(cD)) \cap \mathrm{Sp}(L/C_1) = \emptyset$. On the other hand, the obvious inclusion $C_1/D \leq C_{L/D}(cD)$ implies that $(L/D)/C_{L/D}(cD)$ is a nilpotent minimax group and $\mathrm{Sp}((L/D)/C_{L/D}(cD)) \subseteq \mathrm{Sp}(L/C_1)$. Thus the Sylow q-subgroup of $(L/D)/C_{L/D}(cD)$ is finite for all $q \in \Pi((L/D)/C_{L/D}(cD))$. In particular, the set $\Pi((L/D)/C_{L/D}(cD))$ is finite and hence $(L/D)/C_{L/D}(cD)$ is likewise finite. This means that $C_2/D \leq FC(L/D)$ where $FC(L/D)$ is the FC-center of L/D. The factor-group C_2/C_1 has finite rank, therefore C_2 includes a finite set of elements $c_1, \ldots, c_t$ such that $C_2/\langle c_1, \ldots, c_t\rangle C_1$ is periodic. Put $F/D = \langle c_1 D, \ldots, c_t D\rangle^{L/D}$, so that F/D is a finitely generated subgroup and $(F/D) \cap (C_1/D)$ is finite. It follows that C_2/F is periodic and $\Pi(C_2/F)$ is infinite. Proceeding in the same way, after finitely many steps we construct a normal subgroup E such that L/E is a periodic nilpotent group with $\Pi(L/E)$ infinite. By [**MOS**, Corollary 2.7], G has finite central dimension. This contradiction proves that $\Pi(C_1/B)$ is finite.

Suppose B has infinite rank. Let $p \notin \Pi(C_1/B)$. Since B is a free abelian subgroup, $B \neq B^p = U$. Furthermore B/U is an infinite elementary abelian p-group. It follows that the Sylow p-subgroup of C_1/U is infinite elementary abelian. Let Q/U be the Sylow p'-subgroup of C_1/U. Then C_1/Q is an infinite elementary abelian. Applying Corollary 2.4, we see that G has a normal subgroup X such that G/X is an infinite elementary abelian p-group, which contradicts Corollary 2.2. This contradiction shows that Λ is finite and hence B is finitely generated.

Let $p \in \Pi(C_1/B)$ and let P/B be the Sylow p-subgroup of C_1/B. As above we can prove that $(P/B)/(P/B)^p$ is finite. Then

$$P/B = (V/B) \times (K/B),$$

where K/B is divisible and V/B is finite [**KL2**, Lemma 3]. Suppose that K/B is not Chernikov. Then P/V is a divisible p-subgroup which is not Chernikov. By Lemma 2.5, G has a normal subgroup W such that G/W is a divisible p-subgroup which is not Chernikov, which contradicts Corollary 2.2. This contradiction shows that K/B and P/B are

Chernikov subgroups. This and the finiteness of $\Pi(C_1/B)$ give that C_1/B is Chernikov. Since B is a finitely generated abelian subgroup, this means that C_1 is minimax. Thus Z_1/H is minimax and consequently G/H is likewise minimax. □

Corollary 2.7. *Let G be a nilpotent subgroup of $GL(V,F)$ of infinite central dimension. If G satisfies either Wmin-icd or Wmax-icd, then G is minimax.*

3. Locally nilpotent linear groups with Wmin-icd or Wmax-icd

In the sequel, we usually restrict ourselves to the case of char $F=p>0$. Other different techniques are needed to develop the case of characteristic 0. The first result is interesting in its own right and will be very useful because it gives the structure of finite central dimensional subgroups of an infinite central dimensional locally nilpotent linear group satisfying Wmin-icd or Wmax-icd. We remark that, in prime characteristic, finite central dimensional subgroups of these groups are soluble (see [**MOS**, Lemma 4.1] or at the beginning of Section 4 of [**DEK**]).

Proposition 3.1. *Let G be a locally nilpotent subgroup of $GL(V,F)$ of infinite central dimension, where* $\operatorname{char} F = p > 0$. *Suppose that G satisfies either Wmin-icd or Wmax-icd. If H is a normal subgroup of G of finite central dimension and P is the Sylow p-subgroup of H, then*

(1) *P is nilpotent and bounded,*

(2) *H/P is minimax and abelian-by-finite.*

In particular, $H/t(H)$ is an abelian minimax group and the Sylow p'-subgroup of H is Chernikov.

Proof: Note that $C = C_V(H)$ is an FG-submodule because H is normal in G. Since $\operatorname{centdim}_F H$ is finite, C has finite codimension in V. Therefore the series $C \leq V$ can be refined to a composition FG-series. That is, V has a finite series of FG-submodules

$$\langle 0\rangle = C_0 \leq C_1 = C \leq C_2 \leq \cdots \leq C_n = V$$

such that $C_2/C_1, \dots,$ and C_n/C_{n-1} are simple FG-modules. For each $1 \leq j \leq n-1$, we put $C_G(C_{j+1}/C_j) = D_{j+1}$. Each factor-group G/D_{j+1} can be embedded into $GL(C_{j+1}/C_j, F)$ and, since C_{j+1}/C_j has finite dimension over F, G/D_{j+1} is soluble (see [**WB**, Corollary 3.8]). By a result due to Malcev (see [**WB**, Lemma 3.5]), G/D_j is abelian-by-finite and then Lemma 2.1 yields that G/D_{j+1} is minimax.

Put

$$U = D_2 \cap D_3 \cap \cdots \cap D_n.$$

By Remak's Theorem

$$G/U \hookrightarrow G/D_2 \times \cdots \times G/D_n$$

and so G/U is an abelian-by-finite minimax group. For each $1 \le j \le n-1$, let T_{j+1}/D_{j+1} be the periodic part of G/D_{j+1}. Thus T_{j+1}/D_{j+1} is a p'-group (see [**KOS**, Theorem 3.2]). Moreover, since G/T_{j+1} is a torsion-free abelian-by-finite locally nilpotent group, G/T_{j+1} is abelian. Hence, if T/U is the torsion subgroup of G/U, T/U is a minimax p'-subgroup (in particular, it is Chernikov) and G/T is abelian.

By construction, $U \cap H$ stabilizes the series

$$\langle 0 \rangle = C_0 \le C_1 = C \le C_2 \le \cdots \le C_n = V.$$

Therefore $U \cap H$ is a bounded nilpotent p-subgroup (see [**KW**, Theorem 1.C.1 and Proposition 1.C.3] and [**FL**, Section 43]). Since

$$H/U \cap H \cong HU/U,$$

$U \cap H = P$ is the Sylow p-subgroup of H and H/P is abelian-by-finite minimax. Thus $H/t(H)$ is minimax abelian and the Sylow p'-subgroup of H is Chernikov. □

From this, we easily deduce the "minimaximality" of certain locally nilpotent linear groups whose torsion-free factor-groups are soluble.

Corollary 3.2. *Let G be a locally nilpotent subgroup of $GL(V,F)$ of infinite central dimension, where* $\operatorname{char} F = p > 0$*. Suppose that G satisfies either Wmin-icd or Wmax-icd. If H is a normal subgroup of G such that $H/t(H)$ is soluble, then $H/t(H)$ is minimax. In particular, if $G/t(G)$ is soluble, then $G/t(G)$ is minimax.*

Proof: Put $T = t(H)$. If H has finite central dimension, then H/T is minimax by Proposition 3.1. Suppose that $\operatorname{centdim}_F H$ is infinite. Let

$$T = H_0 \le H_1 \le \cdots \le H_{n-1} \le H_n = H$$

be a series of G-invariant subgroups whose factors are abelian. There exist a number k which is the least number such that $\operatorname{centdim}_F H_k$ is infinite. By Corollary 2.2, the factors $H_k/H_{k-1}, \dots, H_n/H_{n-1}$ are minimax. Proposition 3.1 yields that H_{k-1}/T is abelian minimax. It follows that H/T is minimax. □

We need two auxiliary lemmas in order to prove Theorem B. The first deals with arbitrary locally nilpotent groups, and we do not need to apply any previous result about linear groups. The second one shows that locally nilpotent torsion-free linear groups with Wmin-icd or Wmax-icd are hyperabelian.

Lemma 3.3. *Let G be a locally nilpotent group. Suppose that G has an ascending series of normal subgroups*

$$\langle 1\rangle = C_0 \leq C_1 \leq \cdots \leq C_{n-1} \leq C_n \leq \cdots C = \bigcup_{n\in\mathbb{N}} C_n$$

whose factors C_k/C_{k-1} are all non-trivial torsion-free abelian minimax groups. Then C has a G-invariant subgroup B such that C/B is periodic and $\Pi(C/B)$ is infinite.

Proof: Proceeding by induction on n, we are going to construct an ascending series of G-invariant subgroups

$$0 = B_0 \leq B_1 \leq B_2 \leq \cdots \leq B_n \leq \cdots$$

and an infinite set of distinct primes $\{p_n \mid n \in \mathbb{N}\}$ satisfying the following conditions:

(a) $B_n \leq C_n$,
(b) C_n/B_n is finite and $\Pi(C_n/B_n) = \{p_1, \ldots, p_n\}$, where $p_1, p_2, \ldots, p_n$ are distinct primes and
(c) $B_n \cap C_{n-1} = B_{n-1}$, for all $n \in \mathbb{N}$.

Since C_1 is nilpotent minimax, then we can choose a prime q such that $C_1^q \neq C_1$. Put $p_1 = q$ and $B_1 = C_1^q$. It easy to see that B_1 satisfies the above conditions. Suppose that we have already found primes $p_1, \ldots, p_k$ and constructed G-invariant subgroups

$$\langle 1\rangle = B_0 \leq B_1 \leq \cdots \leq B_k$$

satisfying conditions (a)-(b)-(c). We will find a new prime p_{k+1} and construct a subgroup B_{k+1} in such a way (a)-(b)-(c) hold.

C_{k+1}/B_k is an extension of a finite subgroup C_k/B_k by a torsion-free abelian group C_{k+1}/C_k. Being locally nilpotent, C_{k+1}/B_k is nilpotent. It is easy to see that $t(C_{k+1}/B_k) = C_k/B_k$. On the other hand, by [**HK**, Proposition 2], C_{k+1}/B_k has a torsion-free normal subgroup D_{k+1}/B_k such that $(C_{k+1}/B_k)/(D_{k+1}/B_k)$ is bounded. Let E_{k+1}/D_{k+1} be the Sylow $\{p_1, \ldots, p_k\}'$-subgroup of C_{k+1}/D_{k+1}. Since $\Pi(C_{k+1}/B_k) = \Pi(C_k/B_k) = \{p_1, \ldots, p_k\}$, E_{k+1}/B_k is torsion-free. Furthermore, E_{k+1}/B_k is abelian minimax, so there exists a prime $r \notin \{p_1, \ldots, p_k\}$ such that $(E_{k+1}/B_k)^r \neq E_{k+1}/B_k$. Put $p_{k+1} = r$ and let $L_{k+1}/B_k = (E_{k+1}/B_k)^r$. Clearly L_{k+1}/B_k is a characteristic subgroup of E_{k+1}/B_k, and then it is normal in C_{k+1}/B_k. Obviously C_{k+1}/L_{k+1} is bounded and

$\Pi(C_{k+1}/L_{k+1}) = \{p_1, \dots, p_k, p_{k+1}\}$. Since C_{k+1} is minimax, it has finite special rank. Therefore C_{k+1}/L_{k+1} is finite.

Let $m = |C_{k+1}/L_{k+1}|$ and put $B_{k+1}/B_k = (C_{k+1}/B_k)^m$. We will see that B_{k+1} satisfies the claimed conditions. Clearly B_{k+1} is G-invariant, and, by construction, $B_{k+1} \leq C_{k+1}$. The factor-group C_{k+1}/B_{k+1} is bounded and hence it is finite, because C_{k+1} has finite special rank. Furthermore, $\Pi(C_{k+1}/B_{k+1}) = \{p_1, \dots, p_k, p_{k+1}\}$. The inclusions

$$B_k \leq B_{k+1} \leq L_{k+1} \leq E_{k+1}$$

together with the equation $E_{k+1} \cap C_k = B_k$ imply that $B_{k+1} \cap C_k = B_k$. Hence the claimed construction have just been carried out.

Put now $B = \cup_{n\in\mathbb{N}} B_n$. From (c) we have

$$B_{k+2} \cap C_k = B_{k+2} \cap (C_{k+1} \cap C_k) = (B_{k+2} \cap C_{k+1}) \cap C_k = B_{k+1} \cap C_k = B_k.$$

By induction, $B_{k+s} \cap C_k = B_k$ for every $s \in \mathbb{N}$. It follows

$$B \cap C_k = (\cup_{n\in\mathbb{N}} B_n) \cap C_k = \cup_{n\in\mathbb{N}} (B_n \cap C_k) = \cup_{n\in\mathbb{N}} (B_{k+n} \cap C_k) = B_k.$$

Finally, the isomorphism $C_k/B_k = C_k/(B \cap C_k) \cong BC_k/B$ shows that $\Pi(BC_k/B) = \{p_1, \dots, p_k\}$. This implies that $\Pi(C/B) = \{p_n \mid n \in \mathbb{N}\}$ is infinite. □

Lemma 3.4. *Let G be a locally nilpotent subgroup of $GL(V, F)$. Suppose that G satisfies either Wmin-icd or Wmax-icd. Then $G/t(G)$ has an ascending series of normal subgroups whose factors are pure abelian. In particular, $G/t(G)$ is hyperabelian.*

Proof: Put $T = t(G)$. If G has finite central dimension, then by [**MOS**, Lemma 4.1] G is soluble. Therefore we can assume that $\operatorname{centdim}_F G$ is infinite. Since G/T is a torsion-free locally nilpotent group, G/T has a central series $\mathfrak{Z}$ with pure subgroups ([**GV**, Theorem 7]). In particular, G/T has a non-identity normal pure subgroup U/T. If U has finite central dimension, then again by [**MOS**, Lemma 4.1] U is soluble. Therefore U/T is soluble and so U/T has a non-identity characteristic abelian subgroup A/T. Hence A/T is a non-identity abelian normal subgroup of G/T. Put $L/A = t(G/A)$. Then L/T is a pure abelian normal subgroup of G/T (see [**KA**, §66 and §67]).

Suppose now that $\operatorname{centdim}_F U$ is infinite for every normal pure subgroup U/T of G/T. We remark that if U/T and W/T are two pure normal subgroups of G/T with the property $U/T < W/T$, then the factor W/U has to be infinite. It is rather easy to see that if G satisfies Wmin-icd (respectively Wmax-icd), then G/T satisfies the minimal condition for pure normal subgroups (respectively the maximal condition for pure normal subgroups).

Let $\mathfrak{Z}$ be a central series of G. If G satisfies Wmin-icd, then a set of subgroups of $\mathfrak{Z}$ linearly ordered by inclusion satisfies the minimal condition. Then the set of subgroups of $\mathfrak{Z}$ is completely ordered by inclusion. In particular, $\mathfrak{Z}$ has a minimal element B/T. Since $\mathfrak{Z}$ is a central series, $B/T \leq \zeta(G/T)$. In particular, G/T has a pure abelian normal subgroup.

If G satisfies Wmax-icd, then G/T is a nilpotent group of finite special rank by [**GV**, Theorem 10]. Therefore, in this case G/T has a pure abelian normal subgroup too. It suffices to apply transfinite induction to obtain the required result. □

After these two lemmas, we are now in a position to show the remainder of the main results of this paper.

Theorem 3.5. *Let G be a locally nilpotent subgroup of $GL(V, F)$ of infinite central dimension. Suppose that* $\operatorname{char} F = p > 0$. *If G satisfies either Wmin-icd or Wmax-icd, then $G/t(G)$ is minimax.*

Proof: Put $T = t(G)$. If G/T is soluble, the result follows from Corollary 3.2. Therefore, we suppose that G is insoluble. By Lemma 3.4, G/T is hyperabelian. Therefore we can construct an ascending series of normal subgroups

$$T = C_0 \leq C_1 \leq \cdots C_\alpha \leq C_{\alpha+1} \leq \cdots C_\gamma = G$$

such that $C_{\alpha+1}/C_\alpha$ is a maximal abelian normal subgroup of G/C_α, for every ordinal $\alpha < \gamma$. Proceeding just as in the proof of Lemma 3.4, we see that every subgroup C_α/T is pure in G/T. Since G is insoluble, $\gamma \geq \omega$, where ω is the first infinite ordinal. Put $C = C_\omega$. Every subgroup C_n is soluble for each $n \in \mathbb{N}$. Therefore C_n/T is minimax for each $n \in \mathbb{N}$ by Corollary 3.2. Lemma 3.3 shows that C has a G-invariant subgroup B such that C/B is periodic and $\Pi(C/B)$ is infinite. Since G is locally nilpotent, C/B is a direct product of infinitely many Sylow subgroups. By [**MOS**, Corollary 2.7], $\operatorname{centdim}_F C$ is finite, and, by [**MOS**, Lemma 4.1], C must be soluble, a contradiction. This contradiction shows that G/T is soluble, as required. □

Theorem 3.6. *Let G be a locally nilpotent subgroup of $GL(V, F)$ of infinite central dimension. Suppose that* $\operatorname{char} F = p > 0$. *If G satisfies either Wmin-icd or Wmax-icd, then $G/G^{\mathfrak{F}}$ is minimax and nilpotent.*

Proof: Let $\mathfrak{M}$ be the family of all normal subgroups H such that G/H is finite, and let $L = G^{\mathfrak{F}}$ be the finite residual of G, so that $L = \bigcap\{M \mid M \in \mathfrak{M}\}$. Let q be a prime and put

$$\mathfrak{M}(q) = \{H \in \mathfrak{M} \mid G/H \text{ is a finite } q\text{-group}\}.$$

Put $L(q) = \bigcap\{H \mid H \in \mathfrak{M}(q)\}$. By [**KNKL**, Lemma 2], $G/L(q)$ is nilpotent and $t(G/L(q))$ is finite. Also we note that, by [**MOS**, Corollary 2.7], the factor-group $G/[G,G]G^q$ has to be finite.

Let Q/L be the Sylow q-subgroup of G/L. If $1 \neq xL \in Q/L$, then there exists a subgroup $H \in \mathfrak{M}$ such that $xL \notin H/L$. Let U/H be the Sylow p'-subgroup of G/H. Then G/U is a finite q-group and $x \notin U$. In other words, for every element $xL \in Q/L$ there is a subgroup $U_x \in \mathfrak{M}(q)$ such that $xL \notin U_x/L$. It follows that $Q \cap L(q) = L$, that is

$$Q/L \cong Q/(Q \cap L(q)) \cong QL(q)/L(q).$$

Therefore the finiteness of $t(G/L(q))$ implies that Q/L is finite. This yields that Q/L has a maximal G-invariant subgroup $R(q)/L$. Since G is locally nilpotent, $Q/R(q)$ is a G-central factor (see [**KA**, §63]). Let T/L be the periodic part of G/L and put

$$R = \mathrm{Dr}_{q\in\Pi(G/L)}\, R(q).$$

Then $T/R \leq \zeta(G/R)$. By Theorem 3.5 G/T is minimax, so that it has finite special rank. Applying [**MN**], G/T is nilpotent and consequently so too is G/R. By Theorem 2.6, G/R is minimax. In this case T/R is finite. Therefore $\Pi(G/L)$ is finite. Since every Sylow q-subgroup of G/L is finite, T/L is finite. Then G/L is nilpotent, and applying Theorem 2.6 again, we conclude that G/L is minimax. □

Corollary 3.7. *Let G be a locally nilpotent subgroup of $GL(V,F)$ of infinite central dimension. Suppose that* $\operatorname{char} F = p > 0$. *If G satisfies either Wmin-icd or Wmax-icd, then $G/G^{\mathfrak{N}}$ is minimax.*

Proof: Let $\mathfrak{M}$ be the family of all normal subgroups H such that G/H is nilpotent, and let L be the nilpotent residual of G, so that $L = \bigcap\{M \mid M \in \mathfrak{M}\}$. By Theorem 3.5, $G/t(G)$ is minimax, and, in particular, $G/t(G)$ has finite special rank. We recall that torsion-free locally nilpotent groups of finite special rank are nilpotent ([**MN**]); whence $t(G) \in \mathfrak{M}$. Therefore $L \leq t(G)$.

Put $T = t(G)$. If T has infinite central dimension, then T is a Chernikov subgroup by [**MOS**, Corollary D]. If T has finite central dimension, then, by Proposition 3.1, the Sylow p-subgroup Q of T is bounded nilpotent and T/Q is abelian-by-finite Chernikov. Hence $(G/L)/(QL/L)$ is minimax and QL/L is bounded nilpotent. Let $H \in \mathfrak{M}$, then G/H is minimax by Theorem 2.6. Since a bounded minimax group is finite, the isomorphisms

$$(QL/L)/((H/L) \cap (QL/L)) \cong QL/(H \cap QL) \cong QH/H \leq G/H$$

show that $(QL/L)/((H/L) \cap (QL/L))$ is finite. Put $U/L = (H/L) \cap (QL/L)$. We have

$$T/L = QL/L \times R/L,$$

where R/L is the Sylow p'-subgroup of T/L. Put $W/L = (U/L)(R/L)$. It follows that $(T/L)/(W/L)$ is a finite p-group. Since G/T is nilpotent and T/W is finite, G/W is nilpotent too. Then it has a torsion-free normal subgroup D/W such that G/D is bounded ([**HK**, Proposition 2]). Since G/W has finite special rank, G/D, being bounded, is finite. Furthermore, D/W is torsion-free, so $(T/L) \cap (D/L) = W/L$. The equation $L = \bigcap \mathfrak{M}$ shows that for each element $xL \in QL/L$ there is a normal subgroup W/L of finite index such that $xL \notin W/L$. It follows that

$$(QL/L) \cap (G^{\mathfrak{F}}/L) = \langle 1 \rangle.$$

In other words, QL/L is isomorphic to some subgroup of $G/G^{\mathfrak{F}}$. By Theorem 3.6, $G/G^{\mathfrak{F}}$ is minimax. Thus QL/L is finite. Since $(G/L)/(QL/L)$ is minimax, G/L is likewise minimax. □

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