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A CARLESON TYPE CONDITION FOR INTERPOLATING SEQUENCES IN THE HARDY SPACES OF THE BALL OF $\mathbb{C}^n$

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Abstract

In this work we prove that if S is a dual bounded sequence of points in the unit ball $\mathbb{B}$ of $\mathbb{C}^n$ for the Hardy space $H^p(\mathbb{B})$, then S is $H^s(\mathbb{B})$ interpolating with the linear extension property for any $s \in [1, p[$.

1. Introduction

Let $\mathbb{B}$ be the unit ball of $\mathbb{C}^n$ and σ the Lebesgue measure on $\partial\mathbb{B}$. As usual we define the Hardy spaces $H^p(\mathbb{B})$ as the closure in $L^p(\sigma)$ of the holomorphic polynomials and $H^\infty(\mathbb{B})$ as the algebra of all bounded holomorphic functions in $\mathbb{B}$.

If $a \in \mathbb{B}$ we note $k_a(z)$ its Cauchy kernel and $k_{a,p}(z)$ its normalized Cauchy kernel in $H^p(\mathbb{B})$:

$$k_a(z) := \frac{1}{(1 - \overline{a} \cdot z)^n}, \quad k_{a,p}(z) := \frac{k_a}{\|k_a\|_p}.$$

We know that $\|k_a\|_p = c(a,p)(1 - |a|^2)^{-n/p'}$ with $\frac{1}{p} + \frac{1}{p'} = 1$, $0 < \alpha \leq c(a,p) \leq \beta$ and α, β independent of $a \in \mathbb{B}$ and of $1 \leq p \leq \infty$.

We shall need some definitions.

Definition 1.1. We say that the sequence $S \subset \mathbb{B}$ is dual bounded in $H^p(\mathbb{B})$ if a dual system $\{\rho_a\}_{a \in S} \subset H^p(\mathbb{B})$ for $\{k_{a,p'}\}_{a \in S}$ exists and if this sequence is bounded in $H^p(\mathbb{B})$, i.e.

$$\exists\, C > 0 \text{ s.t. } \forall\, a \in S,\ \|\rho_a\|_p \leq C,\ \forall\, a, b \in S,\ \langle \rho_a, k_{b,p'} \rangle = \delta_{a,b}.$$

One can easily see that this is strongly related to the notion of *uniform minimality* introduced by N. K. Nikol'skiĭ [**9**, p. 131] to study Carleson's interpolation theorem:

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S is dual bounded in $H^p(\mathbb{B})$ if and only if the sequence $\{k_{a,p'}\}_{a\in S}$ is uniformly minimal in $H^{p'}(\mathbb{B})$.

Definition 1.2. We say that a sequence $S \subset \mathbb{B}$ is $H^p(\mathbb{B})$ interpolating for $1 \leq p < \infty$, $S \in IH^p(\mathbb{B})$, with interpolating constant $C_I > 0$ if

$$\forall\, \lambda \in \ell^p,\ \exists\, f \in H^p(\mathbb{B}) \text{ s.t. } \forall\, a \in S,\ f(a) = \lambda_a \left\|k_a\right\|_{p'} \ \text{ and } \ \left\|f\right\|_p \leq C_I \left\|\lambda\right\|_p .$$

We say that $S \subset \mathbb{B}$ is $H^\infty(\mathbb{B})$ interpolating, $S \in IH^\infty(\mathbb{B})$, with interpolating constant C_I if

$$\forall\, \lambda \in \ell^\infty,\ \exists\, f \in H^\infty(\sigma) \text{ s.t. } \forall\, a \in S,\ f(a) = \lambda_a \text{ and } \left\|f\right\|_\infty \leq C_I \left\|\lambda\right\|_\infty .$$

Clearly if S is $H^p(\mathbb{B})$ interpolating, then S is dual bounded in $H^p(\mathbb{B})$. In one variable, L. Carleson [**7**] proved the converse for $p = \infty$ and H. S. Shapiro and A. L. Shields [**11**] did the same for $H^p(\mathbb{D})$, $1 \leq p < \infty$.

For the Bergman spaces of the unit disc, which can be viewed as the functions in $H^p(\mathbb{B}_2)$ not depending of the second variable, by the subordination lemma [**1**], A. P. Schuster and K. Seip [**10**] proved that the dual bounded condition in $A^p(\mathbb{D})$ implies the $A^p(\mathbb{D})$ interpolation.

For $p = \infty$, B. Berndtsson [**6**] proved that the necessary and sufficient condition for interpolation in the unit disc is still a sufficient condition in the ball of $\mathbb{C}^n$, $n > 1$, but it is no longer necessary.

Definition 1.3. We say that the $H^p(\mathbb{B})$ interpolating sequence S has the linear extension property (LEP) if there is a bounded linear operator $E\colon \ell^p \longrightarrow H^p(\mathbb{B})$ such that $\|E\| < \infty$ and $\forall\, \lambda \in \ell^p$, $E\lambda$ interpolates the sequence λ in $H^p(\mathbb{B})$ on S.

In [**3**] we proved by functional analytic methods still in the general setting of uniform algebras, that if S is dual bounded in H^p then for any s such that $1 \leq s < p$, S is H^s interpolating with the LEP, *provided that* $p = \infty$ *or* $p \leq 2$. Hence it remains a gap, the values of p in $]2, \infty[$.

The aim of this work is to fill this gap in the special case of the ball.

Theorem 1.4. *Let $S \subset \mathbb{B}$ be dual bounded in $H^p(\mathbb{B})$, then S is $H^s(\mathbb{B})$ interpolating with the LEP, for any $s \in [1, p[$.*

Two natural questions remain open in general:

(1) let $S \subset \mathbb{B}$ be dual bounded in $H^p(\mathbb{B})$, for a $p \in [1, \infty]$, is S $H^p(\mathbb{B})$ interpolating?
(2) let S be $H^p(\mathbb{B})$ interpolating, for a $p \in [1, \infty]$, does S have the linear extension property in $H^p(\mathbb{B})$?

The answer is yes in one variable for any $p \in [1, \infty]$ and for Hardy and Bergman spaces for these two questions. The answer is still yes for any n for $p = 1$ still for the two questions. But in general this is open.

In [**2**] we proved, in the general setting of uniform algebras, that if S is H^p interpolating then S is H^s interpolating for any s such that $1 \leq s < p$ provided that some structural hypotheses are verified. This is valid here, in the case of the ball, so Theorem 1.4 improves strongly this result in the case of the ball because interpolation implies dual boundedness, but the converse is unknown.

2. Carleson sequences

Remember that $k_{a,q}$ is the normalized reproducing kernel for the point $a \in \mathbb{B}$ in $H^q(\mathbb{B})$.

Definition 2.1. We say that the sequence $S \subset \mathbb{B}$ is a Carleson sequence if, for any q such that $1 \leq q < \infty$, we have

$$\exists\, D_q > 0,\ \forall\, \mu \in \ell^q,\ \left\| \sum_{a\in S} \mu_a k_{a,q} \right\|_q \leq D_q \left\|\mu\right\|_q .$$

In fact, up to the duality $H^p(\mathbb{B})-H^{p'}(\mathbb{B})$, it is proved by Hörmander [**8**] that this condition, for a $q > 1$, is equivalent to the fact that the measure $\chi := \sum\limits_{a\in S} (1-|a|^2)^n \delta_a$ is a Carleson measure, hence if the condition is true for a $q > 1$ it is true for all q's.

P. J. Thomas [**12**] proved that if the sequence S is $H^p(\mathbb{B})$ interpolating for a $p \geq 1$, then χ is a Carleson measure, hence S is a Carleson sequence. As a corollary of this result we have

Lemma 2.2. *If S is dual bounded in $H^p(\mathbb{B})$ for a $p \geq 1$ then S is a Carleson sequence.*

Proof: The dual system $\{\rho_a\}_{a\in S} \subset H^p(\mathbb{B})$ exists and is bounded in $H^p(\mathbb{B})$. Let $\rho_{a,1} := \rho_a k_{a,p'}$ with p' the conjugate exponent for p. Then we have

$$\forall\, a \in S,\ \left\|\rho_{a,1}\right\|_1 \leq \left\|\rho_a\right\|_p \left\|k_{a,p'}\right\|_{p'} \leq \left\|\rho_a\right\|_p \leq C.$$

Moreover $\rho_{a,1}(b) = \rho_a(b) k_{a,p'}(b) = \delta_{ab}\rho_a(a) k_{a,p'}(a) = \delta_{ab}(1-|a|^2)^{-n}$. Hence $\{\rho_{a,1}\}_{a\in S}$ is a dual system for $\{k_{a,\infty}\}_{a\in S}$ and this means that S is also dual bounded in $H^1(\mathbb{B})$. But then it is clear that a dual bounded sequence in $H^1(\mathbb{B})$ is $H^1(\mathbb{B})$ interpolating and we can apply Thomas' theorem to conclude. $\square$

Remark 2.3. Thomas' theorem is valid not only for holomorphic functions but also for harmonic ones; in [**4**] I give a much easier proof of it based on Wirtinger inequality, hence valid only for holomorphic functions.

3. The main result

We are now in position to prove the Theorem 1.4.

We shall need the following lemma which was proved for the Poisson Szegö kernel in the ball in [**5**]. We put on $\partial\mathbb{B}$ the pseudo-distance $\delta(\zeta, z) := \left|1 - \overline{\zeta} \cdot z\right|$ and set D its constant in the quasi triangular inequality, i.e.

$$\forall\, a, b \in \partial\mathbb{B},\ \delta(a, b) \leq D(\delta(a, c) + \delta(c, b)).$$

Lemma 3.1. *The kernel* $K_t(\zeta, z) := \frac{t^{n(p-1)}}{\left|1-(1-t)\overline{\zeta}\cdot z\right|^{np}}$ *verifies for* $p > 1$*:*

(H2) $|K_t(\zeta, z)| \leq Ct^{\alpha}(\delta(\zeta, z) + t)^{-\alpha-n}$, *with* $\alpha = n(p-1) > 0$.

(H3) *For any* ζ, z_0, z, t *such that* $\delta(z_0, z) \leq \frac{1}{2D}(t + \delta(\zeta, z_0))$ *we have*

$$|K_t(\zeta, z) - K_t(\zeta, z_0)| \leq C\delta(z_0, z)^{1/2}(\delta(\zeta, z_0) + t)^{-n-1/2}.$$

Proof: The condition is for small t and we have, with $\alpha = n(p-1)$

$$K_t(\zeta, z) \simeq \frac{t^{n(p-1)}}{(t + \delta(\zeta, z))^{np}} = t^{\alpha}((t + \delta(\zeta, z))^{-n-\alpha},$$

and the condition (H2).

For (H3) we have

$$\text{(3.1)}\quad |K_t(\zeta, z_0) - K_t(\zeta, z)|$$

$$\simeq \frac{t^{n(p-1)}}{(t + \delta(\zeta, z_0))^{np}(t + \delta(\zeta, z))^{np}} \left|(t + \delta(\zeta, z_0))^{np} - (t + \delta(\zeta, z))^{np}\right|.$$

The function $x \longrightarrow f(x) := x^{np}$ verifies

$$|f(a) - f(b)| = |b - a|\, |f'(c)| = |b - a|\, np\, |c|^{np-1}, \quad c \in]a, b[$$

by the mean value property, hence here we get

$$|(t + \delta(\zeta, z_0))^{np} - (t + \delta(\zeta, z))^{np}|$$

$$\leq |\delta(\zeta, z_0) - \delta(\zeta, z)| \max\left((t + \delta(\zeta, z_0))^{np-1}, (t + \delta(\zeta, z))^{np-1}\right).$$

We have

$$t^{n(p-1)} \leq \min\left((t + \delta(\zeta, z_0))^{n(p-1)}, (t + \delta(\zeta, z))^{n(p-1)}\right),$$

because $n(p-1) \geq 0$.

Let $A := \max((t + \delta(\zeta, z_0)), (t + \delta(\zeta, z)))$, $B := \min((t + \delta(\zeta, z_0)), (t + \delta(\zeta, z)))$ then putting the last two inequalities in (3.1) we get

$$|K_t(\zeta, z_0) - K_t(\zeta, z)| \leq |\delta(\zeta, z_0) - \delta(\zeta, z)|\, A^{-1}B^{-n}.$$

By the quasi triangular inequality we have

$$\delta(\zeta,z_0)\leq D\left(\delta(\zeta,z)+\delta(z,z_0)\right)\Longrightarrow t+\delta(\zeta,z)\geq D^{-1}\left(t+\delta(\zeta,z_0)\right)-\delta(z_0,z),$$

because $D\geq 1$. But we have $\delta(z_0,z)\leq\frac{1}{2D}(t+\delta(\zeta,z_0))$ hence, $t+\delta(\zeta,z)\geq\frac{1}{2D}\left(t+\delta(\zeta,z_0)\right)$ so we always have $A^{-1}B^{-n}\leq C(t+\delta(\zeta,z_0))^{-n-1}$.

Now we shall take advantage that $d(a,b):=\sqrt{\delta(a,b)}$ is a distance on $\partial\mathbb{B}$. We have

$$\begin{aligned}|\delta(\zeta,z_0)-\delta(\zeta,z)| &= \left|d^2(\zeta,z_0)-d^2(\zeta,z)\right| \\ &= |d(\zeta,z_0)-d(\zeta,z)|\,(d(\zeta,z_0)+d(\zeta,z)).\end{aligned}$$

Because d is a distance, $|d(\zeta,z_0)-d(\zeta,z)|\leq d(z_0,z)=\delta^{1/2}(z_0,z)$. On the other hand

$$\begin{aligned}d(\zeta,z)\leq d(\zeta,z_0)+d(z_0,z)\Longrightarrow d(\zeta,z_0)+d(\zeta,z)& \\ \leq 2d(\zeta,z_0)+d(z_0,z)\leq C\sqrt{t+\delta(\zeta,z_0)},&\end{aligned}$$

again with $\delta(z_0,z)\leq\frac{1}{2D}(t+\delta(\zeta,z_0))$. So finally we get

$$|K_t(\zeta,z_0)-K_t(\zeta,z)|\leq C\delta^{1/2}(z_0,z)(t+\delta(\zeta,z_0))^{-n-1/2},$$

and the lemma. $\square$

3.1. Proof of the theorem. Suppose that the sequence S is dual bounded in $H^p(\mathbb{B})$, i.e. there is a sequence $\{\rho_a\}_{a\in S}\subset H^p(\mathbb{B})$ such that

$$\forall\, a\in S,\ \|\rho_a\|_p\leq C,\ \langle\rho_a,k_{b,p'}\rangle=\delta_{ab}.$$

Now fix $s<p$ and let $\nu\in\ell^s$. We have to show that we can interpolate the sequence ν in $H^s(\mathbb{B})$.

Let q be such that $\frac{1}{s}=\frac{1}{p}+\frac{1}{q}$ and write $\forall\, a\in S$, $\nu_a=\lambda_a\mu_a$ with $\lambda_a:=\frac{\nu_a}{|\nu_a|}|\nu_a|^{s/p}$ and $\mu_a:=|\nu_a|^{s/q}$; then $\lambda\in\ell^p$, $\mu\in\ell^q$ and $\|\nu\|_s=\|\lambda\|_p\|\mu\|_q$. Let $k_{a,q}$ the normalized reproducing kernel for a in $H^q(\mathbb{B})$ and set $\gamma_a:=\frac{c(a,q)c(a,s')}{c(a,p')c(a,2)^2}$ which is bounded above and below by strictly positive constants independent of $a\in\mathbb{B}$. In particular we have $\exists\, C>0$, $\forall\, a\in\mathbb{B}$, $\gamma_a\leq C$.

Let $h:=\sum_{a\in S}\nu_a\gamma_a\rho_a k_{a,q}=\sum_{a\in S}\gamma_a\lambda_a\rho_a\mu_a k_{a,q}$; h takes the "good" values on S:

$$\forall\, b\in S,\ h(b)=\sum_{a\in S}\gamma_a\nu_a\rho_a(b)k_{a,q}(b)=\gamma_b\nu_b\rho_b(b)k_{b,q}(b)=\nu_b\,\|k_b\|_{s'}\,,$$

because

$$\gamma_b\rho_b(b)k_{b,q}(b)=\gamma_b\left\|k_b\right\|_{p'}\frac{\left\|k_b\right\|_2^2}{\left\|k_b\right\|_q}=\gamma_b c(b,p')\frac{c(b,2)^2}{c(b,q)}(1-|b|^2)^{-n/s}=\left\|k_b\right\|_{s'},$$

by use of $\frac{1}{s}=\frac{1}{p}+\frac{1}{q}$.

Moreover h depends linearly on ν.

It remains to estimate its $H^s(\mathbb{B})$ norm. We have, using Hölder inequalities

$$|h|\leq C\left(\sum_{a\in S}|\lambda_a|^p\,|\rho_a|^p\right)^{1/p}\left(\sum_{a\in S}|\mu_a|^{p'}\,|k_{a,q}|^{p'}\right)^{1/p'}.$$

Let $g:=\left(\sum_{a\in S}|\lambda_a|^p\,|\rho_a|^p\right)^{1/p}$, we have

$$\|g\|_p^p=\int_{\partial\mathbb{B}}\sum_{a\in S}|\lambda_a|^p\,|\rho_a|^p\,d\sigma=\sum_{a\in S}|\lambda_a|^p\,\|\rho_a\|_p^p\leq C\,\|\lambda\|_p^p,$$

hence $g\in L^p(\sigma)$.

Let $f:=\left(\sum_{a\in S}|\mu_a|^{p'}\,|k_{a,q}|^{p'}\right)^{1/p'}$, we have $|h|\leq Cgf$ hence

$$\|h\|_s^s=\int_{\partial\mathbb{B}}|h|^s\,d\sigma\leq C^s\int_{\partial\mathbb{B}}|g|^s\,|f|^s\,d\sigma.$$

Applying Hölder inequality with exponents p/s, q/s, because $\frac{1}{s}=\frac{1}{p}+\frac{1}{q}$, we get

$$\|h\|_s^s\leq C^s\,\|g\|_p^s\,\|f\|_q^s.$$

To have h in $L^s(\sigma)$ it remains to prove that

$$f=\left(\sum_{a\in S}|\mu_a|^{p'}\,|k_{a,q}|^{p'}\right)^{1/p'}\in L^q(\sigma).$$

So let $\phi:=f^{p'}=\sum_{a\in S}|\mu_a|^{p'}\,|k_{a,q}|^{p'}$ we shall show that $\phi\in L^{q/p'}(\sigma)$ $(q>p'\Longrightarrow\frac{q}{p'}>1)$.

From $k_{a,q}=\frac{(1-|a|^2)^{n/q'}}{(1-\overline{a}\cdot z)^n}$, we get

$$|k_{a,q}|^{p'}=\frac{(1-|a|^2)^{\frac{np'}{q'}}}{|1-\overline{a}\cdot z|^{np'}}=(1-|a|^2)^{-\frac{np'}{q}}\frac{(1-|a|^2)^{n(p'-1)}}{|1-\overline{a}\cdot z|^{np'}}(1-|a|^2)^n.$$

Now the measure $\chi := \sum_{a\in S}(1-|a|^2)^n\delta_a$ is Carleson by Lemma 2.2 and the kernel

$$K(a,z) := \frac{(1-|a|^2)^{n(p'-1)}}{|1-\overline{a}\cdot z|^{np'}}$$

verifies (H2) and (H3) by Lemma 3.1, then we can apply the results in [**5**]: if χ is a Carleson measure and $\alpha \in L^s(\chi)$ then the balayage of the measure $\alpha\, d\chi$ by the kernel $K(a,z)$ is in the space $L^s(\sigma)$.

Here, let $\eta_a(z)$ be a smooth function with support in the hyperbolic ball centered at a and with radius $r > 0$ so small that these balls are disjoint for $a \in S$ and such that $\eta_a(a) = 1$. Let

$$\alpha := \sum_{a\in S} |\mu_a|^{p'} (1-|a|^2)^{-\frac{np'}{q}} \eta_a(z),$$

if $\alpha \in L^{q/p'}(\chi)$, we have that its balayage, which is precisely ϕ, is in $L^{q/p'}(\sigma)$.

But we have

$$\int |\alpha|^{q/p'}\, d\chi = \sum_{a\in S}(1-|a|^2)^n\, |\mu_a|^q\, (1-|a|^2)^{-n} = \sum_{a\in S} |\mu_a|^q < \infty,$$

and we are done. □

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