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SUBMANIFOLDS WITH NONPARALLEL FIRST NORMAL BUNDLE REVISITED

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Abstract: In this paper, we analyze the geometric structure of a Euclidean submanifold whose osculating spaces form a nonconstant family of proper subspaces of the same dimension. We prove that if the rate of change of the osculating spaces is small, then the submanifold must be a (submanifold of a) ruled submanifold of a very special type. We also give a sharp estimate of the dimension of the rulings.

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The osculating space of a Euclidean submanifold M^n at a point is the subspace of Euclidean space that is spanned by the tangent and curvature vectors of all smooth curves in M^n through that point. If all osculating spaces along M^n coincide with a fixed subspace H, it is an elementary fact that M^n is contained in an affine subspace parallel to H. Thus, it is a natural problem to study for which submanifolds the osculating spaces form a nonconstant family of proper subspaces of the same dimension. In this paper, we show that if the rate of change of the osculating spaces is small, in a sense to be made precise below, then the submanifold must be contained in a ruled submanifold of a very special type.

Let $f\colon M^n \to \mathbb{R}^N$ denote an isometric immersion of an n-dimensional connected Riemannian manifold into Euclidean space. The *first normal space* of f at $x \in M^n$ is the normal subspace $N_1^f(x) \subset N_f M(x)$ spanned by the image of its second fundamental form α_f at x, that is,

$$N_1^f(x) = \operatorname{span}\{\alpha_f(X,Y) : X, Y \in T_x M\}.$$

The *osculating space* of f at $x \in M^n$ is defined as $f_* T_x M \oplus N_1^f(x)$. It is easy to see that all osculating spaces of f have the same dimension and are parallel to a fixed proper subspace of $\mathbb{R}^N$ if and only if the first normal spaces form a proper normal subbundle N_1^f that is parallel in the normal

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connection; see [**1**] or [**7**]. Then f reduces codimension to $p = \operatorname{rank} N_1^f$, that is, it can be seen as a substantial isometric immersion into an affine subspace $\mathbb{R}^{n+p}$ of $\mathbb{R}^N$.

A rather simple argument shows that N_1^f must be parallel in the normal connection if $p < n$ and at any $x \in M^n$ the s-nullities ν_s of f satisfy

$$\nu_s(x) < n - s \tag{1}$$

for all $1 \leq s \leq p$; see [**1**], [**4**] or (7) below. Recall that

$$\nu_s(x) = \max_{U^s \subset N_1^f(x)} \dim \mathcal{N}(\alpha_{U^s}),$$

where $U^s \subset N_1^f(x)$ is any s-dimensional vector subspace and

$$\mathcal{N}(\alpha_U(x)) = \{Y \in T_xM : \alpha_U(Y, X) = 0 \text{ for all } X \in T_xM\}$$

for $\alpha_U = \pi_U \circ \alpha_f$ and $\pi_U \colon N_1^f \to U$ the orthogonal projection. Notice that $\nu_p(x)$ is the standard *index of relative nullity* $\nu_f(x) = \dim \mathcal{N}(\alpha_f(x))$, that is, the dimension of the *relative nullity* subspace of f at $x \in M^n$.

Consider the subspace $\mathcal{S}(x)$ of $N_1^f(x)$ spanned by the projections onto $N_1^f(x)$ of the derivatives $\tilde{\nabla}_X \mu$ in the ambient space, with $X \in T_xM$, of local sections $\mu \in (N_1^f)^\perp$ of its orthogonal complement in the normal bundle N_fM. If all subspaces $\mathcal{S}(x)$ have the same dimension along M^n, and thus form a vector subbundle $\mathcal{S} = \mathcal{S}_f$, we may say that the rank s of $\mathcal{S}$ measures to what extent the first normal bundle N_1^f fails to be parallel.

If $\mathcal{S}$ coincides with N_1^f and $p \leq 6$, it turns out that condition (1) fails for the relative nullity, i.e., $\nu_f \geq n - p > 0$ at any point. The latter has strong well-known geometric consequences, namely, the submanifold carries a ν_f-dimensional totally geodesic foliation whose leaves are open subsets of affine subspaces in $\mathbb{R}^N$.

Our main result is that there is a single class of submanifolds for which $\mathcal{S}$ is a proper subbundle of N_1^f of rank $s \leq 6$, any other example being a submanifold of an element of this class. These are ruled submanifolds, with rulings of dimension at least $n - s$, for which $\mathcal{S}$ is constant in the ambient space along the rulings. In particular, the rulings belong to the kernel of $\alpha_{\mathcal{S}}$, and therefore condition (1) is violated for s. Examples of such submanifolds, showing that the preceding estimate on the dimension of the rulings is sharp, are constructed in the last section.

As discussed in the next section, the results of this paper generalize those in [**5**] for $p \leq 3$. We also point out that, although stated for submanifolds of Euclidean space, our results can easily be extended to ambient spaces of constant sectional curvature.

1. The result

In this section, we first give a precise statement of our main result and then discuss some particular cases.

Let $f\colon M^n \to \mathbb{R}^N$ denote a locally substantial isometric immersion of a connected Riemannian manifold, i.e., there is no open subset $U \subset M^n$ such that $f(U)$ is contained in a proper affine subspace of $\mathbb{R}^N$. Assume that f is 1-regular, i.e., the first normal spaces $N_1^f(x)$ have constant dimension p. Thus, these subspaces form a vector subbundle N_1^f of the normal bundle N_fM which we assume to be proper, i.e., $p < N - n$.

Assume $p < n$ and let ϕ: $(N_1^f)^\perp \oplus TM \to N_1^f$ be the tensor defined by

$$\phi(\mu, X) = (\nabla_X^\perp \mu)_{N_1^f},$$

where $(\ )_{N_1^f}$ denotes the N_1^f-component. We say that f has *nonparallel first normal bundle at* $x \in M^n$ if $\phi(x) \neq 0$, i.e., if the dimension $s(x)$ of the normal vector subspace $\mathcal{S}(x) \subset N_1^f(x)$ given by

$$\mathcal{S}(x) = \text{span}\{\phi(\mu, X) : \mu \in (N_1^f)^\perp(x) \text{ and } X \in T_xM\}$$

is nonzero. Thus, along each connected component of the open dense subset of M^n where $s(x) = s$ is constant, the vector subspaces $\mathcal{S}(x)$ form a vector subbundle $\mathcal{S}$ of N_1^f.

In the following statement, that an isometric immersion $F\colon N^m \to \mathbb{R}^N$, $m > n$, is an *extension* of the isometric immersion $f\colon M^n \to \mathbb{R}^N$ means that there exists an isometric embedding $i\colon M^n \to N^m$ such that $f = F \circ i$. Also, by f being d-ruled we understand that there exists a d-dimensional integrable distribution in M^n whose leaves are (mapped by f into) open subsets of affine subspaces in the ambient space.

Theorem 1. *Let $f\colon M^n \to \mathbb{R}^N$ be a 1-regular locally substantial isometric immersion such that $s(x) = s$ is constant with $0 < s < n$ and $s \leq 6$. Then, either*

(i) *$s = p$ and f has index of relative nullity $\nu_f \geq n - p$, or*

(ii) *$1 = s < p$ and f has an extension $F\colon N^{n+p-1} \to \mathbb{R}^N$ such that $\nu_F = n + p - 2$ and N_1^F is nonparallel of rank one, or*

(iii) $1 < s < p$ *and there is an open dense subset of* M^n, *the union of open subsets* $U_{k,d}$ *with* $d \geq n - s$ *and* $n - d \leq k \leq q := n - d + p - s$, *such that:*
  (a) $f|_{U_{q,d}}$ *is* d*-ruled and* $\mathcal{S}_f$ *is constant in* $\mathbb{R}^N$ *along the rulings, and*
  (b) $f|_{U_{k,d}}$, $k < q$, *has a ruled extension* $F \colon N^{n+q-k} \to \mathbb{R}^N$ *such that* N_1^F *is nonparallel of rank* $p + k - q$ *and* $\mathcal{S}_F$ *is constant along the rulings. The rulings have dimension* $n + p - k - s$ *and coincide with* $\mathcal{N}(\alpha_F)$ *if* $k = n - d$.

  Moreover, if $s = 2$ *we have that* $U_{k,d} = \emptyset$ *for* $k \geq 5$.

Observe that the ruled extensions in parts (ii) and (b) of (iii) are as in (i) and (a) of (iii), respectively.

For a ruled Euclidean submanifold, it is easily seen that for any vector X tangent to a ruling the Ricci curvature satisfies $\mathrm{Ric}(X) \leq 0$, with equality if and only if X belongs to the relative nullity subspace. Hence, we have the following immediate consequence of Theorem 1.

Corollary 2. *Under the assumptions of Theorem 1, cases (i) and (iii)(a) cannot occur if* $\mathrm{Ric}_M > 0$. *If* $\mathrm{Ric}_M \geq 0$ *then* $f|_{U_{q,d}}$ *in case (iii)(a) satisfies* $\nu_f = d$.

To illustrate Theorem 1 we discuss next the cases $p = 1$, 2 and 3. Notice that these are the cases that have already been considered in [**4**].

Example 3. *The case* $p = 1$. Here, the only possibility is that $s = 1$, and hence $\nu_f = n - 1$. In particular, the manifold M^n is flat.

Submanifolds as above can be easily described parametrically. For instance, consider the image under the normal exponential map of a parallel normal subbundle of the normal bundle of a curve with nonvanishing curvature; see also Theorem 1 in [**5**].

Example 4. *The case* $p = 2$. We only have the following two possibilities:

(i) $s = 2$, and hence $\nu_f = n - 2$.
(ii) $s = 1$, in which case f admits an extension $F \colon N^{n+1} \to \mathbb{R}^N$ such that $\nu_F = n$ (hence N^{n+1} is flat) and N_1^F is nonparallel of rank one.

The submanifolds in case (i) have been studied in [**2**] and [**3**], where a parametric classification has been obtained in most cases.

Example 5. *The case $p = 3$.* Then one of the following holds:

(i) $s = 3$ and f satisfies $\nu_f \geq n - 3$.

(ii) $s = 1$ and f has an extension $F\colon N^{n+2} \to \mathbb{R}^N$ such that $\nu_F = n+1$ (N^{n+2} is flat) and N_1^F is nonparallel of rank one.

(iii) $s = 2$ and either f is $(n-2)$-rules and $\mathcal{S}$ is constant along the rulings or f has an extension $F\colon N^{n+1} \to \mathbb{R}^N$ such that $\nu_F = n-1$ and N_1^F has rank two.

Observe that F in (ii) of Example 4 and Example 5 is as f in Example 3. Also, the extension F in (iv) of Example 5 is as f in (i) of Example 4.

2. A class of ruled extensions

In this section of independent interest, we find sufficient conditions for an Euclidean submanifold to admit a ruled extension carrying a normal subbundle that is constant in the ambient space along the rulings. We point out that a special case was already considered in [**5**].

Let $f\colon M^n \to \mathbb{R}^N$ be an isometric immersion satisfying the following conditions:

(i) Its normal bundle splits orthogonally and smoothly into two vector subbundles
$$N_f M = L \oplus P$$
such that the rank ℓ of L satisfies $0 < \ell < N - n$.

(ii) The subspaces
$$D(x) = \mathcal{N}(\alpha_P(x)) \subset T_x M$$
have constant dimension $d > 0$ on M^n (thus form a tangent subbundle $D \subset TM$).

(iii) The subbundle P is parallel along D in the normal connection, thus in $\mathbb{R}^N$. Hence, also L is parallel along D in the normal connection.

Let $\gamma\colon E \oplus P \to E \oplus L$ be the tensor given by

$$\gamma(Y, \mu) = (\tilde{\nabla}_Y \mu)_{E\oplus L} = -A_\mu Y + (\nabla^\perp_Y \mu)_L, \tag{2}$$

where the subbundle $E \subset TM$ of rank $n-d$ is defined by the orthogonal splitting $TM = D \oplus E$ and $\tilde{\nabla}$ denotes the connection in $\mathbb{R}^N$.

At $x \in M^n$, let $\Gamma(x) \subset E(x) \oplus L(x)$ be the subspace defined by

$$\Gamma(x) = \operatorname{span}\{\gamma(Y, \mu) : Y \in E \text{ and } \mu \in P\}. \tag{3}$$

Since E is spanned by the vectors $A_\mu Y$ for $\mu \in P$ and $Y \in E$, it follows from (2) that

$$n - d \leq \dim \Gamma(x) \leq n - d + \ell. \tag{4}$$

Assume further that

(iv) $\dim \Gamma(x) = k$ is constant on M^n.

Let $\pi \colon \Lambda \to M^n$ be the affine vector bundle of rank $r = n - d + \ell - k$ that is defined by the orthogonal splitting

$$\Gamma^k \oplus \Lambda^r = E^{n-d} \oplus L^\ell.$$

Lemma 6. *The distribution D is integrable and $\Lambda \cap TM = \{0\}$ holds.*

Proof: Take $\mu \in P$ and $Z, Y \in D$. Since P is parallel along D in $\mathbb{R}^N$, we have from

$$0 = \widetilde{R}(Y, Z)\mu = \tilde{\nabla}_Y \tilde{\nabla}_Z \mu - \tilde{\nabla}_Z \tilde{\nabla}_Y \mu - \tilde{\nabla}_{[Y,Z]}\mu \tag{5}$$

that $\tilde{\nabla}_{[Y,Z]}\mu \in P$. Hence $A_\mu [Y, Z] = 0$, and thus D is integrable.

Take $Z \in \Lambda \cap TM$. Then $Z \in E$ and

$$0 = \langle Z, \tilde{\nabla}_X \mu \rangle = -\langle A_\mu Z, X \rangle$$

for any $\mu \in P$ and $X \in TM$. Thus $Z \in D$ and hence $Z = 0$. □

The affine subspaces $\Delta(x)$ defined by

$$\Delta(x) = D(x) \oplus \Lambda(x)$$

form an affine bundle over M^n of rank $d + r = n + \ell - k$.

Lemma 7. *The bundle Δ is parallel in $\mathbb{R}^N$ along the leaves of D.*

Proof: It suffices to show that the orthogonal complement $\Gamma \oplus P$ of Δ in $\mathbb{R}^N$ is parallel in $\mathbb{R}^N$ along the leaves of D. First observe that

$$\Gamma \oplus P = \operatorname{span}\{\tilde{\nabla}_X \mu : X \in TM \text{ and } \mu \in P\}.$$

Then, we have from (5) that

$$\tilde{\nabla}_Y \tilde{\nabla}_X \mu = \tilde{\nabla}_X \tilde{\nabla}_Y \mu + \tilde{\nabla}_{[Y,X]}\mu \in \Gamma \oplus P$$

for any $\mu \in P$, $Y \in D$ and $X \in TM$, and the assertion follows. □

Define $F \colon N^{n+r} \to \mathbb{R}^N$ as the restriction of the map

$$\lambda \in \Lambda \mapsto f(\pi(\lambda)) + \lambda$$

to a tubular neighborhood N^{n+r} of the 0-section $j \colon M^n \hookrightarrow N^{n+r}$ of Λ where it is an immersion. Then $f = F \circ j$ and

$$T_{j(x)}N = j_* T_x M \oplus \Lambda(x) \tag{6}$$

for any $x \in M^n$.

Lemma 7 yields that F is ruled with $\Delta(\lambda) := \Delta(\pi(\lambda))$ as the ruling through $\lambda \in \Lambda$. For $\lambda \in \Lambda$, $\mu \in P$ and $X \in TM$, it follows from

$$\langle \tilde{\nabla}_X \lambda, \mu \rangle = -\langle \lambda, \tilde{\nabla}_X \mu \rangle = 0$$

that $\mathcal{P} \subset N_F N$ where $\mathcal{P}(\lambda) = P(\pi(\lambda))$. Moreover, we have that

$$\Delta = \mathcal{N}(\alpha^F_{\mathcal{P}}).$$

In fact, the inclusion $\Delta \subset \mathcal{N}(\alpha^F_{\mathcal{P}})$ holds because $\mathcal{P}$ is constant along Δ. For the opposite inclusion observe that $\alpha^F_{\mathcal{P}}|_{TM\times TM} = \alpha_P$. We easily obtain from (6) that equality is satisfied along M^n. To conclude the proof observe that the dimension of $\mathcal{N}(\alpha^F_{\mathcal{P}})$ can only decrease along $\Delta \subset N^{n+r}$ from its value on M^n if N^{n+r} is taken small enough.

We summarize the above facts in the following statement.

Proposition 8. *Let $f\colon M^n \to \mathbb{R}^N$ be an isometric immersion satisfying (i)–(iv) above. Then f admits a ruled extension $F\colon N^{n+r} \to \mathbb{R}^N$, $r = n - d + \ell - k$, with the following properties:*

(a) *The distribution Δ of rulings of F satisfies $D^d(x) = \Delta^{d+r}(x) \cap T_xM$ at any $x \in M^n$.*

(b) *There is an orthogonal splitting $N_F N = \mathcal{L} \oplus \mathcal{P}$ so that* $\operatorname{rank}\mathcal{L} = \ell - r$, *$\Delta = \mathcal{N}(\alpha^F_{\mathcal{P}})$ and $\mathcal{P}$ is constant in $\mathbb{R}^N$ along Δ.*

Moreover, we have:

(c) *If $r = 0$ then f is d-ruled and P is constant in $\mathbb{R}^N$ along the rulings.*

(d) *If $r = \ell$ then Δ is the relative nullity distribution of F.*

3. The proof

A key ingredient in the proof of Theorem 1 is a basic property of regular elements of a bilinear form observed by Moore [**6**]. It is stated below as Proposition 9.

Let $\beta\colon V \times U \to W$ be a bilinear form between finite dimensional real vector spaces. We call $Z \in V$ a (left) *regular element* of β if the map $\beta_Z = \beta(Z, \cdot)\colon U \to W$ satisfies

$$\dim \beta_Z(U) = \max\{\dim \beta_Y(U) : Y \in V\},$$

and denote by $RE(\beta)$ the subset of regular elements of β. It is a well-known fact that the set $RE(\beta)$ is open and dense in V.

Proposition 9. *If $\beta\colon V\times U\to W$ is a bilinear form and $Z\in RE(\beta)$, then*

$$\beta(V,\ker\beta_Z)\subset\beta_Z(U).$$

With the notations from Section 1, consider a 1-regular locally substantial isometric immersion $f\colon M^n\to\mathbb{R}^N$ such that $s(x)$ has a constant value $0<s<n$.

Lemma 10. *It holds that $\mathcal{N}(\phi)=\mathcal{N}(\alpha_{\mathcal{S}})$.*

Proof: Let $\mu_1\in RE(\phi)$ be a globally defined unit vector field and set $\phi_{\mu_1}=\phi(\mu_1,\cdot)$. Without loss of generality, we may assume that the subspaces $S_1(x)\subset\mathcal{S}(x)$ defined by

$$S_1(x)=\phi_{\mu_1}(T_xM)$$

have constant dimension $1\le s_1\le s$. Hence the tangent subspaces $D_1(x)=\ker\phi_{\mu_1}(x)$ satisfy $\dim D_1(x)=n-s_1$. It suffices to show that

$$D_1=\mathcal{N}(\alpha_{S_1}), \tag{7}$$

i.e., that $Y\in D_1$ if and only if $A_{\nabla^\perp_X\mu_1}Y=0$ for any $X\in TM$. But this follows from the Codazzi equation

$$A_{\nabla^\perp_X\delta}Y=A_{\nabla^\perp_Y\delta}X \tag{8}$$

for any $\delta\in(N_1^f)^\perp$. □

Lemma 11. *Suppose that $s\le 6$. Then $D=\mathcal{N}(\phi)$ satisfies*

$$\dim D\ge n-s. \tag{9}$$

Proof: Let μ_1 be as in the previous lemma. Again, we may assume that $S_1(x)$ has constant dimension $1\le s_1\le s$ on M^n. In view of Lemma 10, the assertion holds if $s_1=s$. If $s_1<s$, consider the orthogonal splitting

$$\mathcal{S}=S_1\oplus S_1^\perp$$

and let $\psi\colon(N_1^f)^\perp\oplus TM\to S_1^\perp$ denote the bilinear form defined by

$$\psi(\mu,X)=(\nabla^\perp_X\mu)_{S_1^\perp}.$$

Take $\mu_2\in RE(\phi)\cap RE(\psi)$ and set $t=\dim\psi(\mu_2,TM)$. Then $S_2=\phi_{\mu_2}(TM)$ satisfies

$$\dim(S_1+S_2)=s_1+t\quad\text{and}\quad\dim S_1\cap S_2=s_1-t.$$

It follows using Proposition 9 that

$$\dim D_1\cap D_2\ge\dim D_1-\dim S_1\cap S_2\ge n-2s_1+t. \tag{10}$$

If $t = s_1$ then $S_1 \cap S_2 = 0$. Thus $D_1 = D_2$. In particular (9) holds if $s_1 = 1$ since this forces $t = 1$. Therefore, we may assume

$$s_1 \geq 2. \tag{11}$$

We first analyze the case $t = 1$. In this case, we have that $H = \ker \psi(\mu_2, \cdot)$ is a hyperplane in TM. From (8) we obtain

$$A_{\nabla^{\perp}_{Z}\mu_2} X = A_{\nabla^{\perp}_{X}\mu_2} Z = 0$$

for any $Z \in D_1$ and $X \in H$. This implies that $\dim \phi_{\mu_2}(D_1) \leq 1$. Otherwise, there would exist a two-dimensional plane in S_1 such that the corresponding shape operators would have the same kernel of codimension one. But then a vector in this plane would belong to $(N_1^f)^{\perp}$, and this is a contradiction. It follows that $\dim D_1 \cap D_2 \geq n - s_1 - 1$.

If $\mathcal{S} = S_1 + S_2$ then (9) holds since $s = s_1 + 1$ and $D = D_1 \cap D_2$. If otherwise, we just repeat the process and obtain subspaces $S_1, \ldots, S_m$ and $D_1, \ldots, D_m$, $m = s - s_1 + 1$, such that $\mathcal{S} = S_1 + \cdots + S_m$ and $\dim D_1 \cap \cdots \cap D_m \geq n - s_1 - m + 1 - s$. Then $D = D_1 \cap \cdots \cap D_m$, and (9) follows.

By the above, we may assume that $t \geq 2$. We argue for the case $s = 6$, the other cases being similar and easier. If $t = s_1$ then $s_1 = 2, 3$. In these cases we have seen that $D_1 = D_2$, and thus (9) holds. Hence, in view of (11) and $t \geq 2$ we may assume that $s_1 > t \geq 2$. Thus, it remains to consider the cases $(s_1, t) = (3, 2)$ and $(s_1, t) = (4, 2)$. In the latter case, we have that $\mathcal{S} = S_1 + S_2$, and (9) follows from (10). In the first case, we have $\dim(S_1 + S_2) = 5$, $\dim S_1 \cap S_2 = 1$ and $\dim D_1 \cap D_2 \geq n - 4$. We now repeat the process and obtain S_3 such that $\mathcal{S} = S_1 + S_2 + S_3$ and $\dim S_i \cap S_j = 1$ if $i \neq j$. In this case, it is now clear that $\dim D \geq n - 5$. □

Remark 12. Our proof does not work for $s = 7$. In fact, in this case we may have $s_1 = 5$ and $t = 2$. Thus $\mathcal{S} = S_1 + S_2$ and (10) only yields $\dim D \geq n - 8$.

Now consider the global smooth orthogonal splitting $N_1^f = L^{p-s} \oplus \mathcal{S}^s$. Then, we have the global orthogonal splitting

$$N_f M = L^{p-s} \oplus P, \tag{12}$$

where $P = \mathcal{S}^s \oplus (N_1^f)^{\perp}$.

Lemma 13. *The subbundle P is parallel along D in the normal connection.*

Proof: By the Ricci equation, we have

$$\nabla^\perp_Y \nabla^\perp_X \mu_1 - \nabla^\perp_X \nabla^\perp_Y \mu_1 - \nabla^\perp_{[Y,X]} \mu_1 = 0.$$

Take $Y \in D_1$ and $X \in TM$. Then,

$$\nabla^\perp_Y (\nabla^\perp_X \mu_1)_{S_1} + \nabla^\perp_Y (\nabla^\perp_X \mu_1)_{(N_1^f)^\perp} = \nabla^\perp_X \nabla^\perp_Y \mu_1 + \nabla^\perp_{[Y,X]} \mu_1 \in P.$$

By Proposition 9, the second term on the left-hand-side belongs to P. It follows that $\nabla^\perp_Y \delta \in P$ for any $Y \in D_1$ and $\delta \in S_1$. □

Proof of Theorem 1: Assume first that $s = p$, that is, that $\mathcal{S} = N_1^f$. Then, Lemma 10 and Lemma 11 imply that $\nu_f \geq n - p$.

Suppose now that $s < p$. For each positive integer d, let U_d denote the interior of the subset of all $x \in M^n$ such that the subspace $D(x)$ has dimension d. It follows from Lemma 11 that $d \geq n - s$. By the lower semi-continuity of the dimension, we have that $\cup_d U_d$ is (open and) dense in M^n. Now let $U_{k,d}$ be the interior of the subset of all $x \in U_d$ such that the subspace $\Gamma(x)$ given by (3), with respect to the splitting (12), has dimension k. Then (4) with $\ell = p - s$ gives $n - d \leq k \leq q$. Again by the lower semi-continuity of the dimension, we have that $\cup_k U_{k,d}$ is (open and) dense in U_d.

In view of Lemma 10 and Lemma 13, we can apply Proposition 8 for $f|_{U_{k,d}}$. If $k = q$, we obtain from Proposition 8(c) that $f|_{U_{q,d}}$ is d-ruled and P (hence $\mathcal{S}$) is constant in $\mathbb{R}^N$ along the rulings.

If $k < q$, it follows from Proposition 8 that f admits a ruled extension $F\colon N^{n+r} \to \mathbb{R}^N$, $r = n - d + \ell - k = q - k$, with rulings of dimension $n + \ell - k = n + p - k - s$. Moreover, there is an orthogonal splitting $N_F N = \mathcal{L} \oplus \mathcal{P}$, where $\mathcal{P}$ is the parallel extension (in $\mathbb{R}^N$) of P along the rulings, such that $\operatorname{rank} \mathcal{L} = p - s - r$. In particular, $\operatorname{rank} N_1^F = p - r = p + k - q$.

Finally, if $k = n - d$ then the rulings of F coincide with its relative nullity distribution by Proposition 8(d).

The global assertion in (ii) for the case $1 = s < p$ is due to the fact that $s = 1$ implies $d = 1$, and also $k = 1$, as follows from (2). It is also a consequence of (2) that $k \leq 4$ if $s = 2$, hence in this case $U_{k,d} = \emptyset$ for $k \geq 5$. □

4. Examples

In this section we give examples of Euclidean submanifolds satisfying the conditions in part (iii)(a) of Theorem 1. More precisely, we construct ruled submanifolds M^{2m} in $\mathbb{R}^{2m+6}$ with four dimensional first normal bundle such that $\mathcal{S}$ has rank two and is constant along the codimensional two rulings. These examples show that the result cannot be improved since the rulings are not in the relative nullity distribution and their dimension achieve the minimum possible value given by the estimate.

Let $g\colon L^2 \to \mathbb{R}^{2(m+3)}$, $m \geq 2$, be a substantial elliptic surface in the sense of [**2**], i.e., there exists a (unique up to sign) almost complex structure J on L^2 such that

$$\alpha_g(Z,Z) + \alpha_g(JZ,JZ) = 0$$

for any $Z \in TL$. For instance, the surface can be minimal, which is equivalent to J being orthogonal. Then, it turns out that the normal bundle of g splits orthogonally as

$$N_gL = N_1^g \oplus \cdots \oplus N_{m+2}^g,$$

where each plane bundle N_k^g, $1 \leq k \leq m+2$, is its k^{th}-normal bundle; see [**2**] for details. Recall that the k^{th}-*normal space* N_k^h, $k \geq 2$, of an isometric immersion $h\colon M^n \to \mathbb{R}^N$ at $x \in M^n$ is defined as

$$N_k^h(x) = \operatorname{span}\{\alpha_h^{k+1}(X_1,\dots,X_{k+1}) : X_1,\dots,X_{k+1} \in T_xM\},$$

where $\alpha_h^\ell\colon TM \times \cdots \times TM \to N_hM$, $\ell \geq 3$, is the ℓ^{th}-fundamental form given by

$$\alpha_h^\ell(X_1,\dots,X_\ell) = \pi^{\ell-1}(\nabla^\perp_{X_\ell} \cdots \nabla^\perp_{X_3}\alpha(X_2,X_1)).$$

Here π^ℓ is the orthogonal projection onto $(N_1^h \oplus \cdots \oplus N_{\ell-1}^h)^\perp \cap N_hM$.

Define $f\colon M^{2m} \to \mathbb{R}^{2(m+3)}$ as the restriction of the map

$$\xi \in N_1^g \oplus \cdots \oplus N_{m-1}^g \mapsto g(\pi(\xi)) + \xi$$

to a tubular neighborhood of the 0-section L^2 of $\pi\colon N_1^g \oplus \cdots \oplus N_{m-1}^g \to L^2$ where it is an immersion. Given $\xi \in M^{2m} \setminus L^2$, we claim that

$$f_*T_\xi M \oplus N_1^f(\xi) = g_*T_xL \oplus N_1^g(x) \oplus \cdots \oplus N_{m+1}^g(x), \quad x = \pi(\xi).$$

Let $\tilde\xi$ be a local section of $N_1^g \oplus \cdots \oplus N_{m-1}^g$ on a neighborhood U of x such that $\tilde\xi(U) \subset M^{2m}$ and $\tilde\xi(x) = \xi$. Then

$$f_*\tilde\xi_*X = g_*X + \tilde\nabla_X\tilde\xi \tag{13}$$

for any $X \in T_xL$. On the other hand, for a vertical vector $V \in T_\xi M$ we have

$$f_*V = V.$$

Hence $N_1^g(x) \oplus \cdots \oplus N_{m-1}^g(x) \subset f_*T_\xi M$ and $f_*T_\xi M \subset g_*T_xL \oplus N_1^g(x) \oplus \cdots \oplus N_m^g(x)$. Regarding the local section $\tilde\xi$ as a vertical vector field of M^{2m}, we obtain

$$\tilde{\nabla}_X \tilde{\xi} = \tilde{\nabla}_{\tilde{\xi}_* X} f_* \tilde{\xi} \in f_* T_\xi M \oplus N_1^f(\xi). \tag{14}$$

Thus $N_m^g(x) \subset f_* T_\xi M \oplus N_1^f(\xi)$, hence also $g_* T_x L \subset f_* T_\xi M \oplus N_1^f(\xi)$ by (13). Differentiating (13) yields

$$\tilde{\nabla}_{\tilde{\xi}_* Y} f_* \tilde{\xi}_* \tilde{X} = \tilde{\nabla}_Y g_* \tilde{X} + \tilde{\nabla}_Y \tilde{\nabla}_X \tilde{\xi}$$

for all $X, Y \in T_x L$, where $\tilde{X}$ is any vector field on a neighborhood of x with $\tilde{X}(x) = X$. Thus $N_1^f(\xi) \subset g_* T_x L \oplus N_m^g(x) \oplus N_{m+1}^g(x)$ and $N_{m+1}^g(x) \subset N_1^f(\xi)$, and the claim follows.

Note also that the rulings of f are not in its relative nullity distribution. In fact, it follows from (14) that

$$\begin{aligned} \operatorname{span}\{\alpha_f(Z, V) : Z, V \in T_\xi M \text{ and } V \text{ vertical}\} \\ = (g_* T_x L \oplus N_m^g(x)) \cap N_1^f(\xi). \end{aligned} \tag{15}$$

We have from the claim that $N_f M = N_1^f \oplus N_{m+2}^g$. Thus, the immersion f is ruled by $N_1^g \oplus \cdots \oplus N_{m-1}^g$ and $\mathcal{S} = N_{m+1}^g$ has rank two and is constant in the ambient space along the rulings. Moreover, by (15) the rulings are not in the relative nullity distribution and their dimension satisfy the equality in the estimate given in part (iii)(a) of Theorem 1.

References

[1] M. Dajczer, *"Submanifolds and isometric immersions"*, Based on the notes prepared by M. Antonucci, G. Oliveira, P. Lima-Filho and R. Tojeiro, Mathematics Lecture Series **13**, Publish or Perish, Inc., Houston, TX, 1990.

[2] M. Dajczer and L. A. Florit, A class of austere submanifolds, *Illinois J. Math.* **45(3)** (2001), 735–755.

[3] M. Dajczer and P. Morais, Parabolic submanifolds of rank two, *Mat. Contemp.* **34** (2008), 195–232.

[4] M. Dajczer and L. Rodríguez, Substantial codimension of submanifolds: global results, *Bull. London Math. Soc.* **19(5)** (1987), 467–473. DOI: 10.1112/blms/19.5.467.

[5] M. Dajczer and R. Tojeiro, Submanifolds with nonparallel first normal bundle, *Canad. Math. Bull.* **37(3)** (1994), 330–337. DOI: 10.4153/CMB-1994-049-3.

[6] J. D. Moore, Submanifolds of constant positive curvature. I, *Duke Math. J.* **44(2)** (1977), 449–484. DOI: 10.1215/S0012-7094-77-04421-0.

[7] M. Spivak, *"A comprehensive introduction to differential geometry"*, Vols. I–V, Second edition, Publish or Perish, Inc., Wilmington, Del., 1979.

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