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A CHARACTERIZATION OF HYPERBOLIC KATO SURFACES

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Abstract: We give a characterization of hyperbolic Kato surfaces in terms of the existence of an automorphic Green function on a cyclic covering. This is achieved by analysing a naturally defined Levi-flat foliation, and by perturbing certain Levi-flat leaves to strictly pseudoconvex hypersurfaces.

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1. Introduction

Kato surfaces are important examples of compact complex surfaces with first Betti number equal to 1, discovered by Kato more that 30 years ago [**Kat**], see also [**Dlo**] and [**Nak**, §5]. They occupy a central place in the current speculative theory of non-Kählerian surfaces. It is customary today to divide the class of Kato surfaces into three disjoint subclasses:

(1) Enoki surfaces ($\sigma_n = 2n$ in the terminology of [**Dlo**]), which can be also described as exceptional compactifications of affine bundles over elliptic curves [**Nak**, §4.5];
(2) Kato surfaces of intermediate type ($2n < \sigma_n < 3n$);
(3) Inoue-Hirzebruch surfaces ($\sigma_n = 3n$), discovered by Inoue prior to Kato by a very different construction [**Nak**, §4.1 and §4.3].

There is a fundamental difference between class (1) and classes (2) and (3), coming from the results of [**D-O**]. If S belongs to class (2) or (3), then S admits a *Green function*, i.e. a multiplicatively automorphic negative function on the universal covering $\widetilde{S}$ which is pluriharmonic outside an analytic subset $Z \subset \widetilde{S}$, along which the function has logarithmic

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poles. Such a function, which is moreover unique up to a multiplicative constant, cannot exist on surfaces of class (1).

Due to this dichotomy, we shall refer to surfaces of class (2) or (3) as *hyperbolic* Kato surfaces, and to those of class (1) as *parabolic* Kato surfaces.

Our aim is to give a characterization of hyperbolic Kato surfaces in terms of existence of a Green function with *nonempty* polar set. This is analogous to the result of [**Br2**] giving a characterization of Inoue surfaces (another important class of non-Kählerian surfaces, see [**Nak**, §3]) in terms of existence of a Green function *without* poles.

Theorem 1.1. *Let S be a compact connected complex surface of algebraic dimension zero. Suppose that there exists an infinite cyclic covering $\pi\colon \widetilde{S} \to S$ and a negative plurisubharmonic function $F\colon \widetilde{S} \to [-\infty, 0)$ such that:*

(i) *$F \circ \varphi = \lambda \cdot F$ for some positive λ, where $\varphi\colon \widetilde{S} \to \widetilde{S}$ is a generator of the group of deck transformations;*
(ii) *$dd^c F$ is supported on an analytic subset $Z \subset \widetilde{S}$;*
(iii) *$Z \neq \emptyset$.*

Then S is a (possibly blown up) Kato surface.

Even if this statement is formally close to [**Br2**] (where $Z = \emptyset$), its proof follows a completely different path. The main idea is the following. The function F induces on S a (singular) codimension one real foliation $\mathcal{H}$, the leaves of which are noncompact Levi-flat hypersurfaces, spiralling around an arboreal cycle of rational curves. Using the nonemptyness of Z, we shall be able to "approximate" such a Levi-flat leaf by a compact strictly pseudoconvex hypersurface in S, not homologous to zero. Then the conclusion will be a consequence of [**Br1**].

The main technical point is therefore a result on the approximation of Levi-flat hypersurfaces with corners by smooth strictly pseudoconvex hypersurfaces, whose statement and proof will be given in the last part of the paper (Theorem 3.1), under a degree of generality going far beyond the context of Theorem 1.1.

2. Proof of Theorem 1.1 modulo Theorem 3.1

Let S satisfy the assumptions of Theorem 1.1. As in [**Br2**, §1], we may also suppose that S is minimal, and then Enriques-Kodaira classification leads to see that S belongs to the class VII_0, that is $b_1(S) = 1$ and $\mathrm{kod}(S) = -\infty$. Moreover, S cannot be a Hopf surface nor an Enoki surface: indeed, those surfaces contain a Zariski-open subset uniformised by $\mathbb{C}^2$, and this fact clearly prevents the existence of a nonconstant negative plurisubharmonic function on some covering (Liouville's theorem).

Denote by $\{Z_j\}_{j\in I}$ the irreducible components of Z, so that

$$dd^c F = \sum_{j\in I} \mu_j \cdot \delta_{Z_j}$$

for suitable positive real numbers $\{\mu_j\}_{j\in I}$. Here, as usual, δ_{Z_j} denotes the integration current on Z_j. The function F is equal to $-\infty$ on Z (logarithmic poles) and is finite and pluriharmonic on $\widetilde{S}\setminus Z$.

Since $F\circ\varphi = \lambda\cdot F$, the set Z is φ-invariant, and so it projects to an analytic subset $C \subset S$. By results of Kodaira, Enoki and Nakamura [**Nak**, §6 and §7], each irreducible component of C is a rational curve, and each connected component is either a tree of rational curves or a cycle of rational curves with possibly some trees attached (an *arboreal* cycle). Moreover, the intersection form of such a component is negative definite, so that the component is contractible to a normal singularity.

Lemma 2.1. *Every connected component of C is a (arboreal) cycle of rational curves.*

Proof: Suppose, by contradiction, that C_0 is a connected component of tree type. Then C_0 is simply connected, and some neighbourhood U of C_0 lifts isomorphically to $\widetilde{S}$. We obtain from F a function $F_0\colon U \to [-\infty, 0)$ such that $dd^c F_0$ is supported on C_0. This means that a nontrivial linear combination of the irreducible components of C_0 is dd^c-cohomologous to zero. But this contradicts the fact that the intersection form of C_0 is negative definite. □

In particular, and since $C \neq \emptyset$, it follows from a result of Nakamura [**Nak**, §8] that S is a degeneration of blown up Hopf surfaces, i.e. there exists a family of surfaces over the disc $\{S\}_{t\in\mathbb{D}}$ such that $S_0 = S$ and S_t is a blown up Hopf surface for every $t \neq 0$. This fact will allow later to use the result of [**Br1**].

Still by Nakamura's results [**Nak**, §7] we may (and will) suppose that C is connected, and that the inclusion of C into S induces an isomorphism between fundamental groups (otherwise S is an Inoue-Hirzebruch surface and the proof is done).

Referee's note: this is obtained from a suitable version of the so-called "Federer Support Theorem", cf. Corollary 2.14, p. 143 in: J. P. Demailly, *Complex Analytic and Differential Geometry*, `http://www-fourier.ujf-grenoble.fr/~demailly/manuscripts/agbook.pdf`.

The function $F|_{\widetilde{S}\setminus Z}$ induces a real analytic (singular) fibration

$$f = \log(-F)\colon S \setminus C \longrightarrow \mathbb{S}^1 = \mathbb{R}/(\log \lambda) \cdot \mathbb{Z}.$$

The level sets of f and the curve C may be informally thought as a singular codimension one foliation $\mathcal{H}$ on S. A regular level set of f is a leaf (or a union of leaves) which is properly embedded in $S \setminus C$ and accumulates to the full C, by spiralling around it.

Let us be more precise. Take an embedded real analytic circle $\gamma \subset C$, not passing through the singularities (nodes) of C, such that $C \setminus \gamma$ is connected. This is possible since C is an arboreal cycle; thus γ is contained in one of the irreducible components of C which compose the cycle inside C, and in that component γ separates the two nodes corresponding to the intersection with the other components of the cycle. Because γ is homotopic to zero in S (being contained in a rational curve), the function F induces a function F_0 on a neighbourhood of γ, uniquely defined up to multiplication by λ^n, $n \in \mathbb{Z}$. This function F_0 has a logarithmic pole along C, and we can find holomorphic coordinates (z, w) on a neighbourhood V of γ such that:

(1) $C \cap V = \{w = 0\}$;
(2) $\gamma = \{w = 0, |z| = 1\}$;
(3) $F_0 = \mu \cdot \log |w| + \beta \cdot \log |z|$, for some $\mu > 0$ and $\beta \in \mathbb{R}$.

Here z takes values in an annulus $\{1 - \varepsilon < |z| < 1 + \varepsilon\}$ and w in a disc $\{|w| < \varepsilon\}$, for some $\varepsilon > 0$ (and $\varepsilon < 1$, since F_0 is negative). The existence of those coordinates follows from the easy fact that a closed logarithmic 1-form with poles on C can be put in the normal form $\mu\frac{dw}{w} + \beta\frac{dz}{z}$, around γ.

Let Γ denote the solid torus embedded in V corresponding to the points with $|z| = 1$. Remark that it is a Levi-flat hypersurface, foliated by the holomorphic discs $\{z = c\}$, $c \in \gamma$.

Take now a regular level set of f, $W = W_t = f^{-1}(t)$. Since F is pluriharmonic on $\widetilde{S} \setminus Z$, W is a real analytic Levi-flat hypersurface, properly embedded in $S \setminus C$. It intersects V along the level sets $F_0^{-1}(-\lambda^n e^t)$, $n \in \mathbb{Z}$, and Γ along the countable collection of tori

$$T_n = \{|z| = 1, \, |w| = r_n\}, \quad n \geq 0,$$

where $\{r_n\} \subset (0, \varepsilon)$ is a sequence decreasing to 0 (and, more explicitely, given recursively by $\log r_n = \lambda^{\pm n} \cdot \log r_0$, with the sign $\pm$ chosen depending on the sign of $\lambda - 1$).

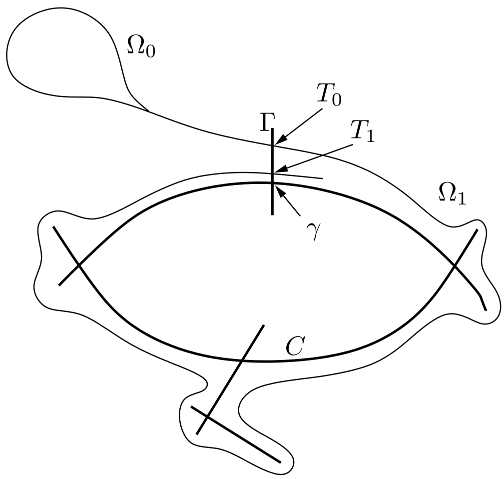
Lemma 2.2. *We have*

$$W \setminus \bigcup_{n\geq 0} T_n = \bigcup_{n\geq 0} \Omega_n,$$

where each Ω_n *is open and relatively compact in* W*, and moreover:*

(i) $\partial\Omega_0 = T_0$*;*
(ii) Ω_n *is connected and* $\partial\Omega_n = T_n - T_{n-1}$ *for every* $n \geq 1$*.*

Proof: Let $\eta\colon [0,1] \to C$ be a closed path generating $\pi_1(C)$, starting and ending at some point of γ. It can be lifted to W, as a (nonclosed) path starting at some point of T_n (n large) and ending at some point of $T_{n+\ell}$: this is just the holonomy of $\mathcal{H}$ along η, in a mildly singular context. Now, η is also a generator of $\pi_1(S)$, and this means that if we prolonge F_0 along η then, when we return to V, we get $\lambda^{\pm 1}F_0$. In other words, we must have $\ell = \pm 1$. It follows that T_n and T_{n-1} bound a relatively compact domain $\Omega_n \subset W$, and T_0 bounds a relatively compact open subset $\Omega_0 \subset W$. □



It is also easy to control the topology of the domains Ω_n, $n \geq 1$: each one is diffeomorphic to the boundary of a small tubular neighbourhood of C from which the intersection with Γ has been removed. This topology may be relatively complicated, because of the possible presence of the arborification of C (if C is just a cycle, without attached trees, then Ω_n is diffeomorphic to $\mathbb{T}^2 \times (0,1)$, for every $n \geq 1$). Remark that we do not claim, in the previous lemma, that Ω_0 is connected too: W could have a compact connected component, far from C. But, anyway,

Ω_0 contains precisely one noncompact connected component, the one with boundary T_0, and this component will be denoted by N.

Consider now the Levi foliation on W, restricted to the compact connected threefold with boundary

$$\overline{N} = N \cup T_0.$$

Denote this foliation by $\mathcal{F}$, and note that it is transverse to the boundary, where it restricts to a linear foliation with slope β/μ (see (3) above).

Lemma 2.3. *Every leaf of $\mathcal{F}$ intersects the boundary.*

Proof: On a neighbourhood of $\overline{N}$ in S the function F induces a function F_1 (as usual, unique up to multiplication by a power of λ). The Levi foliation on $\overline{N}$ is then defined by the *closed* 1-form $\omega = d^c F_1|_{\overline{N}}$. It is then a standard fact that, depending on the periods of ω, either $\mathcal{F}$ is a proper fibration over the circle, or every leaf of $\mathcal{F}$ is dense in $\overline{N}$. In both cases the conclusion is evident. □

The torus T_0 is also boundary of a solid torus $B_0 \subset \Gamma \subset S$. Take now

$$M = N \cup T_0 \cup B_0.$$

It is a compact Levi-flat hypersurface with corners, which satisfies all the requirements of Theorem 3.1 of the next section. Hence, we deduce from that theorem that M can be perturbed to a smooth strictly pseudoconvex hypersurface $\widetilde{M}$. Moreover, $\widetilde{M}$ is certainly not homologous to zero in S: its intersection number with a generator of $\pi_1(C)$ (the loop η of Lemma 2.2) is equal to ± 1. We can therefore apply the result of [**Br1**] and conclude that S is a Kato surface (of hyperbolic type, since we have already excluded the Enoki case).

3. Levi-flat hypersurfaces with corners

Here we switch to a more general context, and work on an arbitrary (not necessarily compact) complex surface X.

We shall say that a closed connected subset $M \subset X$ is a *Levi-flat hypersurface with corners* if for every $p \in M$ there exists a neighbourhood U_p of p in X, with coordinates (z, w), such that $M \cap U_p$ is expressed either by

$$\{\operatorname{Im} z = 0\}$$

Referee's note: the one whose closure has T_0 as boundary.

Referee's note: cf. Theorem 9.1.4, p. 206 in: A. Candel, L. Conlon, *Foliations I*, GSM vol. 23, Amer. Math. Soc., 2000.

or by

$$\{\operatorname{Im} z = 0,\, \operatorname{Im} w \leq 0\} \cup \{\operatorname{Im} w = 0,\, \operatorname{Im} z \leq 0\}.$$

Thus, there is a smooth part M^0, where the first representation holds, and a singular part M^1. If M^1_j is a connected component of M^1, then we have locally (on M^1_j) two smooth pieces of M adjacent to M^1_j. To simplify the matter, we shall suppose that even globally we have *two* different connected components of M^0 adjacent to M^1_j (instead of only one adjacent two times). Therefore, the closure $\overline{M^0_k}$ of a connected component M^0_k of M^0 is an embedded connected threefold with boundary $\partial M^0_k = \cup_{j \in I_k} M^1_j$.

Every $\overline{M^0_k}$ is a Levi-flat hypersurface with boundary. We shall denote by $\mathcal{F}_k$ its Levi foliation; note that it is transverse to the boundary. Moreover, if M^1_j is a commont component of ∂M^0_k and ∂M^0_h, then the two boundary foliations $\mathcal{F}_k|_{M^1_j}$ and $\mathcal{F}_h|_{M^1_j}$ are mutually transverse.

Example 3.1. Take the standard totally real torus $T = \{|z| = |w| = 1\} \subset \mathbb{C}^2$. Take two real analytic mutually transverse foliations $\mathcal{L}_1$ and $\mathcal{L}_2$ on T. These foliations can be complexified: there exists, on some neighbourhood V of T, two holomorphic foliations $\mathcal{F}_1$ and $\mathcal{F}_2$ whose traces on T are $\mathcal{L}_1$ and $\mathcal{L}_2$. Let now M_j be the saturation of T by $\mathcal{F}_j$, i.e. the union of the leaves of $\mathcal{F}_j$ intersecting T. Then, if V is sufficiently small, M_1 and M_2 are real analytic Levi-flat hypersurfaces in V, intersecting transversely along T. By taking only "half" of M_1 and M_2 we get a Levi-flat hypersurface with corners in V.

Remark that a Levi-flat hypersurface with corners is, in particular, a topological manifold, and so the notion of orientability makes sense. If M is orientable, then every connected component of M^1 is also orientable (being two-sided in M), and hence, when compact, it is a totally real torus. It is then not difficult to see that the previous example gives a universal local model for M around such a component of M^1.

If M is orientable and $U \subset X$ is a tubular neighbourhood, then $U \setminus M$ has two connected components. If $V_j \subset U$ is a tubular neighbourhood of M^1_j, then $V_j \setminus (M \cap V_j)$ has also two connected components, a "pseudoconvex" one V_j^+ and a "pseudoconcave one" V_j^-: if M is locally expressed by $\{\operatorname{Im} z = 0,\, \operatorname{Im} w \leq 0\} \cup \{\operatorname{Im} w = 0,\, \operatorname{Im} z \leq 0\}$ around a point of M^1_j, then V_j^+ is the component containing $\{\operatorname{Im} z < 0,\, \operatorname{Im} w < 0\}$. We shall say that M *has a pseudoconvex side* if all the $\{V_j^+\}$ are contained in the same connected component of $U \setminus M$, which will henceforth be denoted by U^+ (and the other by U^-).

It is rather evident that if M does not have a pseudoconvex side, then there is no hope for approximating M by strictly pseudoconvex smooth hypersurfaces. In order to do such an approximation when a pseudoconvex side exists, we shall need an additional dynamical assumption, which will allow to propagate the strict pseudoconvexity concentrated on M^1 to the full M. This is somewhat related to the approximation of confoliations by contact structures [**E-T**].

Theorem 3.1. *Let $M \subset X$ be a compact, connected, orientable, Levi-flat hypersurface with corners, admitting a pseudoconvex side. Suppose that for every smooth piece $\overline{M_k^0}$ the Levi foliation $\mathcal{F}_k$ has the following property: every leaf of $\mathcal{F}_k$ intersects the boundary ∂M_k^0. Then M can be C^0-approximated by smooth strictly pseudoconvex hypersurfaces.*

For the proof we shall need the following general fact.

Proposition 3.1. *Let N be a compact, connected, orientable manifold with boundary (of arbitrary dimension n), equipped with a smooth foliation by Riemann surfaces $\mathcal{F}$, transverse to the boundary and such that every leaf intersects the boundary. Then there exists a smooth function φ on N which is strictly subharmonic along the leaves of $\mathcal{F}$.*

Proof: It is sufficient to prove the following: for every $p \in N$ there exists $\varphi \in C^\infty(N)$ such that $\Delta_{\mathcal{F}}\varphi \geq 0$ and $(\Delta_{\mathcal{F}}\varphi)(p) > 0$. Here $\Delta_{\mathcal{F}}$ is the foliated laplacian with respect to a hermitian metric on the leaves.

Set $P = \{z \in \mathbb{C} \mid |z| < 2, \operatorname{Re} z \geq 0\}$, $Q = \{z \in \mathbb{C} \mid |z| < 1, \operatorname{Re} z \geq 0\}$. Since the leaf through p intersects ∂N, we can find an embedding (foliated chart) $j\colon P \times \mathbb{R}^{n-2} \to N$, sending boundary to boundary, such that $j^*(\mathcal{F})$ is the horizontal foliation (with leaves $P \times \{t\}$, $t \in \mathbb{R}^{n-2}$) and $j(\frac{1}{2}, 0) = p$. We may also require that $j|_{P\times\{t\}}$ is holomorphic for every t, with respect to the complex structure on the leaves of $\mathcal{F}$.

Take on P a smooth function φ_0 such that:

(i) $\operatorname{Supp}(\varphi_0) \subset Q$;
(ii) $\Delta\varphi_0 \geq 0$ on P;
(iii) $(\Delta\varphi_0)(\frac{1}{2}) > 0$.

It is evident that such a function exists. Let $\chi \in C^\infty_{\text{cpt}}(\mathbb{R}^{n-2})$ be a nonnegative function with $\chi(0) > 0$. Set $\varphi_1(z,t) = \chi(t) \cdot \varphi_0(z)$. Then $j_*\varphi_1$ can be extended, by zero, to the full N. The resulting $\varphi \in C^\infty(N)$ satisfies $\Delta_{\mathcal{F}}\varphi \geq 0$ and $(\Delta_{\mathcal{F}}\varphi)(p) > 0$. □

Remark 3.1. A variant of this construction produces $\varphi \in C^\infty(N \setminus \partial N)$ strictly subharmonic along the leaves and exhaustive ($\varphi(p) \to +\infty$ as p tends to ∂N).

Let us now turn to the proof of Theorem 3.1.

Take a smooth piece $\overline{M_k^0}$ and a small tubular neighbourhood $U_k \subset X$ of it; we may assume that $\overline{M_k^0}$ prolonges to a real analytic hypersurface $\widetilde{M_k^0}$ (without boundary) properly embedded in U_k, so that M_k^0 is a relatively compact domain in $\widetilde{M_k^0}$. Up to shrinking U_k around $\overline{M_k^0}$ (an operation that we will do again without further noticing), we dispose on U_k of a strictly plurisubharmonic function: it is sufficient to take

$$\psi = g_0^2 + \varepsilon\varphi,$$

where g_0 is any defining function of $\widetilde{M_k^0}$, φ is the function provided by Proposition 3.1, smoothly extended to U_k in any way, and $\varepsilon > 0$ is sufficiently small. Indeed, at points of $\overline{M_k^0}$ we have $dd^c g_0 = 2dg_0 \wedge d^c g_0$, which is semipositive with Kernel $T^{\mathbb{C}}\overline{M_k^0}$, and $dd^c\varphi$ is strictly positive on that Kernel.

Lemma 3.1. *There exists a defining function g of $\widetilde{M_k^0}$ such that $-\log|g|$ is strictly plurisubharmonic on $U_k \setminus \widetilde{M_k^0}$.*

Proof: Start with the defining function g_0 already used before. Around a point of $\widetilde{M_k^0}$ choose coordinates (z, w) such that $\widetilde{M_k^0}$ corresponds to $\{\operatorname{Im} z = 0\}$, so that $g_0 = e^{-h}\cdot\operatorname{Im} z$ for some h smooth. Then $dd^c(-\log|g_0|) = dd^c h + dd^c(-\log|\operatorname{Im} z|)$ admits a lower bound outside $\operatorname{Im} z = 0$, since the first term is bounded and the second one is positive. Thus, by compactness of $\overline{M_k^0}$, there exists a (possibly negative) constant C such that $dd^c(-\log|g_0|) > C \cdot dd^c\psi$ on $U_k \setminus \widetilde{M_k^0}$. It is then sufficient to take $g = e^{-|C|\cdot\psi} \cdot g_0$. □

The level sets

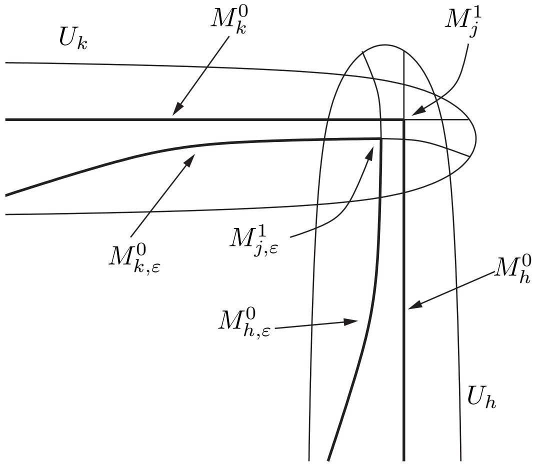
$$\widetilde{M_{k,\varepsilon}^0} = \{g = \varepsilon\}, \quad \varepsilon \neq 0$$

are therefore smooth strictly pseudoconvex hypersurfaces converging to $\widetilde{M_k^0}$ as $\varepsilon \to 0$, from both sides.

Remark 3.2. The domains $\{|g| < \varepsilon\}$, $\varepsilon > 0$, constitute a basis of strictly *pseudoconcave* neighbourhoods of $\widetilde{M_k^0}$. As observed in [**F-L**], the existence of a basis of strictly *pseudoconvex* neighbourhoods is a more subtle problem, involving for instance the holonomy of the foliation.

We apply Lemma 3.1 to every piece $\overline{M_k^0}$ of M, getting strictly pseudoconvex hypersurfaces $\widetilde{M_{k,\varepsilon}^0}$ in the *pseudoconvex side* of M.

Referee's note: $dd^c(g_0^2) = 2dg_0 \wedge d^c g_0$.



If $\widetilde{M_k^0}$ and $\widetilde{M_h^0}$ intersect along some torus M_j^1, then $\widetilde{M_{k,\varepsilon}^0}$ and $\widetilde{M_{h,\varepsilon}^0}$ intersect along some still totally real torus $M_{j,\varepsilon}^1$. We define $M_{k,\varepsilon}^0 \subset \widetilde{M_{k,\varepsilon}^0}$ as the relatively compact domain bounded by those tori, so that the subset

$$M_\varepsilon = \left(\bigcup_k M_{k,\varepsilon}^0\right) \cup \left(\bigcup_j M_{j,\varepsilon}^1\right)$$

is a topological manifold with corners close to M, strictly pseudoconvex in its smooth part.

It remains to smooth the corners of M_ε. To do so, observe that in some neighbourhood W of $M_{j,\varepsilon}^1$ the hypersurface M_ε can be expressed by

$$\{\mathrm{Max}(g_k, g_h) = 0\},$$

where g_k and g_h are smooth strictly plurisubharmonic functions defining $\widetilde{M_{k,\varepsilon}^0}$ and $\widetilde{M_{h,\varepsilon}^0}$. Let $\mathrm{Max}_\delta \colon \mathbb{R}^2 \to \mathbb{R}$ be a regularized maximum function (i.e. a smooth convex function equal to Max outside an euclidean δ-neighbourhood of the diagonal). Then $\mathrm{Max}_\delta(g_k, g_h)$ is smooth and strictly plurisubharmonic, and for δ sufficiently small its zero set is a smooth and strictly pseudoconvex hypersurface in W, close to $M_\varepsilon \cap W$ and equal to it ouside a compact subset of W. It is sufficient now to replace $M_\varepsilon \cap W$ with that smooth hypersurface.

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