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MARCINKIEWICZ INTERPOLATION THEOREMS FOR ORLICZ AND LORENTZ GAMMA SPACES

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Abstract: Fix the indices α and β, $1 < \alpha < \beta < \infty$, and suppose ϱ is an Orlicz gauge or Lorentz gamma norm on the real-valued functions on a set X which are measurable with respect to a σ-finite measure μ on it. Set

$$M(\gamma, X) := \{f \colon X \to \mathbb{R} \text{ with } \sup_{\lambda>0} \lambda\mu(\{x \in X : |f(x)| > \lambda\})^{\frac{1}{\gamma}} < \infty\},$$

$\gamma = \alpha, \beta$. In this paper we obtain, as a special case, simple criteria to guarantee that a linear operator T satisfies $T \colon L_\varrho(X) \to L_\varrho(X)$, whenever $T \colon M(\alpha, X) \to M(\alpha, X)$ and $T \colon M(\beta, X) \to M(\beta, X)$.

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1. Introduction

A generalization of the Marcinkiewicz interpolation theorem to Orlicz spaces contains the conditions

$$\int_{b^{-1}}^{t} \frac{A(s)}{s^{\alpha+1}}\, ds \le \frac{A(Kt)}{t^\alpha},$$
$$(1.1)\qquad \int_{t}^{\infty} \frac{A(s)}{s^{\beta+1}}\, ds \le \frac{A(Kt)}{t^\beta},$$

where $1 < \alpha < \beta < \infty$, $0 < b \le \infty$, A is a Young function and $K > 0$ is a constant independent of $t \in (b^{-1}, \infty)$; see [**20**, Vol. II, Chapter XII, Theorem 4.22]. One of the consequences of a principal result of this paper is that if $L_A = L_A(X)$ is an Orlicz space defined with respect

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to a σ-finite measure μ on X, $\mu(X) = b$, then the conditions (1.1) are necessary and sufficient for L_A to be an interpolation space between the Marcinkiewicz spaces $M(\alpha) = M(\alpha, X)$ and $M(\beta) = M(\beta, X)$. Recall that $f \in M(\alpha)$, say, is equivalent to

$$\varrho_{M(\alpha,X)}(f) := \sup_{\lambda>0} \lambda \mu_f(\lambda)^{\frac{1}{\alpha}} < \infty,$$

in which

$$\mu_f(\lambda) := \mu\big(\{x \in X : |f(x)| > \lambda\}\big).$$

We will work in the general setting of rearrangement-invariant (r.i.) norms, ϱ, on the class $\mathfrak{M}(X)$ of μ-measurable functions on X. Such a norm determines an r.i. space

$$L_\varrho = L_\varrho(X) := \{f \in \mathfrak{M}(X) : \varrho(|f|) < \infty\}.$$

See Section 2 below for details. We only mention here that the key property of an r.i. norm is

$$\varrho(f) = \varrho(g)$$

whenever f and g are equimeasurable, in the sense that $\mu_f = \mu_g$.

Two families of r.i. norms will be of special interest to us, namely, the Orlicz gauge norms and the Lorentz gamma norms. The former norms are defined in terms of a Young function, A, by

$$\varrho_A(f) := \inf\left\{\lambda > 0 : \int_X A\left(\frac{|f(x)|}{\lambda}\right) dx \le 1\right\}.$$

The latter norms are given in terms of an index p, $1 < p < \infty$, and a positive, locally integrable (weight) function, ϕ, on $I_b = (0, b)$, $b = \mu(X)$, by

$$\varrho_{p,\phi}(f) := \left[\int_{I_b} f^{**}(t)^p \phi(t)\, dt\right]^{\frac{1}{p}}, \quad f \in \mathfrak{M}(X);$$

here,

$$f^{**}(t) := t^{-1} \int_0^t f^*(s)\, ds,$$

with

$$f^* = \mu_f^{-1},$$

the inverse being in a generalized sense; again, see Section 2 below. We require

$$\int_1^\infty \phi(t) t^{-p}\, dt < \infty, \text{ when } b = \infty, \text{ and } \int_{I_b} \phi(t) t^{-p}\, dt = \infty, \text{ when } b < \infty;$$

otherwise, $\Gamma_{p,\phi} := L_{\varrho_{p,\phi}}$ would consist only of the zero function in the first case and would be identical to the space $L_1 = L_1(I_b)$ of Lebesgue-integrable functions on I_b in the second case. Such weights ϕ will be called nontrivial.

We first state a result in which the boundedness of certain operators T is asserted to follow from that of the supremum operators, S_α and T_β, $\alpha, \beta > 1$, defined at Lebesgue-measurable f on I_b and $t \in I_b$ by

$$(S_\alpha f)(t) := t^{-\frac{1}{\alpha}} \sup_{0<s\le t} s^{\frac{1}{\alpha}} f^*(s),$$

$$(T_\beta f)(t) := t^{-\frac{1}{\beta}} \sup_{t\le s<b} s^{\frac{1}{\beta}} f^*(s),$$

respectively. This result is proved in a more general setting by Dmitriev and Kreĭn [**7**], though the authors state that an earlier version in our context is due to Peetre. We give here a new proof (see Section 3) that emphasizes the role of the operators S_α and T_β, which role is only implicit in the work of the previous authors.

Theorem 1.1 (Dmitriev-Kreĭn-Peetre). *Let (X_1, μ_1) and (X_2, μ_2) be σ-finite measure spaces for which $\mu_1(X_1) = \mu_2(X_2) = b$. Suppose the quasilinear operator T satisfies*

$$T\colon M(\alpha, X_1) \to M(\alpha, X_2) \text{ and } T\colon M(\beta, X_1) \to M(\beta, X_2)$$

for indices α and β, with $1 < \alpha < \beta < \infty$. Define the r.i. norms, ϱ_i, on $\mathfrak{M}(X_i)$ in terms of given r.i. norms, $\bar{\varrho}_i$, on $\mathfrak{M}(I_b)$ by

$$\varrho_i(f) = \bar{\varrho}_i(f^*)$$

and suppose

$$M(\alpha, X_i) \cap M(\beta, X_i) \subset L_{\varrho_i}(X_i) \subset M(\alpha, X_i) + M(\beta, X_i), \quad i = 1, 2.$$

Then,

$$T\colon L_{\varrho_1}(X_1) \to L_{\varrho_2}(X_2)$$

whenever

$$S_\alpha\colon L_{\bar{\varrho}_1}(I_b) \to L_{\bar{\varrho}_2}(I_b) \text{ and } T_\beta\colon L_{\bar{\varrho}_1}(I_b) \to L_{\bar{\varrho}_2}(I_b). \tag{1.2}$$

Our paper is devoted to obtaining simple criteria to guarantee (1.2) when ϱ_1 and ϱ_2 are both Orlicz gauge norms or both Lorentz gamma norms. These criteria, asserting that it suffices to test the boundedness of S_α and T_β on characteristic functions of sets, are given in Theorems A and B, which we now state.

Theorem A. *Let* (X_1, μ_1) *and* (X_2, μ_2) *be* σ*-finite measure spaces with* $\mu_1(X_1) = \mu_2(X_2) = b$*. Fix the indices* α *and* β*,* $1 < \alpha < \beta < \infty$*. Suppose* A_1 *and* A_2 *are Young functions satisfying*

$$M(\alpha, X_i) \cap M(\beta, X_i) \subset L_{A_i}(X_i) \subset M(\alpha, X_i) + M(\beta, X_i), \quad i = 1, 2.$$

Assume, in addition, that $t^{-\frac{1}{\alpha}} \notin L_{A_2}(I_b)$*,*

$$A_2(t) = 0, \quad t \in I_{b^{-1}},$$

when $b < \infty$*,*

$$\int_0^1 A_2(t) t^{-1-\alpha}\, dt < \infty$$

when $b = \infty$ *and*

$$\int_1^\infty A_2(t) t^{-1-\beta}\, dt < \infty,$$

for all b*.*

Then, given any quasilinear operator T *such that*

$$T\colon M(\alpha, X_1) \to M(\alpha, X_2) \text{ and } T\colon M(\beta, X_1) \to M(\beta, X_2),$$

one has

$$T\colon L_{A_1}(X_1) \to L_{A_2}(X_2),$$

whenever

$$\begin{aligned} &\int_{b^{-1}}^t \frac{A_2(s)}{s^{\alpha+1}}\, ds \le \frac{A_1(Kt)}{t^\alpha}, \\ &\int_t^\infty \frac{A_2(s)}{s^{\beta+1}}\, ds \le \frac{A_1(Kt)}{t^\beta}, \end{aligned} \tag{1.3}$$

the constant $K > 0$ *being independent of* $t \in (b^{-1}, \infty)$*.*

In particular, the first condition in (1.3) *is necessary and sufficient in order that*

$$S_\alpha\colon L_{A_1}(I_b) \to L_{A_2}(I_b),$$

while the second condition is necessary and sufficient for

$$T_\beta\colon L_{A_1}(I_b) \to L_{A_2}(I_b).$$

Theorem B. *Let* (X_1, μ_1) *and* (X_2, μ_2) *be* σ*-finite measure spaces for which* $\mu_1(X_1) = \mu_2(X_2) = b$*. Fix the indices* α *and* β*, with* $1 < \alpha < \beta < \infty$*. Suppose the index* p*,* $1 < p < \infty$*, and the nontrivial weight functions,* ϕ_1 *and* ϕ_2*, are such that*

$$M(\alpha, X_i) \cap M(\beta, X_i) \subset \Gamma_{p,\phi_i}(X_i) \subset M(\alpha, X_i) + M(\beta, X_i), \quad i = 1, 2.$$

Then, given any quasilinear operator T *such that*

$$T\colon M(\alpha, X_1) \to M(\alpha, X_2) \text{ and } T\colon M(\beta, X_1) \to M(\beta, X_2),$$

one has

$$T\colon \Gamma_{p,\phi_1}(X_1) \to \Gamma_{p,\phi_2}(X_2),$$

whenever

$$(1.4)\quad \begin{aligned} \int_0^t s^{\frac{p}{\alpha}-1} \int_s^b \phi_2(y) y^{-\frac{p}{\alpha}}\, dy\, ds &\le K \int_0^t s^{p-1} \int_s^b \phi_1(y) y^{-p}\, dy\, ds, \\ t^{\frac{p}{\beta}} \int_0^t s^{\frac{p}{\beta'}-1} \int_s^b \phi_2(y) y^{-p}\, dy\, ds &\le K \int_0^t s^{p-1} \int_s^b \phi_1(y) y^{-p}\, dy\, ds, \end{aligned}$$

in which $\beta' = \frac{\beta}{\beta-1}$ *and the constant* $K > 0$ *is independent of* $t \in I_b$.

In particular, the first condition in (1.4) *is necessary and sufficient in order that*

$$S_\alpha\colon \Gamma_{p,\phi_1}(I_b) \to \Gamma_{p,\phi_2}(I_b),$$

while the second one is necessary and sufficient for

$$T_\beta\colon \Gamma_{p,\phi_1}(I_b) \to \Gamma_{p,\phi_2}(I_b).$$

The proofs of Theorems A and B appear in Sections 4 and 5, respectively, following the proof of Theorem 1.1 in Section 3. The final section has a number of applications and examples and, as well, a brief discussion of operators on spaces between pairs of the original Lorentz spaces, introduced in [**15**]. Section 2 to follow outlines the necessary background on r.i. norms and interpolation theory. In particular, it discusses certain r.i. norms whose Boyd and fundamental indices coincide.

2. Background

Suppose (X, μ) is a σ-finite measure space. Let $\mathfrak{M}(X) = \mathfrak{M}(X, \mu)$ be the class of real-valued μ-measurable functions on X. Given $f \in \mathfrak{M}(X)$, we define the decreasing rearrangement, f^*, of f on $I_b := (0, b)$, $b = \mu(X)$, by

$$f^*(t) := \inf\{\lambda > 0 : \mu_f(\lambda) \le t\}, \quad t \in I_b,$$

where

$$\mu_f(\lambda) := \mu\big(\{x \in X : |f(x)| > \lambda\}\big), \quad \lambda \in \mathbb{R}_+.$$

It satisfies the following inequality of Hardy and Littlewood:

$$\int_X |f(x)g(x)|\, d\mu(x) \le \int_{I_b} f^*(t) g^*(t)\, dt, \quad f, g \in \mathfrak{M}(X).$$

The operation of rearrangement is not sublinear though it satisfies

$$(2.1)\quad (f+g)^*(t_1+t_2) \le f^*(t_1) + g^*(t_2), \quad f, g \in \mathfrak{M}(X), \quad 0 < t_1+t_2 < b.$$

One does have, however,

$$(f+g)^{**}(t) \le f^{**}(t) + g^{**}(t), \quad f, g \in \mathfrak{M}(X), \quad t \in I_b;$$

here, the Hardy average, h^{**}, of h^*, is as defined in the introduction.

Definition 2.1. A rearrangement-invariant (r.i.) Banach function norm, $\bar{\varrho}$, on the class, $\mathfrak{M}(I_b)$, of Lebesgue-measurable functions on I_b satisfies the following seven axioms:

(A1) $\bar{\varrho}(f) = \bar{\varrho}(|f|) \ge 0$ with $\bar{\varrho}(f) = 0$ if and only if $f = 0$ a.e. on I_b;

(A2) $\bar{\varrho}(cf) = c\bar{\varrho}(f)$, $c \ge 0$;

(A3) $\bar{\varrho}(f+g) \le \bar{\varrho}(f) + \bar{\varrho}(g)$;

(A4) $f_n \uparrow f$ implies $\bar{\varrho}(f_n) \uparrow \bar{\varrho}(f)$;

(A5) $\bar{\varrho}(\chi_E) < \infty$ for all measurable subsets, E, of I_b with $|E| < \infty$;

(A6) $\int_E |f(t)|\, dt \le C_E \bar{\varrho}(f)$, for all measurable subsets, E, of I_b with $|E| < \infty$;

(A7) $\bar{\varrho}(f) = \bar{\varrho}(f^*)$ or, equivalently, $\mu_f = \mu_g$ implies $\bar{\varrho}(f) = \bar{\varrho}(g)$.

Using such a $\bar{\varrho}$ one can define an r.i. norm, ϱ, on a general $\mathfrak{M}(X)$, with $\mu(X) = b$, by

$$\varrho(f) = \bar{\varrho}(f^*), \quad f \in \mathfrak{M}(X). \tag{2.2}$$

For details on this and, indeed, all things related to r.i. spaces, we refer to [**1**, Chapters 1 and 2].

A basic tool for working with r.i. norms ϱ is the Hardy-Littlewood-Pólya (HLP) Principle, (see [**1**, Chapter 2, Theorem 4.6]) which asserts that

$$f^{**} \le g^{**} \text{ implies } \varrho(f) \le \varrho(g). \tag{2.3}$$

The Köthe dual of an r.i. norm ϱ is another such norm, ϱ', with

$$\varrho'(g) := \sup_{\varrho(h) \le 1} \int_X |g(x)h(x)|\, d\mu(x), \quad g, h \in \mathfrak{M}(X).$$

It obeys the Principle of Duality; that is,

$$\varrho'' := (\varrho')' = \varrho.$$

Further, the Hölder inequality

$$\int_X |f(x)g(x)|\, d\mu(x) \le \varrho(f)\varrho'(g)$$

holds for every $f, g \in \mathfrak{M}(X)$. We observe that if ϱ is defined in terms of $\bar{\varrho}$, as in (2.2), then

$$\varrho'(f) = \bar{\varrho}'(f^*), \quad f \in \mathfrak{M}(X).$$

Corresponding to an r.i. norm ϱ is the set

$$L_\varrho(X) := \{f \in \mathfrak{M}(X) : \varrho(f) < \infty\},$$

which becomes a Banach space with

$$||f||_{L_\varrho(X)} := \varrho(f);$$

indeed, it is a so-called rearrangement-invariant Banach function space or, for short, an r.i. space.

The Orlicz gauge norm is defined in terms of a Young function

$$A(t) := \int_0^t a(s)\,ds, \quad t \geq 0,$$

in which $a(s)$ is a strictly increasing function on $\mathbb{R}_+$, with $a(0+) = 0$ and $\lim_{s\to\infty} a(s) = \infty$. We have

$$\varrho_A(f) := \inf\left\{\lambda > 0 : \int_X A\left(\frac{|f(x)|}{\lambda}\right) d\mu(x) = \int_{I_b} A\left(\frac{f^*(t)}{\lambda}\right) dt \leq 1\right\}, \quad f \in \mathfrak{M}(X),$$

and

$$L_A(X) = L_{\varrho_A}(X) := \{f \in \mathfrak{M}(X) : \varrho_A(f) < \infty\}.$$

The Köthe dual of ϱ_A is, essentially, the gauge norm $\varrho_{\tilde{A}}$, where

$$\tilde{A}(t) := \int_0^t a^{-1}(s)\,ds, \quad t \in \mathbb{R}_+,$$

is called the Young function complementary to A; in fact,

$$\varrho_{\tilde{A}}(g) \leq \varrho'_A(g) \leq 2\varrho_{\tilde{A}}(g), \quad g \in \mathfrak{M}(X).$$

Given an index p, $1 < p < \infty$, and a nontrivial weight ϕ on I_b, the Lorentz gamma norm, $\varrho_{p,\phi}$, is defined by

$$\varrho_{p,\phi}(f) := \left[\int_{I_b} f^{**}(t)^p \phi(t)\,dt\right]^{\frac{1}{p}}, \quad f \in \mathfrak{M}(X).$$

This norm determines the Lorentz gamma space

$$\Gamma_{p,\phi}(X) = L_{\varrho_{p,\phi}} := \{f \in \mathfrak{M}(X) : \varrho_{p,\phi}(f) < \infty\}.$$

As mentioned in the introduction, we require

$$\int_1^\infty \phi(t) t^{-p}\,dt < \infty, \text{ when } b = \infty, \text{ and } \int_{I_b} \phi(t) t^{-p}\,dt = \infty, \text{ when } b < \infty.$$

The Köthe dual of $\varrho_{p,\phi}$ is equivalent to the Lorentz gamma norm $\varrho_{p',\psi}$, with $p' = \frac{p}{p-1}$ and

$$\psi(t) := \frac{t^{p'+p-1} \int_0^t \phi(s)\, ds \int_t^b \phi(s) s^{-p}\, ds}{\left(\int_0^t \phi(s)\, ds + t^p \int_t^b \phi(s) s^{-p}\, ds\right)^{p'+1}}, \quad t \in I_b,$$

provided

$$\int_0^1 \phi(t) t^{-p}\, dt = \int_1^\infty \phi(t)\, dt = \infty, \quad \text{if } b = \infty.$$

See [**10**, Theorem 6.2].

The dilation operator, E_s, $s \in \mathbb{R}_+$, given at $f \in \mathfrak{M}(I_b)$, $0 < b \le \infty$, and $t \in I_b$, by

$$(E_s f)(t) := \begin{cases} f(t/s), & \text{if } 0 < t < bs, \\ 0, & \text{if } bs \le t < b, \end{cases}$$

is bounded on any r.i. space $L_{\bar{\varrho}}(I_b)$ [**1**, Chapter 3, Proposition 5.11]. Denote the norm of E_s on $L_{\bar{\varrho}}(I_b)$ by $h_{\bar{\varrho}}(s)$ and define the lower and upper Boyd indices of $L_{\bar{\varrho}}(I_b)$ as

$$i_{\bar{\varrho}} := \lim_{s\to\infty} \frac{\log s}{\log h_{\bar{\varrho}}(s)} \text{ and } I_{\bar{\varrho}} := \lim_{s\to 0+} \frac{\log s}{\log h_{\bar{\varrho}}(s)}, \tag{2.4}$$

respectively. They satisfy

$$1 \le i_{\bar{\varrho}} \le I_{\bar{\varrho}} \le \infty;$$

also

$$i_{\bar{\varrho}'} = \frac{I_{\bar{\varrho}}}{I_{\bar{\varrho}} - 1} \text{ and } I_{\bar{\varrho}'} = \frac{i_{\bar{\varrho}}}{i_{\bar{\varrho}} - 1}.$$

See [**14**, Vol. II, pp. 131–132].

If in (2.4) we replace $h_{\bar{\varrho}}(s)$ by the norm, $k_{\bar{\varrho}}(s)$ of E_s on characteristic functions of sets of finite measure, we obtain the so-called fundamental indices.

The following result is proved in [**3**].

Theorem 2.2. *Fix α, β and b with $1 < \alpha < \beta < \infty$ and $0 < b \le \infty$. Set $(P_\alpha f)(t) := t^{-\frac{1}{\alpha}} \int_0^t f(s) s^{\frac{1}{\alpha}-1}\, ds$ and $(Q_\beta f)(t) := t^{-\frac{1}{\beta}} \int_t^b f(s) s^{\frac{1}{\beta}-1}\, ds$ for suitable $f \in \mathfrak{M}(I_b)$ and $t \in I_b$. Let $\bar{\varrho}$ be an r.i. norm on $\mathfrak{M}(I_b)$. Then,*

$$P_\alpha \colon L_{\bar{\varrho}}(I_b) \to L_{\bar{\varrho}}(I_b) \text{ if and only if } i_{\bar{\varrho}} > \alpha;$$

again,

$$Q_\beta \colon L_{\bar{\varrho}}(I_b) \to L_{\bar{\varrho}}(I_b) \text{ if and only if } I_{\bar{\varrho}} < \beta.$$

In case $\bar{\varrho} = \varrho_A$ is an Orlicz norm, one has

$$h_\varrho(s) \approx \lim_{t \to 0+} \frac{A^{-1}(1/t)}{A^{-1}(1/st)}.$$

This reflects the fact that the norm of E_s on an Orlicz space is essentially determined on characteristic functions of sets of finite measure and that $\varrho_A(\chi_E) = \frac{1}{A^{-1}(|E|^{-1})}$. The same is true for Lorentz gamma spaces. This is the content of the following result from [**8**].

Theorem 2.3. *Let (X, μ) be a σ-finite measure space with $\mu(X) = b$. Fix an index p, $1 < p < \infty$, and suppose ϕ is a nontrivial weight function on I_b. Take $\bar{\varrho}(f) = \varrho_{p,\phi}(f)$, $f \in \mathfrak{M}(I_b)$. Set*

$$h_{\bar{\varrho}} = \sup \frac{\bar{\varrho}\,(E_t f)}{\bar{\varrho}(f)}, \quad t \in \mathbb{R}_+,\ 0 \neq f \in \mathfrak{M}(I_b),$$

and define the Boyd indices $i_{\bar{\varrho}}$ and $I_{\bar{\varrho}}$ as in (2.4). Then, these indices can be computed by using the formula

$$h_{\bar{\varrho}}(s) \approx \sup_{0<t<b} \left[\frac{\int_0^{st} \phi(y)\,dy + s^p t^p \int_{st}^b \phi(y) y^{-p}\,dy}{\int_0^t \phi(y)\,dy + t^p \int_t^b \phi(y) y^{-p}\,dy} \right]^{\frac{1}{p}}.$$

We now describe certain parts of Interpolation Theory used later on.

Let (X_0, X_1) be a pair of Banach spaces compatible in the sense that they are continuously imbedded in a common Hausdorff topological vector space H. Their K-functional is defined for each f in the vector sum $X_0 + X_1$ by

$$K(t, f; X_0, X_1) := \inf_{f=g+h} \left[||g||_{X_0} + t\,||h||_{X_1} \right], \quad t \in \mathbb{R}_+.$$

The K-functional is a nonnegative, increasing, concave function of t on $\mathbb{R}_+$; see [**1**, Proposition 2, p. 294]. So,

$$K(t, f; X_0, X_1) = K(0+, f; X_0, X_1) + \int_0^t k(s, f; X_0, X_1)\,ds, \quad t \in \mathbb{R}_+,$$

in which the k-functional, $k(t, f; X_0, X_1)$, is a uniquely defined nonnegative, right-continuous, decreasing function on $\mathbb{R}_+$. According to [**1**, Proposition 1.15, p. 303],

$$K(0+, f; X_0, X_1) = 0 \text{ for all } f \in X_0 + X_1$$

if and only if $X_0 \cap X_1$ is dense in X_0.

Next, we restrict attention to r.i. spaces of functions in the context of a σ-finite measure space (X,μ), with $\mu(X)=b$. Such spaces are continuously imbedded in the Hausdorff topological vector space consisting of the set $\mathfrak{M}(X)$ together with the (metrizable) topology of convergence on sets of finite measure.

A special case of [**1**, Theorem 1.19, pp. 305–306] is

Theorem 2.4. *Let ϱ_0, ϱ_1, σ_0 and σ_1 be r.i. norms on $\mathfrak{M}(X)$ defined in terms of the norms $\bar\varrho_0$, $\bar\varrho_1$, $\bar\sigma_0$ and $\bar\sigma_1$ on $\mathfrak{M}(I_b)$. Given the r.i. norm λ on $\mathfrak{M}(\mathbb{R}_+)$, $g\in\mathfrak{M}(I_b)$ and $f\in\mathfrak{M}(X)$, set*

$$\bar\varrho(g):=\lambda\Big(k\big(t,g;L_{\bar\varrho_0}(I_b),L_{\bar\varrho_1}(I_b)\big)\Big)$$

and

$$\bar\sigma(g):=\lambda\Big(k\big(t,g;L_{\bar\sigma_0}(I_b),L_{\bar\sigma_1}(I_b)\big)\Big),$$

also

$$\varrho(f):=\bar\varrho(f^*)\text{ and }\sigma(f):=\bar\sigma(f^*).$$

Then, $L_\varrho=L_\varrho(X)$ and $L_\sigma=L_\sigma(X)$ are r.i. spaces of functions in $\mathfrak{M}(X)$ with the norms $||f||_\varrho:=\varrho(f)$ and $||f||_\sigma:=\sigma(f)$. Moreover, if T is any linear operator on $L_{\varrho_0}+L_{\varrho_1}$ satisfying

$$T\colon L_{\varrho_0}\to L_{\sigma_0}\text{ and }T\colon L_{\varrho_1}\to L_{\sigma_1},$$

then, $T\colon L_\varrho\to L_\sigma$. In particular, L_ϱ is an interpolation space between L_{ϱ_0} and L_{ϱ_1} in the sense that, for any linear operator T,

$$T\colon L_{\varrho_0}\to L_{\varrho_0}\text{ and }T\colon L_{\varrho_1}\to L_{\varrho_1},$$

implies $T\colon L_\varrho\to L_\varrho$; similarly, L_σ is an interpolation space between L_{σ_0} and L_{σ_1}.

Lastly, we recall that, for $1<p\le\infty$, $1\le q\le\infty$, the Lorentz norms, $\varrho_{p,q}$, are defined at $f\in\mathfrak{M}(X)$, $\mu(X)=b$, by

$$\varrho_{pq}(f):=\left(\int_{I_b}\left[t^{\frac1p-\frac1q}f^{**}(t)\right]^q dt\right)^{\frac1q},\ \text{when } q<\infty,$$

and

$$\varrho_{p\infty}(f):=\sup_{0<t<b}t^{\frac1p}f^{**}(t);$$

see [**11**]. We will write $L_{\varrho_{pq}}(X)$ as $\Lambda(p,q,X)$, using the special notation $\Lambda(p,X)$ when $q=1$ and $M(p,X)$ when $q=\infty$.

3. The proof of Theorem 1.1

Proof: Given (1.2), the fundamental result on K-functionals [**1**, Chapter 5, Theorem 1.11, p. 301] and the Holmstedt formula [**1**, Chapter 5, Theorem 2.1, pp. 307–309]

$$K\big(t, g; M(\alpha, X_i), M(\beta, X_i)\big) \approx \sup_{0<s\le t^\gamma} s^{\frac{1}{\alpha}} g^*(s) + t \sup_{t^\gamma \le s < b} s^{\frac{1}{\beta}} g^*(s),$$

where $g \in M(\alpha, X_i) + M(\beta, X_i)$, $i = 1, 2$ and $\frac{1}{\gamma} = \frac{1}{\alpha} - \frac{1}{\beta}$, one has

$$\sup_{0<s\le t} s^{\frac{1}{\alpha}} (Tf)^*(s) + t^{\frac{1}{\gamma}} \sup_{t\le s<b} s^{\frac{1}{\beta}} (Tf)^*(s) \le C \sup_{0<s\le Ct} s^{\frac{1}{\alpha}} f^*(s) + C t^{\frac{1}{\gamma}} \sup_{Ct\le s<b} s^{\frac{1}{\beta}} f^*(s),$$

with $C > 1$ independent of t, $0 < t < \frac{b}{C}$. Hence, by [**12**, (3.19)],

$$\sup_{0<s\le t} s^{\frac{1}{\alpha}} (Tf)^{**}(s) \approx \sup_{0<s\le t} s^{\frac{1}{\alpha}} (Tf)^*(s) \le C \sup_{0<s\le Ct} s^{\frac{1}{\alpha}} f^*(s) + C t^{\frac{1}{\gamma}} \sup_{Ct\le s<b} s^{\frac{1}{\beta}} f^*(s)$$

and so, for some $K > C$,

$$t^{\frac{1}{\alpha}} (Tf)^{**}(t) \le K \sup_{0<s\le Ct} s^{\frac{1}{\alpha}} f^*(s) + K t^{\frac{1}{\gamma}} \sup_{Ct\le s<b} s^{\frac{1}{\beta}} f^*(s), \quad 0 < t < \frac{b}{C}.$$

Dividing both sides by $t^{\frac{1}{\alpha}}$, we arrive at

$$(Tf)^{**}(t) \le K^2 (S_\alpha f + T_\beta f)(Ct) \le K^2 (S_\alpha f + T_\beta f)^{**}(Ct), \quad 0 < t < \frac{b}{C}.$$

From this, HLP, (1.2) and the continuity of the dilation operator yield

$$\varrho_2(Tf) = \bar{\varrho}_2\big((Tf)^*\big) \le K^2 \bar{\varrho}_2\big((S_\alpha f + T_\beta f)^*(Ct)\big) \le M \bar{\varrho}_1(f^*) = M \varrho_1(f),$$

in which $M = K^2 h_{\bar{\varrho}_2}(C) \big[||S_\alpha||_{L_{\bar{\varrho}_1}(I_b) \to L_{\bar{\varrho}_2}(I_b)} + ||T_\beta||_{L_{\bar{\varrho}_1}(I_b) \to L_{\bar{\varrho}_2}(I_b)} \big]$. □

4. The proof of Theorem A

Lemma 4.1. *Fix $\alpha > 1$ and $b \in (0, \infty]$. Let A be a Young function satisfying $t^{-\frac{1}{\alpha}} \notin L_A(I_b)$,*

(4.1) $$A(t) = 0, \quad t \in I_{b^{-1}},$$

when $b < \infty$, and

$$\int_0^1 A(t) t^{-1-\alpha} \, dt < \infty$$

when $b=\infty$*. Then,*

$$E_\alpha(t) := \alpha t^\alpha \int_{b^{-1}}^{t} \frac{A(s)}{s^{\alpha+1}}\, ds$$

is a strictly increasing function of t *on* (b^{-1},∞)*, with*

$$\varrho_A\big(s^{-\frac{1}{\alpha}}\chi_{(t,b)}(s)\big) = \frac{t^{-\frac{1}{\alpha}}}{E_\alpha^{-1}(t^{-1})}$$

for all $t\in\mathbb{R}_+$ *when* $b=\infty$ *and for sufficiently small* t *when* $b<\infty$*.*

Proof: This is essentially a modification of (4.44) in [**5**, p. 63]. We deal only with the case $b<\infty$, the proof being, in fact, simpler when $b=\infty$.

Now, $\varrho_A(s^{-\frac{1}{\alpha}}\chi_{(t,b)}(s))$ is, by definition, the number λ such that

$$\int_t^b A\left(\frac{s^{-\frac{1}{\alpha}}}{\lambda}\right) ds = 1$$

or, with $y=\frac{s^{-\frac{1}{\alpha}}}{\lambda}$,

$$\frac{\alpha}{\lambda^\alpha}\int_{\max\{b^{-1},b^{-\frac{1}{\alpha}}\lambda^{-1}\}}^{t^{-\frac{1}{\alpha}}\lambda^{-1}} \frac{A(y)}{y^{\alpha+1}}\, dy = 1. \tag{4.2}$$

Since $t^{-\frac{1}{\alpha}}\notin L_A(I_b)$, one has $\lim_{t\to 0_+}\varrho_A(s^{-\frac{1}{\alpha}}\chi_{(t,b)}(s))=\infty$. Hence, for sufficiently small t, we obtain $b^{-\frac{1}{\alpha}}\lambda^{-1}\le b^{-1}$, and (4.2) becomes

$$\frac{\alpha}{\lambda^\alpha}\int_{b^{-1}}^{t^{-\frac{1}{\alpha}}\lambda^{-1}} \frac{A(y)}{y^{\alpha+1}}\, dy = 1;$$

that is,

$$E_\alpha\left(\frac{1}{\lambda t^{\frac{1}{\alpha}}}\right) = t^{-1}.$$

Thus,

$$\frac{1}{\lambda t^{\frac{1}{\alpha}}} = E_\alpha^{-1}(t^{-1}),$$

or

$$\lambda = \frac{t^{-\frac{1}{\alpha}}}{E_\alpha^{-1}(t^{-1})}. \qquad \square$$

Lemma 4.2. *Fix* $\beta>1$ *and let* A *be a Young function satisfying*

$$\int_1^\infty A(t)t^{-1-\beta}\, dt < \infty.$$

Then,

$$F_\beta(t) := \beta t^\beta \int_t^\infty \frac{A(s)}{s^{\beta+1}}\, ds$$

is a strictly increasing function of t on $\mathbb{R}_+$, with

$$\varrho_A(s^{-\frac{1}{\beta}}\chi_{(0,t)}(s)) = \frac{t^{-\frac{1}{\beta}}}{F_\beta^{-1}(t^{-1})}, \quad t \in \mathbb{R}_+.$$

Proof: Similar to that of Lemma 4.1. □

Proof of Theorem A: Theorem 1.1 guarantees

$$T\colon L_{A_1}(X_1) \to L_{A_2}(X_2)$$

whenever

$$S_\alpha\colon L_{A_1}(I_b) \to L_{A_2}(I_b) \text{ and } T_\beta\colon L_{A_1}(I_b) \to L_{A_2}(I_b).$$

We will prove the equivalence of the boundedness of S_α and the first of the conditions in (1.3), namely,

$$\int_{b^{-1}}^{t} \frac{A_2(s)}{s^{\alpha+1}}\, ds \le \frac{A_1(Kt)}{t^\alpha}, \quad t > b^{-1}. \tag{4.3}$$

The proof that the boundedness of T_β is equivalent to the second condition in (1.3) is similar.

To begin, assume

$$S_\alpha\colon L_{A_1}(I_b) \to L_{A_2}(I_b) \tag{4.4}$$

and let $t \in I_b$. A simple calculation shows

$$\big(S_\alpha \chi_{(0,t)}\big)(s) = \chi_{(0,t)}(s) + t^{\frac{1}{\alpha}} s^{-\frac{1}{\alpha}} \chi_{(t,b)}(s), \quad s \in I_b.$$

Therefore,

$$\varrho_{A_2}\big(S_\alpha \chi_{(0,t)}\big) \ge t^{\frac{1}{\alpha}} \varrho_{A_2}\big(s^{-\frac{1}{\alpha}} \chi_{(t,b)}(s)\big), \quad t \in I_b,$$

so, with $f = \chi_{(0,t)}$, (4.4) ensures

$$t^{\frac{1}{\alpha}} \varrho_{A_2}\big(s^{-\frac{1}{\alpha}} \chi_{(t,b)}(s)\big) \le C \varrho_{A_1}\big(\chi_{(0,t)}\big) = \frac{C}{A_1^{-1}(t^{-1})}, \tag{4.5}$$

$C > 0$ being independent of $t \in I_b$. In view of Lemma 4.1, (4.5) implies

$$\frac{1}{E_\alpha^{-1}(t^{-1})} \le \frac{C}{A_1^{-1}(t^{-1})}$$

for sufficiently small t, in which E_α is defined with respect to A_2. Since E_α is increasing, we conclude that there exists some $t_0 \ge b^{-1}$ such that

$$E_\alpha(t) \le \alpha^{-1} A_1(Ct), \quad t \ge t_0.$$

Since $\alpha > 1$, this yields

$$E_\alpha(t) \le A_1(Ct), \quad t \ge t_0.$$

Setting

$$C' := \sup_{t\in[b^{-1},t_0]} \frac{A_1^{-1}(E_\alpha(t))}{t}$$

and

$$K := \max\{C, C'\},$$

we get (4.3).

Suppose now that (4.3) holds. Fix $0 \le f \in L_{A_1}(X_1)$, $\varrho_{A_1}(f) = 1$, and, for $t \in \mathbb{R}_+$, define

$$f_t(s) = \min[f^*(s), t] \text{ and } f^t(s) = f^*(s) - f_t(s), \quad s \in I_b.$$

Then, f^t and f_t are nonnegative and decreasing,

$$(S_\alpha f_t)(s) \le t, \quad s \in I_b, \tag{4.6}$$

and, since, by (2.1),

$$f^*(2s) = (f_t + f^t)(2s) \le f_t(s) + f^t(s), \quad 0 < s < \tfrac{b}{2},$$

we have

$$(S_\alpha f)(2s) \le (S_\alpha f_t)(s) + (S_\alpha f^t)(s), \quad 0 < s < \tfrac{b}{2}. \tag{4.7}$$

We observe that, by the argument of [**12**, Lemma 3.5], one has

$$t^\alpha \left|\{S_\alpha g > t\}\right| \le C \sup_{s\in\mathbb{R}_+} s^\alpha \left|\{|g|\, g > s\}\right|, \quad g \in \mathfrak{M}(I_b), \quad t \in \mathbb{R}_+. \tag{4.8}$$

Thus, with $A_2' = a_2$,

$$
\begin{aligned}
&\int_0^{\frac{b}{2}} A_2\big(\tfrac{1}{2}(S_\alpha f)(2t)\big)\,dt \\
&= \int_{\mathbb{R}_+} a_2(t)\,\big|\{s\in(0,\tfrac{b}{2}) : (S_\alpha f)(2s) > 2t\}\big|\,dt \\
&\le \int_{\mathbb{R}_+} a_2(t)\,\big|\{s\in(0,\tfrac{b}{2}) : (S_\alpha f_t)(s) > t\}\big|\,dt \\
&\quad + \int_{\mathbb{R}_+} a_2(t)\,\big|\{s\in(0,\tfrac{b}{2}) : (S_\alpha f^t)(s) > t\}\big|\,dt, \qquad \text{by (4.7)}, \\
&= \int_{\mathbb{R}_+} a_2(t)\,\big|\{s\in(0,\tfrac{b}{2}) : (S_\alpha f^t)(s) > t\}\big|\,dt, \qquad \text{by (4.6)}, \\
&\le C\int_{\mathbb{R}_+} a_2(t)t^{-\alpha} \sup_{s\in\mathbb{R}_+} s^\alpha\,\big|\{y\in I_b : f^t(y) > s\}\big|\,dt, \qquad \text{by (4.8)}, \\
&= C\int_{\mathbb{R}_+} a_2(t)t^{-\alpha} \sup_{s\ge t}(s-t)^\alpha\,|\{y\in I_b : f^*(y) > s\}|\,dt \\
&\le C\int_0^{b^{-1}} a_2(t)t^{-\alpha} \sup_{s\ge t} s^\alpha\,|\{y\in I_b : f^*(y) > s\}|\,dt \\
&\quad + C\int_{b^{-1}}^\infty a_2(t)t^{-\alpha} \sup_{s\ge t} s^\alpha\,|\{y\in I_b : f^*(y) > s\}|\,dt.
\end{aligned}
$$

Now, the first term is no bigger than

$$C\varrho_{M(\alpha,X_1)}(f)\int_0^{b^{-1}} a_2(t)t^{-\alpha}\,dt,$$

which, in turn, using the inequality $ta_2(t)\le A_2(2t)$, is majorized by

$$C\varrho_{M(\alpha,X_1)}(f)\int_0^{b^{-1}} A_2(2t)t^{-\alpha-1}dt = 2^\alpha C\varrho_{M(\alpha,X_1)}(f)\int_0^{2b^{-1}} A_2(t)t^{-\alpha-1}dt,$$

this being finite by assumption. We observe that, if $b<\infty$, one has

$$M(\alpha,X_i)+M(\beta,X_i) = M(\alpha,X_i), \quad i=1,2,$$

while, if $b=\infty$, the first term is zero.

For the second term we have

$$C\int_{b^{-1}}^\infty a_2(t)t^{-\alpha}\sup_{s\ge t} s^\alpha\,|\{y\in I_b : f^*(y) > s\}|\,dt \le C\int_{b^{-1}}^\infty a_2(t)\big(T_{\frac{1}{\alpha}}h\big)(t)\,dt,$$

where

$$h(t) := |\{y\in I_b : f^*(y) > t\}|$$

and

$$\big(T_{\frac{1}{\alpha}}h\big)(t) := t^{-\alpha}\sup_{s\geq t} s^{\alpha}h(s), \quad t > b^{-1}.$$

As $a_2(s) \leq s^{-1}A_2(2s)$, (4.3) implies

$$t^{\alpha}\int_{b^{-1}}^{t} \frac{a_2(s)}{s^{\alpha}}\,ds \leq A_1(2Kt), \quad t > b^{-1}.$$

A slight modification of [**9**, Theorem 3.2] guarantees there exists a $K > 0$ such that, with $A_1' = a_1$,

$$\int_{b^{-1}}^{\infty} a_2(t)\big(T_{\frac{1}{\alpha}}h\big)(t)\,dt \leq \int_{b^{-1}}^{\infty} a_1(Kt)h(t)\,dt \leq \int_{I_b} A_1\big(Kf(t)\big)\,dt, \qquad 0 \leq f \in \mathfrak{M}(I_b).$$

Altogether, then,

$$\int_0^{\frac{b}{2}} A_2\big(\tfrac{1}{2}(S_\alpha f)(2t)\big)\,dt \leq 2^{\alpha}C\varrho_{M(\alpha,X_1)}(f)\int_0^{2b^{-1}} A_2(t)t^{-\alpha-1}\,dt + C\int_{I_b} A_1(Kf(t))\,dt,$$

or

$$\int_{I_b} A_2\Big(\frac{1}{2}(S_\alpha f)(t)\Big)\,dt \leq 2^{\alpha+1}C\varrho_{M(\alpha,X_1)}(f)\int_0^{2b^{-1}} A_2(t)t^{-\alpha-1}\,dt + 2C\int_{I_b} A_1(Kf(t))\,dt,$$

$0 \leq f \in \mathfrak{M}(I_b)$, from which (4.4) follows by a standard argument. □

5. The proof of Theorem B

Proof of Theorem B: We proceed as in the proof of Theorem A. Thus,

$$T\colon \Gamma_{p,\phi_1}(X_1) \to \Gamma_{p,\phi_2}(X_2)$$

follows from

$$S_\alpha\colon \Gamma_{p,\phi_1}(I_b) \to \Gamma_{p,\phi_2}(I_b) \text{ and } T_\beta\colon \Gamma_{p,\phi_1}(I_b) \to \Gamma_{p,\phi_2}(I_b).$$

The connection of the latter to (1.4) will be achieved by our showing

$$S_\alpha\colon \Gamma_{p,\phi_1}(I_b) \to \Gamma_{p,\phi_2}(I_b) \tag{5.1}$$

if and only if

$$\int_0^t s^{\frac{p}{\alpha}-1}\int_s^b \phi_2(y)y^{-\frac{p}{\alpha}}\,dy\,ds \leq K\int_0^t s^{p-1}\int_s^b \phi_1(y)y^{-p}\,dy\,ds, \quad t\in I_b, \tag{5.2}$$

and

$$T_\beta \colon \Gamma_{p,\phi_1}(I_b) \to \Gamma_{p,\phi_2}(I_b) \tag{5.3}$$

if and only if

$$t^{\frac{p}{\beta}} \int_0^t s^{\frac{p}{\beta'}-1} \int_s^b \phi_2(y) y^{-p}\, dy\, ds \le K \int_0^t s^{p-1} \int_s^b \phi_1(y) y^{-p}\, dy\, ds, \quad t \in I_b. \tag{5.4}$$

We observe that for $f \in M(I_b)$, $t \in I_b$, one has

$$S_\alpha f(t) = \sup_{0<y\le 1} y^{\frac{1}{\alpha}} f^*(ty),$$

whence $S_\alpha f$ is nonincreasing on I_b, that is, $S_\alpha f = (S_\alpha f)^*$. Thus, $(S_\alpha f)(t) \le (S_\alpha f)^{**}(t)$, $t \in I_b$. Using this and [**12**, Theorem 3.6], we conclude that (5.1) is equivalent to

$$\int_{I_b} (S_\alpha f)(s)^p \phi_2(s)\, ds \le C \int_{I_b} f^{**}(s)^p \phi_1(s)\, ds, \quad f \in \mathfrak{M}_+(I_b). \tag{5.5}$$

Taking $f = \chi_{(0,t)}$, this reads

$$\int_{I_b} (S_\alpha \chi_{(0,t)})(s)^p \phi_2(s)\, ds \le C \int_0^t \chi_{(0,t)}^{**}(s)^p \phi_1(s)\, ds, \quad t \in I_b.$$

But,

$$\begin{aligned}\int_{I_b} (S_\alpha \chi_{(0,t)})(s)^p \phi_2(s)\, ds &= \int_0^t \phi_2(s)\, ds + t^{\frac{p}{\alpha}} \int_t^b \phi_2(s) s^{-\frac{p}{\alpha}}\, ds \\ &= \frac{p}{\alpha} \int_0^t s^{\frac{p}{\alpha}-1} \int_s^b \phi_2(y) y^{-\frac{p}{\alpha}}\, dy, \qquad t \in I_b.\end{aligned}$$

Further,

$$\chi_{(0,t)}^{**}(s) = \min\left[1, \tfrac{t}{s}\right],$$

so,

$$\begin{aligned}\int_{I_b} \chi_{(0,t)}^{**}(s)^p \phi_1(s)\, ds &= \int_0^t \phi_1(s)\, ds + t^p \int_t^b \phi_1(s) s^{-p}\, ds \\ &= p \int_0^t s^{p-1} \int_s^b \phi_1(y) y^{-p}\, dy\, ds.\end{aligned}$$

Therefore, when (5.1) holds, we get (5.2) with $K = C\alpha$.

Suppose, next, that (5.2) holds. We claim

$$(S_\alpha f(s))^p \le \tfrac{p}{\alpha} s^{-\frac{p}{\alpha}} \int_0^s f^*(y)^p y^{\frac{p}{\alpha}-1}\, dy, \quad f \in \mathfrak{M}_+(I_b),\ s \in I_b.$$

Indeed, for each z, $0 < z \le s$,

$$\int_0^s f^*(y)^p y^{\frac{p}{\alpha}-1}\,dy \ge \int_0^z f^*(y)^p y^{\frac{p}{\alpha}-1}\,dy \ge f^*(z)^p \int_0^z y^{\frac{p}{\alpha}-1}\,dy = \tfrac{\alpha}{p} z^{\frac{p}{\alpha}} f^*(z)^p,$$

and hence

$$\tfrac{p}{\alpha} s^{-\frac{p}{\alpha}} \int_0^s f^*(y)^p y^{\frac{p}{\alpha}-1}\,dy \ge s^{-\frac{p}{\alpha}} \sup_{0<z\le s} z^{\frac{p}{\alpha}} f^*(z)^p = (S_\alpha f)(s)^p.$$

Thus, (5.1) will follow once we show

$$\int_{I_b} s^{-\frac{p}{\alpha}} \int_0^s f^*(y)^p y^{\frac{p}{\alpha}-1}\,dy \phi_2(s)\,ds \le C \int_{I_b} f^{**}(s)^p \phi_1(s)\,ds, \quad f \in \mathfrak{M}_+(I_b).$$

Interchanging the order of integration on the left hand side this becomes

$$\int_{I_b} f^*(y)^p y^{\frac{p}{\alpha}-1} \int_y^b s^{-\frac{p}{\alpha}} \phi_2(s)\,ds \le C \int_{I_b} f^{**}(s)^p \phi_1(s)\,ds, \quad f \in \mathfrak{M}_+(I_b).$$

According to [**17**, Theorem 3.2],

$$\int_{I_b} f^*(y)^p y^{p-1} \int_y^b \phi_1(s) s^{-p}\,ds\,dy \le C \int_{I_b} f^{**}(s)^p \phi_1(s)\,ds,$$

so (5.5) would be a consequence of

$$\int_{I_b} f^*(y)^p y^{\frac{p}{\alpha}-1} \int_y^b \phi_2(s) s^{-\frac{p}{\alpha}}\,ds\,dy \le C \int_{I_b} f^*(y)^p y^{p-1} \int_y^b \phi_1(s) s^{-p}\,ds\,dy,$$
$$f \in \mathfrak{M}_+(I_b).$$

Finally, [**18**, Remark (i), p. 148] asserts that this last inequality holds if and only if (5.2) is satisfied.

It remains to prove the equivalence of (5.3) and (5.4). To begin, (5.3) ensures that, for every $t \in I_b$,

$$\begin{aligned}\int_{I_b} (T_\beta \chi_{(0,t)})^{**}(s)^p \phi_2(s)\,ds &\le C \int_{I_b} \chi_{(0,t)}^{**}(s)^p \phi_1(s)\,ds \\ &= Cp \int_0^t s^{p-1} \int_s^b \phi_1(y) y^{-p}\,dy\,ds.\end{aligned}$$

But,

$$(T_\beta \chi_{(0,t)})(s) = \left(\frac{t}{s}\right)^{\frac{1}{\beta}} \chi_{(0,t)}(s), \quad t \in I_b,$$

so,

$$\begin{aligned}(T_\beta \chi_{(0,t)})^{**}(s) &= s^{-1}\int_0^s \left(\tfrac{t}{s}\right)^{\frac{1}{\beta}}\, dy\chi_{(0,t)}(s) + s^{-1}\int_0^s \left(\tfrac{t}{y}\right)^{\frac{1}{\beta}}\, dy\chi_{(t,b)}(s)\\ &= \tfrac{\beta}{\beta-1}\left[\left(\tfrac{t}{s}\right)^{\frac{1}{\beta}}\chi_{(0,t)}(s) + \tfrac{t}{s}\chi_{(t,b)}(s)\right]\end{aligned}$$

and, therefore,

$$\begin{aligned}\int_{I_b}(T_\beta\chi_{(0,t)})^{**}(s)\phi_2(s)\,ds &= \left(\tfrac{\beta}{\beta-1}\right)^p\left[t^{\frac{p}{\beta}}\int_0^t \phi_2(s)s^{-\frac{p}{\beta}}\,ds + t^p\int_t^b \phi_2(s)s^{-p}\,ds\right]\\ &= \frac{p}{\beta'}\left(\tfrac{\beta}{\beta-1}\right)^p t^{\frac{p}{\beta}}\int_0^t s^{\frac{p}{\beta'}-1}\int_s^b \phi_2(y)y^{-p}\,dy\,ds.\end{aligned}$$

Thus, (5.3) implies (5.4).

Conversely, assume (5.4) is satisfied. By [**12**, Theorem 3.8], we have

$$(T_\beta f)^{**}(t) \le 2(T_\beta f^{**})(t), \quad t \in I_b.$$

Therefore, in order to obtain (5.3), we need only show

$$\int_{I_b}(T_\beta f^{**})(t)^p\phi_2(t)\,dt \le C\int_{I_b} f^{**}(s)^p\phi_1(s)\,ds, \quad f \in \mathfrak{M}_+(I_b).$$

In the remainder of the proof we suppose $b < \infty$; the argument in case $b = \infty$ is even simpler.

Now, an elementary calculation yields

$$(T_\beta f^{**})(t)^p \le \begin{cases} 2^{\frac{p}{\beta}}t^{-\frac{p}{\beta}}\sup_{t\le s<\frac{b}{2}} s^{\frac{p}{\beta}}f^{**}(s)^p, & 0<t<\frac{b}{2},\\ 2^{\frac{p}{\beta}}f^{**}(\frac{b}{2})^p, & \frac{b}{2}\le t<b.\end{cases}$$

Further, when $0 < t < \frac{b}{2}$, one has

$$\sup_{t\le y<\frac{b}{2}} y^{\frac{p}{\beta}}f^{**}(y)^p \le 2^{p+1}\int_t^b f^{**}(s)^p s^{\frac{p}{\beta}-1}\,ds, \quad t \in I_{\frac{b}{2}},$$

since, given $t < y < \frac{b}{2}$,

$$\begin{aligned}\int_t^b f^{**}(s)^p s^{\frac{p}{\beta}-1}\,ds &\ge \int_y^{2y} f^{**}(s)^p s^{\frac{p}{\beta}-1}\,ds\\ &\ge f^{**}(2y)^p y^{\frac{p}{\beta}}\log 2 \ge (2^{-p}\log 2)y^{\frac{p}{\beta}}f^{**}(y).\end{aligned}$$

Thus,

$$\int_0^{\frac{b}{2}} (T_\beta f^{**})(t)^p \phi_2(t)\, dt \le 2^{p+1} \int_0^b t^{\frac{p}{\beta}} \int_t^b f^{**}(s)^p s^{\frac{p}{\beta}-1}\, ds \phi_2(t)\, dt$$
$$= 2^{p+1} \int_{I_b} f^{**}(s)^p s^{\frac{p}{\beta}-1} \int_0^s \phi_2(t) t^{\frac{p}{\beta}}\, dt\, ds.$$

We conclude

$$\int_0^{\frac{b}{2}} (T_\beta f^{**})(t)^p \phi_2(t)\, dt \le C \int_{I_b} f^{**}(s)^p \phi_1(s)\, ds,$$

provided

$$\int_{I_b} f^{**}(s)^p s^{\frac{p}{\beta}-1} \int_0^s \phi_2(y) y^{\frac{p}{\beta}}\, dy\, ds \le 2^{-p-1} C \int_{I_b} f^{**}(s)^p \phi_1(s)\, ds,$$

which according to [**19**, Theorem 3.3] is equivalent to (5.4).

Again, taking $t = \frac{b}{2}$ in (5.4), there follows

$$\begin{aligned} 2^{-p} \frac{\beta'}{p} &\le \left(\tfrac{b}{2}\right)^{\frac{p}{\beta}} \int_0^{\frac{b}{2}} s^{\frac{p}{\beta'}-1}\, ds \int_{\frac{b}{2}}^b \phi_2(y) y^{-p}\, dy \\ &\le \left(\tfrac{b}{2}\right)^{\frac{p}{\beta}} \int_0^{\frac{b}{2}} s^{\frac{p}{\beta'}-1} \int_s^b \phi_2(y) y^{-p}\, dy\, ds \\ &\le K \int_0^{\frac{b}{2}} s^{p-1} \int_s^b \phi_1(y) y^{-p}\, dy\, ds \\ &\le K \left[\int_0^{\frac{b}{2}} \phi_1(s)\, ds + \int_{\frac{b}{2}}^b \phi_1(s)\, ds \right]. \end{aligned}$$

Altogether, then, with $C = \frac{p}{\beta'} 2^{p[1+\frac{1}{\beta}]} K$,

$$\begin{aligned} \int_{\frac{b}{2}}^b (T_\beta f^{**})(s)^p \phi_2(s)\, ds &\le 2^{\frac{p}{\beta}} \int_{\frac{b}{2}}^b f^{**}(\tfrac{b}{2})^p \phi_2(s)\, ds \\ &\le C \left[\int_0^{\frac{b}{2}} f^{**}(\tfrac{b}{2})^p \phi_1(s)\, ds + \int_{\frac{b}{2}}^b f^{**}(\tfrac{b}{2})^p \phi_1(s)\, ds \right] \\ &\le 2^p C \int_{I_b} f^{**}(s) \phi_1(s)\, ds. \end{aligned}$$

This completes the proof. □

6. Applications and Examples

The Marcinkiewicz space, $M(\alpha, X)$, is the Köthe dual of the original Lorentz space, $\Lambda(\alpha', X)$, $\alpha' = \frac{\alpha}{\alpha-1}$. Accordingly, one obtains

Theorem 6.1. *Let (X_1, μ_1) and (X_2, μ_2) be σ-finite measure spaces for which $\mu_1(X_1) = \mu_2(X_2) = b$. Suppose the linear operator T satisfies*

$$T\colon \Lambda(\alpha, X_1) \to \Lambda(\alpha, X_2) \text{ and } T\colon \Lambda(\beta, X_1) \to \Lambda(\beta, X_2),$$

for indices α and β, with $1 < \alpha < \beta < \infty$. Define the r.i. norms, ϱ_i, on $\mathfrak{M}(X_i)$ in terms of the r.i. norms $\bar{\varrho}_i$ on $\mathfrak{M}(I_b)$ by

$$\varrho_i(f) := \bar{\varrho}_i(f^*)$$

and suppose

$$\Lambda(\alpha, X_i) \cap \Lambda(\beta, X_i) \subset L_{\varrho_i}(X_i) \subset \Lambda(\alpha, X_i) + \Lambda(\beta, X_i), \quad i = 1, 2.$$

Then,

$$T\colon L_{\varrho_1}(X_1) \to L_{\varrho_2}(X_2),$$

whenever

$$S_{\alpha'} : L_{\bar{\varrho}_2'}(I_b) \to L_{\bar{\varrho}_1'}(I_b) \text{ and } T_{\beta'} : L_{\bar{\varrho}_2'}(I_b) \to L_{\bar{\varrho}_1'}(I_b), \tag{6.1}$$

where $\alpha' = \frac{\alpha}{\alpha-1}$, $\beta' = \frac{\beta}{\beta-1}$.

Theorem A'. *Let (X_1, μ_1) and (X_2, μ_2) be σ-finite measure spaces with $\mu_1(X_1) = \mu_2(X_2) = b$. Fix the indices α and β, $1 < \alpha < \beta < \infty$. Suppose A_1 and A_2 are Young functions satisfying*

$$\Lambda(\alpha, X_i) \cap \Lambda(\beta, X_i) \subset L_{A_i}(X_i) \subset \Lambda(\alpha, X_i) + \Lambda(\beta, X_i), \quad i = 1, 2.$$

Then, given any linear operator T such that

$$T\colon \Lambda(\alpha, X_1) \to \Lambda(\alpha, X_2) \text{ and } T\colon \Lambda(\beta, X_1) \to \Lambda(\beta, X_2),$$

one has

$$T\colon L_{A_1}(X_1) \to L_{A_2}(X_2),$$

whenever

$$\begin{aligned} &\int_{b^{-1}}^{t} \frac{\tilde{A}_1(s)}{s^{\alpha'+1}}\, ds \le \frac{\tilde{A}_2(Kt)}{t^{\alpha'}}, \\ &\int_{t}^{\infty} \frac{\tilde{A}_1(s)}{s^{\beta'+1}}\, ds \le \frac{\tilde{A}_2(Kt)}{t^{\beta'}}, \end{aligned} \tag{6.2}$$

in which $\tilde{A}_i$ is the Young function complementary to A_i, $i = 1, 2$, $\alpha' = \frac{\alpha}{\alpha-1}$, $\beta' = \frac{\beta}{\beta-1}$ and the constant $K > 0$ is independent of $t > b^{-1}$.

In particular, the first condition in (6.2) *is necessary and sufficient in order that*

$$S_{\alpha'}\colon L_{\tilde{A}_2}(I_b) \to L_{\tilde{A}_1}(I_b),$$

while the second condition is necessary and sufficient for

$$T_{\beta'}\colon L_{\tilde{A}_2}(I_b) \to L_{\tilde{A}_1}(I_b).$$

Theorem B'. *Let* (X_1, μ_1) *and* (X_2, μ_2) *be* σ*-finite measure spaces, with* $\mu_1(X_1) = \mu_2(X_2) = b$. *Fix the indices* α *and* β *satisfying* $1 < \alpha < \beta < \infty$. *Suppose the index* p, $1 < p < \infty$, *and the nontrivial weight functions,* ϕ_1 *and* ϕ_2, *are such that*

$$\Lambda(\alpha, X_i) \cap \Lambda(\beta, X_i) \subset \Gamma_{p,\phi_i}(X_i) \subset \Lambda(\alpha, X_i) + \Lambda(\beta, X_i), \quad i = 1, 2.$$

Assume, in addition,

$$\int_0^1 \phi_i(t) t^{-p}\, dt = \int_1^\infty \phi_i(t)\, dt = \infty, \quad i = 1, 2,$$

if $b = \infty$. *Then, given any linear operator* T *for which*

$$T\colon \Lambda(\alpha, X_1) \to \Lambda(\alpha, X_2) \text{ and } T\colon \Lambda(\beta, X_1) \to \Lambda(\beta, X_2),$$

one has

$$T\colon \Gamma_{p,\phi_1}(X_1) \to \Gamma_{p,\phi_2}(X_2),$$

whenever

$$\begin{aligned} &\int_0^t \psi_1(s)\, ds + t^{\frac{p'}{\alpha'}} \int_t^b \psi_1(s) s^{-\frac{p'}{\alpha'}}\, ds \le K \int_0^t s^{p'-1} \int_s^b \psi_2(y) y^{-p'}\, dy\, ds, \\ &t^{\frac{p'}{\beta'}} \int_0^t \psi_1(s) s^{-\frac{p'}{\beta'}}\, ds + t^{p'} \int_t^b \psi_1(y) y^{-p'}\, dy \le K \int_0^t s^{p'-1} \int_s^b \psi_2(y) y^{-p'}\, dy\, ds \end{aligned} \tag{6.3}$$

in which

$$\psi_i(t) := \frac{t^{p'+p-1} \int_0^t \phi_i(s)\, ds + \int_t^b \phi_i(s) s^{-p}\, ds}{\left(\int_0^t \phi_i(s)\, ds + t^p \int_t^b \phi_i(s) s^{-p}\, ds\right)^{p'+1}}, \quad i = 1, 2,$$

$p' = \frac{p}{p-1}$, $\alpha' = \frac{\alpha}{\alpha-1}$, $\beta' = \frac{\beta}{\beta-1}$ *and the constant* $K > 0$ *is independent of* $t \in I_b$.

In particular, the first condition in (6.3) *is necessary and sufficient in order that*

$$S_\alpha\colon \Gamma_{p',\psi_2}(I_b) \to \Gamma_{p',\psi_1}(I_b),$$

while the second one is necessary and sufficient for

$$T_\beta\colon \Gamma_{p',\psi_2}(I_b) \to \Gamma_{p',\psi_1}(I_b).$$

We next consider what happens when $X_1 = X_2 = X$, $\mu_1 = \mu_2 = \mu$ and $\varrho_1 = \varrho_2 = \varrho$ in our theorems.

Theorem 6.2. *Let (X,μ) be a σ-finite measure space, with $\mu(X)=b$. Define the r.i. norm ϱ on $\mathfrak{M}(X)$ in terms of the r.i. norm $\bar{\varrho}$ on $\mathfrak{M}(I_b)$ by*

$$\varrho(f) := \bar{\varrho}(f^*).$$

Fix indices α and β satisfying $1<\alpha<\beta<\infty$. Then, the conditions

$$S_\alpha\colon L_{\bar{\varrho}}(I_b) \to L_{\bar{\varrho}}(I_b) \text{ and } T_\beta\colon L_{\bar{\varrho}}(I_b) \to L_{\bar{\varrho}}(I_b) \tag{6.4}$$

are equivalent to $L_\varrho(X)$ being an interpolation space between $M(\alpha,X)$ and $M(\beta,X)$. Again, the conditions

$$S_{\alpha'}\colon L_{\bar{\varrho}'}(I_b) \to L_{\bar{\varrho}'}(I_b) \text{ and } T_{\beta'}\colon L_{\bar{\varrho}'}(I_b) \to L_{\bar{\varrho}'}(I_b)$$

are equivalent to $L_\varrho(X)$ being an interpolation space between $\Lambda(\alpha,X)$ and $\Lambda(\beta,X)$; as usual, $\alpha'=\frac{\alpha}{\alpha-1}$ and $\beta'=\frac{\beta}{\beta-1}$.

Proof: Taking $X_1=X_2=X$, $\mu_1=\mu_2=\mu$ and $\varrho_1=\varrho_2=\varrho$ in the Dmitriev-Kreĭn-Peetre Theorem, it is seen that (6.4) implies $L_\varrho(X)$ is an interpolation space between $M(\alpha,X)$ and $M(\beta,X)$. As for the converse, we observe that, according to [**13**, Theorem 2.3], the conditions (6.4) are equivalent to $L_{\varrho'}(X)$ being an interpolation space for both $\big(L_1(X),\Lambda(\alpha',X)\big)$ and $\big(\Lambda(\beta',X),L_\infty(X)\big)$ which, in turn, amounts to $L_\varrho(X)$ being an interpolation space for $\big(L_1(X),M(\beta,X)\big)$ and $\big(M(\alpha,X),L_\infty(X)\big)$. But, $M(\alpha,X)$ and $M(\beta,X)$ are interpolation spaces for the couples $\big(L_1(X),M(\beta,X)\big)$ and $\big(M(\alpha,X),L_\infty(X)\big)$, respectively. This means that whenever $L_\varrho(X)$ is an interpolation space for $\big(M(\alpha,X), M(\beta,X)\big)$ it is also an interpolation space for $\big(L_1(X),M(\beta,X)\big)$ and $\big(M(\alpha,X),L_\infty(X)\big)$ and hence the conditions (6.4) hold.

The second assertion follows by an argument similar to the one above. □

Theorem 6.3. *Fix indices α and β satisfying $1<\alpha<\beta<\infty$. Let ϱ be an Orlicz norm or a Lorentz gamma norm on $\mathfrak{M}(I_b)$ having Boyd indices $i_{\bar{\varrho}}$ and $I_{\bar{\varrho}}$. Then, the following are equivalent:*

(i) $S_\alpha\colon L_{\bar{\varrho}}(I_b) \to L_{\bar{\varrho}}(I_b)$;
(ii) $t^{\frac{1}{\alpha}}\,\bar{\varrho}(s^{-\frac{1}{\alpha}}\chi_{(t,b)}(s)) \le C\bar{\varrho}(\chi_{(0,t)})$, *with $C>0$ independent of $t\in I_b$;*
(iii) $\alpha < i_{\bar{\varrho}}$.

Again, the following are equivalent:

(iv) $T_\beta\colon L_{\bar{\varrho}}(I_b) \to L_{\bar{\varrho}}(I_b)$;
(v) $t^{\frac{1}{\beta}}\,\bar{\varrho}(s^{-\frac{1}{\beta}}\chi_{(0,t)}(s)) \le C\bar{\varrho}(\chi_{(0,t)})$, *with $C>0$ independent of $t\in I_b$;*
(vi) $I_{\bar{\varrho}} < \beta$.

Proof: The equivalence of (i) and (ii) and of (iv) and (v) has been established in Theorems A and B. Again, the estimate

$$\begin{aligned}(S_\alpha f)(t) &= t^{-\frac{1}{\alpha}} \sup_{0<s\le t} s^{\frac{1}{\alpha}} f^*(s)\\ &\le t^{-\frac{1}{\alpha}} \sup_{0<s\le t} s^{\frac{1}{\alpha}-1} \int_0^s f^*(y)\,dy\\ &\le t^{-\frac{1}{\alpha}} \sup_{0<s\le t} \int_0^s f^*(y) y^{\frac{1}{\alpha}-1}\,dy\\ &= (P_\alpha f^*)(t),\end{aligned}$$

together with Theorem 2.2, give (i) from (iii). A similar argument shows (vi) entails (iv).

Now, in [**16**, Theorem 11.8, pp. 90–91], the assertions (i) implies (iii) and (iv) implies (vi) are proved in the case ϱ is an Orlicz norm. The argument used is quite general and, indeed, works for Lorentz gamma norms as well, in view of Theorem 2.2. □

Remark 6.4. When the norm ϱ in Theorem 6.3 is an Orlicz norm, ϱ_A, (ii) and (v), together, are equivalent to the conditions (1.1) from the interpolation theorem of Zygmund. In view of (iii) and (vi), and [**3**, Theorem 1], the weak-type assumptions of that theorem can be replaced by the less demanding restricted weak-type requirements

$$T\colon \Lambda(\alpha, X) \to M(\alpha, X) \text{ and } T\colon \Lambda(\beta, X) \to M(\beta, X).$$

The next example shows the conditions (1.2) and (6.1) in Theorems 1.1 and 6.1, respectively, are not necessary to guarantee the conclusions of those theorems.

Example 6.5. Fix β, $1 < \beta < \infty$. One readily verifies that

$$T_\beta\colon \Lambda(\alpha, q, I) \to \Lambda(\alpha, q, I), \quad I = (0,1),$$

if and only if $1 < \alpha < \beta$ and $1 \le q \le \infty$ or $\alpha = \beta$ and $q = \infty$, in which case $\Lambda(\beta, q, I) = M(\beta, I)$. Again,

$$S_{\beta'}\colon \Lambda(\alpha', q, I) \to \Lambda(\alpha', q, I)$$

if and only if $1 < \alpha < \beta$ and $1 \le q \le \infty$ or $\alpha = \beta$ and $q = \infty$, when $\Lambda(\beta', q, I) = M(\beta', I)$.

However, for the linear operator T given by

$$f \to t^{-\frac{1}{\beta}}\left(\log \tfrac{1}{t}\right)^{-\frac{1}{\gamma}} \int_0^1 f(s)\,ds, \quad 1 < \gamma < \infty,$$

with associate operator T' sending

$$g \to \int_0^1 g(t) t^{-\frac{1}{\beta}}\left(\log \tfrac{1}{t}\right)^{-\frac{1}{\gamma}}\,dt,$$

one has

$$T\colon \Lambda(\beta, q, I) \to \Lambda(\beta, q, I)$$

if and only if $\gamma < q \leq \infty$, and

$$T'\colon \Lambda(\beta', q, I) \to \Lambda(\beta', q, I)$$

if and only if $1 \leq q < \gamma'$.

Our goal now is to use results already obtained to study operators, T, satisfying more general conditions than those considered so far, namely,

$$T\colon M(\alpha_1, X_1) \to M(\alpha_2, X_2) \text{ and } T\colon M(\beta_1, X_1) \to M(\beta_2, X_2)$$

where $1 < \alpha_i < \beta_i < \infty$, $i = 1, 2$. In particular, we seek an explicit connection between norms ϱ_1 and ϱ_2 in an inequality of the form

$$\varrho_2(Tf) \leq C\varrho_1(f).$$

This connection is supplied by Theorem 2.4.

Theorem 6.6. *Let (X_1, μ_1) and (X_2, μ_2) be σ-finite measure spaces with $\mu_1(X) = \mu_2(X) = b$. For $i = 1, 2$, fix indices α_i and β_i satisfying $1 < \alpha_i < \beta_i < \infty$. Given an r.i. norm, $\bar{\varrho}$, on $\mathfrak{M}(I_b)$ define the r.i. functionals*

$$\varrho_1(g) := \bar{\varrho}\Big(t^{-1} \sup_{0<s\leq t^{\gamma_1}} s^{\frac{1}{\alpha_1}} g^*(s) + \sup_{t^{\gamma_1}<s<b} s^{\frac{1}{\beta_1}} g^*(s)\Big),$$
$$g \in \mathfrak{M}(X_1), \quad \frac{1}{\gamma_1} = \frac{1}{\alpha_1} - \frac{1}{\beta_1},$$

and

$$\varrho_2(h) := \bar{\varrho}\Big(t^{-1} \sup_{0<s\leq t^{\gamma_2}} s^{\frac{1}{\alpha_2}} h^*(s) + \sup_{t^{\gamma_2}<s<b} s^{\frac{1}{\beta_2}} h^*(s)\Big),$$
$$h \in \mathfrak{M}(X_2), \quad \frac{1}{\gamma_2} = \frac{1}{\alpha_2} - \frac{1}{\beta_2}.$$

Then, ϱ_1 and ϱ_2 are equivalent to r.i. norms on $\mathfrak{M}(X_1)$ and $\mathfrak{M}(X_2)$, respectively. Moreover, if the linear operator T satisfies

$$T\colon M(\alpha_i, X_i) \to M(\beta_i, X_i), \quad i = 1, 2,$$

and if $i_{\bar{\varrho}} > 1$, one has

$$\varrho_2(Tf) \le C\varrho_1(f), \tag{6.5}$$

where $C > 0$ is independent of $f \in \mathfrak{M}(X_1)$, $\varrho_1(f) < \infty$.

Proof: Theorem 2.4 ensures the inequality

$$\begin{aligned} \bar{\varrho}\Big(k\big(t, (Tf)^*; M(\alpha_2, X_2), M(\beta_2, X_2)\big)\Big) \\ \le C\bar{\varrho}\Big(k\big(t, f^*; M(\alpha_1, X_1), M(\beta_1, X_1)\big)\Big) \end{aligned} \tag{6.6}$$

in which $C > 0$ is independent of $f \in \mathfrak{M}(X_1)$. Again, $i_{\bar{\varrho}} > 1$ means

$$\bar{\varrho}(Pg^*) \approx \bar{\varrho}(g^*), \quad g \in \mathfrak{M}(I),$$

so, (6.6) implies (6.5). □

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