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DYNAMICS OF (PSEUDO) AUTOMORPHISMS OF 3-SPACE: PERIODICITY VERSUS POSITIVE ENTROPY

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Abstract: We study the iteration of the family of maps given by 3-step linear fractional recurrences. This family was studied earlier from the point of view of finding periodicities. In this paper we finish that study by determining all possible periods within this family. The novelty of our approach is that we apply the methods of complex dynamical systems. This leads to two classes of interesting pseudo automorphisms of infinite order. One of the classes consists of completely integrable maps. The other class consists of maps of positive entropy which have an invariant family of $K3$ surfaces.

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0. Introduction

We consider the family of birational maps of 3-space which may be written in affine coordinates as

$$f_{\alpha,\beta}\colon (x_1,x_2,x_3)\mapsto \left(x_2,x_3,\frac{\alpha_0+\alpha_1x_1+\alpha_2x_2+\alpha_3x_3}{\beta_0+\beta_1x_1+\beta_2x_2+\beta_3x_3}\right). \tag{0.1}$$

The algebraic iterates $f^n_{\alpha,\beta}:=f_{\alpha,\beta}\circ\cdots\circ f_{\alpha,\beta}$ are rational maps for all $n\in\mathbf{Z}$. Here we study the dynamics of $f=f_{\alpha,\beta}$, by which we mean the behavior of f^n as $n\to\pm\infty$. We have invertible dynamics since f has a rational inverse, but it does not behave like a diffeomorphism (or even a homeomorphism). There are two difficulties if we want to regard f as a mapping of points. First, there is the set of indeterminacy $\mathcal{I}(f)$; f blows up each point of $\mathcal{I}(f)$ to a variety of positive dimension. Second, there can be hypersurfaces E which are exceptional, in the sense that the codimension of $f(E-\mathcal{I}(f))$ is at least 2. We will say that f is a pseudo-automorphism if neither f nor f^{-1} has an exceptional hypersurface. In dimension 2, every pseudo-automorphism is in fact an automorphism.

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However, for pseudo-automorphisms, indeterminate behaviors are possible in higher dimension which have no analogue in dimension 2.

Given a rational map $f\colon X \dashrightarrow X$ there is a well-defined pullback map on cohomology, $f^*\colon H^*(X) \to H^*(X)$. Passage to cohomology, however, may not be compatible with iteration because the identity $(f^*)^n = (f^n)^*$ may not be valid. Given a birational map f in dimension 2, Diller and Favre [**DiF**] showed that there is a new manifold $\pi\colon Y \to X$ such that the iterates of the induced map f_Y behave naturally on cohomology, in the sense that $(f_Y^*)^n = (f_Y^n)^*$. In dimension greater than 2, however, no such theorem is known.

Given a rational map of $\mathbf{P}^n$ we may consider modifications $\pi\colon X \to \mathbf{P}^n$, where π is a morphism which is birational. This induces a rational map $f_X := \pi^{-1} \circ f \circ \pi$ of X, which might have pointwise properties which are different from those of the original f. If f_X is a pseudo-automorphism, then f_X acts naturally on $H^{1,1}(X)$. The exponential rate of growth of f^n on $H^{p,p}$: $\delta_p(f) := \lim_{n\to\infty} ||f^{n*}|_{H^{p,p}(X)}||^{1/n}$ is known as the p^{th} dynamical degree and is a birational invariant (see [**DS**]).

Within the family (0.1) we find the first known examples of pseudo-automorphisms of positive entropy on blowups of $\mathbf{P}^3$:

Theorem 1. *Suppose that $\alpha = (a, 0, \omega, 1)$ and $\beta = (0, 1, 0, 0)$ where $a \in \mathbf{C} \setminus \{0\}$ and ω is a non-real cube root of the unity. Then there is a modification $\pi\colon Z \to \mathbf{P}^3$ such that f_Z is a pseudo-automorphism. The dynamical degrees $\delta_1(f) = \delta_2(f) \cong 1.28064 > 1$ are equal and are given by the largest root of $t^8 - t^5 - t^4 - t^3 + 1$. The entropy of f_Z is the logarithm of the dynamical degree and is thus positive.*

A mapping f is said to be reversible if it is conjugate to f^{-1}. Many maps that arise in mathematical physics are reversible because the relevant physical laws are invariant under time reversal. For the mappings in Theorems 1 and 3, f is reversible on the level of cohomology: f_Z^* is conjugate to $(f_Z^{-1})^* = (f_Z^*)^{-1}$. The identity $\delta_1(f) = \delta_2(f)$ for such maps is a consequence of the duality between $H^{1,1}$ and $H^{2,2}$, so they are not cohomologically hyperbolic in the terminology of [**G2**].

Theorem 2. *For the mappings in Theorem 1, there is a 1-parameter family of surfaces $S_c \subset Z$, $c \in \mathbf{C}$ which have the invariance $fS_c = S_{\omega c}$. For generic c, S_c is $K3$, and the restriction $f^3|_{S_c}$ is an automorphism. For generic c and c', the surfaces S_c and $S_{c'}$ are biholomorphically inequivalent, and the automorphisms $f^3|_{S_c}$ and $f^3|_{S_{c'}}$ are not smoothly conjugate.*

The surface S_0 is invariant, and the restriction f_{S_0} is an automorphism which has the same entropy as f. This is smaller than the entropy of the automorphism constructed in [**M2**, Theorem 1.2] and is thus the smallest known entropy for a *projective* $K3$ surface automorphism.

Closely related to the dynamics of f_Z is the Green current, a $(1,1)$-current T^+ which is expanded by f_Z^*, and a current T^- for f_Z^{-1}. The existence of the Green current is given by Diller and Guedj [**DG**] in the case where the expanded cohomology class is nef. For our case, we use a result of Bayraktar [**Ba**]. In §7 we obtain T^+, as well as the invariant $(2,2)$-current $T^+ \wedge T^-$. The slices of $T^\pm$ and $T^+ \wedge T^-$ on the surfaces S_c give the expanded/contracted currents, as well as the unique invariant measure of maximal entropy for the automorphism $f|_{S_c}$.

The following mappings have quadratic degree growth and complete integrability:

Theorem 3. *Suppose that* $\beta = (0,1,0,0)$ *and either* $\alpha = (0,0,\omega,1)$ *or* $\alpha = (a,0,1,1)$ *where* $a \in \mathbf{C} \setminus \{1\}$*,* $\omega \neq 1$*, and* $\omega^3 = 1$*. Then the degree of* f^n *grows quadratically in* n*. Further, there is a modification* $\pi\colon Z \to \mathbf{P}^3$ *such that* f_Z *is a pseudo-automorphism. There is a two-parameter family of surfaces* S_c*,* $c = (c_1, c_2) \in \mathbf{C}^2$ *which are invariant under* f^3*. For generic* c *and* c'*,* S_c *is a smooth* $K3$ *surface, and* $S_c \cap S_{c'}$ *is a smooth elliptic curve.*

For each of these maps, the family of invariant $K3$ surfaces becomes singular at an invariant 8-cycle $\mathcal{R}$ of rational surfaces (see (7.2)). We show that the restriction $f|_{\mathcal{R}}$ is not birationally conjugate to a surface automorphism: see Appendix C for the maps in Theorem 1 and Proposition 8.2 for the maps in Theorem 3. By Corollary 1.6, then, we have:

Theorem 4. *Let* f *be a map from Theorems 1 and 3. If* $a \neq 1$*, then* f *is not birationally conjugate to an automorphism.*

We note that for birational surface maps, the degree growth of the iterates determines whether the map is birationally conjugate to an automorphism: This occurs if and only if either (i) the degrees are bounded or degree growth is quadratic (see [**DiF**]), or (ii) if the dynamical degree is a Salem number (see [**BC**]). Theorem 4 shows that this result does not hold in dimension 3.

We will also determine which mappings $f_{\alpha,\beta}$ are periodic, or finite order, in the sense that $f^p = \mathrm{id}$ for some $p > 0$. In contrast to Theorem 4, it was shown by de Fernex and Ein [**dFE**] that if f is a rational map of finite order, then there is a modification f_X as above, which is an automorphism of X. If f_X is periodic, then f_X^* will also be periodic.

In (4.1) and (4.2) we identify conditions which are necessary for f to be periodic and are sufficient for the existence of a space $Z = Z_{\alpha,\beta}$ such that f_Z is a pseudo-automorphism. We show that for a map in (0.1), if f_Z^* is periodic, then f also turns out to be periodic. The birational map (0.1) may also be considered as a 3-step linear fractional recurrence: given z_0, z_1, z_2, we define a sequence $\{z_n\}$ by

$$z_{n+3} = \frac{\alpha_0 + \alpha_1 z_n + \alpha_2 z_{n+1} + \alpha_3 z_{n+2}}{\beta_0 + \beta_1 z_n + \beta_2 z_{n+1} + \beta_3 z_{n+2}}. \tag{0.2}$$

The recurrence (0.2) is said to be periodic if the sequence $\{z_n\}$ is periodic for all choices of initial terms z_0, z_1 and z_2. Equivalently, $f^p_{\alpha,\beta} = \mathrm{id}$ for some p. For all $r > 0$ there are r-step recurrences of the form (0.2). In [**BK2**] we determined the possible periods for 2-step linear fractional recurrences. McMullen [**M1**] has explained the periods that arise by showing that the corresponding (2-dimensional) $f_{\alpha,\beta}$ represent certain Coxeter elements.

Here we determine all possible periods for 3-step recurrences (0.2). To rule out trivial cases, we assume that the coefficients satisfy (2.3), and we have:

Theorem 5. *The only nontrivial periods for* (0.2) *are* 8 *and* 12. *Each periodic recurrence is equivalent to one of the following:*

$$\text{(period 8)} \qquad z_{n+3} = \frac{1 + z_{n+1} + z_{n+2}}{z_n}, \quad z_{n+3} = \frac{-1 - z_{n+1} + z_{n+2}}{z_n},$$

$$\text{(period 12)} \qquad z_{n+3} = \frac{\eta/(1-\eta) + \eta z_{n+1} + z_{n+2}}{\eta^2 + z_n}, \quad \eta^3 = -1.$$

In the notation of (0.1), *the first case corresponds to* $\beta = (0,1,0,0)$, $\alpha = (\pm 1, 0, \pm 1, 1)$, *and the second case to* $\beta = (\eta^2, 1, 0, 0)$, $\alpha = (\eta/(1-\eta), 0, \eta, 1)$.

Each of these mappings has a different structure; these structures are described in Theorems 6.9 and 6.10. The first period 8 recurrence above was found by Lyness [**L**], and the second one was found by Csörnyei and Laczkovich [**CL**] (see also [**CGM, CGMs**]). We note that the period 12 recurrences are the case $k = 3$ of a general phenomenon exhibited in [**BK3**]: *For each k, there are k-step linear fractional recurrences with period $4k$.* There is a literature dealing with r step recurrences of the form (0.2). We refer to the books [**KoL**], [**KuL**], [**GL**], [**CaL**] and the extensive bibliographies they contain. That direction of research is largely concerned with the case where the structural parameters α, β, as well as the dynamical points, are real and positive. This avoids the difficulty that the denominator in (0.2) might vanish, causing the expression to be undefined; but the restriction to positive numbers leads to a subdivision into a large number of distinct cases to be treated separately.

In working with the family $f_{\alpha,\beta}$, we work with the pointwise iterates as much as possible, but this runs into difficulties if the orbit enters the indeterminacy locus, which is frequently the case with the orbits of "interesting" points. We can often deal with this by blowing up certain subsets. In this way we convert these subsets into hypersurfaces, and we then deal with the hypersurfaces by passing to f^* on Pic. This allows us to convert many difficulties with indeterminate orbits into more tractable problems of Linear Algebra. Such a procedure is useful for covering the whole parameter space, since it allows us to determine properties of f^* for all mappings in a large family.

This paper is organized as follows. §1 assembles some general information about rational maps and the geometry of blowing up. §2 gives the specific behaviors of the maps (0.1). It is evident, then, that there are two possibilities, defined by (3.1), which we call "critical" and "non-critical," and in §3 we show that any periodic map must be critical. We study the structure of general critical maps in §4. In Theorem 5.1 we show that if f is a critical map satisfying (5.1), then f_Z is a pseudo-automorphism. Pseudo-automorphisms are discussed in §5, together with the possibilities for the induced map f_Z^* on cohomology. In §6 we determine the periodic mappings and give the proof of Theorem 5. In §7 we give the proof of Theorems 1 and 2. At the end of §7 we present a different pseudo-automorphism with positive entropy; it has properties similar to those given in Theorems 1 and 2, but we do not discuss it in detail. The proof of Theorem 3 is given in §8.

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1. Rational maps

A rational map $f\colon \mathbf{P}^d \dashrightarrow \mathbf{P}^d$ is given by a $(d+1)$-tuple of homogeneous polynomials, all of the same degree: $f = [f_0 : \cdots : f_d]$. We may divide f by $\mathrm{g.c.d.}(f_0, \ldots, f_d)$ so that f_i's have no common polynomial factor. We define the degree of f, $\deg(f)$, to be the (common) degree of the f_j's. The *indeterminacy locus* of f is defined by

$$\mathcal{I}(f) = \{x \in \mathbf{P}^d : f_0(x) = \cdots = f_d(x) = 0\}$$

and is a subvariety of codimension at least 2, and f defines a holomorphic mapping $f\colon \mathbf{P}^d \setminus \mathcal{I}(f) \to \mathbf{P}^d$. If S is an irreducible subvariety of $\mathbf{P}^d$, and

$S \not\subset \mathcal{I}(f)$, we define the *strict transform*, written simply as $f(S)$, to be the closure of $f(S - \mathcal{I}(f))$. We say that an irreducible variety V is *exceptional* for a rational mapping f if $V \not\subset \mathcal{I}(f)$, and if the dimension of $f(V - \mathcal{I}(f))$ is strictly less than the dimension of V. Following [**DO**, p. 64], we say that $f\colon X \dashrightarrow Y$ is a *pseudo-isomorphism* if f is birational, and if neither f nor f^{-1} has an exceptional hypersurface. It follows that if f is a pseudo-isomorphism, then $f\colon X \setminus \mathcal{I}(f) \to Y \setminus \mathcal{I}(f^{-1})$ is biholomorphic. If $X = Y$, we say that f is a *pseudo-automorphism.*

Theorem 1.1. *If $f\colon X \dashrightarrow Y$ is a pseudo-isomorphism between 3-dimensional manifolds, then the indeterminacy locus has no isolated points.*

Proof: Suppose that there is an isolated point $p \in \mathcal{I}(f)$. Since f^{-1} has no exceptional hypersurfaces, f must blow p up to a curve $C' \subset Y$. Now we consider the behavior of f^{-1} on C'. We must have $C' \subset \mathcal{I}(f^{-1})$, for if f^{-1} is regular at a point $q \in C'$, then f^{-1} must map an open subset of C' to p. Thus the jacobian of f^{-1} must vanish at q. Since the jacobian vanishes on a hypersurface, f^{-1} would have an exceptional hypersurface containing q. Thus q must be indeterminate. Since the total transform of q under f^{-1} is given by $\bigcap_{\epsilon>0} \overline{(f^{-1}(B(q,\epsilon) - \mathcal{I}(f^{-1})))}$, it must be connected, and it must be a curve C containing p. But since p was an isolated point of $\mathcal{I}(f)$, there are nearby points $p' \in C - \mathcal{I}(f)$. Since f is regular at these points, it must map them to q, and thus f must have an exceptional hypersurface. By this contradiction, we see that $\mathcal{I}(f)$ has no isolated points. $\square$

For a rational map $f\colon X \dashrightarrow X$, we consider the iterates $f^j = f \circ \cdots \circ f$, $j > 0$. If Σ is an irreducible hypersurface, then $\Sigma \not\subset \mathcal{I}(f^j)$ for reasons of dimension, so we may consider the sequence of varieties $V_j := f^j(\Sigma)$, for $j > 0$. Since we will be interested in knowing to what extent the iterates of f behave like a pointwise-defined dynamical system, we note: *If $S \not\subset \mathcal{I}(g)$ is irreducible and if $g(S) \not\subset \mathcal{I}(f)$, then $S \not\subset \mathcal{I}(f \circ g)$, and* $f(g(S)) = (f \circ g)(S)$. We may also define f at points of indeterminacy. Let $\gamma_f = \{(x,y) \in (\mathbf{P}^d - \mathcal{I}) \times \mathbf{P}^d : y = f(x)\}$ denote the graph of f at its regular points, and we let Γ denote the closure of γ_f inside $\mathbf{P}^d \times \mathbf{P}^d$. It follows that Γ is an irreducible variety of dimension d, and there are holomorphic projections $\pi_j\colon \Gamma \to \mathbf{P}^d$, $j = 1, 2$, onto the first and second factors, respectively, and we have $f = \pi_2 \circ \pi_1^{-1}$ on $\mathbf{P}^d - \mathcal{I}$. For a point $p \in \mathbf{P}^d$, we define the *total transform* to be $f_*p := \pi_2(\pi_1^{-1}p)$, and then we define $f_*(S) := \bigcup_{p \in S} f_*p$. It is easily seen that we have: *If Σ is an irreducible hypersurface, then $f_*(g(\Sigma)) \supset (f \circ g)(\Sigma)$.*

Proposition 1.2. *Suppose that* $f: X \dashrightarrow X$ *is rational, and suppose that for each exceptional hypersurface* E *and for* $m > 0$*, we have* $f^m(E-\mathcal{I}) \not\subset \mathcal{I}$*. It follows that* $(f^*)^n = (f^n)^*$ *on* $H^{1,1}(X)$ *for* $n \geq 0$*.*

Proof: It is sufficient to show that $(f^*)^2 = (f^2)^*$ on $\mathrm{Pic}(X)$. If D is a divisor, then f^*D is the divisor on X which is the same as $f^{-1}D$ on $X - \mathcal{I}$. Since $\mathcal{I}$ has codimension at least 2, we also have $(f^2)^*D = f^*(f^*D)$ on $X - \mathcal{I} - f^{-1}(\mathcal{I})$. By our hypothesis $f^{-1}(\mathcal{I})$ has codimension at least 2. Thus we have $(f^2)^*D = (f^*)^2D$ on X. □

In a similar way, we may define $f^*: H^{p,q}(X) \to H^{p,q}(X)$. That is, if β is a (p,q) form on X, then the pullback $\pi_2^*\beta$ is a smooth form on Γ. We may let $[\pi_2^*\beta]$ denote the reinterpretation of the form as a current, and we may push it forward to obtain a current $f^*\beta = \pi_{1*}[\pi_2^*\beta]$ on X. This pulls smooth forms back to currents and is well defined at the level of cohomology classes. If $\alpha \in H^{p',q'}$ is an element of the dual cohomology group, then we have $\langle \alpha, f^*\beta \rangle = \langle \pi_1^*\alpha, \pi_2^*\beta \rangle$. Now if f is birational and $g = f^{-1}$, then

$$\langle g^*\alpha, \beta \rangle = \langle \pi_1^*\alpha, \pi_2^*\beta \rangle = \langle \alpha, f^*\beta \rangle. \tag{1.1}$$

If we have $(f^n)^* = (f^*)^n$ on $H^{p,q}$ for $n \geq 0$, then this gives us $(g^n)^* = (g^*)^n$ on $H^{p',q'}$. If f and g have no exceptional hypersurfaces, then, as in Proposition 1.2, we have $(fg)^* = g^*f^*$. Taking $g = f^{-1}$, we have

Proposition 1.3. *If* f *is a pseudo-automorphism, then* $(f^n)^* = (f^*)^n$ *on both* $H^{1,1}$ *and* $H^{d-1,d-1}$ *for all* $n \in \mathbf{Z}$*.*

From this we get the following:

Proposition 1.4. *Let* $f: X \dashrightarrow X$ *is a pseudo-automorphism on a d-dimensional manifold. If* f^* *and* $(f^*)^{-1}$ *are conjugate as linear transformations of* $H^{1,1}$*, then we have equality of the dynamical degrees* $\delta_1(f) = \delta_{d-1}(f)$*.*

Proof: If f^* and $(f^*)^{-1}$ are conjugate on $H^{1,1}$, then f^* on $H^{1,1}$ is conjugate to f^* on $H^{d-1,d-1}$. Since f is a pseudo-automorphism, the dynamical degree δ_1 is equal to the modulus of the largest eigenvalue of f^* on $H^{1,1}$. Similarly, δ_{d-1} is given by the largest eigenvalue of f^* acting on $H^{d-1,d-1}$. Since these are linearly conjugate, they eigenvalues are the same. □

Now let us define some specific blowup situations. This will serve to define the constructions we will use in the sequel, and it allows us to exhibit the models of indeterminate behavior that we will encounter.

Blowing up a point and a line which contains it. We use $(x_0, x_1, x_2) \mapsto [x_0 : x_1 : x_2 : 1]$ as local coordinates in a neighborhood of $e_3 := [0 : 0 : 0 : 1] \in \mathbf{P}^3$. Let X_1 be the space obtained by blowing up a point e_3 and we let E_3 denote the fiber over e_3. We may use

$$\pi_1 \colon X_1 \ni (s_0, s_1, \xi_2)_1 \mapsto [\xi_2 s_0 : \xi_2 s_1 : \xi_2 : 1] \in \mathbf{P}^3 \tag{1.2}$$

as a local coordinate system for a neighborhood of $E_3 \cap \{x_0 = x_1 = 0\}$ in X_1. It follows that the exceptional fiber $E_3 = \{\xi_2 = 0\}$ in this coordinate system.

Let $\Sigma_{01} = \{x_0 = x_1 = 0\} \subset \mathbf{P}^3$ denote the x_2-axis. The strict transform of Σ_{01} inside X_1 may be written as $\Sigma_{01} = \{s_0 = s_1 = 0\}$. Thus $\Sigma_{01} \cap E_3 = \{s_0 = s_1 = \xi_2 = 0\}$. Let X_2 be a complex manifold obtained by blowing up Σ_{01} in X_1. We can define a local coordinate system of X_2 via $\pi_2 \colon X_2 \ni (t_0, \eta_1, \xi_2)_2 \mapsto (t_0\eta_1, \eta_1, \xi_2)_1 \in X_1$. Thus $\pi_2 \circ \pi_1 \colon X_2 \to \mathbf{P}^3$ is given, in this coordinate neighborhood, by

$$\pi_1 \circ \pi_2 \colon X_2 \ni (t_0, \eta_1, \xi_2)_2 \mapsto [t_0\eta_1\xi_2 : \eta_1\xi_2 : \xi_2 : 1] \in \mathbf{P}^3. \tag{1.3}$$

The inverse of π_1 (resp. π_2) gives a model of indeterminate behavior that blows up the point $(0, 0, 0)$ (resp. the line $\{x_1 = x_2 = 0\}$) to a hyperplane:

$$\begin{aligned} \pi_1^{-1} &\colon (x_1, x_2, x_3) \mapsto (x_1/x_3, x_2/x_3, x_3), \\ \pi_2^{-1} &\colon (x_1, x_2, x_3) \mapsto (x_1/x_2, x_2, x_3). \end{aligned} \tag{1.4}$$

Blowing up two intersecting lines. Let $\pi_1 \colon Z_1 \to \mathbf{P}^3$ be the blowup of the x_1-axis $\Sigma_{02} = \{x_0 = x_2 = 0\} \subset \mathbf{P}^3$. We use local coordinate system in Z_1

$$\pi_1 \colon Z_1 \ni (\xi, x, s)_{Z1} \mapsto [s\xi : x : s : 1] \in \mathbf{P}^3.$$

Let us denote the blowup fiber over the point $o = \Sigma_{01} \cap \Sigma_{02} = [0 : 0 : 0 : 1] \in \mathbf{P}^3$ as $\mathcal{F}_o^1$ then in this coordinate system we have $\mathcal{F}_o^1 = \{s = x = 0\}$. The strict transform of the x_2-axis in Z_1 is given by $\ell_2 = \{\xi = x = 0\}$ and $\mathcal{F}_o^1 \cap \ell_2 = (0, 0, 0)_{Z_1}$. Now let Z_2 be the blowup of ℓ_2 with a local coordinate system

$$\begin{aligned} \pi := \pi_1 \circ \pi_2 \colon (t, \eta, s)_{Z_2} \in Z_2 \mapsto (t, t\eta, s)_{Z_1} &\in Z_1 \\ \mapsto [ts : t\eta : s : 1] &\in \mathbf{P}^3. \end{aligned}$$

We denote the second (new) fiber over o as $\mathcal{F}_o^2$, so $\mathcal{F}_o^2 = (0, \eta, 0)_{Z_2}$. Let us also use $\mathcal{F}_o^1$ for its strict transform in Z_2, so $\mathcal{F}_o^1 \cup \mathcal{F}_o^2 = \pi_2^{-1} \circ \pi_1^{-1}\{o\}$ and $\mathcal{F}_o^1 = (t, 0, 0)_{Z_2}$.

Let $\tau[x_0 : x_1 : x_2 : x_3] = [x_0 : x_2 : x_1 : x_3]$ be the involution that interchanges the x_1- and x_2-axes. It follows that τ induces the involution $\tilde{\tau} = \pi^{-1} \circ \tau \circ \pi$ on Z_2. In coordinates, we have

$$\tilde{\tau} \colon (t, \eta, s) \mapsto (s/\eta, \eta, t\eta), \tag{1.5}$$

which will serve as our third model of indeterminate behavior. We note that $\tilde{\tau}$ is regular on $\mathcal{F}_o^2 - \mathcal{F}_o^1$, while each point of $\mathcal{F}_o^1$ blows up to the variety $\mathcal{F}_o^1$.

Similarly we can blow up the x_2- axis first and then the strict transform of x_1-axis. Performing similar computations, we obtain a blowup space $\hat{\pi} \colon Y_2 \to \mathbf{C}^3$. The identity map ι on $\mathbf{P}^3$ lifts to a map $\tilde{\iota} \colon Z_2 \to Y_2$, which in local coordinates is similar to $\tilde{\tau}$.

Remark. Suppose that γ' and γ'' are curves in $\mathbf{P}^3$ which intersect transversally at points $\{p_1, \ldots, p_N\}$. We have local coordinate systems for $1 \le j \le N$ so that p_j is the origin, and γ' (resp. γ'') coincides with the x-axis (resp. the y-axis) in a neighborhood of p_j. Since the operation of blowing up the axes is local near p_j, we may construct a blowup space $\pi \colon W \to \mathbf{P}^3$ in which γ' and γ'' are both blown up, and over each p_j we are free to choose whether γ' or γ'' was blown up first, independently of the choices over p_k for $k \ne j$.

Theorem 1.5. *Let f be a birational map of X. Let $X_0 \subset X$ be a hypersurface such that the strict transform is $f(X_0) = X_0$. Let $\varphi \colon X \to Y$ is a birational map which conjugates (f, X) to an automorphism (g, Y). Then there is a birational map $\hat{\varphi} \colon X \to \hat{Y}$ such that the strict transform $\hat{Y}_0 := \hat{\varphi}(X_0)$ is a nonsingular hypersurface, and the induced map $\hat{g} := \hat{\varphi} \circ f \circ \hat{\varphi}^{-1}$ gives an automorphsm of $\hat{Y}$.*

Proof: We may assume that X_0 is irreducible. Since X_0 is a hypersurface, we may take its strict transform $\varphi(X_0)$. If $\varphi(X_0)$ is a point in Y, then it is fixed by g. If $\pi_1 \colon Y_1 \to Y$ is the blowup of the point $\varphi(X_0)$, then g lifts to an automorphism of Y_1. Let $\varphi_1 := \pi_1^{-1} \circ \varphi$. If $\varphi_1(X_0)$ is again a point, we can repeat this blowing-up process until $\varphi_1(X_0)$ has dimension > 0, which we may assume to be 1. If the singular locus of $\varphi_1(X_0)$ is nonempty, it is finite and invariant under f_1. Now we can blow up the singular set of $\varphi_1(X_0)$ finitely many times and have a new blowup space $\pi_2 \colon Y_2 \to Y_1$. Since we were blowing up invariant sets, the induced birational map g_2 of Y_2 is again an automorphism. Now the image $\varphi_2(X_0)$ must be a nonsingular curve, which must be invariant. We can blow up this curve, and repeat the process finitely many times so that $\varphi_3(X_0)$ has dimension > 1. We continue this process until $\varphi_N(X_0)$ is a nonsingular hypersurface in Y_N, and now we set $\hat{Y} = Y_N$. □

Corollary 1.6. *Let f be a birational map of X. Let $X_0 \subset X$ be a hypersurface for which the strict transform is $f(X_0) = X_0$. Let $\varphi\colon X \to Y$ is a birational map which conjugates (f, X) to an automorphism (g, Y). Then the restriction (f_{X_0}, X_0) is birationally conjugate to an automorphism.*

Proof of Theorem 4: Let f be as in Theorem 4. In Appendix C we study the restriction of f^8 to the plane $\Sigma_3 = \{[x_0 : x_1 : x_2 : x_3] \in \mathbf{P}^3 : x_3 = 0\}$. There we show that this restricted mapping is not birationally equivalent to an automorphism of Σ_3. Thus Theorem 4 is a consequence of Corollary 1.6. □

2. Linear fractional recurrences

The maps (2.2) are among the Cremona transformations of 3-space which are discussed in Chapter 10 of [**H**]. We discuss general properties of these transformations, and for the generic parameters (2.10) we construct a new space $\pi\colon X \to \mathbf{P}^3$, such that passing to the induced map f_X effectively eliminates one of the exceptional components.

For $\{i_1, \ldots, i_k\} \subset \{0, 1, 2, 3\}$, we use the notation

$$\Sigma_{i_1 \cdots i_k} = \{x \in \mathbf{P}^3 : x_{i_j} = 0,\ 1 \le j \le k\}, \tag{2.1}$$

and for a vector $A = (A_0, \ldots, A_3)$ we will write $A \cdot x = A_0x_0 + A_1x_1 + A_2x_2 + A_3x_3$. In homogeneous coordinates the maps (0.1) take the form

$$f[x_0 : x_1 : x_2 : x_3] = [x_0\beta\cdot x : x_2\beta\cdot x : x_3\beta\cdot x : x_0\alpha\cdot x], \tag{2.2}$$

where $\alpha = (\alpha_0, \alpha_1, \alpha_2, \alpha_3)$, $\beta = (\beta_0, \beta_1, \beta_2, \beta_3)$. In the sequel, we will assume

$$\alpha \ne \lambda\beta, \quad \beta \ne (\beta_0, 0, 0, 0), \quad (\alpha_1, \beta_1) \ne (0, 0). \tag{2.3}$$

Note that if one of the first two conditions does not hold, then f is linear, and if the third condition does not hold, then f is independent of x_1 and thus f is actually a 2-step recurrence. If we set $\gamma = \beta_1\alpha - \alpha_1\beta$ and $\beta_1 \ne 0$, then we have

$$\mathcal{I} = \Sigma_{\beta\gamma} \cup \Sigma_{0\beta} \cup \{e_1\},$$

where $\Sigma_\beta = \{\beta\cdot x = 0\}$, $\Sigma_\gamma = \{\gamma\cdot x = 0\}$, $\Sigma_0 = \{x_0 = 0\}$, $\Sigma_{\beta\gamma} = \Sigma_\beta \cap \Sigma_\gamma$, $\Sigma_{0\beta} = \Sigma_0 \cap \Sigma_\beta$, and $e_1 = [0 : 1 : 0 : 0] = \Sigma_{023}$.

The Jacobian determinant of f is given by $2x_0(\gamma\cdot x)(\beta\cdot x)^2$. Thus we see that the exceptional hypersurfaces are $\mathcal{E} = \{\Sigma_0, \Sigma_\beta, \Sigma_\gamma\}$. The action of f on the exceptional varieties is given as follows: for $\lambda_2, \lambda_3 \in \mathbf{C}$, $(\lambda_2, \lambda_3) \neq (0, 0)$,

$$(2.4)\qquad f\colon \begin{array}{l} \Sigma_\beta \mapsto e_3, \\ \Sigma_0 \cap \{\lambda_2 x_2 = \lambda_3 x_3\} \mapsto [0 : \lambda_3 : \lambda_2 : 0], \\ \Sigma_\gamma \cap \{\lambda_2 x_2 = \lambda_3 x_3\} \mapsto \Sigma_{BC} \cap \{\lambda_2 x_1 = \lambda_3 x_2\}, \end{array}$$

where we set $\check{\alpha} = (\alpha_0, \alpha_2, \alpha_3, 0)$, $\check{\beta} = (\beta_0, \beta_2, \beta_3, 0)$, and

$$B = (-\alpha_1, 0, 0, \beta_1), \quad C = \beta_1\check{\alpha} - \alpha_1\check{\beta}.$$

Thus Σ_β is blown down to a point. The pencil of lines in Σ_γ passing through $e_1 \in \Sigma_0 \cap \Sigma_\gamma$ are all mapped to points in Σ_{BC}. The pencil of lines in Σ_0 passing through e_1 are all mapped to points on the line Σ_{03}, which is again one of the exceptional lines. We have strict transforms:

$$(2.5)\qquad f\colon \Sigma_0 \mapsto \Sigma_{03} \mapsto e_1.$$

The inverse is given by

$$(2.6)\quad f^{-1}[x_0 : x_1 : x_2 : x_3] = [x_0 B\cdot x : x_0\check{\alpha}\cdot x - x_3\check{\beta}\cdot x : x_1 B\cdot x : x_2 B\cdot x],$$

and the indeterminacy locus is $\mathcal{I}(f^{-1}) = \Sigma_{0B} \cup \Sigma_{BC} \cup \{e_3\}$. The Jacobian of f^{-1} is $2x_0(C\cdot x)(B\cdot x)^2$, so the exceptional hypersurfaces are $\mathcal{E}(f^{-1}) = \{\Sigma_0, \Sigma_B, \Sigma_C\}$ and for $\mu_1, \mu_2 \in \mathbf{C}$, $(\mu_1, \mu_2) \neq (0, 0)$,

$$(2.7)\qquad f^{-1}\colon \begin{array}{l} \Sigma_B \mapsto e_1, \\ \Sigma_0 \cap \{\mu_1 x_1 = \mu_2 x_2\} \mapsto \Sigma_{0\beta} \cap \{\mu_1 x_2 = \mu_2 x_3\}, \\ \Sigma_C \cap \{\mu_1 x_1 = \mu_2 x_2\} \mapsto \Sigma_{\beta\gamma} \cap \{\mu_1 x_2 = \mu_2 x_3\}. \end{array}$$

Now let us construct the space $\pi_1\colon X_1 \to \mathbf{P}^3$ by blowing up a point e_1, and then the space $\pi_2\colon X \to X_1$ obtained by blowing up a line Σ_{03}. We set

$$(2.8)\qquad \pi = \pi_1 \circ \pi_2\colon X \to \mathbf{P}^3.$$

Let S_{03} denote the blowup fiber over the strict transform of Σ_{03} in X_1 and E_1 for the strict transform of $\pi_1^{-1}e_1$ in X_1. For the induced map on X, the orbit of Σ_0 becomes

$$(2.9)\qquad f_X\colon \Sigma_0 \to S_{03} \to E_1 \to \Sigma_B.$$

If X and Y are irreducible, we will say that a rational map $f\colon X \dashrightarrow Y$ is *dominant* if the rank of df is equal to the dimension of Y on a dense open set. Let us define a generic condition:

$$(2.10)\qquad \beta_1 \neq 0, \quad \beta_1\alpha_2 \neq \alpha_1\beta_2, \quad \text{and} \quad \beta_1\alpha_3 \neq \alpha_1\beta_3.$$

For simplicity we use the same notation for both a variety and its strict transform, if there is no possibility of confusion.

Proposition 2.1. *If* (2.10) *holds, then all the maps in* (2.9) *are dominant, so* $\mathcal{E}(f_X) = \{\Sigma_\beta, \Sigma_\gamma\}$. *Further,* $\mathcal{I}(f_X) = \Sigma_{\beta 0} \cup \Sigma_{\beta\gamma}$.

Proof: Let us first consider the restriction to S_{03}. We may use the local coordinates for a neighborhood of S_{03}, $(s_0, x_2, \xi_3)_{S_{03}} \mapsto [s_0 : 1 : x_2 : s_0\xi_3]$. For the neighborhood of the exceptional fiber E_1 over e_1, we use $(t_0, \zeta_2, \zeta_3)_{E_1} \mapsto [t_0 : 1 : t_0\zeta_2 : t_0\zeta_3]$. It follows that $S_{03} = \{(0, x_2, \xi_3)_{S_{03}}\}$ and $E_1 = \{(0, \zeta_2, \zeta_3)_{E_1}\}$. Using these local coordinates we have

$$f_X|_{S_{03}} \colon (0, x_2, \xi_3)_{S_{03}} \mapsto \left(0, \xi_3, \frac{\alpha_1 + \alpha_2 x_2}{\beta_1 + \beta_2 x_2}\right)_{E_1}.$$

To have a dominant map, it is required that $\beta_1\alpha_2 \neq \alpha_1\beta_2$. For the restrictions of the induced birational map to Σ_0 and E_1 are given by linear maps:

$$f_X \colon \Sigma_0 \ni [0 : x_1 : x_2 : x_3] \mapsto \left(0, \frac{x_3}{x_2}, \frac{\alpha_1 x_1 + \alpha_2 x_2 + \alpha_3 x_3}{\beta_1 x_1 + \beta_2 x_2 + \beta_3 x_3}\right)_{S_{03}} \in S_{03},$$
$$f_X \colon E_1 \ni (0, \zeta_2, \zeta_3)_{E_1} \mapsto [\beta_1 : \beta_1\zeta_2 : \beta_1\zeta_3 : \alpha_1] \in \Sigma_B.$$

We see that $f_X|_{E_1}$ is dominant because $\beta_1 \neq 0$. And since $\beta_1\alpha_2 \neq \alpha_1\beta_2$ and $\beta_1\alpha_3 \neq \alpha_1\beta_3$, we see that $f_X|_{\Sigma_0}$ is dominant. □

Thus in passing to f_X, we have removed one exceptional hypersurface and one point of indeterminacy. There is a group of linear conjugacies acting on the family (0.2). For $(\lambda, c, \mu) \in \mathbf{C}^* \times \mathbf{C}^* \times \mathbf{C}$, we set

(2.11a) $\quad (\alpha, \beta) \mapsto (\lambda\alpha, \lambda\beta),$

(2.11b)
$$(\alpha, \beta) \mapsto (\alpha_0, c\alpha_1, c\alpha_2, c\alpha_3, c\beta_0, c^2\beta_1, c^2\beta_2, c^2\beta_3),$$
$$(\alpha, \beta) \mapsto (\alpha'_0, \alpha'_1, \alpha'_2, \alpha'_3, \beta'_0, \beta_1, \beta_2, \beta_3),$$

(2.11c)
$$\alpha'_0 = \alpha_0 + \mu(\alpha_1 + \alpha_2 + \alpha_3) + \mu(\beta_0 + \mu\beta_1 + \mu\beta_2 + \mu\beta_3),$$
$$\alpha'_1 = \alpha_1 - \mu\beta_1,$$
$$\alpha'_2 = \alpha_2 - \mu\beta_2,\ \alpha'_3 = \alpha_3 - \mu\beta_3,\ \beta'_0 = \beta_0 + \mu(\beta_1 + \beta_2 + \beta_3).$$

The first action corresponds to the homogeneity of f. The action (2.11b) corresponds to a scaling of (x_1, x_2, x_3) in affine coordinates, and (2.11c) comes from translation by the vector (μ, μ, μ). Note that these actions preserve the form of the recurrence relation.

3. Non-critical maps

A map f of the form (2.2) is *critical* if (3.1) holds:

$$\beta_2 = \beta_3 = 0, \quad \text{and} \quad \beta_1\alpha_2\alpha_3 \neq 0. \tag{3.1}$$

In this section we establish the following:

Theorem 3.1. *If f is not critical, then f is not periodic.*

We will use the following criterion:

Proposition 3.2. *Suppose that $f\colon X \to X$ is periodic, i.e., f_X^p is the identity for some $p > 1$. If $E \subset X$ is an exceptional hypersurface, then $f^j E \subset \mathcal{I}(f_X)$ for some $1 \leq j < p$.*

Proof: Since E is exceptional, then $\operatorname{codim}(f(E)) \geq 2$. Let us consider the sequence of varieties $V_j := f_X^j(E)$. If $V_j \not\subset \mathcal{I}(f_X)$ for all j, then applying the strict transform of f repeatedly, we have $f_X^{j+1}(E) = f_X(V_j)$ for all j, so $\operatorname{codim}(f(V_j)) \geq 2$ for all j. On the other hand, we must have $f_X^p(E) = E = V_p$. □

The proof of Theorem 3.1 will involve several cases, so we start with some lemmas.

Lemma 3.3. *Let $\pi\colon X \to \mathbf{P}^3$ be the complex manifold defined in* (2.8). *If $\beta_1 = 0$, then there is a exceptional hypersurface $V \subset X$ and a positive integer k such that $(f_X^k)^n V \not\subset \mathcal{I}(f_X^k)$ either for all $n \geq 1$ or for all $n \leq -1$.*

Proof: Note that from the third assumption in (2.3), we have $\alpha_1 \neq 0$. We use two local coordinate systems for the exceptional divisor S_{03}:

$$(s,x,\xi)_{S_{03}^{(1)}} \mapsto [s\xi : x : 1 : s], \quad \text{and} \quad (s,x,\xi)_{S_{03}^{(2)}} \mapsto [s : x : 1 : s\xi]$$

for the blowup divisor E_1 we use the following two local coordinate systems:

$$(t,\zeta_2,\zeta_3)_{E_1^{(0)}} \mapsto [t : 1 : t\zeta_2 : t\zeta_3], \quad \text{and} \quad (\zeta_0,\zeta_2,t)_{E_1^{(3)}} \mapsto [t\zeta_0 : 1 : t\zeta_2 : t].$$

Since $\beta \neq (\beta_0, 0, 0, 0)$, either $\beta_3 \neq 0$ or $\beta_2 \neq 0$.

Case $\beta_3\beta_2 \neq 0$: In this case, the orbit of Σ_β is given by

$$\begin{aligned} f_X\colon \Sigma_\beta \mapsto e_3 \mapsto (0,0,\alpha_3/\beta_3)_{S_{03}^{(2)}} &\mapsto (0,\alpha_3/\beta_3,\alpha_2/\beta_2)_{E_1^{(0)}} \\ &\mapsto e_3 \in \Sigma_0 \setminus \mathcal{I}(f_X). \end{aligned}$$

Thus the orbit of Σ_β is pre-periodic and $f_X^n \Sigma_\beta$ is a regular point for all $n \geq 1$.

Case $\beta_3 \neq 0$ *and* $\beta_2 = 0$*:* Since $\beta_3 \neq 0$, we may assume that $\beta_3 = 1$. Notice that since both β_1 and β_2 are equal to zero, the second condition (2.10) is not satisfied and we have

$$f_X: \begin{array}{l} \Sigma_0 \ni [0:x_1:x_2:x_3] \mapsto (0, x_2/x_3, (\alpha_1 x_1 + \alpha_2 x_2 + \alpha_3 x_3)/x_3)_{S_{03}^{(2)}} \in S_{03}, \\ S_{03} \ni (0,x,\xi)_{S_{03}^{(2)}} \mapsto [0:\beta_0+\xi:0:\alpha_2+\alpha_1 x] \in \Sigma_0, \\ \Sigma_{02} \ni [0:x_1:0:x_3] \mapsto (0,0,(\alpha_1 x_1 + \alpha_3 x_3)/x_3)_{S_{03}^{(2)}} \in S_{03} \cap \Sigma_1 = \pi^{-1}(e_2). \end{array}$$

It follows that

$$f_X: \Sigma_0 \mapsto S_{03} \mapsto \Sigma_{02} \mapsto S_{03} \cap \Sigma_1.$$

We also have

$$f_X: E_1 \ni (0, \zeta_2, \zeta_3)_{E_1^{(0)}} \mapsto [0:0:0:1] = e_3 \in \Sigma_0.$$

(i) $\beta_3 \neq 0$, $\beta_2 = 0$, and $\alpha_2 \neq 0$: With these parameters, we have a two-cycle between Σ_{02} and the fiber over e_2, $\{(0,0,\xi)_{S_{03}^{(2)}}\}$. Since $e_3 \in \Sigma_{02}$ we have

$$f_X: \Sigma_\beta \mapsto e_3 \mapsto (0,0,\alpha_3)_{S_{03}^{(2)}} \mapsto [0:\beta_0+\alpha_3:0:\alpha_2] \mapsto \cdots$$

Both Σ_{02} and the fiber over e_2 in S_{03} are disjoint from $\mathcal{I}(f_X)$, the forward orbit of Σ_β consists of points in $X - \mathcal{I}(f_X)$. In fact the curve $S_{03} \cap \Sigma_1 = \pi^{-1}(e_2)$ is invariant under f_X^2:

$$f_X^2: S_{03} \cap \Sigma_1 \ni (0,0,\xi)_{S_{03}^{(2)}} \mapsto (0,0,(\alpha_2\alpha_3 + \alpha_1\beta_0 + \alpha_1\xi)/\alpha_2)_{S_{03}^{(2)}} \in S_{03} \cap \Sigma_1.$$

Thus for all $n \geq 1$, $(f_X^2)^n \Sigma_\beta$ is a regular point in S_{03}.

(ii) $\beta_3 \neq 0$, $\beta_2 = 0$, $\alpha_2 = 0$, and $\beta_0 + \alpha_3 \neq 0$: In this case the point e_3 is periodic of period 3:

$$f_X: \Sigma_\beta \mapsto e_3 \mapsto (0,0,\alpha_3)_{S_{03}^{(2)}} \mapsto E_1 \cap \{x_2 = \alpha_3 x_1\} \mapsto e_3.$$

Thus we have a hypersurface

$$\begin{aligned} V = \{&(\alpha_0\alpha_3 + 2\alpha_0\beta_0 + \beta_0^3)x_0^2 + (\alpha_1\alpha_3 + \alpha_1\beta_0)x_0x_1 + \alpha_1\beta_0 x_0 x_2 \\ &+ (\alpha_0 + \alpha_3^2 + \alpha_3\beta_0 + \beta_0^2)x_0x_3 + \alpha_1 x_2 x_3 = 0\} \end{aligned}$$

such that $(f_X^3)^n V = e_3$ for all $n \geq 1$.

(iii) $\beta_3 \neq 0$, $\beta_2 = 0$, $\alpha_2 = 0$, and $\beta_0 + \alpha_3 = 0$: For this case, let us consider f^2. We see that

$$f_X^2: S_{03} \ni (0,x,\xi)_{S_{03}^{(1)}} \mapsto \left(0, 0, \frac{x\xi}{1+\beta_0\xi - \beta_0 x\xi}\right)_{S_{03}^{(1)}} \in S_{03}.$$

It follows that S_{03} is still exceptional in this case and the point $(0,0,0)_{S_{03}^{(1)}} \notin \mathcal{I}(f_X^2)$ is fixed under f_X^2. Thus $(f_X^2)^n S_{03} \not\subset \mathcal{I}(f_X^2)$ for all $n \geq 1$.

Case $\beta_3 = 0$ and $\beta_2 \neq 0$: Under the backward map, the hypersurface Σ_0 is exceptional. We see

$$f_X^{-1}\colon \begin{array}{l} \Sigma_0 \ni [0 : x_1 : x_2 : x_3] \mapsto (0, x_1/x_2, 0)_{E_1^{(3)}} \in E_1 \cap \Sigma_0, \\ E_1 \ni (\zeta_0, \zeta_2, 0)_{E_1^{(3)}} \mapsto (0, (\beta_2 - \alpha_2\zeta_0)/(\alpha_1\zeta_0), \zeta_2/\zeta_0)_{S_{03}^{(2)}} \in S_{03}, \\ S_{03} \ni (0, x, \xi)_{S_{03}^{(2)}} \mapsto [0 : -\alpha_3 - \alpha_2 x + \beta_2 x\xi : \alpha_1 x : \alpha_1] \in \Sigma_0. \end{array}$$

Let us set $p := \Sigma_0 \cap E_1 \cap S_{03}$. It follows that the point p is fixed under f_X^{-1}. Since $p \in f_X^{-n}\Sigma_0$ for all $n \geq 1$, we see that $f_X^{-n}\Sigma_0 \not\subset \mathcal{I}(f_X^{-1})$ for all $n \geq 1$. □

Now let us suppose that $\beta_1 \neq 0$. Using actions (2.11a)–(2.11c), we may assume that $\beta_1 = 1$ and $\alpha_1 = 0$.

Lemma 3.4. *Suppose that $\beta_1 = 1$, $\alpha_1 = 0$. If either $\beta_2 \neq 0$ or $\beta_3 \neq 0$, then Σ_0 is exceptional and pre-periodic for f^{-1}.*

Proof: Let us first assume that $\beta_3 \neq 0$. Then

$$f^{-1}\colon \Sigma_0 \ni [0 : x_1 : x_2 : x_3] \mapsto [0 : -(\beta_2 x_1 + \beta_3 x_2) : x_1 : x_2] \in \Sigma_{0\beta}$$

and $\Sigma_{0\beta}$ is invariant under f^{-1}:

$$\begin{aligned} f^{-1}\colon &[0 : -(\beta_2 x_2 + \beta_3 x_3) : x_2 : x_3] \\ &\mapsto [0 : \beta_2(\beta_2 x_2 + \beta_3 x_3) - \beta_3 x_2 : -(\beta_2 x_2 + \beta_3 x_3) : x_2]. \end{aligned}$$

Now suppose $\beta_3 = 0$ and $\beta_2 \neq 0$. In this case we have

$$\begin{aligned} f^{-1}\colon \Sigma_0 \ni [0 : x_1 : x_2 : x_3] &\mapsto [0 : -\beta_2 x_1 : x_1 : x_2] \\ &\mapsto [0 : \beta_2^2 : -\beta_2 : 1] \in \Sigma_{0\beta} \setminus \mathcal{I}(f^{-1}) \end{aligned}$$

and this last point is fixed under f^{-1}. □

Let $\pi\colon Z \to \mathbf{P}^3$ be the complex manifold obtained by blowing up e_2 and Σ_{02} and let E_2 and S_{02} be the corresponding blowup divisors. In the following lemma, we use the local coordinates $(s_0, x_1, \xi_2)_{S_{02}} \mapsto [s_0 : x_1 : s_0\xi_2 : 1]$ in a neighborhood of $S_{02} = \{s_0 = 0\}$ and $(u_0, \eta_1, \eta_3)_{E_2} \mapsto [u_0 : \eta_1 u_0 : 1 : \eta_3 u_0]$ in a neighborhood of $E_2 = \{u_0 = 0\}$.

Lemma 3.5. *Suppose that $\beta_1 = 1$, $\alpha_1 = \beta_2 = \beta_3 = 0$. If either $\alpha_2 = 0$ or $\alpha_3 = 0$, then S_{02} is exceptional and pre-periodic for f_Z^2 or f_Z^{-2}.*

Proof: If $\alpha_2 = \alpha_3 = 0$, then the mapping is basically one-dimensional. In other word, the recurrence defined in (0.2) can be reduced to one-step recursion

$$w_{k+1} = \frac{\alpha_0}{\beta_0 + w_k} \quad \text{where} \quad w_k = z_{i+3k}, \quad i = 0, 1, 2.$$

Let us consider two cases separately:

(i) *Case $\alpha_3 = 0$ and $\alpha_2 \neq 0$:*

$$f_Z^2 \colon (0, t, \xi_2)_{S_{02}} \mapsto \left(0, \frac{\beta_0 + \xi_2}{\alpha_2}, 0\right)_{S_{02}} \mapsto \left(0, \frac{\beta_0}{\alpha_2}, 0\right)_{S_{02}} \mapsto \left(0, \frac{\beta_0}{\alpha_2}, 0\right)_{S_{02}}.$$

(ii) *Case $\alpha_2 = 0$ and $\alpha_3 \neq 0$:*

$$f_Z^{-2} \colon (0, t, \xi_2)_{S_{02}} \mapsto \left(0, \frac{\alpha_3}{\xi_2}, -\beta_0\right)_{S_{02}} \mapsto \left(0, -\frac{\alpha_3}{\beta_0}, -\beta_0\right)_{S_{02}}$$
$$\mapsto \left(0, -\frac{\alpha_3}{\beta_0}, -\beta_0\right)_{S_{02}}. \qquad \square$$

Theorem 3.6. *If f is not critical, then there exists a complex manifold X such that either there are a positive integer k and an exceptional hypersurface $E \subset X$ for an induced birational map f_X^k such that $(f_X^k)^n E \not\subset \mathcal{I}(f_X^k)$ for $n = 1, 2, \ldots$, or the analogous statement holds for f_X^{-1}.*

Proof: Let X denote either the space X or Z in the lemmas above. This theorem follows from the Lemmas 3.3–3.5. $\square$

Proof of Theorem 3.1: If f is not critical, then Theorem 3.6 says that in each case that there are a positive integer k and an exceptional hypersurface that does not map into $\mathcal{I}(f_X^k)$. By Proposition 3.2, then, f^k is not periodic and therefore f is not periodic. $\square$

4. Critical maps

Here we study critical maps in general. Let us recall the condition for f being critical: $\beta_2 = \beta_3 = 0$ and $\beta_1\alpha_2\alpha_3 \neq 0$. Using the action (2.11a)–(2.11c) we may assume that a critical map satisfies:

$$(4.1) \qquad \beta_1 = 1, \quad \beta_2 = \beta_3 = 0, \quad \alpha_1 = 0, \quad \alpha_2 \neq 0, \quad \alpha_3 = 1.$$

In this section, we show (Lemma 4.2) that for every critical map there is a blowup space $\pi \colon Y \to X$ such that the induced map f_Y has only one exceptional hypersurface, which is Σ_γ. We determine the indeterminacy locus of f_Y (Corollary 4.6) and the dynamical degree for the generic case (Theorem 4.8).

Proposition 4.1. *If f is critical, then f^{-1} is conjugate to a critical map.*

Proof: Let $\beta = (\beta_0, 1, 0, 0)$ and $\alpha = (\alpha_0, 0, \alpha_2, 1)$ be parameters of a critical map f. We consider a linear map ϕ: $[x_0 : x_1 : x_2 : x_3] \mapsto [x_0 : x_3 : x_2 : x_1]$. It follows that we have

$$\phi^{-1} \circ f^{-1} \circ \phi\colon [x_0 : x_1 : x_2 : x_3] \mapsto [x_0x_1 : x_2x_1 : x_3x_1 : x_0(\alpha_0x_0 - \beta_0x_1 + x_2 + \alpha_2x_3)].$$

Thus f^{-1} is conjugate to a critical map of the form (2.2) with parameter values $\beta' = (0, 1, 0, 0)$ and $\alpha' = (\alpha_0, -\beta_0, 1, \alpha_2)$ which satisfy the condition (3.1). □

Remark. By Proposition 4.1, each result for f corresponds to a result for f^{-1}. The translation between f and f^{-1} is guided by notation: $\beta \leftrightarrow B$, $\gamma \leftrightarrow C$, $1 \leftrightarrow 3$: thus $f^{-1}\Sigma_C = \Sigma_{\beta\gamma}$, etc.

If (4.1) holds, it follows that $e_3 = \Sigma_0 \cap \Sigma_\beta \cap \{x_2 = 0\} \in \mathcal{I}$, and

$$f\colon \Sigma_\beta \dashrightarrow e_3 \rightsquigarrow \Sigma_{01} \rightsquigarrow \Sigma_0 \dashrightarrow \Sigma_{03} \dashrightarrow e_1 \rightsquigarrow \Sigma_B. \tag{4.2}$$

We define a new complex manifold $\pi_Y : Y \to \mathbf{P}^3$ by blowing up e_1 and e_3, then the strict transform of Σ_{01}, followed by the strict transform of Σ_{03}. (Equivalently, we start with X and blow up the strict transform of e_1 and Σ_{03}.) For $j = 1, 3$, we denote the exceptional divisor over e_j by E_j and the exceptional divisor over Σ_{0j} by S_{0j} for $j = 1, 3$. The induced birational map $f_Y : Y \to Y$ maps

$$f_Y : \Sigma_\beta \to E_3 \to S_{01} \to \Sigma_0 \to S_{03} \to E_1 \to \Sigma_B. \tag{4.3}$$

Lemma 4.2. *The maps in (4.4) are dominant; Σ_γ is the unique exceptional hypersurface for f_Y, and Σ_C is the unique exceptional hypersurface for f_Y^{-1}.*

Proof: Using the local coordinates defined by $(t, \xi_1, \xi_2)_{E_3} \mapsto [t : t\xi_1 : t\xi_2 : 1]$ and $(s, \eta, x_2)_{S_{01}} \mapsto [s : s\eta : x_2 : 1]$, we have

$$\begin{aligned} &\Sigma_\beta \setminus \Sigma_{\beta\gamma} \ni [x_0 : -\beta_0x_0 : x_2 : x_3] \mapsto (0, \tfrac{x_2}{x_0}, \tfrac{x_3}{x_0})_{E_3} \in E_3,\\ f_Y :\ &E_3 \ni (0, \xi_1, \xi_2)_{E_3} \mapsto (0, \xi_2, \beta_0 + \xi_1)_{S_{01}} \in S_{01},\\ &S_{01} \setminus \Sigma_{\beta\gamma} \ni (0, \eta_1, x_2)_{S_{01}} \mapsto [0 : x_2(\beta_0 + \eta_1) : (\beta_0 + \eta_1) : 1 + \alpha_2x_2] \in \Sigma_0. \end{aligned}$$

In Proposition 2.1 we showed that the maps $\Sigma_0 \to S_{03} \to E_1 \to \Sigma_B$ are dominant. It follows that Σ_γ is the only exceptional hypersurface for f_Y, and Σ_C is the only one for f_Y^{-1}. □

For $p \in \mathbf{P}^3$, we will say that a point of $\pi_Y^{-1}p$ is at *level 1* if it could have been obtained by blowing up a point or a curve in $\mathbf{P}^3$. Thus the points of all fibers are of level 1, unless they lie over e_1, e_3, or $e_2 = \Sigma_{01} \cap \Sigma_{03}$. The fibers $E_1 \cap S_{03}$ and $E_3 \cap S_{01}$ represent the points of E_1 and E_3 which are at level 2. Over e_2, we define $\mathcal{F}^1_{e_2} := S_{01} \cap \pi_Y^{-1} e_2$ and $\mathcal{F}^2_{e_2} := S_{03} \cap \pi_Y^{-1} e_2$. We see that $\mathcal{F}^i_{e_2}$, for $i = 1, 2$ is at level i.

Now we determine the indeterminacy locus of f_Y. In the construction of Y, we see that new indeterminacy locus is generated when two centers of blowups intersect. Thus we have

$$\mathcal{I}(f_Y) \subset \Sigma_{\beta\gamma} \cup (E_1 \cap S_{03}) \cup (E_3 \cap S_{01}) \cup \mathcal{F}^1_{e_2} \cup \mathcal{F}^2_{e_2} \cup \Sigma_{02} \cup \mathcal{F}_{0\beta\gamma},$$

where $\mathcal{F}_{0\beta\gamma} := \pi_Y^{-1}(\Sigma_{\beta\gamma} \cap \Sigma_{01})$. We see that the three curves on level 2 are not indeterminate:

Lemma 4.3. *If f is critical, then the indeterminacy loci $\mathcal{I}(f_Y)$ and $\mathcal{I}(f_Y^{-1})$ do not contain $E_1 \cap S_{03}$, $E_3 \cap S_{01}$, or $\mathcal{F}^2_{e_2}$.*

Proof: Let us first consider the blowup fiber over $E_3 \cap \Sigma_{01}$. For this fiber let us use a local coordinate $(\xi_0, \xi_1, t_2)_{E_3} \mapsto [t_2\xi_0 : t_2\xi_1 : t_2 : 1] \in \mathbf{P}^3$. It follows that the strict transform of $\Sigma_{01} = \{(0, 0, t_2)_{E_3}\}$ and $E_3 \cap \Sigma_{01} = (0, 0, 0)_{E_3}$. The local coordinates in a neighborhood of the second blowup fiber over $E_3 \cap S_{01}$ are given by $(\eta_0, u_1, t_2)_{E_3^{01}} \mapsto (\eta_0 u_1, u_1, t_2)_{E_3} \mapsto [\eta_0 u_1 t_2 : u_1 t_2 : t_2 : 1] \in \mathbf{P}^3$ and we have the second blowup fiber $E_3 \cap S_{01} = \{(\eta_0, 0, 0)_{E_3^{01}}\}$. With these coordinates, we see that

$$f_Y \colon (\eta_0, 0, 0)_{E_3^{01}} \mapsto (0, 0, \eta_0)_{S_{01}} = S_{01} \cap \Sigma_0,$$

where $(\xi, t, x_3)_{S_{01}} \mapsto [\xi t : t : 1 : x_3]$ gives local coordinates near S_{01}. It follows that the second blowup fiber $E_3 \cap S_{01}$ is not indeterminate for f_Y. The computations for f_Y^{-1} and for $E_1 \cap S_{03}$ are essentially the same, and we see that $E_1 \cap S_{03}$ and $E_3 \cap S_{01}$ are not indeterminate for f_Y or f_Y^{-1}.

To consider the second blowup fiber $\mathcal{F}^2_{e_2}$, we use local coordinates $(\xi, s, x_3)_{01} \to [\xi s : s : 1 : x_3]$. In this coordinates we see that $S_{01} = \{s = 0\}$ and the strict transform of $\Sigma_{03} = \{\xi = 0, x_3 = 0\}$. Thus the local coordinates near the blowup of Σ_{03} are given by $(\eta, s, t)_{03} \mapsto (\eta t, s, t)_{01} \mapsto [\eta t s : s : 1 : t]$ and we have $\mathcal{F}^2_{e_2} = \{(\eta, 0, 0)_{03}\}$. With these coordinates, we have

$$f_Y \colon \mathcal{F}^2_{e_2} \ni (\eta_0, 0, 0)_{03} \mapsto (0, 0, \alpha_2 \eta)_{E_1} = E_1 \cap \Sigma_0,$$

where $(\xi_0, t_2, \xi_3)_{E_1} \mapsto [\xi_0 t_2 : 1 : t_2 : \xi_3 t_2]$ is local coordinates near E_1. Similarly we see that $f_Y^{-1} \mathcal{F}^2_{e_2} = E_3 \cap \Sigma_0$ and the mapping is dominant. $\square$

Recall from §2 that in $\mathbf{P}^3$ each point on $\Sigma_{\beta\gamma}$ blows up to a line in Σ_C. Note that $[0:0:1:-\alpha_2] = \Sigma_{\beta\gamma} \cap \Sigma_{01}$, and let $\mathcal{F}_{0\beta\gamma} := \pi_Y^{-1}(\Sigma_{\beta\gamma} \cap \Sigma_{01})$. Note that the base point is the intersection of Σ_{01} and $\Sigma_{\beta\gamma}$, two indeterminate lines. Similarly, we write $\mathcal{F}_{0BC} := \pi_Y^{-1}(\Sigma_{BC} \cap \Sigma_{03}) = \pi_Y^{-1}[0:1:-\alpha_2:0]$.

Lemma 4.4. *If f is critical, the fiber curve $\mathcal{F}_{0\beta\gamma}$ is a component of $\mathcal{I}(f_Y)$. Further $f_Y\colon \mathcal{F}_{0\beta\gamma} - \Sigma_{\beta\gamma} \rightsquigarrow \mathcal{F}_{0BC}$ in the following senses:*

(i) *If $p \in \mathcal{F}_{0\beta\gamma} - \Sigma_{\beta\gamma}$, then $(f_Y)_*(p) = \mathcal{F}_{0BC}$.*

(ii) *If $\pi\colon T_{ii} \to Y$ is the blowup of p with exceptional divisor E_p, and if $f_{T_{ii}}$ is the induced map, then the strict transform of $f_{T_{ii}}(E_p) = \mathcal{F}_{0BC}$.*

(iii) *If $\pi\colon T_{iii} \to Y$ is the blowup of $\mathcal{F}_{0\beta\gamma}$ and $\mathcal{F}_{0BC}$, with exceptional divisors $E_{0\beta\gamma}$ and E_{0BC}, then the induced map $f_{T_{iii}}\colon E_{0\beta\gamma} \dashrightarrow E_{0BC}$ is dominant.*

(iv) *If $\pi\colon T_{iv} \to Y$ is the blowup of $p \in \mathcal{F}_{0\beta\gamma}$ and $\mathcal{F}_{0BC}$, then the image of the induced map $f_{T_{iv}}\colon E_p \dashrightarrow E_{0BC}$ is a curve $\gamma \subset E_{0BC}$.*

(v) *If $\hat{\pi}\colon T_v \to T_{iv}$ is the blowup of the curve γ in (iv), and let E_γ denote the exceptional divisor. Then the induced map $f_{T_v}\colon E_p \dashrightarrow E_\gamma$ is dominant.*

Proof: Let us consider local coordinates in a neighborhood of the fiber $\mathcal{F}_{0\beta\gamma}$ and local coordinates in a neighborhood of $\mathcal{F}_{0BC}$:

$$(s_0, \eta_1, x)_{S_{01}} \sim [s_0 : \eta_1 s_0 : 1 : x] \in \mathbf{P}^3 \quad \text{and} \quad \mathcal{F}_{0\beta\gamma} = \{s_0 = 0, x = -\alpha_2\},$$
$$(s_0, x, \eta_3)_{S_{03}} \sim [s_0 : 1 : x : \eta_3 s_0] \in \mathbf{P}^3 \quad \text{and} \quad \mathcal{F}_{0BC} = \{s_0 = 0, x = -\alpha_2\}.$$

Thus

$$\begin{aligned} f_Y(s_0, \eta_1, x)_{S_{01}} &= [s_0(\beta_0+\eta_1) : \beta_0+\eta_1 : x(\beta_0+\eta_1) : \alpha_0 s_0+\alpha_2+x] \\ &= \left[s_0 : 1 : x : \frac{\alpha_0 s_0 + \alpha_2 + x}{\beta_0 + \eta_1}\right] \\ &= \left(s_0, x, \frac{\alpha_0 s_0 + \alpha_2 + x}{s_0(\beta_0 + \eta_1)}\right)_{S_{03}}. \end{aligned} \tag{4.4}$$

The condition for $p \in \mathcal{F}_{0\beta\gamma} - \Sigma_{\beta\gamma}$ is that $p = (0, \hat{\eta}_1, -\alpha_2)_{S_{01}}$ with $\beta_0 + \hat{\eta}_1 \neq 0$. For such points p, we see from (4.4) that we get all points of $\mathcal{F}_{0BC}$ as limits as we let $s_0 \to 0$ and $x \to -\alpha_2$. This proves (i).

For (ii), we consider the map $\pi(t, \xi_2, \xi_3) = (t, t\xi_2 + \hat{\eta}_1, t\xi_3 - \alpha_2)_{S_{01}} = (s_0, \eta_1, x)_{S_{01}}$, which gives the blowup at p. By (4.4), the induced map $f_T = f_Y \circ \pi$ is given in coordinates as

$$f_T(t, \xi_2, \xi_3) = (t, -\alpha_2 + t\xi_3, (\alpha_0 + \xi_3)/(\beta_0 + \hat{\eta}_1 + t\xi_2))_{S_{03}}.$$

On $E_p=\{t=0\}$, we have $f_T(t,\xi_2,\xi_3)=(0,-\alpha_2,(\alpha_0+\xi_3)/(\beta_0+\hat\eta_1))_{S_{03}}$, whose image is $\mathcal{F}_{0BC}$.

For (iii), we consider the blowup $\pi\colon T\to Y$, with coordinate system

$$\pi_{0\beta\gamma}(r_1,r_2,\mu)=(s_0=r_1,\eta_1=\mu,x=-\alpha_2+r_1r_2)_{S_{01}}$$

near $\mathcal{F}_{0\beta\gamma}$ and exceptional divisor $E_{0\beta\gamma}=\{r_1=0\}$ and local coordinates

$$\pi_{0BC}(u_1,u_2,v)=(s_0=u_1,x=-\alpha_2+u_1u_2,\eta_3=v)_{S_{03}}$$

near $\mathcal{F}_{0BC}$ and exceptional divisor $E_{0BC}=\{u_1=0\}$. It follows that

$$f_T(r_1,r_2,\mu)=\pi_{0BC}^{-1}\circ f_Y\circ\pi_{0\beta\gamma}(r_1,r_2,\mu)=\left(r_1,r_2,\frac{\alpha_0+r_2}{\beta_0+\mu}\right).$$

When we set $r_1=0$, we have a dominant map from $E_{0\beta\gamma}$ to E_{0BC}.

For (iv) we use the blowups defined earlier; $f_T=\pi_{0BC}^{-1}\circ f\circ\pi$. In coordinates, this map is

$$f_T(t,\xi_2,\xi_3)=\left(t,\xi_3,\frac{\alpha_0+\xi_3}{\beta_0+\hat\eta_1+t\xi_2}\right)_{0BC}.$$

On the exceptional divisor $E_p=\{t=0\}$, we have $f_T(0,\xi_2,\xi_3)=(0,\xi_3,(\alpha_0+\xi_3)/(\beta_0+\hat\eta_1))_{0BC}$.

For (v) we parametrize the curve γ as $\xi_3\mapsto(0,\xi_3,\phi(\xi_3))_{0BC}$, where $\phi(s)=(\alpha_0+s)/(\beta_0+\hat\eta_1)$. The choose coordinates (v_1,s,v_2) such that the blowup map $\hat\pi\colon T_v\to T_{iv}$ is given by $\hat\pi(v_1,s,v_2)=(v_1,s,v_1v_2+\phi(s))_{T_{iv}}$, and the exceptional fiber is $E_\gamma=\{v_1=0\}$. The induced map is given by

$$f_{T_v}(t,\xi_2,\xi_3)=\hat\pi^{-1}(f_{T_{iv}}(t,\xi_2,\xi_3))=\left(t,\xi_3,\frac{1}{t}\left(\frac{\alpha_0+\xi_3}{\beta_0+\hat\eta_1+t\xi_2}-\phi(\xi_3)\right)\right).$$

Taking the limit as $t\to0$ gives the d/dt derivative of the last coordinate at $t=0$, and provides the map $f_{T_v}(0,\xi_2,\xi_3)=(0,\xi_3,-\xi_2(\alpha_0+\xi_3)/(\beta_0+\hat\eta_1)^2)$, so $f_{T_v}\colon E_p\to E_\gamma$ is dominant. □

Let us define a set $S\subset Y$ to be *totally invariant* if it is completely invariant for the total transform, or if for all $p\in S$, we have $(f_Y)_*p\subset S$ and for all $p\notin S$ we have $(f_Y)_*p\cap S=\emptyset$.

Lemma 4.5. *If f is critical, then Σ_{02} is indeterminate for f_Y. Each point of Σ_{02} blows up to $\mathcal{F}^1_{e_2}$, and $\mathcal{F}^1_{e_2}$ is mapped smoothly to Σ_{02}. The set $\Sigma_{02}\cup\mathcal{F}^1_{e_2}$ is totally invariant.*

Proof: Recall that $f\Sigma_{02} = e_2$, and the point e_2 was blown up. We consider points $[s : 1 : s\xi : x]$ which are close to Σ_{02} when s is small. We see that

$$f_Y : [s : 1 : s\xi : x] \mapsto \left[\frac{s}{x} : \frac{s\xi}{x} : 1 : s\frac{\alpha_0 s + \alpha_2 s\xi + x}{x(\beta_0 s + 1)}\right]$$
$$= \left(\frac{1}{\xi}, \frac{s\xi}{x}, s\frac{\alpha_0 s + \alpha_2 s\xi + x}{x(\beta_0 s + 1)}\right)_{01}.$$

Letting $s \to 0$ we see that $f_Y[0 : 1 : 0 : x] \rightsquigarrow \{(\eta, 0, 0)_{01}\}$. Using the same local coordinates we also see that

$$f_Y : \mathcal{F}^1_{e_2} \ni (\eta, 0, 0)_{01} \mapsto \left[0 : 1 : 0 : \frac{\alpha_2 \eta}{\beta_0 \eta + 1}\right] \in \Sigma_{02}.$$

For the second statement, we notice that from (4.4) $f_Y((\Sigma_{02} \cup \mathcal{F}^1_{e_2})^c - \mathcal{I}(f_Y))$ is disjoint from the set $\Sigma_{02} \cup \mathcal{F}^1_{e_2}$. Since $\mathcal{I}(f_Y) = \Sigma_{\beta\gamma} \cup \mathcal{F}_{0\beta\gamma} \cup \Sigma_{02}$ and $\Sigma_{\beta\gamma} \cap \Sigma_{02} = \emptyset$, we see that the sets $\Sigma_{02} \cup \mathcal{F}^1_{e_2}$ and $\Sigma_{\beta\gamma} \cup \mathcal{F}_{0\beta\gamma}$ are disjoint. It follows that $\Sigma_{02} \cup \mathcal{F}^1_{e_2}$ is totally invariant. □

From Lemma 4.3 we see that $\mathcal{I}(f_Y) \subset \Sigma_{\beta\gamma} \cup \mathcal{F}^1_{e_2} \cup \Sigma_{02} \cup \mathcal{F}_{0\beta\gamma}$. In the proof of Lemma 4.5 we see that $\mathcal{F}^1_{e_2} \cap \mathcal{I}(f_Y) = \emptyset$. It follows that

Corollary 4.6. *If f is critical, then $\mathcal{I}(f_Y) = \Sigma_{\beta\gamma} \cup \mathcal{F}_{0\beta\gamma} \cup \Sigma_{02}$ has pure dimension* 1.

The behavior of f_Y at Σ_{02} is, in suitable coordinates, given by the third model (1.5). The behavior of f_Y at $\mathcal{F}_{0\beta\gamma}$, as seen in Lemma 4.4, is different from the model (1.5). Further, we note that by Proposition 4.1 and the remark following it, the analogues of Lemmas 4.2–4.5 all hold for f_Y^{-1}. For instance, Σ_C is the unique exceptional hypersurface for f_Y^{-1}, $\mathcal{I}(f_Y^{-1}) = \mathcal{F}^1_{e_2} \cup \mathcal{F}_{0BC} \cup \Sigma_{BC}$, and each point of $\mathcal{F}_{0BC} - \Sigma_{BC}$ blows up under f_Y^{-1} to $\mathcal{F}_{0\beta\gamma}$.

Corollary 4.7. *If f is critical, then $f_Y^j \Sigma_\gamma \cap (\Sigma_{02} \cup \mathcal{F}^1_{e_2}) = \emptyset$ for all $j \geq 0$.*

Proof: By Lemma 4.5, it suffices to consider the case $j = 0$. By (4.1), $e_2 \notin \Sigma_\gamma$ in $\mathbf{P}^3$, so the fiber over e_2 remains disjoint from Σ_γ inside Y. Now $e_1 = \Sigma_{02} \cap \Sigma_\gamma$ in $\mathbf{P}^3$ and we see that Σ_{02} and Σ_γ are separated when we blow up e_1 to make Y. □

Recall that the degree complexity is $\delta(f) = \lim_{n\to\infty}(\deg(f^n))^{1/n}$. If $\delta(f) > 1$, then the degrees of the iterates f^n grow exponentially in n. In particular, f cannot be periodic if $\delta(f) > 1$.

Theorem 4.8. *If f is critical, and if $f_Y^n\Sigma_\gamma \not\subset \Sigma_{\beta\gamma}\cup\mathcal{F}_{0\beta\gamma}$, then the first dynamical degree is $\delta(f)\sim 1.32472$, the largest root of x^3-x-1.*

Proof: Using Corollary 4.7 we see that $f_Y^m\Sigma_\gamma\cap\mathcal{I}(f_Y)=\emptyset$ for all $m\ge 1$. Thus by Proposition 1.2 we have $(f_Y^*)^m=(f_Y^m)^*$ for all m. Thus $\delta(f)$ is the spectral radius of f_Y^*. Inside the Picard group $\mathrm{Pic}(Y)$, we let H_Y be the class of a generic hyperplane in Y, and we have

$$f_Y^*:\quad \begin{array}{l} H_Y\to 2H_Y-E_1-E_3-S_{01},\\ S_{01}\to E_3\to\Sigma_\beta=H_Y-E_3-S_{01},\\ E_1\to S_{03}\to\Sigma_0=H_Y-E_1-E_3-S_{01}-S_{03}.\end{array}\tag{4.5}$$

The computation of f_Y^* is standard; see [**BK1**], [**BK2**]. The characteristic polynomial of this transformation is $(x^2+1)(x^3-x-1)$, so $\delta(f)$ is as claimed. □

Now we give the existence of Green currents, which are invariant currents with the equidistribution properties given in the following:

Theorem 4.9. *If f is as in Theorem 4.8, then there is a positive closed current T_Y^+ in the class of α_Y^+ with the properties: $f_Y^*T_Y^+=T_Y^+$, and if Ξ^+ is a smooth form which represents α_Y^+, then $\lim_{n\to\infty}\delta_1(f)^{-n}f_Y^{n*}\Xi_Y^+=T_Y^+$ in the weak sense of currents on Y.*

Proof: Recall from Corollary 4.6 that $\mathcal{I}(f_Y)=\Sigma_{02}\cup\Sigma_{\beta\gamma}\cup\mathcal{F}_{0\beta\gamma}$. The total forward image of this set is $\pi_2\pi_1^{-1}\mathcal{I}(f_Y)=\mathcal{F}^1_{e_2}\cup\mathcal{F}_{0BC}\cup\Sigma_C$. We will show that if $\sigma\subset\pi_2\pi_1^{-1}\mathcal{I}(f_Y)$ is any curve, then $\alpha_Y^+\cdot\sigma\ge 0$. The theorem will then be a consequence of Theorem 1.3 of [**Ba**].

Up to a scalar multiple, we may write $\alpha_Y^+=H_Y-c_1E_1-c_3E_3-c_{01}S_{01}-c_{03}S_{03}$. Then since f_Y^* is given by (4.5) we have $1>c_1>c_3>c_{01}=c_{03}>0$, $c_1>c_{01}+c_{03}$, and $c_1+c_3=1$. Let us start with $\mathcal{F}_{0\beta\gamma}\subset\mathcal{I}(f_Y)$. Points of this curve are blown up to $\mathcal{F}_{0BC}$. The curve $\sigma=\mathcal{F}_{0BC}$ is the exceptional fiber inside S_{03} over the point $\Sigma_{BC}\cap\Sigma_{03}\in\mathbf{P}^3$. Thus $\sigma\cdot S_{03}=-1$, so $\alpha_Y^+\cdot\sigma=c_{03}>0$. Points of the indeterminate curve Σ_{02} blow up to $\sigma=\mathcal{F}^1_{e_2}$. In this case, we have that $\sigma\cdot S_{01}$ and $\sigma\cdot S_{03}$, are ±1, with opposite signs, depending on the order of blowup of Σ_{01} and Σ_{03}. Thus $\sigma\cdot\alpha_Y^+=\pm c_{01}\mp c_{03}=0$.

The other possibility is that $\sigma\subset\Sigma_C$. In this case, we have $\sigma\cdot H=\deg(\sigma)$. Further, if we let m_3 denote the multiplicity of σ at e_3, then σ is represented by $\deg(\sigma)L-m_1\mathcal{F}^1_{01}-m_2\mathcal{F}^1_{03}-m_3\epsilon_3$, where $\mathcal{F}^1_{01}$ and $\mathcal{F}^1_{03}$ represent fibers of S_{01} and S_{03}, and $\epsilon_3=E_3\cap\Sigma_C$. The multiplicities m_1, m_2, m_3 are bounded above by $\deg(\sigma)$. Since $\mathcal{F}^1_{01}\cdot S_{01}=\mathcal{F}^1_{03}\cdot S_{03}=\epsilon_3\cdot E_3=-1$, we have $\sigma\cdot\alpha_Y^+\ge\deg(\sigma)(1-c_{01}-c_{03}-c_3)>0$. □

5. Pseudo-automorphisms

In this section we assume that f is critical. We let $\mu_j := f_Y^j \Sigma_\gamma$ denote the strict transforms of Σ_γ under the iterates of f_Y and we consider the condition:

$$f_Y^N \Sigma_\gamma = \Sigma_{\beta\gamma} \quad \text{for some} \quad N > 1. \tag{5.1}$$

Theorem 5.1. *If a critical map f satisfies* (5.1)*, then there is a blowup space $\pi\colon Z \to Y$ such that f_Z is a pseudo-automorphism.*

Before giving the proof of Theorem 5.1 we give the statements of some lemmas that we will use. The proofs of these lemmas involve blowup computations similar to the proof of Lemma 4.4, so we omit them.

Lemma 5.2. *Let $\pi\colon T \to Y$ be the blowup of the curve Σ_{BC}, and let E denote the exceptional fiber. Then the induced map f_T gives a dominant map $f_T\colon \Sigma_\gamma \dashrightarrow E$.*

Lemma 5.3. *Let L be a line in Σ_γ passing through e_1, so L is exceptional for f_Y and is mapped to a point $p \in \Sigma_{BC}$. Let $\pi\colon T \to Y$ be the space obtained by blowing up p and then the line L. Let E_p and E_L denote the corresponding blowup divisors. Then the induced map $f_T\colon E_L \dashrightarrow E_p$ is dominant.*

The indeterminacy locus is $\mathcal{I}(f_Y) = \Sigma_{\beta\gamma} \cup \mathcal{F}_{0\beta\gamma} \cup \Sigma_{02}$. Note that the each point of Σ_{02} blows up to $\mathcal{F}^1_{e_2}$ and $\Sigma_{02} \cup \mathcal{F}^1_{e_2}$ is totally invariant by Lemma 4.5. If $p \in \Sigma_{\beta\gamma} - \mathcal{F}_{0\beta\gamma}$, then there is a line $L_p \subset \Sigma_C$, passing through e_3 such that $f_* p = L_p$. If $p = \Sigma_{\beta\gamma} \cap \mathcal{F}_{0\beta\gamma}$, then $f_* p = L_p \cup \mathcal{F}_{0BC}$.

Lemma 5.4. *Suppose that $p \in \Sigma_{\beta\gamma}$. Let $\pi\colon T \to Y$ be the space obtained by blowing up p and L_p; let E_p and E_L denote the corresponding exceptional divisors. Then it follows that f_T induces a dominant map $f_T\colon E_p \dashrightarrow E_L$.*

Proof of Theorem 5.1: With N as in (5.1) and $1 \le j \le N$, we consider the strict transform $\mu_j := f_Y^j(\Sigma_\gamma)$. Let $\Lambda_0 = \{\mu_j : \dim(\mu_j) = 0\}$ and let $\Lambda_1 = \{\mu_j : \dim(\mu_j) = 1\}$. Let $\pi\colon Z \to X$ denote the space obtained by blowing up first the points in Λ_0 and then the curves $\mu_j \in \Lambda_1$, in the order of increasing j. Let M_j denote the exceptional divisor over μ_j.

Now let $f_Z\colon Z \dashrightarrow Z$ denote the induced map. We will show that the exceptional set is $\mathcal{E}(f_Z) = \emptyset$. This will be sufficient, since a similar argument will show that $\mathcal{E}(f_Z^{-1}) = \emptyset$. Clearly, $\mathcal{E}(f_Z) \subset \Sigma_\gamma \cup \bigcup_{1 \le j \le N} M_j$. By Lemma 5.2, we have that $f_Z\colon \Sigma_\gamma \dashrightarrow M_1$ is dominant. Thus Σ_γ is not exceptional for f_Z. Now we will move forward in the space Z, following above the μ_j. As we move forward we see that for each j,

$f_Z\colon M_j \dashrightarrow M_{j+1}$ is dominant if there is a point of μ_j where f_Y is a local diffeomorphism. We may continue in this fashion unless one of the following happens: (a) $\mu_j \subset \Sigma_\gamma$, or (b) $\mu_j \subset \mathcal{I}(f_Z)$.

Case 1: $\mu_j \cap (\mathcal{F}_{0\beta\gamma} - \Sigma_{\beta\gamma}) = \emptyset$ for all j. If (a) occurs, then μ_{j+1} is a point of $\mu_1 = \Sigma_{BC}$. We know that $f_Z\colon M_j \dashrightarrow M_{j+1}$ is dominant by Lemma 5.3. Now for $j < \ell$, if $\mu_\ell \notin \mathcal{I}(f_Y)$, then f_Y is a local diffeomorphism in a neighborhood of μ_ℓ. Thus f_Z establishes an isomorphism between M_ℓ and $M_{\ell+1}$, and in particular is a dominant map. It is also possible that $\mu_\ell \in \mathcal{I}(f_Y)$. By the assumption of this case, we must have $\mu_\ell \in \Sigma_{\beta\gamma}$. Now we know that $f_Z\colon M_\ell \dashrightarrow M_{\ell+1}$ is dominant by Lemma 5.4. Now we continue with dominant maps until we reach $M_N \dashrightarrow \Sigma_C$, which is dominant by the proof of Lemma 5.2, applied to f_Y^{-1}.

Case 2: $\mu_j = \mathcal{F}_{0\beta\gamma}$ for the first j for which $\mu_j \cap (\mathcal{F}_{0\beta\gamma} - \Sigma_{\beta\gamma}) \neq \emptyset$. By Lemma 4.4(iii), we have that $f_Z\colon M_j = E_{0\beta\gamma} \dashrightarrow M_{j+1} = E_{0BC}$ is dominant. Thus we will continue to have dominant maps $M_\ell \dashrightarrow M_{\ell+1}$ until either (a) or (b) occurs. The only possibility that was not dealt with in Case 1 above is that μ_ℓ might be a point of $\mathcal{F}_{0\beta\gamma} - \Sigma_{\beta\gamma}$. Now we will apply Lemma 4.4(ii) with $p = \mu_\ell$, so we conclude that $f_Z(E_p)$ is a curve in $E_{0BC} = M_{j+1}$, and thus we must have $\mu_{\ell+1} = \mu_{j+1}$. On the other hand, we have continued by dominant maps of the blowup divisors M_k, $1 \le k \le \ell$. Thus we have that $f_Y^{\ell+1}\colon \Sigma_\gamma \dashrightarrow E_{0BC} = M_{\ell+1}$ is dominant. On the other hand, we already had a dominant map $f_Y^{j+1}\colon \Sigma_\gamma \dashrightarrow E_{0BC}$. This is not possible since f_Y is birational. We conclude that μ_ℓ is back in Case 1 for all $\ell > j$, which finishes the proof of Case 2.

Case 3: For some $\ell \ge 1$, $f_Y(\mathcal{F}_{0BC}), f_Y^2(\mathcal{F}_{0BC}), \ldots, f^\ell(\mathcal{F}_{0BC})$ are curves, and $f_Y^\ell(\mathcal{F}_{0BC}) = \mathcal{F}_{0\beta\gamma}$. Let us start with $\pi'\colon Y' \to Y$ which is the space Y blown up at the orbit of the curves $f_Y^k \mathcal{F}_{0BC}$ for $0 \le k \le \ell$. Now we follow the proof of Cases 1 and 2 above, except that we define the $\mu_j := f_{Y'}^j(\Sigma_\gamma)$ in terms of iteration of $f_{Y'}$. The only difference now is that for (b), we might have a point $\mu_j \in \mathcal{F}_{0\beta\gamma} - \Sigma_{\beta\gamma}$. In this case, we see from Lemma 4.4(ii) that μ_{j+1} must be a curve in E_{0BC}, and by Lemma 4.4(v), the induced map on the blowup divisors is dominant. Thus we are effectively back in Case 1.

Case 4: None of the above. In this case, μ_j is a point of $\mathcal{F}_{0\beta\gamma} - \Sigma_{\beta\gamma}$ for some j, and $\mathcal{F}_{0BC}$ is not part of an invariant cycle of curves. We will show that this case does not occur. Suppose that j is the first time that μ_j is in $\mathcal{F}_{0\beta\gamma} - \Sigma_{\beta\gamma}$. Then by Lemma 4.4(i), we have $\mu_{j+1} = \mathcal{F}_{0BC}$. Now the possibilities of the subsequent future of μ_ℓ, $\ell \geq j+1$ are either (a) or (b). Possibility (b) cannot happen first, for this would mean that μ_ℓ is a curve, and the only way that μ_ℓ can be contained in the indeterminacy locus $\mathcal{I}(f_Y)$ is that $\mu_\ell = \mathcal{F}_{0\beta\gamma}$ or $\mu_\ell = \Sigma_{\beta\gamma}$. The first option is exactly Case 3, which is assumed not to happen. The with the second option, we now consider f_Y^{-1}, starting with Σ_C, going backwards through $\Sigma_{\beta\gamma}$, and continuing until we reach $\mu_{j+1} = \mathcal{F}_{0BC}$. Thus we are in Case 2, with f_Y replaced by f_Y^{-1}. However, in this case μ_j cannot be a point of $\mathcal{F}_{0\beta\gamma}$.

This shows that the future of μ_ℓ must first encounter possibility (a). Thus there exists ℓ_0 such that μ_{ℓ_0} is an exceptional line in Σ_γ, which means that μ_{ℓ_0+1} is a point in μ_1. Now we consider the subsequent future of $\mu_{\ell+1}$. The orbit of μ_1 enters $\mathcal{F}_{0\beta\gamma}$ after j steps. We conclude that μ_ℓ must be contained in $\hat{\mu} := \bigcup_{1\leq k\leq \ell_0} \mu_k$ unless we have $\mu_\ell \in \mathcal{I}(f_Y)$ at some stage. The only possibilities for this are to have $\mu_\ell \in \Sigma_{\beta\gamma}$. There is no loss of generality for us to start with $\pi'\colon Y' \to Y$, which is the blowup of $\mathcal{F}_{0\beta\gamma}$ and $\mathcal{F}_{0BC}$, and we let $\mu_j := f^j_{Y'}(\Sigma_\gamma)$ be the strict transform inside of Y'. In this case, we have dominant maps on all of the blowup divisors M_ℓ. Thus if $\mu_\ell \in \Sigma_{\beta\gamma}$, then by Lemma 5.4, we have $\mu_{\ell+1} = L_{\mu_\ell}$, as in the notation given just before Lemma 5.4. Thus $\mu_{\ell+1}$ is a curve, and as we move forward, this curve cannot become $\mu_{\ell_1} = \Sigma_{\beta\gamma}$ for some ℓ_1 for the reason given in the previous paragraph. The other possibilities are either: (a) in which case we have $\mu_{\ell_1+1} \subset \mu_1$, or (b) $\mu_{\ell_1} = \mathcal{F}_{0\beta\gamma}$, in which case we have $\mu_{\ell_1+1} = \mathcal{F}_{0BC}$. With both of these possibilities, we are back inside of $\hat{\mu}$. Thus (5.1) cannot hold. $\square$

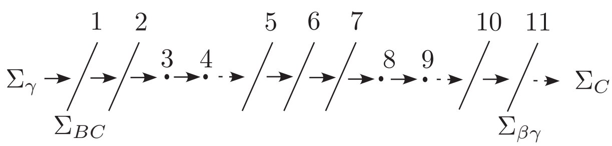

Figure 1. A hypothetical orbit: $d_1=2$, $u_1=4$, $d_2=7$, $u_2=9$, $m_d=m_u=2$, $N=11$.

Now let us introduce some notation to describe how the dimensions of the varieties μ_j can change. Let $m_{\mathcal{F}}$ be the number such that $\mu_{m_{\mathcal{F}}+2} = \mathcal{F}_{0\beta\gamma} \cap \Sigma_{\beta\gamma}$ if this case occurs. Otherwise we set $m_{\mathcal{F}} = \infty$. Similarly we set $m_{c\mathcal{F}}$ be the number such that $\mu_{m_{c\mathcal{F}}+1} = \mathcal{F}_{0BC} \cap \Sigma_{BC}$ if this case occurs and $m_{c\mathcal{F}} = \infty$ otherwise. Let m_d be the number of positive integers $d_1 < d_2 < \cdots < d_{m_d}$ denote the iterates for which $1 = \dim \mu_{d_j} >$

$\dim \mu_{d_j+1} = 0$ and $\mu_{d_j+1} \neq \Sigma_{BC} \cap \mathcal{F}_{0BC}$. Similarly, we let m_u be the number of positive integers $u_1 < u_2 < \cdots < u_{m_u}$ for which the dimensions go up, i.e., $0 = \dim \mu_{u_j} < \dim \mu_{u_j+1} = 1$ and $\mu_{u_j} \neq \Sigma_{\beta\gamma} \cap \mathcal{F}_{0\beta\gamma}$. We also let m_s be the number of positive integers $s_1 < s_2 < \cdots < s_{m_s}$ such that $\mu_{s_j+2} \subset \mathcal{F}_{0\beta\gamma}$ and $\mu_{s_j+2} \neq \mathcal{F}_{0\beta\gamma} \cap \Sigma_{\beta\gamma}$.

To illustrate this numbering scheme, a hypothetical orbit of Σ_γ is given in Figure 1. Here we have assumed that we are in the simpler case $m_{\mathcal{F}} = m_{c\mathcal{F}} = \infty$ and $m_s = 0$, which means that the orbit never enters $\mathcal{F}_{0\beta\gamma}$.

We use the numbers m_s, m_u, m_d, s_j, u_j, d_j, $m_{\mathcal{F}}$, $m_{c\mathcal{F}}$ and N to define four Laurent polynomials:

$$Q_1 := -1 - \sum_{j=1}^{m_d} \frac{1}{t^{d_j}} + \frac{1}{t} \sum_{j=1}^{m_s} \frac{1}{t^{s_j}} + \frac{1}{t^{m_{\mathcal{F}}}} \left(\frac{1}{t} + \frac{1}{t^2} \right) - \frac{1}{t^{m_{c\mathcal{F}}}},$$

$$Q_2 := \sum_{j=1}^{m_s} \frac{1}{t^{s_j}} \left(\frac{1}{t} + \frac{1}{t^2} + \frac{1}{t^3} + \frac{1}{t^4} \right) + \frac{1}{t^{m_{\mathcal{F}}}} \left(\frac{1}{t} + \frac{1}{t^2} \right) + \frac{1}{t^{m_{c\mathcal{F}}}} \left(\frac{1}{t} + \frac{1}{t^2} \right),$$

$$Q_3 := -1 - \sum_{j=1}^{m_d} \frac{1}{t^{d_j}} + \sum_{j=1}^{m_s} \frac{1}{t^{s_j}} \left(1 + \frac{1}{t} - \frac{1}{t^4} \right)$$
$$+ \frac{1}{t^{m_{\mathcal{F}}}} \left(1 + \frac{1}{t} + \frac{1}{t^2} \right) - \frac{1}{t^{m_{c\mathcal{F}}}} \left(1 + \frac{1}{t^2} \right),$$

$$Q_4 := -t - t\sum_{j=1}^{m_d} \frac{1}{t^{d_j}} - t\sum_{j=1}^{m_u} \frac{1}{t^{u_j}} - \frac{1}{t^{N-1}} - \sum_{j=1}^{m_s} \frac{1}{t^{s_j}} \left(\frac{1}{t} + \frac{1}{t^3} \right)$$
$$+ \frac{1}{t^{m_{\mathcal{F}}+1}} - \frac{t}{t^{m_{c\mathcal{F}}}} \left(1 + \frac{1}{t^2} \right).$$

With the Q_j, we can write the characteristic polynomial for f_Z^*:

Theorem 5.5. *If f is critical and* (5.1) *holds, then the dynamical degree of f is given by the largest root of the polynomial*

$$\chi_f(t) := t^{N-1}(t^2+1) \left[(Q_1 - Q_4)t^3 + (2Q_1 - Q_2 - Q_3 - Q_4)t^2 + (Q_1 - Q_3)t + Q_4 \right]. \tag{5.2}$$

The calculation to establish (5.2) is lengthy, so we defer it to Appendix A.

6. Periodic maps

In this section, we determine all possible periodic 3-step recurrences. By §3, we may assume (4.1). The question of periodicities for maps (4.1) with $\beta_0 = 0$ has been answered by Csörnyei and Laczkovich [**CL**]: they have shown that the only periodicities in this case are the two period 8 maps given in the theorem stated in the Introduction. We will consider the general case where β_0 is possibly nonzero. We start by giving a necessary condition for a map to be periodic.

Proposition 6.1. *If f is periodic and if E is an exceptional hypersurface, then there is an exceptional hypersurface E' for f^{-1} such that $f^n E = E'$ for some $n > 0$ and the co-dimension of $f^j E$ is ≥ 2 for all $j = 1, \dots, n-1$.*

Proof: Suppose f has period p. Since $f^p E = E$ and $\operatorname{codim} fE \geq 2$, it follows that there exists $0 < n \leq p$ such that $\operatorname{codim} f^{n-1} E \geq 2$ and $\operatorname{codim} f^n E = 1$. Thus $f^n E$ is an exceptional for f^{-1}. □

Since f is critical, $\dim f^j \Sigma_\beta < 2$ for $j = 1, 2$, and $f^3 \Sigma_\beta = \Sigma_0$; further, $\dim f^j \Sigma_0 < 2$ for $j = 1, 2$, and $f^3 \Sigma_0 = \Sigma_B$. By Lemma 4.2 the only exceptional hypersurface for f_Y is Σ_γ, and the only exceptional hypersurface for f_Y^{-1} is Σ_C. This gives us the following necessary condition for f to be periodic.

Corollary 6.2. *If f is periodic, then f is critical and there is some $n > 0$ such that $f_Y^n \Sigma_\gamma = \Sigma_{\beta\gamma}$ and $f_Y^{-n} \Sigma_C = \Sigma_{BC}$.*

Proof: If f is periodic, then so is f_Y. Since both f_Y and f_Y^{-1} have unique exceptional hypersurfaces, there exists $n \geq 0$ such that $f_Y^n \Sigma_\gamma = \Sigma_{\beta\gamma}$ which blows up to a hypersurface Σ_C. If f is periodic, then so is f^{-1} and thus $f_Y^{-n} \Sigma_C = \Sigma_{BC}$. □

A polynomial $p(z) = \sum_{i=0}^k a_i z^i$, $a_i \in \mathbf{C}$ is said to be *self-reciprocal* if $p(z) = \pm z^k \overline{p(1/\overline{z})}$.

Lemma 6.3. *If f is periodic, then $\chi_f(t)$ is self-reciprocal, and $\chi_f = \chi_{f^{-1}}$.*

Proof: If f is periodic, then the characteristic polynomial of f_Z^*, $\chi(t)$ is a product of cyclotomic factors and thus $\chi(t)$ is self-reciprocal. Furthermore by Theorem 5.1 f_Z is a pseudo-automorphism and therefore $(f_Z^*)^{-1} = (f_Z^{-1})^*$. It follows that χ_f and $\chi_{f^{-1}}$ are integer polynomials with the same roots. □

Lemma 6.4. *If f is periodic, then $m_{\mathcal{F}} = m_{c\mathcal{F}} = \infty$ and there is a non-negative integer m such that*

(i) $m = m_u = m_d < N$, $1 < d_1 < u_1 < \cdots < d_m < u_m < N$ *and*
(ii) $N - u_j = d_{m+1-j}, N - d_j = u_{m+1-j}$ *for* $j = 1, \dots, m$.

Proof: From (5.2) we see that the characteristic polynomial $\chi = \chi_f(t)$ is given by $\chi(t) = t^{N-1}(t^2+1)\varphi(t)$, where

$$\begin{aligned}\varphi(t) = T_s(t-1)\left(t+1+\frac{1}{t}\right)+\frac{1}{t^{N-1}}\left(t^N(t^3-t-1)+(t^3+t^2-1)\right)\\ + t(T_d(t^3-t-1)+T_u(t^3+t^2-1))\\ + \frac{1}{t^{m_{\mathcal{F}}+1}}(t^3+t^2-1)+\frac{1}{t^{m_{c\mathcal{F}}-1}}(t^3-1),\end{aligned} \tag{6.1}$$

where $T_s = \sum_{j=1}^{m_s}(1/t^{s_j})$, $T_d = \sum_{j=1}^{m_d}(1/t^{d_j})$, and $T_u = \sum_{j=1}^{m_u}(1/t^{u_j})$. By Lemma 6.3, $\chi(t)$ should be self-reciprocal. Since the first part of χ and the first line of (6.1) are self-reciprocal, it suffices to consider the case $m_s = 0$ and $m_u m_d \neq 0$. In this case $\dim f_Z^j\Sigma_\gamma = \dim f_Z^{j+1}\Sigma_\gamma$ if and only if $j \notin \{u_i, i = 1, \dots, m_u\} \cup \{d_i, i = 1, \dots, m_d\} \cup \{m_{\mathcal{F}}+2, m_{c\mathcal{F}}\}$. Thus it is clear that we have $m = m_u + 1 = m_d + 1 < N$ such that $\hat{d}_j \in \{d_i, i = 1, \dots, m_d\} \cup \{m_{c\mathcal{F}}\}$, $\hat{u}_j \in \{u_i, i = 1, \dots, m_u\} \cup \{m_{\mathcal{F}}+2\}$, and $1 < \hat{d}_1 < \hat{u}_1 < \cdots < \hat{d}_m < \hat{u}_m < N$ for some positive integer m. Thus we have

$$\begin{aligned}f_Z\colon \Sigma_\gamma \to \Sigma_{BC} \to \cdots \to f_Z^{\hat{d}_1}\Sigma_\gamma \to p_1 \in \Sigma_{BC} \to \cdots \to q_1 \in \Sigma_{\beta\gamma}\\ \rightsquigarrow f_Z^{\hat{u}_1+1}\Sigma_\gamma \subset \Sigma_C \to \cdots \to f_Z^{\hat{d}_2}\Sigma_\gamma \to \cdots \to f_Z^N\Sigma_\gamma = \Sigma_{\beta\gamma} \rightsquigarrow \Sigma_C.\end{aligned}$$

By interchanging the roles of Σ_β, Σ_γ and Σ_B, Σ_C, we see that the characteristic polynomial for f^{-1} is given by $\hat{\chi}_{f^{-1}}(t) = t^{N-1}(t^2+1)\hat{\varphi}(t)$ where

$$\begin{aligned}\hat{\varphi}(t) = \frac{1}{t^{N-1}}\left(t^N(t^3-t-1)+(t^3+t^2-1)\right)\\ + t(T_u(t^3-t-1)+T_d(t^3+t^2-1))\\ + \frac{1}{t^{m_{c\mathcal{F}}+1}}(t^3+t^2-1)+\frac{1}{t^{m_{\mathcal{F}}-1}}(t^3-1).\end{aligned} \tag{6.2}$$

Since both f and f^{-1} have the same characteristic polynomial, by comparing χ_f and $\chi_{f^{-1}}$ we see that $m_{\mathcal{F}} = m_{c\mathcal{F}} = \infty$ and d_i, u_i satisfy conditions (i) and (ii). □

Lemma 6.5. *Suppose f is periodic.*

(i) *If m is even, then for all $j = 1, \ldots, m$, $2 \le u_j - d_j \le d_1$.*
(ii) *If m is odd, then $1 \le u_{(m+1)/2} - d_{(m+1)/2} \le d_1$ and for all $j \neq (m+1)/2$, $2 \le u_j - d_j \le d_1$.*

Proof: Suppose j_* is the smallest positive integer such that $u_{j_*} - d_{j_*} > d_1$. Then we have the following: (1) $f_Y^{d_{j_*}} \Sigma_\gamma$ is an exceptional line in Σ_γ; (2) $f_Y^{d_{j_*}+i} \Sigma_\gamma$ is a point in $f_Y^i \Sigma_\gamma$ for $i = 1, \ldots, d_1$; and (3) $f_Y^{d_{j_*}+d_1+1} \Sigma_\gamma = f_Y^{d_1+1} \Sigma_\gamma$, which is a point in Σ_{BC}. It follows that the exceptional hypersurface Σ_γ is pre-periodic, which contradicts to the hypothesis f is periodic. If $u_j - d_j = 1$, then $f_Y^{d_j+1} \Sigma_\gamma = \Sigma_{BC} \cap \Sigma_{\beta\gamma} = f_Y^{u_j} \Sigma_\gamma$. Thus the situation $u_j - d_j = 1$ can happen at most once, and by Lemma 6.4 we see that $(N - d_j) - (N - u_j) = u_{m-j+1} - d_{m-j+1} = 1$. It follows that $j = m - j + 1$ and thus $j = (m+1)/2$. □

Lemma 6.6. *Suppose f is critical and $m \ge 1$ then*

(i) *$d_1 \neq 1, 3, 4$.*
(ii) *If $d_1 = 2$, then $m = 1$, and either (i) $\alpha_0 = \alpha_2 = 1$ and $\beta_0 = 0$ or (ii) $\alpha_0 = \eta^2 \alpha_2 = \eta$, $\beta = \eta^2$ where $\eta^2 - \eta + 1 = 0$.*
(iii) *If $m \ge 2$ is odd, then for $j = 1, \ldots, m-1$, $d_{j+1} - u_j \ge 5$.*
(iv) *If $m \ge 2$ is even, then for $1 \le j \le m-1$ and $j \neq m/2$, $d_{j+1} - u_j \ge 5$ and $d_{m/2+1} - u_{m/2} \ge 4$.*

Proof: (i) $d_1 = 1$ means Σ_{BC} is a line through e_1 in Σ_γ. Since $\Sigma_{BC} = \{x_3 = 0, \alpha_0 x_0 + \alpha_2 x_1 + x_2 = 0\}$ and $\alpha_2 \neq 0$, it follows that $e_1 \notin \Sigma_{BC}$ and thus $d_1 \neq 1$. Since $\Sigma_{BC} \subset \Sigma_3$, we have $f_Y^3 \Sigma_\gamma = f_Y^2 \Sigma_{BC} \subset \Sigma_1$ which doesn't contain e_1. Furthermore $f_Y \Sigma_1 = \{[\beta_0 x_0 : \beta_0 x_2 : \beta_0 x_3 : \alpha_0 x_0 + \alpha_2 x_2 + x_3]\}$ if $\beta_0 \neq 0$ and $f_Y \Sigma_1$ is a line in the blowup fiber E_3 if $\beta_0 = 0$. It follows that $f_Y^4 \Sigma_\gamma$ does not contain e_1. Therefore $d_1 \neq 3$ or 4. The statement for (ii) can be confirmed by direct computation. For each $j \le m-1$, $f_Y^{u_j+1} \Sigma_\gamma$ is a line through e_3 in Σ_C which can be parametrized as $t \mapsto \{[1 : \mu : -\alpha_0 - \alpha_2 \mu : t]\}$ for some fixed $\mu \in \mathbf{C} \cup \{\infty\}$. By computing the forward iteration of $[1 : \mu : -\alpha_0 - \alpha_2 \mu : t]$ we see that $d_{j+1} - u_j \neq 1, 2$, or 3. Furthermore $d_{j+1} - u_j = 4$ if and only if $f_Y^{u_j+1} \Sigma_\gamma = \Sigma_C \cap \{(1 + \alpha_2 \beta_0) x_0 + \alpha_2 x_1 = 0\}$. It follows that $d_{j+1} - u_j = 4$ occurs only once. Suppose $d_{j+1} - u_j = 4$ for some $1 \le j \le m-1$. By Lemma 6.4 we see that $(N - u_j) - (N - d_{j+1}) = d_{m-j+1} - u_{m-j} = 4$. It follows that $j + 1 = m - j + 1$ and thus $j = m/2$. The statements (iii) and (iv) follow. □

Direct computation shows the following properties:

Lemma 6.7. *Suppose f is critical, then $\hat\varphi(t)$ defined in (6.2) satisfies*
(i) $\hat\varphi(1) = 0$, *and*
(ii) $\hat\varphi'(1) = 7(m+1) - (N + \sum_{j=1}^m (u_j - d_j))$.

Lemma 6.8. *Suppose that $m \geq 2$ and that f is critical satisfying* (5.1). *Then*

$$N + \sum_{j=1}^m (u_j - d_j) \quad \begin{array}{ll} \geq 9m+3 & \text{if } m \text{ is odd,} \\ \geq 9m+4 & \text{if } m \text{ is even.} \end{array}$$

Thus $\hat\varphi'(1) < 0$, so $\hat\varphi$ has a root greater than 1.

Proof: Suppose m is even. By Lemma 6.4 we see that

$$N+\sum_{j=1}^m (u_j - d_j) = 2d_1 + 4(u_1 - d_1) + 2(d_2 - u_1) + \cdots + 2(d_{m/2} - u_{m/2-1}) + 4(u_{m/2} - d_{m/2}) + (d_{m/2+1} - u_{m/2}).$$

By Lemma 6.6, (i) and (ii), we have $d_1 \geq 5$. Applying Lemma 6.5(i) and Lemma 6.6(iv) we have

$$N + \sum_{j=1}^m (u_j - d_j) \geq 2\cdot 5 + 4\cdot 2 + 2\cdot 5 + \cdots + 4\cdot 2 + 4 = 9m+4.$$

Similarly when m is odd

$$N+\sum_{j=1}^m (u_j - d_j) = 2d_1 + 4(u_1 - d_1) + 2(d_2 - u_1) + \cdots \cdots + 2(d_{(m+1)/2} - u_{(m-1)/2}) + 2(u_{(m+1)/2} - d_{(m+1)/2}).$$

Again applying Lemma 6.5(ii) and Lemma 6.6(iii) we have

$$N + \sum_{j=1}^m (u_j - d_j) \geq 2\cdot 5 + 4\cdot 2 + 2\cdot 5 + \cdots + 2\cdot 5 + 2 = 9m+3. \qquad \square$$

Theorem 6.9. *If f is periodic with $m = 0$ and $m_s = 0$, then f is one of the following:*

- $\alpha = (-1, 0, -1, 1)$, $\beta = (0, 1, 0, 0)$: *$f_{\alpha\beta}$ has period 8 and there is a conic Q such that*
  $$f_Y : \Sigma_\gamma \to \Sigma_{BC} \to Q \to \Sigma_{\beta\gamma} \rightsquigarrow \Sigma_C.$$
- $\alpha = (-1/2, 0, -1, 1)$, $\beta = (1, 1, 0, 0)$: *$f_{\alpha\beta}$ has period 12, and*
  $$f_Y : \Sigma_\gamma \to \Sigma_{BC} \to L_1 \to L_2 \to \Sigma_{\beta\gamma} \rightsquigarrow \Sigma_C,$$
  where we set $L_1 = \Sigma_2 \cap \{x_0 + x_3 = 0\}$ and $L_2 = \Sigma_1 \cap \{x_0 + x_2 = 0\}$.

Proof: The polynomial defined in (5.2) is also given by

$$\chi(t) = (t^2+1)\left(t^N(t^3-t-1)+t^3+t^2-1\right).$$

It follows that $\chi(t)$ has a root bigger than 1 if and only if $N \geq 8$ and in case $N=7$ the matrix representation of f_Z^* has 3×3 Jordan block with eigenvalue 1. Thus we need to check the situation $f^{n+1}\Sigma_\gamma = \Sigma_{\beta\gamma}$ only for $n \leq 5$. For this, let us parametrize $\Sigma_{BC} = \{[1:t:-\alpha_0-\alpha_2 t:0]\}$ and let $[f_0^{(n)}:f_1^{(n)}:f_2^{(n)}:f_3^{(n)}]$ denote the n^{th} iteration of Σ_{BC}. If $f^{n+1}\Sigma_\gamma = \Sigma_{\beta\gamma}$, then for all t we have

$$\beta_0 f_0^{(n)} + f_1^{(n)} = 0, \quad \text{and} \quad \alpha_0 f_0^{(n)} + \alpha_2 f_2^{(n)} + f_3^{(n)} = 0. \tag{6.3}$$

Since equations in (6.3) are polynomials in t whose coefficients are integer polynomials in the variables β_0, α_0, and α_2, we may use the computer show that for $0 \leq n \leq 5$, the only two possibilities are those listed above. □

Theorem 6.10. *If f is periodic with $m=1$ and $m_s=0$, then f is one of the following:*

- $\alpha = (1,0,1,1)$, $\beta = (0,1,0,0)$*: f has period* 8, $\Sigma_{BC}\cap\Sigma_{\beta\gamma} \neq \emptyset$, *and*

$$f_Y : \Sigma_\gamma \to \Sigma_{BC} \to \Sigma_\gamma \cap \Sigma_2 \to \Sigma_{BC}\cap\Sigma_{\beta\gamma} \rightsquigarrow \Sigma_C \cap \Sigma_2 \to \Sigma_{\beta\gamma} \rightsquigarrow \Sigma_C.$$

- $\alpha = (\eta/(1-\eta),0,\eta,1)$, $\beta = (\eta^2,1,0,0)$ *and* $\eta^3=-1$, $\eta \neq -1$*: $f_{\alpha\beta}$ has period* 12, *and*

$$f_Y : \Sigma_\gamma \to \Sigma_{BC} \to \Sigma_\gamma\cap\Sigma_2 \to p_1 \in \Sigma_{BC} \to p_2 \in \Sigma_{\beta\gamma}$$
$$\rightsquigarrow \Sigma_C \cap \Sigma_1 \to \Sigma_{\beta\gamma} \rightsquigarrow \Sigma_C,$$

  where $p_1 = [1:0:-\eta^2:0] \in \Sigma_{BC}$ *and* $p_2 = [1:-\eta^2:0:-\eta^2] \in \Sigma_{\beta\gamma}$.

Proof: From (5.2) the characteristic polynomial of f_Z^* is given by

$$\chi(t) = t^{N-(u_1+d_1)}(t^2+1)(t^{d_1}+1)\left(t^{u_1}(t^3-t-1)+t^3+t^2-1\right).$$

It follows that $\chi(t)$ has a root bigger than 1 if and only if $u_1 \geq 8$. If $u_1 = 7$, the f_Z^* has a 3×3 Jordan block. Thus if f_Z^* is periodic, then $d_1 \leq 5 < u_1$. By direct computation of $f^n\Sigma_\gamma = f^{n-1}\Sigma_{BC}$ for $n=1,\dots,5$, we can easily check the two conditions (i) $f^{n-1}\Sigma_{BC} \subset \Sigma_\gamma$, (ii) $f^{n-1}\Sigma_{BC} \subset \{x_3 = \lambda x_2\}$ for some $\lambda \in \mathbf{C}$ and thus we see that there are only two possibilities listed in this theorem. □

Theorem 6.11. *If $m \geq 2$, $m_s = 0$, then f has exponential degree growth (and is not periodic).*

Proof: By Lemmas 6.7 and 6.8 we see that $\chi_N(1) = 0$ and $\chi'_N(1) = 2\hat{\varphi}'(1) < 0$. Since the leading coefficient of χ_N is 1, there exist a real root which is strictly bigger than 1. It follows that the dynamical degree of f is strictly bigger than 1. □

Theorem 6.12. *If $1 \le m_s < \infty$, then f is not periodic.*

Proof: By Lemmas 6.7 and 6.8 we see that $\chi_N(1)=0$ and $\chi'_N(1)=2(3+\hat{\varphi}'(1))=2(-2m+1)$. It follows that if $m \ge 1$, then f has positive entropy. Now suppose $m = 0$, we have $\chi_N(t)=(t^2+1)\big(t^N(t^3-t-1)+t^{N-1}T_s(t-1)(t+1+1/t)+t^3+t^2-1\big)$. If f is periodic, then the characteristic polynomial for f_Z^* should be self-reciprocal. It follows that $s_j + s_{m_s+1-j} = N-4$. Thus we have

$$\chi_N(t)=(t^2+1)\left(t^N(t^3-t-1)+t^3+t^2-1\right)$$
$$+\frac{1}{2}(t^2+1)t^{N/2+3}(t-1)(t+1+1/t)\sum_{j=1}^{m_s}\left(t^{N/2-2-s_i}+t^{-N/2+2+s_i}\right).$$

By inspection we see that $s_1 \ge 3$ and it follows that $N \ge 8m_s + 2$. We can also check that $\chi_N(1) = 0$ and $\chi'_N(1) = 14-2N+6m_s \le 10(1-m_s)$. Therefore if $m_s > 1$, then f is not periodic. Now suppose $m_s = 1$. If $s_1 > 3$, then $N > 10$ and therefore $\chi'_N(1)$ is strictly negative. It follows that if $s_1 > 3$, f is not periodic. In case $s_1 = 3$, $m_s = 1$, the matrix representation for f_Z^* has 3×3 Jordan block with eigenvalue 1 and all other eigenvalues have modulus 1. □

Proof of Theorem 5: The statement of Theorem 5 in the Introduction follows from Theorems 6.9–6.12. □

We remark that in the proof of Theorem 6.12, we see that if $m_s = 1$, $s_1 = 3$ and $m = 0$, then the degree of f^n is quadratic in n. This case occurs for $\alpha = (a, 0, 1, 1)$ and $\beta = (0, 1, 0, 0)$, which is the so-called Lyness process and will be discussed in §8.

7. Pseudo-automorphisms with positive entropy

In this section we consider the case

$$\beta = (0, 1, 0, 0) \quad \text{and} \quad \alpha = (a, 0, \omega, 1), \tag{7.1}$$

where $\omega^2 + \omega + 1 = 0$ and $a \in \mathbf{C} \setminus \{0\}$. With this choice of parameters, we see that f is critical and that $\Sigma_B = \Sigma_3$ and $\Sigma_\beta = \Sigma_1$. Since the maps $f\colon \Sigma_3 \to \Sigma_2 \to \Sigma_1$ are dominant, (4.4) gives an 8-cycle of dominant maps

$$f_Y\colon \Sigma_1 \to E_3 \to S_{01} \to \Sigma_0 \to S_{03} \to E_1 \to \Sigma_3 \to \Sigma_2 \to \Sigma_1. \tag{7.2}$$

Since this 8-cycle is fundamental to our understanding of f in this case, we will refer to the union of these 8 hypersurfaces as the *rotor* and denote it as $\mathcal{R}$. Clearly, f_Y^8 fixes each component of the rotor; in addition, it has a relatively simple expression. On Σ_3, or example, we have:

$$\begin{aligned} f_Y^8 \colon \Sigma_3 \ni & [x_0 : x_1 : x_2 : 0] \\ & \mapsto [x_0(ax_0 + \omega x_2)(ax_0 + ax_1 + \omega x_2) \\ & : x_1(x_1x_2 + a\omega x_0^2 + a\omega x_0 x_1 + a\omega x_0 x_2 + \omega^2 x_0 x_2 + \omega^2 x_2^2) \\ & : \omega x_2(ax_0 + \omega x_2)(x_1 + a\omega x_0 + \omega^2 x_2) : 0] \in \Sigma_3. \end{aligned} \tag{7.3}$$

The restriction of f_Y^8 to the rotor is studied in Appendix C.

Note that by (7.1), $\Sigma_{BC} = \Sigma_3 \cap \Sigma_C$ and $\Sigma_{\beta\gamma} = \Sigma_1 \cap \Sigma_\gamma$. Using (7.2) we may verify that f_Y satisfies condition (5.1), which in this case is

$$f_Y^j \Sigma_\gamma \not\subset \mathcal{F}_{0\beta\gamma} \text{ for all } 1 \le j \le 10, \text{ and } f_Y^{11}\Sigma_\gamma = \Sigma_{\beta\gamma}. \tag{7.4}$$

We define the space $\pi_Z \colon Z \to Y$ by successively blowing up the 11 curves $\gamma_j := f_Y^j \Sigma_\gamma$, $1 \le j \le 11$. The dynamical degree, being a birational invariant, is independent of the order in which the γ_j's are blown up.

Theorem 7.1. *The induced map f_Z is a pseudo-automorphism, and the dynamical degree of f is greater than* 1*.*

Proof: From (7.4) we see that f_Y satisfies conditions (5.1) and (5.2) in Theorem 5.1, so f_Z is a pseudo-automorphism. By Lemma 5.3, the characteristic polynomial of f_Z^* is $t^{11}(t^3 - t - 1) + t^3 + t^2 - 1 = (-1 + t)(1 + t)(1 + t^4)(1 - t^3 - t^4 - t^5 + t^8)$. Thus $\delta(f)$ is the largest root of this polynomial, which is approximately 1.28064. □

The space Z has been defined earlier, but now let us be more precise: we define Z as the space obtained by blowing up first $\gamma_{11} \subset Y$, then we blow up the strict transform of γ_{10} in the resulting space, followed by blowing up the strict transform of γ_9, and continuing this way until we blow up the strict transform of γ_1. We will use the notation Γ_j to denote the exceptional divisor of the blowup of γ_j. There are no points where three distinct γ_j's intersect. If $p = \gamma_j \cap \gamma_k$, with $j > k$, then we blow up γ_j first, and we refer to the fiber in Γ_j over p as the *first* fiber over p, and write it as $\mathcal{F}_p^1$. We then blow up the strict transform of γ_k, and the blowup fiber over the point $\gamma_k \cap \Gamma_j$ is equal to $\Gamma_j \cap \Gamma_k$.

Let us describe some of the intersections of the γ_j's. f is constant on each line in Σ_γ passing through e_1. Further, $\gamma_1 \subset \Sigma_3$, $\gamma_2 \subset \Sigma_2$, and $\gamma_6 \subset \Sigma_0$, and $e_1 = \Sigma_0 \cap \Sigma_2 \cap \Sigma_3$. We set $\ell_2 = \Sigma_\gamma \cap \Sigma_3$, $\ell_3 = \Sigma_\gamma \cap \Sigma_2$, and $\ell_7 = \Sigma_\gamma \cap \Sigma_0$. Thus we have $f(\ell_j) = \gamma_1 \cap \gamma_j$ for $j = 2, 3, 7$. The curve $\gamma_9 \subset \Sigma_1$ is a conic, and $\gamma_9 \cap \gamma_1$ consists of two points. We let ℓ_9'

and ℓ_9'' denote the two lines in Σ_γ for which $f(\ell_9' \cup \ell_9'') = \gamma_1 \cap \gamma_9$. This accounts for all the curves γ_j which intersect γ_1. As a consequence of the order of blowup, the first fiber $\mathcal{F}^1_{f(\ell_j)} = \mathcal{F}^1_{\gamma_1 \cap \gamma_j}$, $j = 2, 3, 7$ is contained in Γ_j and similarly for ℓ_9', ℓ_9''.

There is a similar situation for the γ_j's which intersect γ_{11}. The curves γ_5, γ_9 and γ_{10} each intersect γ_{11} in a single point, and γ_3, which intersects γ_{11} in 2 points, and this accounts for all the intersection points between γ_{11} and the other γ_j's.

Let us use the notation $\pi_1 \colon Z_1 \to Y$ for the manifold obtained by blowing up the curve $\gamma_{11} \subset Y$. This is the first blowup performed in the construction of Z. Let $f_{Z_1} : Z_1 \dashrightarrow Z_1$ be the induced map. Since $\mathcal{I}(f_Y) = \gamma_{11} \cup \Sigma_{02} \cup \mathcal{F}^1_{0\beta\gamma}$, it follows that $\mathcal{I}(f_{Z_1}) \subset \Sigma_{02} \cup \mathcal{F}^1_{0\beta\gamma} \cup \gamma_{10} \cup \Gamma_{11}$.

Lemma 7.2. $\mathcal{I}(f_{Z_1}) = \mathcal{F}^1_{0\beta\gamma} \cup \Sigma_{02} \cup \gamma_{10}$.

Proof: We have seen already that the indeterminacy locus is contained in $\Sigma_{02} \cup \mathcal{F}^1_{0\beta\gamma} \cup \gamma_{10} \cup \Gamma_{11}$, so it suffices to show that $\mathcal{I}(f_{Z_1}) \cap \Gamma_{11}$ consists of the two points $\gamma_{10} \cap \Gamma_{11}$ and $\mathcal{F}^1_{0\beta\gamma} \cap \Gamma_{11}$. Thus we look at f_{Z_1} in coordinate charts that cover Γ_{11}. We will look first at $\Gamma_{11} \cap \pi^{-1}(\gamma_{11} - \gamma_{10})$.

In the local coordinates $(s, \zeta, x_3)_{S_{01}} \mapsto [s : s\zeta : 1 : x_3] \in \mathbf{P}^3$ in the neighborhood of $S_{01} - E_1 = \{s = 0, \zeta \neq \infty\}$, we have $\gamma_{11} = \{\zeta = 0,\, as + \omega + x_3 = 0\}$ and $\mathcal{F}^1_{0\beta\gamma} \cap \gamma_{11} = (0, 0, -\omega)_{S_{01}}$. We use the local coordinate charts $(s, t, \eta)'$ on U' and $(s, \eta, t)''$ on U'' so that π_1 is given by

$$\pi' \colon U' \ni (s, t, \eta)' \mapsto (s, t, -as - \omega + t\eta)_{S_{01}},$$
$$\pi'' \colon U'' \ni (s, \eta, t)'' \mapsto (s, t\eta, -as - \omega + t)_{S_{01}}.$$

It is evident that $\Gamma_{11} \supset \{t = 0\}$ in both coordinate charts, and $U' \cup U'' \supset \pi_1^{-1}(\gamma_{11} - \gamma_{10})$. The induced map $f \circ \pi_1 \colon U' \cup U'' \to \mathbf{P}^3$ is given by

$$\begin{aligned} &U' \ni (s, t, \eta)' \mapsto [s : 1 : -as + t\eta - \omega : \eta], \\ &U'' \ni (s, \eta, t)'' \mapsto [s\eta : \eta : \eta(-as + t - \omega) : 1]. \end{aligned} \tag{7.5}$$

So we see that $\{t = 0\}$ is mapped to Σ_C.

From (7.5) we see that the map $f \circ \pi_1 \colon U' \cup U'' \to \mathbf{P}^3$ is everywhere regular. The only points of Σ_C which is blown up in the construction of Y are e_3 and $[0 : 1 : -\omega : 0]$ which is the base point of $\mathcal{F}^1_{0BC}$. By (7.5), the preimage of e_3 is $(0, 0, 0)'' \in U''$, and the preimage $[0 : 1 : -\omega : 0]$ is $(0, 0, 0)' \in U'$. Working in local coordinates in Y over e_3, we find that $f \circ \pi_1 \colon U'' \to Y$ is everywhere regular. Thus we conclude that $f_Y \circ \pi_1 \colon Z_1 \to Y$ is regular on $(U' - (0, 0, 0)') \cup U''$. Now in order to pass to f_{Z_1} we need to consider the point $\gamma_{11} \cap \Sigma_C$ which is blown up. However, this is the image point of $\gamma_{10} \cap \gamma_{11}$, which is not in our coordinate chart. We note that $(0,0,0)'$ is the point $\mathcal{F}^1_{0\beta\gamma} \cap \Gamma_{11}$, so we conclude that f_{Z_1} is regular at all points of $\Gamma_{11} - (\pi_1^{-1}(\gamma_{11} \cap \gamma_{10}) \cup \mathcal{F}^1_{0\beta\gamma})$.

Now we consider $\Gamma_{11} \cap \pi^{-1}(\gamma_{11} - \mathcal{F}^1_{0\beta\gamma})$, which does not lie over any of the centers of blowup in the construction of Y. We use the local coordinates $(s, x_2, \zeta) \mapsto [1 : s : x_2 : -a - \omega x_2 + s\zeta]$ in a neighborhood of $\{s = 0, \zeta \neq \infty\} \subset \Gamma_{11} - \pi_1^{-1}(\mathcal{F}^1_{0\beta\gamma})$, and we get

$$f_{Z_1} : \Gamma_{11} \ni (0, x_2, \zeta) \mapsto [1 : x_2 : -a - \omega x_2 : \zeta] \in \Sigma_C \quad \text{if } (0, x_2, \zeta) \neq (0, 0, a\omega - a). \tag{7.6}$$

Similarly using the local coordinates $(\zeta, x_2, s) \mapsto [1 : s\zeta : x_2 : -a - \omega x_2 + s] \in \Gamma_{11}$, we have

$$f_{Z_1} : \Gamma_{11} \ni (\zeta, x_2, 0) \mapsto [\zeta : x_2\zeta : \zeta(-a - \omega x_2) : 1] \in \Sigma_C \quad \text{if } (\zeta, x_2, 0) \neq \left(\frac{1}{a\omega - a}, 0, 0\right). \tag{7.7}$$

Since both $(0, 0, a\omega - a)$ in (7.6) and $(1/(a\omega - a), 0, 0)$ in (7.7) correspond to the point $\gamma_{11} \cap \gamma_{10}$, combining with the previous conversation about $\Gamma_{11} - \pi_1^{-1}(\gamma_{11} \cap \gamma_{10})$, we conclude that f_{Z_1} is regular at all points of $\Gamma_{11} - (\gamma_{10} \cup \mathcal{F}^1_{0\beta\gamma})$. □

Lemma 7.3. *The three curves γ_5, γ_{11}, $\mathcal{F}^1_{0\beta\gamma}$ intersect transversally inside Y, and γ_1, γ_7, $\mathcal{F}^1_{0BC}$ intersect transversally inside Y. Thus, inside Z_1, the strict transform of $\mathcal{F}^1_{0\beta\gamma}$ is disjoint from the strict transforms of γ_j, $1 \leq j \leq 10$.*

Proof: It suffices to prove the first statement. We may write $\gamma_{11} \subset \mathbf{P}^3$ as $s \mapsto [s : 0 : 1 : -as - \omega]$. This intersects Σ_{01} in the point $[0 : 0 : 1 : -\omega]$. We use the coordinate system $\pi \colon (u, \eta, x_3) \mapsto [u : u\eta : 1 : x_3] \in \mathbf{P}^3$. Thus $\mathcal{F}^1_{0\beta\gamma} = \{u = 0, x_3 = -\omega\}$. In this coordinate system, γ_{11} becomes $s \mapsto (s, 0, -as - \omega)$, so γ_{11} crosses S_{01} when $s = 0$, at the point $(0, 0, -\omega)$. On the other hand, if we map γ_{11} backward under f_Y^{-6}, we find an expression for γ_5. The base point is given by $[0 : 0 : 1 : s]$, and the fiber coordinate is given by $\eta = (1+as)(1+as+\omega(1+a-s))/(as(-1+s+as-\omega(1+a-as))$. Thus when the base point is $[0 : 0 : 1 : -\omega]$, we have $\eta = 0$. Thus all three curves meet at $(u, \eta, x_3) = (0, 0, -\omega)$. The curve γ_{11} is transverse to S_{01}, but γ_5 and $\mathcal{F}^1_{0\beta\gamma}$ are tangential to S_{01}, so γ_{11} is transverse to the other two, and γ_5 is transverse to $\{x_3 = -\omega\}$, while $\mathcal{F}^1_{0\beta\gamma}$ is tangential to this set. □

For $2 \leq j \leq 11$, let $\pi_j\colon Z_j \to Z_{j-1}$ be the blowup of the strict transform of γ_{12-j} inside Z_{j-1} and $\pi\colon Z \to Y = \pi_{11} \circ \pi_{10} \circ \cdots \circ \pi_1$, that is, we blowup γ_{11} first, then γ_{10}, then γ_9, etc. Let $f_{Z_j}: Z_j \dashrightarrow Z_j$, $f_Z\colon Z \dashrightarrow Z$ denote the induced map.

Lemma 7.4. *For $1 \leq j \leq 10$, $\mathcal{I}(f_{Z_j}) = \mathcal{F}^1_{0\beta\gamma} \cup \Sigma_{02} \cup \gamma_{11-j}$.*

Proof: Suppose p is a point of $\gamma_j \cap \gamma_k$, $1 \leq j < k \leq 10$. Because of the order of blowup, γ_k is blown up before γ_j and γ_{k+1} is blown up before γ_{j+1}. Since f_Y is regular at p and the order of blowups at p is consistent with the order of blowups at $f_Y(p)$, the induced map f_{Z_i} is a local biholomorphism in a neighborhood of the exceptional divisor over p for $12 - j \leq i \leq 11$.

Notice that for all $1 \leq j \leq 11$ the strict transform of γ_j does not intersect Σ_{02} in Y. Suppose γ_j intersects $\mathcal{F}^1_{0\beta\gamma}$ at a point q. Using the local coordinates in the neighborhood $(s, \zeta, x_3)_{S_{01}}$, we may assume that $q = (0, \zeta_*, -\omega)_{S_{01}}$ and $\gamma_j(s) = (Q_1(s), Q_2(s) + \zeta_*, Q_3(s) - \omega)_{S_{01}}$, where $\gamma_j = \{\gamma_j(s),\, s \in \mathbf{C}\}$, and $\gamma_j(0) = q$. Consider two local coordinate charts covering the exceptional divisor over the point q:

$$(s, t, \eta) \mapsto (Q_1(s), Q_2(s) + \zeta_* + t, Q_3(s) - \omega + t\eta)_{S_{01}},$$
$$(s, \eta, t) \mapsto (Q_1(s), Q_2(s) + \zeta_* + t\eta, Q_3(s) - \omega + t)_{S_{01}}.$$

With a computation similar to Lemma 7.2, we see that the induced map is regular everywhere on the exceptional divisor over q, $\mathcal{F}(q)$, except the point of intersection $\mathcal{F}(q) \cap \mathcal{F}^1_{0\beta\gamma}$. Now since the curve γ_{11-j} is the pre-image of γ_{12-j}, we have $\mathcal{I}(f_{Z_j}) = \mathcal{F}^1_{0\beta\gamma} \cup \Sigma_{02} \cup \gamma_{11-j}$. □

From the previous lemma we have $\mathcal{I}(f_{Z_{10}}) = \mathcal{F}^1_{0\beta\gamma} \cup \Sigma_{02} \cup \gamma_1$. Since Σ_γ is the pre-image of γ_1, we have $\mathcal{I}(f_Z) \subset \mathcal{F}^1_{0\beta\gamma} \cup \Sigma_{02} \cup \Sigma_\gamma$. From (5.2) we see that for all most every line $\ell \subset \Sigma_\gamma$, through e_1 in Σ_γ, f maps ℓ regularly to a point $q \in \gamma_1$. In our construction of Z, we blew up $\gamma_{11}, \ldots, \gamma_2$ before γ_1. Thus the map f_Z will map ℓ regularly to the fiber of Γ_1 over q unless q is an intersection point of $\gamma_1 \cap \gamma_j$ for some $2 \leq j \leq 11$.

Lemma 7.5. *Suppose $q \in \gamma_1 \cap \gamma_j$ for some $j = 2, \ldots, 11$ and $\ell_j \subset \Sigma_\gamma$ be the line which mapped to q by f_Y. The line $\ell_j \subset \mathcal{I}(f_Z)$ and every point in ℓ_j blows up to the first blowup fiber $\mathcal{F}^1_q$.*

Proof: Let us parametrize $\gamma_1 = \{\gamma_1(t) = [-\frac{1}{a}(1 + \omega t) : t : 1 : 0],\, t \in \mathbf{C}\}$. Let us set $q = \gamma_1(t_*)$ for some $t_* \in \mathbf{C}$ and $\gamma_j = \{\gamma_j(s) = [Q_0(s) - \frac{1}{a}(1 + \omega t_*) : Q_1(s) + t_* : Q_2(s) + 1 : Q_3(s)]\}$. The line ℓ_j is given by the strict transform in Y of the line connecting e_1 and $\tilde{q} = [-\frac{1}{a}(1 + \omega t_*) : 0 : t_* : 1]$ in $\mathbf{P}^3$. To see the image of the line ℓ_j, we consider the set

$U = \{[-\frac{1}{a}(1+\omega t_*) + s\zeta : u : t_* + s : 1]\}$ which has the property that $U \cap \{s = 0\} = \ell_j - \{e_1\}$. Since the point q is blown up twice, let us consider a local coordinate charts for $\pi_{12-j}^{-1}(\gamma_j)$:

$$(v, \xi, s)_{\gamma_j} \mapsto \left[\frac{Q_0(s) - \frac{1}{a}(1+\omega t_*)}{Q_2(s)+1} + v : \frac{Q_1(s) + t_*}{Q_2(s)+1} + v\xi : 1 : \frac{Q_3(s)}{Q_2(s)+1}\right].$$

Using the induced map f_Z, we see that

$$f_Z \colon \ell_j \ni [-\frac{1}{a}(1+\omega t_*) : u : t_* : 1] \rightsquigarrow \{(0,\xi,0), \xi \in \mathbf{C}\} \subset \Gamma_j,$$

that is, each point in ℓ_j blows up to a whole first blowup fiber over q. □

Before Lemma 7.2, we enumerated the possibilities for lines ℓ and points q as in the hypotheses of Lemma 7.5. Thus we may combine Lemmas 7.2–7.5 to have the following theorem:

Theorem 7.6. *The indeterminacy locus $\mathcal{I}(f_Z) = \Sigma_{02} \cup \mathcal{F}^1_{0\beta\gamma} \cup \ell_2 \cup \ell_3 \cup \ell_7 \cup \ell_9' \cup \ell_9''$. If ζ is a point of one of the lines ℓ, then f_Z blows up ζ to the first fiber $\mathcal{F}^1_{f(\ell)}$.*

Now we give the existence of Green currents for the invariant class $\alpha = \alpha_Z^+ \in H^{1,1}(Z)$.

Theorem 7.7. *There is a positive closed current T_Z^+ in the class of α_Z^+ with the property: if Ξ^+ is a smooth form which represents α_Z^+, then $\lim_{n\to\infty} \delta_1(f)^{-n} f_Z^{n*} \Xi_Z^+ = T_Z^+$ in the weak sense of currents on Z.*

Proof: The map f_Z^* is given in Appendix A, where we are in Case (II). Working directly with the matrix (A.1), we see that the invariant class is given by:

$$\alpha = H_Z - c_1 \tilde{E}_1 - c_3 \tilde{E}_3 - c_{01} \tilde{S}_{01} - c_{03} \tilde{S}_{03} - \sum_{j=1}^{11} c_j' \mathcal{F}_j,$$

where $c_1,\ c_3 > 0$, $c_1 + c_3 = 1$, $c_{11}' > c_{10}' > \cdots > c_1' > 0$, and $c_{01} = c_{03} > c_8'$. As in Theorem 4.9, we will show that $\alpha_Z^+ \cdot \sigma \geq 0$ for each curve σ inside the forward image of $\mathcal{I}(f_Z)$. The result will follow from Theorem 1.3 of [**Ba**].

Let us start with $\mathcal{F}_{0\beta\gamma} \subset \mathcal{I}(f_Z)$. Points of this curve are blown up to $\mathcal{F}_{0BC}$. The curve $\sigma = \mathcal{F}_{0BC}$ is the exceptional fiber inside S_{03} over the point $\Sigma_{BC} \cap \Sigma_{03} \in \mathbf{P}^3$. Thus $\sigma \cdot S_{03} = -1$. In the construction of Z, γ_7 will be blown up to create the exceptional divisor Γ_7. At this stage, by Lemma 7.3, σ and γ_1 become separated. Thus $\sigma \cdot \Gamma_1 = 0$, and $\sigma \cdot \Gamma_7 = 1$, so $\alpha \cdot \sigma = c_{03} - c_7 > 0$.

Points of the indeterminate curve Σ_{02} blow up to $\sigma = \mathcal{F}^1_{e_2}$. In this case, we have that $\sigma \cdot S_{01}$ and $\sigma \cdot S_{03}$, are ± 1, with opposite signs, so $\sigma \cdot \alpha^+_Z = \pm c_{01} \mp c_{03} = 0$ as was seen in the proof of Theorem 4.9.

The other possibility is $\ell \subset \mathcal{I}(f_Z)$, for one of the indeterminate lines in Σ_γ. This blows up to one of the first fibers $\sigma = \mathcal{F}^1_\zeta$. In this case, σ crosses Γ_1 transversally, so $\sigma \cdot \Gamma_1 = 1$. On the other hand, $\sigma \subset \Gamma_j$ for some $j > 1$, so we have $\sigma \cdot \Gamma_j = -1$. Thus $\sigma \cdot \alpha^+_Z = c'_j - c'_1 > 0$. □

Remark. Considering the symmetry between f and f^{-1}, we find that $\mathcal{I}(f_Z^{-1}) = \Sigma_{02} \cup \mathcal{F}^1_{0\beta\gamma} \cup \bigcup_\zeta \mathcal{F}^1_\zeta$, where the ζ's are the intersection points of γ_1 with the curves γ_2, γ_3, γ_7, and γ_9.

If we instead blow up the γ_j's in the order $\gamma_1, \gamma_2, \ldots$, and call the resulting space $\hat{Z}$. Then we have $\mathcal{I}(f_{\hat{Z}}) = \Sigma_{02} \cup \mathcal{F}^1_{0\beta\gamma} \cup \bigcup_\zeta \mathcal{F}^1_\zeta$, where the $\zeta \in \gamma_{11}$ are the points of intersection with γ_3, γ_5, γ_9, and γ_{10}. Each of these points ζ is blown up by $f_{\hat{Z}}$ to a line of the pencil in Σ_C passing through e_3.

Thus we can apply a similar argument to α^-_Z to obtain the Green current for f_Z^{-1}.

Corollary 7.8. *There is a positive closed current T^-_Z in the class of α^-_Z with the property: if Ξ^- is a smooth form which represents α^-_Z, then $\lim_{n\to\infty} \delta_1(f)^{-n} f_Z^{-n*} \Xi^-_Z = T^-_Z$ in the weak sense of currents on Z.*

Next we show what happens to the invariant fibration when we lift it to Z. Let us set $P_0 = x_0x_1x_2x_3$, and let P_1 be a homogeneous quartic polynomial defined in Appendix B. For $c \in \mathbf{C}$, let us set $S_c = \{cP_0 + P_1 = 0\}$, so the rotor $\mathcal{R}$ corresponds to $c = \infty$. Since we have $f(S_c) = S_{\omega c}$, the surface S_0 is invariant.

Proposition 7.9. *The variety $S_0 := \{P_1 = 0\} \subset \mathbf{P}^3$ has singular points at e_1, e_3 and the fixed points $p_\pm$. If $p_\pm$ are blown up (in additional to the e_1 and e_3 which were blown up to construct Y), then the strict transform of S_0 is a nonsingular $K3$ surface.*

Proof: Using the computer, we find that the critical points of P_1 occur exactly at e_1, e_3 and $p_\pm = (x_\pm, x_\pm, x_\pm) \in \mathbf{C}^3$ where $x_\pm$ are the roots of $x^2 = a + (1+\omega)x$. (`Mathematica`, for instance, can do this.) Further, $p_\pm$ are singular points of type A_1. The singular points e_1 and e_3 are type A_1 unless $a = (1+2\omega)/(1-\omega)$, in which case they are type A_2. In either case, it follows (see, for instance, [**EJ**, Lemma 3.1 and Remark 3.2]) that S_0 is $K3$. □

Corollary 7.10. *For all but finitely many values of $c \in \mathbf{C}$, the strict transform of S_c in Z is a nonsingular $K3$ surface.*

Let $\mathcal{P} \subset Y$ denote the (finite) set of all intersection points of distinct curves $\gamma_j \cap \gamma_k$. Since the γ_j lie in the rotor, we have $\mathcal{P} \subset \mathcal{R}$. The rotor is the union of 8 smooth hypersurfaces which intersect transversally, so the singular locus of $\mathcal{R}$ is the set where two (or more) of these surfaces intersect. We will write $\mathcal{P}_s$ (resp. $\mathcal{P}_r$) for the points of $\mathcal{P}$ which are contained in the singular (resp. regular) locus of $\mathcal{R}$.

While Z itself depends on the order in which the curves γ_j are blown up, the following propositions are valid for any ordering of the blowups.

Proposition 7.11. *For $p \in \mathcal{P}_r$, there is a unique $c_p \in \mathbf{C}$ such that $S_{c_p} \subset Y$ is singular at p. This is a conical singularity, and the strict transform $S_{c_p} \subset Z$ contains the first fiber $\mathcal{F}^1_{c_p}$.*

Proof: Without loss of generality, we may choose coordinates (x, y, z) so that $p = 0$, $L = z$ near p, and $\mathcal{R} = \{z = 0\}$. Let us suppose that $p \in f^j_Y \Sigma_{BC} \cap f^k_Y \Sigma_{BC}$. Since the curves $f^j_Y \Sigma_{BC}$ are contained in $\mathcal{R}$ and intersect transversally, we may suppose that near p the curves $f^j_Y \Sigma_{BC}$ and $f^k_Y \Sigma_{BC}$ coincide with the x- and y-axes. Thus the tangent to $\{M = 0\}$ at p is given by $z = 0$, so we may suppose that $M = \lambda z + xy + \cdots$. The surfaces are then $S_c = \{M + cL = 0\} = \{\lambda z + xy + cz + \cdots = 0\}$. The surface S_c is singular if $c = -\lambda$. We blow up the x-axis by the coordinate change $(x, s, \eta) \mapsto (x, s, s\eta)$. The first fiber is $\mathcal{F}^1_p = \{x = s = 0\}$. The strict transforms of the surfaces are $S_c = \{(\lambda + c)\eta + x = 0\}$. The strict transform of the y-axis is now the s-axis, which is contained in each S_c. Otherwise, the S_c's are disjoint. The strict transform of $S_{-\lambda}$ contains $\mathcal{F}^1_p$. After we blow up the s-axis, the surfaces are all disjoint and smooth. □

Proposition 7.12. *For $p \in \mathcal{P}_s$, S_c is smooth at p for all $c \in \mathbf{C}$. The first fiber is contained in the rotor: $\mathcal{F}^1_p \subset \mathcal{R} \subset Z$.*

Proof: We may assume that p is a normal crossing of two of the hypersurfaces of $\mathcal{R}$. Thus we may choose coordinates (x, y, z) such that $p = 0$, and $L = xy$ near p. We may assume that $f^j_Y \Sigma_{BC}$ is the x-axis, and $f^k_Y \Sigma_{BC}$ is the y-axis. Since M contains both axes, we may assume that $M = z + \varphi$, where φ is divisible by xy. Thus $S_c = \{M + cL = z + \varphi + cxy = 0\}$ is smooth for all $c \in \mathbf{C}$. When we blow up the x-axis, we use coordinates $(x, s, \eta) \mapsto (x, s, s\eta)$. The strict transforms are then $S_c = \{s\eta + \tilde{\varphi} + csx = 0\}$, where $\tilde{\varphi}$ is divisible by xs. Dividing this equation by s, we have $S_c = \{\eta + \psi(x, s, \eta) + cx = 0\}$, where $\psi(0, s, 0) = 0$, since S_c contains the s-axis (the strict transform of the y-axis). We have $\mathcal{F}^1_p = \{x = s = 0\}$. Now we blow up the s-axis via the coordinates $(\xi, s, t) \mapsto (\xi t, s, t) = (x, s, \eta)$. This gives the new strict transforms $S_c = \{1 + \hat{\psi}(\xi, s, t) + c\xi = 0\}$, where $\hat{\psi}(\xi, s, t) = t^{-1}\psi(\xi t, s, t)$ is regular. The strict transform of $\mathcal{F}^1_p$ is now $\{\xi = s = 0\}$, which is disjoint from the S_cs. □

If $p' \in \gamma_1 \cap \gamma_9$, then there is a unique $c' \in \mathbf{C}$ be such that $S_{c'}$ is singular at p'. Let ℓ'_9 denote the line for which $f(\ell'_9) = p'$. By Theorem 7.6 and Proposition 7.11, it follows that f_Z maps ℓ'_9 to the strict transform of $S_{c'}$ inside Z. Thus the total transform of ℓ'_9 under f^n_Z is contained in $S_{\omega^n c'}$. Let p'' denote the other point of $\gamma_1 \cap \gamma_9$, and let $c'' \in \mathbf{C}$ denote the corresponding parameter. Let $\hat{S} = S_{c'} \cup S_{\omega c'} \cup S_{\omega^2 c'} \cup S_{c''} \cup S_{\omega c''} \cup S_{\omega^2 c''}$. We see that $\hat{S}$ is a f_Z-invariant set which contains $\ell'_9 \cup \ell''_9$. Let $\mathcal{R}$ denote the strict transform of the rotor in Z. The sets $\mathcal{R}$, $\hat{S}$, and $\Sigma_{02} \cup \mathcal{F}^1_{e_2}$ are totally invariant, and we break the indeterminacy locus into three sets:

$$\mathcal{I}(f_Z) = (\Sigma_{02} \cup \mathcal{F}^1_{e_2}) \cup (\mathcal{I}(f_Z) \cap \mathcal{R}) \cup (\mathcal{I}(f_Z) \cap \hat{S})$$

with $\mathcal{I}(f_Z) \cap \mathcal{R} = \ell_2 \cup \ell_3 \cup \ell_7 \cup \mathcal{F}^1_{0\beta\gamma}$, and $\mathcal{I}(f_Z) \cap \hat{S} = \ell'_9 \cup \ell''_9$. We set

$$\Omega = Z - (\hat{S} \cup \mathcal{R} \cup \Sigma_{02} \cup \mathcal{F}^1_{e_2}).$$

By Propositions 7.11 and 7.12, $\mathcal{R}$ is disjoint from the strict transform of each S_c. Thus f_Z is regular on Ω, and Ω is invariant under f_Z.

Proposition 7.13. *For every S_c in Ω the dynamical degree of the restriction is $\delta(f^3_c) = \delta_1(f)^3$.*

Proof: Let us denote Γ a hypersurface in Z whose cohomology class in $H^{1,1}(Z)$ is H_Z. It follows that the degree of $f_Z^{-3n}\Gamma$ grows like $\delta_1(f)^{3n}$. On the other hand $S_c \subset \Omega$ does not contain an irreducible component of the indeterminacy locus for f_Z. It follows that we have $S_c \cap (f^3_Z)^{-n}\Gamma = (f^3_Z)^{-n}(S_c \cap \Gamma)$. Because S_c is non-singular and f^3_Z is pseudo-automorphism, the degree of $(f^3_Z)^{-n}(S_c \cap \Gamma) = (f^3_c)^{-n}(S_c \cap \Gamma)$ is $4\,\delta_1(f)^{3n}$. Thus the dynamical degree of f^3_c is $\delta_1(f)^3$. □

Using the fact that f_Z is regular on the large invariant set Ω, we avoid the difficulties that can occur in defining the entropy of a map (see [**G1**]).

Theorem 7.14. *The entropy of f is $\log \delta_1(f)$.*

Proof: Since f is equivalent to a pseudo automorphism, and f_Z^* is conjugate to $(f_Z^*)^{-1}$, both the first and the second dynamical degrees are equal. Combining the result in [**DS**] and the fact that $h_{\text{top}}(f) \geq h_{\text{top}}(f_0)$, we have the inequality

$$\log \delta_1(f) \geq h_{\text{top}}(f) \geq h_{\text{top}}(f_0) = \log \delta_1(f)$$

which gives the result. □

Since $\Xi^\pm$ and f_Z are regular on Ω, the potential of $g^\pm$ is continuous on Ω. Thus we may define the wedge product $T_2 := T^+ \wedge T^-$ as a positive, closed $(2,2)$-current on Ω, and we have:

Proposition 7.15. $\lim_{n\to\infty} \delta_1(f)^{-2n} {f_Z^*}^n \Xi^+ \wedge {f_Z^*}^{-n} \Xi^- = T_2$ *exists as a* $(2,2)$*-current on* Ω.

We have seen that the restrictions $f^3|_{S_c}$ are automorphisms, and there are invariant currents $\mu_c^\pm$ on S_c, as well as invariant measures $\mu_c := \mu_c^+ \wedge \mu_c^-$ (see [**C**]). The following property leads us to consider $T^\pm$ and T_2 as the "bifurcation currents" for the family $\{f^3|_{S_c}\}$ (see [**DuF**]).

Theorem 7.16. *For* $S_c \subset \Omega$ *the slices by* S_c *are well-defined and give the corresponding dynamical objects:* $T^\pm|_{S_c} = \mu_c^\pm$, *and* $T_2|_{S_c} = \mu_c$.

Proof: If we set $h = f^3$, then the class $[S_c]$ is invariant under h^*. Thus $\alpha^+ \cdot [S_c] \in H^{1,1}(S_c)$ is a class that is expanded by a factor of $\delta_1(f)$. It follows that the restriction $\Xi^+|_{S_c}$ gives the expanded class, and this converges to μ_c^+. Similarly, the normalized pullbacks/push-forwards of $\Xi^+ \wedge \Xi^-$ on S_c will converge to μ_c. □

Theorem 7.17. *For generic* c', c'', *the maps* $f^3|_{S_{c'}}$ *and* $f^3|_{S_{c''}}$ *are not smoothly conjugate, and the surfaces* $S_{c'}$ *and* $S_{c''}$ *are not isomorphic.*

Proof: There is an invariant 6-cycle of curves, Γ_j, $j = 0, \ldots, 5$ for f. For generic c, $\Gamma_j \cap S_c$ is a saddle 2-cycle for $f^3|_{S_c}$. The multipliers of this saddle cycle are not constant in c, so the maps $f^3|_{S_c}$ are not smoothly conjugate. Since the automorphism group of S_c is disconnected, we see that the family $\{S_c\}$ cannot consist of surfaces which are all isomorphic to each other. □

Remark. If $a_2 \neq 1$ is a primitive 5^{th} root of unity and $a_0 = b_0 = 0$, then we may repeat most of the arguments in this section for this map. In particular, we have:

Theorem 7.18. *If a_2 is a primitive 5^{th} root of unity, and $a_0 = b_0 = 0$, then f is equivalent to a pseudo-automorphism, and the dynamical degrees $\delta_1(f) = \delta_2(f) \approx 1.3211018 > 1$ are the largest root of $t^{19}(t^3 - t - 1) + t^3 + t^2 - 1$. The entropy of f is $\log \delta_1(f) > 0$. Furthermore there are two quartic polynomials which are invariant in the sense of* (B.1)*. This gives a family of $K3$ surfaces which are invariant under f^5.*

8. Pseudo-automorphisms which are completely integrable

Let us consider two cases for maps of the form (2.2):

(8.1a) $\alpha = (a, 0, 1, 1), \quad a \neq 1, \qquad$ and $\quad \beta = (0, 1, 0, 0),$

(8.1b) $\alpha = (0, 0, \omega, 1), \quad \omega^3 = 1, \quad \omega \neq 1,\quad$ and $\quad \beta = (0, 1, 0, 0).$

The map (8.1a) has been extensively studied under the name Lyness process. The maps (8.1a) and (8.1b) exhibit similarities to the maps in the previous section: they are critical maps, and the iterates of the critical image Σ_{BC} go "once around" the rotor and land on $\Sigma_{\beta\gamma}$. The difference with §7 is that $f_Y^4 \Sigma_{BC} = \mathcal{F}_{0\beta\gamma}$ is an indeterminate curve, and by Lemma 4.4 this fiber is mapped to $\mathcal{F}_{0BC}$, that is, $f_Y^4 \Sigma_{BC} = \mathcal{F}_{0\beta\gamma} \subset S_{01}$ and $f_Y^5 \Sigma_{BC} = \mathcal{F}_{0BC} \subset S_{03}$. Thus Σ_{BC} arrives at $\Sigma_{\beta\gamma}$ one step faster than was the case in §7.

Let $\pi\colon Z \to Y$ denote the space obtained by blowing up the orbit $f^j \Sigma_{BC}$, $f^{-j} \Sigma_{\beta\gamma}$, $0 \leq j \leq 4$ (one curve less than the construction in §7).

Theorem 8.1. *The induced map f_Z is a pseudo-automorphism, and the iterates of f have quadratic degree growth.*

Proof: Since $f_Y^9 \Sigma_{BC} = \Sigma_{\beta\gamma}$ and $f_Y^4 \Sigma_{BC} = \mathcal{F}_{0\beta\gamma}, f_Y^5 \Sigma_{BC} = \mathcal{F}_{0BC}$, we see that f_Y satisfies the condition in Theorem 5.1. This theorem then follows from Theorems 5.1 and Lemma 5.3. □

Proposition 8.2. *In cases* (8.1a) *and* (8.1b)*, the induced rotor map $f^8|_{\Sigma_3}$ has linear degree growth. This map is not birationally conjugate to a surface automorphism.*

Proof: In the case (8.1b), the restriction of f_Y^8 to Σ_3 is given by setting $a_0 = 0$ in (7.3), so we find the degree 2 birational map:

$$f_Y^8|_{\Sigma_3}\colon [x_0 : x_1 : x_2 : 0] \mapsto [x_0 \omega^2 x_2 : x_1(x_1 + \omega^2 x_0 + \omega^2 x_2) : \omega^2 x_2(x_1 + \omega^2 x_2) : 0].$$

This map has three distinct exceptional lines. Two of exceptional lines are mapped to fixed points $[1 : 0 : -1 : 0]$ and $[0 : 1 : -a : 0]$. The remaining exceptional line is mapped to a point of indeterminacy $e = [1 : -1 : 0 : 0]$. We let W be the blowup space obtained by blowing up Σ_3 at e. The induced map has only two exceptional lines which are mapped to fixed points and therefore the induced map is algebraically stable. The action on Pic is given by the matrix $\left(\begin{smallmatrix} 2 & 1 \\ -1 & 0 \end{smallmatrix}\right)$ which has an eigenvalue 1 with 2×2 Jordan block. It follows that the degree of restriction map grows linearly.

The analysis in the case (8.1a) is essentially the same. The induced rotor map is now:

$$f_Y^8|_{\Sigma_3} : [x_0 : x_1 : x_2 : 0] \mapsto [x_0(ax_0+ax_1+x_2) : x_1(x_0+x_1+x_2) : x_2(ax_0+x_1+x_2) : 0].$$

This map has three exceptional lines. Two of them are mapped to fixed points $[1 : 0 : -1]$ and $[0 : 1 : -a]$. The third exceptional line is mapped to $[1 : -1 : 0]$, which is indeterminate. After we blow up the point $[1 : -1 : 0]$, the induced map is algebraically stable and the action on Pic has an eigenvalue 1 with 2×2 Jordan block.

Finally, since the restriction of f_W to the rotor has linear degree growth. It follows from [**DiF**] that this restriction is not an automorphism. □

We consider first the Lyness map, i.e., case (8.1a). This is known to be integrable, and the invariant polynomials are given in [**CGMs**] and [**KoL**]. These invariant polynomials, which satisfy (B.1) with $t = 1$, are:

$$\begin{aligned} &Q_0 = x_0x_1x_2x_3, \\ (8.2)\quad &Q_1 = (ax_0 + x_1 + x_2 + x_3)(x_0 + x_1)(x_0 + x_2)(x_0 + x_3), \\ &Q_2 = (x_0(ax_0+x_1+x_2+x_3)+x_1x_3)(x_0 + x_1 + x_2)(x_0 + x_2 + x_3). \end{aligned}$$

The set $\{Q_0 = 0\}$ gives an invariant 8-cycle of rational surfaces, which is the rotor $\mathcal{R} \subset Y$. (Although $Q_0 = 0$ consists of 4 irreducible components in $\mathbf{P}^3$, it yields an 8-cycle inside Y because these components map through the indeterminacy locus, which is blown up to yield an additional 4 divisors.) The set $\{Q_1 = 0\}$ gives an invariant 4-cycle, and $\{Q_2 = 0\}$ gives an invariant 3-cycle; the components of the 8-, 4-, and 3-cycles are rational surfaces. As we observed in §4, f_Y induces dominant maps on each of these cycles. And as in Proposition 8.2, we may show that the restriction of f^4 to the 4-cycle, and the restriction of f^3 to the 3-cycle both have linear degree growth.

Let us define the surfaces $S_c = \{Q_c = 0\}$ with $Q_c := c_0Q_0 + c_1Q_1 + c_2Q_2$. If we also write S_c for its strict transform inside Z, we have $fS_c = S_c$.

Theorem 8.3. *For generic c, the surface S_c is an irreducible $K3$ surface.*

Proof: For generic c, we find that S_c has 16 singular points: two of them are e_1, e_3, which are type A_2, and there are 14 more which are of type A_1. In the construction of Z, we blew up e_1 and e_3. Then we blew up $f^j\Sigma_{BC}$, $0 \le j \le 10$, and the other 14 singular points are contained in these curves. It follows that the strict transform of S_c inside Z is smooth and thus $K3$. □

Theorem 8.4. *For generic c and c', the intersection $S_c \cap S_{c'}$ is an elliptic curve. The restriction of f^3 to S_c has quadratic degree growth.*

Proof: Since S_c is a $K3$ surface, it has trivial canonical bundle. Thus the birational map f^3 of S_c must be an automorphism. For generic c and $c' \neq c$, the intersections $S_c \cap S_{c'}$ give an invariant fibration of S_c. Since $f^3|S_c$ is an automorphism, then by [**DiF**] the intersection $S_c \cap S_{c'}$ is an elliptic curve and the restriction of f to the family of $K3$ surfaces has quadratic degree growth. □

The map (8.1b) is similar. In this case the solutions to (B.1) take the form:

$$
\begin{aligned}
R_0 &= x_0x_1x_2x_3,\\
R_1 &= (x_0 + \omega x_1)(x_0 + \omega x_2)(x_0 + \omega x_3)(x_1 + \omega^2 x_2 + \omega x_3),\\
R_2 &= \omega x_1x_3(x_0 + \omega x_1)(x_0 + \omega x_3)\\
&\quad + \omega^2 x_0x_2 \left(x_0(x_1 + \omega x_3) + x_2(\omega x_1 + x_3) + \omega^2 x_0x_2\right),
\end{aligned}
\tag{8.3}
$$

where $t_{R_0} = 1$, $t_{R_1} = \omega^2$, and $t_{R_2} = \omega^2$. As before, we see that f_Z will have an invariant 8-cycle given by the rotor $\mathcal{R} \subset Z$. And $\{R_1 = 0\}$ will give a 4-cycle of rational surfaces. For generic c, the singularities of the surface $S_c = \{\sum c_jR_j = 0\}$ are e_1, e_3 (type A_2) and e_2 (type A_1). As in Theorems 8.3 and 8.4, we have:

Theorem 8.5. *In case* (8.1b)*: for generic c, S_c is a $K3$ surface, f^3 is an automorphism of S_c with quadratic growth, and the intersections $S_c \cap S_{c'}$ are elliptic curves.*

A. Appendix: Computing the characteristic polynomial for f_Z^*

We continue to assume that f is a critical map for which (5.1) holds and let $\pi\colon Z \to \mathbf{P}^3$ be the space constructed in Theorem 5.1. We continue with the notation $\mu_j := f_Y^j(\Sigma_\gamma)$. Recall that in Cases 1 and 2 in the proof of Theorem 5.1, the space Z was constructed by blowing up the varieties μ_j, $1 \le j \le N$. In this situation, we will consider f_Z^* acting on $\mathrm{Pic}(Z)$ in Lemma A.1 below.

The remaining scenario in the proof of Theorem 5.1 is Case 3 (since Case 4 was shown not to happen), and in this case there is a cycle of curves $\gamma_0 := \mathcal{F}_{0BC}$, $\gamma_1 = f_Y(\mathcal{F}_{0BC}), \dots, \gamma_\ell = f_Y^\ell(\mathcal{F}_{0BC}) = \mathcal{F}_{0\beta\gamma}$. The space Z was constructed by blowing up the curves $\gamma_0, \dots, \gamma_\ell$ and the varieties $\mu_1, \dots, \mu_N$. We may choose whether to blow up the γ_i's first and then the μ_j's, or the other way around. For instance, if μ_i is a point of γ_j, we may blow up μ_i first and then blow up the strict transform of γ_j later. On the other hand, we may blow up γ_j first, writing Γ_j as the blowup divisor over γ_j. Then the fiber $\hat\mu_i := \pi^{-1}(\mu_i)$ is a curve in Γ_j, and we may blow up the curve $\hat\mu_i$ later. If Z' and Z'' are obtained by blowing up these varieties in different orders, then the identity map $\iota\colon Y \to Y$ induces a pseudo-isomorphism between Z' and Z'', and this gives a natural identification between $\mathrm{Pic}(Z')$ and $\mathrm{Pic}(Z'')$.

In this case, Γ_k is taken by f_Z^{-1} to Γ_{k-1}, $1 \le k \le \ell$, and Γ_0 is taken to Γ_ℓ. Since f_Z is a pseudo-automorphsm, this is sufficient to determine its action on the cohomology classes $\{\Gamma_k\}$. We define $\hat\Gamma \subset \mathrm{Pic}(Z)$ to be the subspace spanned by the classes of the divisors $\Gamma_0, \dots, \Gamma_\ell$, and thus $\hat\Gamma$ is invariant under f_Z^*. In this scenario, we will consider the quotient map induced by f_Z^* acting on the quotient space $\mathrm{Pic}(Z)/\hat\Gamma$, and it is this quotient map that we will represent in Lemma A.1.

Let m_u, m_d, m_s, d_j, u_j, s_j and N be the numbers defined in §5. We define the $(N+5)\times(N+5)$ matrix

$$\begin{pmatrix} 2 & 0 & 1 & 0 & 1 & 0 & \cdots & 0 & 1 \\ -1 & 0 & -1 & 0 & 0 & 0 & \cdots & 0 & -1 \\ 0 & 1 & -1 & 0 & 0 & 0 & \cdots & 0 & 0 \\ -1 & 0 & -1 & 0 & -1 & 0 & \cdots & 0 & 0 \\ -1 & 0 & -1 & 1 & -1 & 0 & \cdots & 0 & 0 \\ -1 & 0 & 0 & 0 & -1 & 0 & \cdots & 0 & -1 \\ * & & & & & & & & * \\ \vdots & & & & & & & & \vdots \\ * & & & & & & & & * \end{pmatrix}, \tag{A.1}$$

where the $*$'s indicate that the 7^{th} through the $N+5^{\text{th}}$ rows remain to be specified. We will define the j^{th} row r_j in terms of the elements e_k, which are vectors of length $N+5$ in which the k^{th} entry is 1, and all other entries are 0:

(a) if $j = N - d_i$ for some $i = 1, \ldots, m_d$, then $r_{j+6} = e_{j+5} - e_{N+5}$;

(b) if $j = N - u_i$ for some $i = 1, \ldots, m_u$, then $r_{j+6} = -e_1 - e_5 + e_{j+5} - e_{N+5}$;

(c) if $j = N - s_i$, for some $i = 1, \ldots, m_s$,

$$\begin{aligned} r_{j+2} &= -e_1 - e_3 + e_{j+1} - e_{N+5}, & r_{j+3} &= -e_3 + e_{j+2}, \\ r_{j+4} &= -e_1 - e_3 - e_5 + e_{j+3} - e_{N+5}, & r_{j+5} &= -e_1 - e_3 - e_5 + e_{j+4}, \\ r_{j+6} &= -e_5 + e_{j+5}; \end{aligned}$$

(d) if $j = N - m_{\mathcal{F}}$, then

$$\begin{aligned} r_{j+4} &= -2e_1 - e_3 - 2e_5 + e_{j+3} - e_{N+5}, \\ r_{j+5} &= -e_1 - e_3 - e_5 + e_{j+4}, \\ r_{j+6} &= -e_5 + e_{j+5}; \end{aligned}$$

(e) if $j = N - m_{c\mathcal{F}}$, then

$$\begin{aligned} r_{j+4} &= -e_1 - e_3 + e_{j+3} - e_{N+5}, \\ r_{j+5} &= -e_3 + e_{j+2}, \\ r_{j+6} &= -e_1 - e_3 - e_5 + e_{j+3} - e_{N+5}; \end{aligned}$$

(f) otherwise, $r_{j+6} = e_{j+5}$.

Let us define $\beta' := (\beta_0, 0, 1, 0)$ and $\beta'' := (\beta_0, 0, 0, 1)$, so we have

$$\Sigma_{\beta''} \to \Sigma_{\beta'} \to \Sigma_\beta \to E_3 \to S_{01} \to \Sigma_0 \to S_{03} \to E_1 \to \Sigma_3 \to \Sigma_2 \to \Sigma_1.$$

Note that if $\beta_0 = 0$, then $\Sigma_\beta = \Sigma_1$, and this becomes the 8-cycle in (7.2). The following curves are important for computing f_Z^*

$$\text{(A.2)} \quad f_Y: \begin{array}{l} \ell_\beta := \Sigma_\beta \cap \{\alpha_2 x_2 + (1 + \alpha_2\beta_0)x_0 = 0\} \to f(\ell_\beta) \to \mathcal{F}_{0\beta\gamma}, \\ \ell'_\beta := \Sigma_{\beta''} \cap \Sigma_\beta \to E_3 \cap \Sigma_{\beta'} \to S_{01} \cap \Sigma_\beta \to \Sigma_0 \cap E_3. \end{array}$$

Lemma A.1. *There is a basis of* $\operatorname{Pic}(Z)$ *(or* $\operatorname{Pic}(Z)/\hat{\Gamma}$*) with respect to which the matrix* (A.1) *represents* f_Z^*.

Proof: We will use the notation $\mathcal{F}_j$ for the blowup divisor of μ_j. There are three cases to consider.

Case (I): Whenever $1 \le j \le N$ and $\mu_j \subset \Sigma_\beta$, then $\mu_j \subset \Sigma_{\beta\gamma} \cup \ell_\beta$. In this case we have

$$\begin{aligned}
f_Z^* H_Z &= 2\mathcal{H}_Z - E_1 - S_{01} - E_3 - \sum_{i=1}^{m_u} \mathcal{F}_{u_i} \\
&\quad - \sum_{i=1}^{m_s} (\mathcal{F}_{s_i+1} + \mathcal{F}_{s_i+2} + \mathcal{F}_{s_i+4}) - \mathcal{F}_{m_\mathcal{F}+1} - 2\mathcal{F}_{m_\mathcal{F}+2} - \mathcal{F}_{m_{c\mathcal{F}}+2}, \\
\{\Sigma_0\} &= \mathcal{H}_Z - E_1 - S_{03} - S_{01} - E_3 \\
&\quad - \sum_{i=1}^{m_s} (\mathcal{F}_{s_i+1} + \mathcal{F}_{s_i+2} + \mathcal{F}_{s_i+3} + \mathcal{F}_{s_i+4}) \\
&\qquad\qquad - \mathcal{F}_{m_\mathcal{F}+1} - \mathcal{F}_{m_\mathcal{F}+2} - \mathcal{F}_{m_{c\mathcal{F}}+1} - \mathcal{F}_{m_{c\mathcal{F}}+2}, \\
\{\Sigma_\beta\} &= \mathcal{H}_Z - S_{01} - E_3 - \mathcal{F}_N - \sum_{i=1}^{m_u} \mathcal{F}_{u_i} \\
&\quad - \sum_{i=1}^{m_s} (\mathcal{F}_{s_i} + \mathcal{F}_{s_i+1} + \mathcal{F}_{s_i+2}) - \mathcal{F}_{m_\mathcal{F}} - \mathcal{F}_{m_\mathcal{F}+1} - 2\mathcal{F}_{m_\mathcal{F}+2}, \\
\{\Sigma_\gamma\} &= \mathcal{H}_Z - E_1 - \mathcal{F}_N - \sum_{i=1}^{m_s} (\mathcal{F}_{s_i+2} + \mathcal{F}_{s_i+4}) \\
&\quad - \sum_{i=1}^{m_u} \mathcal{F}_{u_i} - \sum_{i=1}^{m_d} \mathcal{F}_{d_i} - \mathcal{F}_{m_\mathcal{F}+2} - \mathcal{F}_{m_{c\mathcal{F}}} - \mathcal{F}_{m_{c\mathcal{F}}+2}.
\end{aligned}$$

Since we have $f_Z^*\colon E_1 \mapsto S_{03} \mapsto \{\Sigma_0\}$, $S_{01} \mapsto E_3 \mapsto \{\Sigma_\beta\}$, $\mathcal{F}_j \mapsto \mathcal{F}_{j-1}$ for all $j = 2, \dots, N$, and $\mathcal{F}_1 \mapsto \{\Sigma_\gamma\}$ using the ordered basis $\{H_Z, E_1, S_{03}, S_{01}, E_3, \mathcal{F}_N, \mathcal{F}_{N-1}, \dots, \mathcal{F}_2, \mathcal{F}_1\}$ for $\mathrm{Pic}(Z)$ we see that (A.1) is the matrix representation for f_Z^*.

Case (II): There are κ positive integers $1 < p_1 < \cdots < p_\kappa < N$ such that for $j = 1, \dots, \kappa$, $\mu_{p_j} \subset \Sigma_\beta \setminus \left(\ell_\beta \cup \Sigma_{\beta\gamma} \cup \ell'_\beta\right)$.

For this case let us use the ordered basis

$$\tilde{\mathcal{B}} = \{H_Z, \tilde{E}_1, \tilde{S}_{03}, \tilde{S}_{01}, \tilde{E}_3, \mathcal{F}_N, \mathcal{F}_{N-1}, \dots, \mathcal{F}_2, \mathcal{F}_1\}$$

for $\mathrm{Pic}(Z)$ where $\tilde{E}_3 = E_3 + \sum_{i=1}^{\kappa} \mathcal{F}_{p_i+1}$, $\tilde{S}_{01} = S_{01} + \sum_{i=1}^{\kappa} \mathcal{F}_{p_i+2}$,$\tilde{S}_{03} = S_{03} + \sum_{i=1}^{\kappa} \mathcal{F}_{p_i+4}$, and $\tilde{E}_1 = E_1 + \sum_{i=1}^{\kappa} \mathcal{F}_{p_i+5}$. Using this new ordered basis we can see that

$$
\begin{aligned}
f_Z^*\colon \tilde{E}_1 \mapsto \tilde{S}_{03} \mapsto \{\Sigma_0\}
&+ \sum_{i=1}^{\kappa} \mathcal{F}_{p_i+3} = \mathcal{H}_Z - \tilde{E}_1 - \tilde{S}_{03} - \tilde{S}_{01} - \tilde{E}_3 \\
&- \sum_{i=1}^{m_s} (\mathcal{F}_{s_i+1} + \mathcal{F}_{s_i+2} + \mathcal{F}_{s_i+3} + \mathcal{F}_{s_i+4}) \\
&\qquad - \mathcal{F}_{m_\mathcal{F}+1} - \mathcal{F}_{m_\mathcal{F}+2} - \mathcal{F}_{m_{c\mathcal{F}}+1} - \mathcal{F}_{m_{c\mathcal{F}}+2}.
\end{aligned}
$$

In a similar way we may compute f_Z^* of H_Z, $\tilde{S}_{01}$, $\tilde{E}_3$ and $\mathcal{F}_N$ and see that the matrix representation with $\tilde{\mathcal{B}}$ is given by (A.1).

Case (III): There are τ integers $1 < q_1 < \cdots < q_\tau < N$ such that $\mu_{q_j} \subset \ell'_\beta$ for $j = 1, \ldots, \tau$.

Let us consider the ordered basis

$$
\hat{\mathcal{B}} = \{H_Z, \hat{E}_1, \hat{S}_{03}, \hat{S}_{01}, \hat{E}_3, \mathcal{F}_N, \mathcal{F}_{N-1}, \ldots, \mathcal{F}_2, \mathcal{F}_1\}
$$

for Pic(Z) where $\hat{E}_3 = \tilde{E}_3 + \sum_{i=1}^{\tau}(\mathcal{F}_{q_i+1} + \mathcal{F}_{q_i+3})$, $\hat{S}_{01} = \tilde{S}_{01} + \sum_{i=1}^{\tau}(\mathcal{F}_{q_i+2} + \mathcal{F}_{q_i+4})$, $\hat{S}_{03} = \tilde{S}_{03} + \sum_{i=1}^{\tau}(\mathcal{F}_{q_i+4} + \mathcal{F}_{q_i+6})$, and $\hat{E}_1 = \tilde{E}_1 + \sum_{i=1}^{\tau}(\mathcal{F}_{q_i+5} + \mathcal{F}_{q_i+7})$. Since $f_Y^2 \ell'_\beta = \Sigma_\beta \cap S_{01}$, we have

$$
\begin{aligned}
\{\Sigma_\beta\} &= \mathcal{H}_Z - \tilde{S}_{01} - \tilde{E}_3 - \sum_{i=1}^{\tau}(\mathcal{F}_{q_i} + \mathcal{F}_{q_i+1} + 2\mathcal{F}_{q_i+2} + \mathcal{F}_{q_i+3} + \mathcal{F}_{q_i+4}) - \mathcal{F}_N \\
&\quad - \sum_{i=1}^{m_u} \mathcal{F}_{u_i} - \sum_{i=1}^{m_s} (\mathcal{F}_{s_i} + \mathcal{F}_{s_i+1} + \mathcal{F}_{s_i+2}) - \mathcal{F}_{m_\mathcal{F}} - \mathcal{F}_{m_\mathcal{F}+1} - 2\mathcal{F}_{m_\mathcal{F}+2} \\
&= \mathcal{H}_Z - \hat{S}_{01} - \hat{E}_3 - \sum_{i=1}^{\tau}(\mathcal{F}_{q_i} + \mathcal{F}_{q_i+2}) - \mathcal{F}_N - \sum_{i=1}^{m_u} \mathcal{F}_{u_i} \\
&\quad - \sum_{i=1}^{m_s} (\mathcal{F}_{s_i} + \mathcal{F}_{s_i+1} + \mathcal{F}_{s_i+2}) - \mathcal{F}_{m_\mathcal{F}} - \mathcal{F}_{m_\mathcal{F}+1} - 2\mathcal{F}_{m_\mathcal{F}+2}.
\end{aligned}
$$

It follows that we have

$$\begin{aligned} f_Z^*\colon \hat{S}_{01} \mapsto \hat{E}_3 \mapsto \{\Sigma_\beta\} \\ + \sum_{i=1}^{\tau} (\mathcal{F}_{q_i} + \mathcal{F}_{q_i+2}) = \mathcal{H}_Z - \hat{S}_{01} - \hat{E}_3 - \mathcal{F}_N \\ - \sum_{i=1}^{m_u} \mathcal{F}_{u_i} - \mathcal{F}_{m_s} - \mathcal{F}_{m_s+1} - \mathcal{F}_{m_s+2}. \end{aligned}$$

For the other basis elements, computations are essentially identical and thus we see that (A.1) represents f_Z^* with respect to the ordered basis $\hat{\mathcal{B}}$.
□

According to the previous lemma, we see that the characteristic polynomial of f_Z^* only depends on m_u, m_d, m_s, d_j, u_j, s_j, $m_{\mathcal{F}}$, $m_{c\mathcal{F}}$ and N.

Lemma A.2. *The characteristic polynomial of f_Z^* is given by*

$$\pm t^{N-1}(t^2{+}1)\left[(Q_1{-}Q_4)t^3{+}(2Q_1 - Q_2 - Q_3 - Q_4)t^2{+}(Q_1 - Q_3)t{+}Q_4\right].$$

Proof: We subtract tI from the matrix (A.1) and perform a sequence of row operations on it. Step (i): we add or subtract the 6^{th} row to the rows whose last entry is 1 or -1 and then (ii) for $j = 1, \dots, N-1$, we subtract $1/t^j$ times the $N+4-j^{\text{th}}$ row from 6^{th} row. This gives

$$\det(f_Z^* - t\,I) = \det\begin{pmatrix} A & 0 \\ * & B \end{pmatrix},$$

where

$$A = \begin{pmatrix} 1-t & 0 & 1 & 0 & 0 & -t \\ 1-t & -t & 0 & 0 & 1 & 0 \\ 0 & 1 & -1-t & 0 & 0 & 0 \\ -1 & 0 & -1 & -t & -1 & 0 \\ -1 & 0 & -1 & 1 & -1-t & 0 \\ Q_1 & 0 & Q_2 & 0 & Q_3 & Q_4 \end{pmatrix},$$

$$B = \begin{pmatrix} -t & 0 & 0 & \cdots & 0 & 0 \\ 1 & -t & 0 & \cdots & 0 & 0 \\ 0 & 1 & -t & \cdots & 0 & 0 \\ \vdots & & \ddots & \ddots & & 0 \\ 0 & & & \ddots & -t & 0 \\ 0 & & & & 1 & -t \end{pmatrix}$$

with Q_1, Q_2, Q_3, Q_4 as in §5. We have

$$\det(f_Z^* - t\,\mathrm{Id}) = (-1)^{N-1} t^{N-1} \det(A),$$

and we evaluate $\det(A)$ to obtain the polynomial given above. □

B. Appendix: Invariant polynomials

We will look for polynomials $P(x) = \sum a_I x^I$ which are invariant in the sense that

$$P \circ f = t \cdot j_f \cdot P, \tag{B.1}$$

where $t \neq 0$ is constant, and $j_f = 2x_0(\gamma \cdot x)(\beta \cdot x)^2$ is the Jacobian determinant. If P and Q are solutions to (B.1) with multipliers t_P and t_Q, then $\varphi = P/Q$ is a rational function with the invariance property: $\varphi \circ f = (t_P\, t_Q^{-1})\,\varphi$. If P is a solution to (B.1), then P defines a meromorphic 3-form Ω_P: on the set $x_0 \neq 0$, it is given by $P(1, x_1, x_2, x_3)^{-1}\, dx_1 \wedge dx_2 \wedge dx_3$. This is invariant in the sense that $f^*\Omega_P = t_P^{-1}\Omega_P$. It follows that $\{P = 0\}$ is an f-invariant surface which represents the canonical class in $\mathbf{P}^3$ and its strict transforms are invariant surfaces which represent the canonical classes in Y and Z.

The equation (B.1) can be rewritten as a system of linear equations for the coefficients of the monomials in P. This system can be solved directly for all the maps in §7 and §8. For instance, in §7, ω is a non-real root of unity and $a_0 = a \neq 0$, and we find a solution for $t = \omega^2$:

$$\begin{aligned} P_1 = {} & (1-\omega)\left(a^2x_0^4 + (1+a)x_0x_1x_2^2 + x_1^2x_3^2 + ax_1x_2x_3^2\right) \\ & - (2+\omega)\left(x_0x_2^3 + (1+a)x_0x_1^2x_3 + ax_1x_2^2x_3 + ax_0^2x_3^2\right) \\ & + (1+2\omega)\left(ax_0^2x_1^2 + ax_0x_1^2x_2 + ax_1^2x_2x_3 + ax_0x_2x_3^2\right) \\ & + ax_0^3x_1(1+a+2\omega-a\omega) \\ & + (1-2a+2\omega-a\omega)\left((1+a)x_0^2x_1x_3 + x_0x_2^2x_3\right) \\ & + x_0^2x_2^2(1-a+2\omega+a\omega) \\ & - (2-a+\omega+a\omega)\left((1+a)x_0^2x_1x_2 + x_0x_1x_3^2\right) + ax_0^3x_3(1-2a-\omega-a\omega) \\ & + (1+a)x_0^2x_2x_3(1+a-\omega+2a\omega) + ax_0^3x_2(2+a+\omega+2a\omega). \end{aligned}$$

C. Appendix: The rotor map

Let $g := f_Z^8|_{\Sigma_3}$ denote the rotor map restricted to Σ_3, which is written in coordinates in (7.3). By factoring the jacobian determinant, we see that there are four exceptional curves.

$$\mathcal{C}_1 = \{ax_0 + \omega x_2 = 0\},$$
$$\mathcal{C}_2 = \{ax_0 + ax_1 + \omega x_2 = 0\},$$
$$\mathcal{C}_3 = \{a\omega x_0 + x_1 + \omega^2 x_2 = 0\},$$
$$\mathcal{C}_4 = \{a\omega x_0^2 + a\omega x_0 x_1 + a\omega x_0 x_2 + \omega^2 x_0 x_2 + x_1 x_2 + \omega^2 x_2^2 = 0\}.$$

Lemma C.1. *If $a \neq \omega^j$ and $a^j \neq \omega^{j\pm 2}$ for all $j \geq 2$, then g is not birationally conjugate to an automorphism.*

Proof: The exceptional curves $\mathcal{C}_2$ and $\mathcal{C}_4$ mapped to a three cycle: $g:\mathcal{C}_2 \mapsto [0:1:-a\omega] \mapsto [0:1:-a] \mapsto [0:1:-a\omega^2] \mapsto [0:1:-a\omega]$ and $g\colon \mathcal{C}_4 \mapsto [1:0-\omega^2] \mapsto [1:0:-\omega] \mapsto [1:0:-1] \mapsto [1:0:-\omega^2]$. For $\mathcal{C}_3$ we see that $g^j\mathcal{C}_3 = [1 : -\omega^2(\omega/a)^{j-1} : 0]$ for all $j \geq 1$. It follows that these three curves have orbits that do not encounter the indeterminacy locus of g. The remaining exceptional curve $\mathcal{C}_1$ mapped to $e_1 = [0:1:0]$, which is indeterminate. We let W be the space obtained by blowing up Σ_3 at e_1, and we let E_1 be the corresponding exceptional divisor. Under the induced map g_W we have $g_W(E_1) = E_1$ and the orbit of the strict transform of $\mathcal{C}_1$ remains in E_1 and does not encounter the indeterminacy locus of g_W.

Now if H denote the class of a generic line in W, then $\langle H, E_1\rangle$ is an ordered basis for $\mathrm{Pic}(W)$. The action on Pic is given by the matrix $g_W^* = \left(\begin{smallmatrix} 3 & 1 \\ -1 & 0 \end{smallmatrix}\right)$. The largest eigenvalue is $\lambda = (3+\sqrt{5})/2$ and invariant class is given by $\theta = \lambda H - E_1$. Since $\theta^2 = \lambda^2 - 1 \neq 0$, it follows from [**DiF**, Theorem 5.4] that g is not birationally conjugate to an automorphism. □

Lemma C.2. *If $a^j = \omega^{j-2}$ for some $j \geq 2$, then g is not birationally conjugate to an automorphism.*

Proof: In case $a^j = \omega^{j-2}$ for some $j \geq 2$, the orbits of three exceptional curves $\mathcal{C}_2$, $\mathcal{C}_3$, and $\mathcal{C}_4$ are the same as the previous lemma. After we blow up e_1 on Σ_3, the strict transform of $\mathcal{C}_1$ mapped to a point of indeterminacy after j-th iteration of g_W. We let W_2 be the space obtained by blowing up W at $g_W^k\mathcal{C}_1$ for $k = 1, \ldots, j$ and we let F_k, $1 \leq k \leq j$ be the corresponding exceptional divisors. Under the induced map g_{W_2}, the exceptional line $\mathcal{C}_1$ is removed and the orbits of remaining three exceptional curves do not encounter the indeterminacy locus of g_{W_2}.

Let $\langle H, F_j, F_{j-1}, \ldots, F_1, E_1\rangle$ be the ordered basis for $\mathrm{Pic}(W_2)$. The characteristic polynomial of the action on Pic is given by $t^{j+2} - 4t^{j+1} + 3t^j + t^2 - 2t + 1$. It follows that the dynamical degree is not a Salem number. Thus by [**DiF**], g is not birationally conjugate to an automorphism. □

Lemma C.3. *If $a^j = \omega^{j+2}$ for some $j \geq 2$, then g is not birationally conjugate to an automorphism.*

Proof: When $a^j = \omega^{j+2}$, the orbit of $\mathcal{C}_3$ is different from Lemma C.1, that is $g^{j+1}\mathcal{C}_3 = [1 : -1 : 0]$, which is indeterminate. We let W_3 be the space obtained by blowing up Σ_3 at e_1 and $g^k\mathcal{C}_3$, $1 \leq k \leq j+1$, and we let E_1 and F_k, $1 \leq k \leq j+1$ be the corresponding exceptional divisors. Using the ordered basis $\langle H, F_{j+1}, F_j, \ldots, F_1, E_1\rangle$ for $\mathrm{Pic}(W_3)$, we see that the characteristic polynomial of the action on Pic is given by $t^{j+3} - 3t^{j+2} + t^{j+1} + t$. Similarly as in Lemma C.2, the dynamical degree is not a Salem number and therefore g is not birationally conjugate to an automorphism. □

Lemma C.4. *If $a = \omega$, then g is not birationally conjugate to an automorphism.*

Proof: In this case we see that $\mathcal{C}_2$ is mapped to a point of indeterminacy under 2 iterations and $\mathcal{C}_4$ is also mapped to a point of indeterminacy under 3 iterations. After we blow up e_1, we can check that the orbits of other two remaining exceptional lines does not encounter the indeterminacy locus. After we blow up the orbit of $\mathcal{C}_2$ and the orbit of $\mathcal{C}_4$, we see that the dynamical degree of g is given by the largest root of the polynomial $t^3 - t^2 - 2t - 1$. Again since this number is not a Salem number we have our result. □

Lemma C.5. *If $a = \omega^2$, then g is not birationally conjugate to an automorphism.*

Proof: If $a = \omega^2$ the each component of g has the same factor $x_0 + x_1 + \omega^2 x_2$. It follows that the restriction of f_Y^8 to Σ_3 is a degree 2 birational map. There are two exceptional lines and both exceptional lines are mapped to points of indeterminacy. After we blow up the points on the orbits of three exceptional lines, we see that the induced map has one exceptional line which is mapped to a point of indeterminacy. Once we blow up this point of indeterminacy, we see that the induced map has no exceptional lines and therefore the induced map is algebraically stable. Furthermore the characteristic polynomial of the action on Pic is $t(1+t)(t-1)^3$ and the action on Pic has 2×2 Jordan block. It follows that the degree of g grows linearly. According to [**DiF**], we have that g is not birationally conjugate to an automorphism. □

Lemma C.6. *If $a = 1$, then the degree of g^n grows quadratically.*

Proof: For this case all four exceptional curves are mapped to points of indeterminacy: $g\colon \mathcal{C}_1 \mapsto e_1$, $\mathcal{C}_2 \mapsto [0:1:-\omega]$, $\mathcal{C}_3 \mapsto [1:-\omega^2:0] \mapsto [1:-1:0]$ and $g\colon \mathcal{C}_4 \mapsto [1:0:-\omega^2]$. We let Z be the space obtained by blowing up Σ_3 at all five points in the orbit of exceptional curves and we let E_1, Q_2, Q_3, Q_4, and Q_5 be the corresponding exceptional divisors. Under the induced map g_Z, there is a unique exceptional line which is the strict transform of $\mathcal{C}_1$. The image $g_Z\mathcal{C}_1$ is a point of indeterminacy of g_Z. We blow up $g_Z\mathcal{C}_1 \in E_1$ and denote the exceptional fiber by Q_1. We use $\langle H, Q_1, Q_2, Q_3, Q_4, Q_5, E_1\rangle$ as the ordered basis of Pic. The characteristic polynomial of the action on Pic is given by $(t-1)^4(t+1)(t^2+t+1)$ and the matrix representation of the action on Pic has 3×3 Jordan block. It follows that the degrees of g^n grow quadratically. □

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