GEOMETRIC SINGULAR PERTURBATION THEORY FOR NON-SMOOTH DYNAMICAL SYSTEMS

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Abstract: In this article we deal with singularly perturbed Filippov systems Z_ε:

$$(1) \qquad \dot{x} = \begin{cases} F(x,y,\varepsilon) & \text{if } h(x,y,\varepsilon) \le 0, \\ G(x,y,\varepsilon) & \text{if } h(x,y,\varepsilon) \ge 0, \end{cases} \quad \varepsilon \dot{y} = H(x,y,\varepsilon),$$

where $\varepsilon \in \mathbb{R}$ is a small parameter, $x \in \mathbb{R}^n$, $n \ge 2$, and $y \in \mathbb{R}$ denote the slow and fast variables, respectively, and F, G, h, and H are smooth maps. We study the effect of singular perturbations at typical singularities of Z_0. Special attention will be dedicated to those points satisfying $q \in \{h(x,y,0) = 0\} \cap \{H(x,y,0) = 0\}$ where F or G is tangent to $\{h(x,y,0) = 0\}$. The persistence and the stability properties of those objects are investigated.

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1. Introduction

Non-smooth dynamical systems (or Filippov systems) occur in various situations like mechanical systems with dry friction or with impacts. They also appear in control theory, electronic, economics, medicine and biology (see for instance [**1**], [**3**], [**6**] and [**7**]). All these motivational models emerge from differential equations with discontinuous right-hand sides.

Let $U \subset \mathbb{R}^n$ be an open set. We denote by $C^r(U, \mathbb{R}^n)$ the set of all vector fields of class C^r defined on U, with $r \ge 1$, endowed with the C^r-topology.

The simplest case of a Filippov system occurs when the phase space is composed by two domains such that for each domain a different ODE (ordinary differential equation) governs the dynamics, namely

$$(2) \qquad \dot{x} = Z(x) = \begin{cases} F(x) & \text{if } h(x) \le 0, \\ G(x) & \text{if } h(x) \ge 0. \end{cases}$$

In equation (2), $F, G \in C^r(U, \mathbb{R}^n)$ and $h \colon U \to \mathbb{R}$ is a smooth function having $0 \in \mathbb{R}$ as a regular value. The common boundary $\mathcal{M} = \{x \in U \mid h(x) = 0\}$ between the domains $\mathcal{M}_- = \{x \in U \mid h(x) \leq 0\}$ and $\mathcal{M}_+ = \{x \in U \mid h(x) \geq 0\}$ is called the *switching manifold*. We will use the notation $Z = (F, G)$ to represent the Filippov system (2) and denote by $\Omega^r(U)$ the set of all vector fields Z of the form (2) defined on U.

We also use $Fh(p) = F(p) \cdot \nabla h(p)$ to denote the scalar product in $\mathbb{R}^n$ between the vector field $F \colon U \to \mathbb{R}^n$ and the gradient of the function $h \colon U \to \mathbb{R}$. For $k \geq 2$, we define inductively $F^k h(p) = F(F^{k-1}h)(p)$.

On the switching manifold $\mathcal{M}$ the following open sets are distinguished:

- *Sewing region*: $\mathcal{M}^1 = \{p \in \mathcal{M} : [Fh(p)][Gh(p)] > 0\}$.
- *Escaping region*: $\mathcal{M}^2 = \{p \in \mathcal{M} : Fh(p) < 0, \, Gh(p) > 0\}$.
- *Sliding region*: $\mathcal{M}^3 = \{p \in \mathcal{M} : Fh(p) > 0, \, Gh(p) < 0\}$.

The definitions of these three regions exclude the so-called tangency points, that is, points where one of the two vector fields F or G is tangent to $\mathcal{M}$. They are characterized by $p \in \mathcal{M}$ such that $Fh(p) = 0$ or $Gh(p) = 0$. Generically speaking, these points are on the boundary of the regions $\mathcal{M}^1$, $\mathcal{M}^2$, and $\mathcal{M}^3$, which we denote by $\partial\mathcal{M}^1$, $\partial\mathcal{M}^2$, and $\partial\mathcal{M}^3$, respectively. Tangency points include the case $F(p) = 0$ or $G(p) = 0$, that is, when one of the two vector fields has an equilibrium point at $\mathcal{M}$. We define two types of tangency between a smooth vector field and a manifold, which will be used in the paper.

Definition 1.1. We say that a smooth vector field F has a fold or quadratic tangency with $\mathcal{M} = \{h(x) = 0\}$ at a point $p \in \mathcal{M}$ provided $Fh(p) = 0$ and $F^2 h(p) \neq 0$.

Definition 1.2. A smooth vector field F has a cusp or cubic tangency with $\mathcal{M} = \{h(x) = 0\}$ at a point $p \in \mathcal{M}$ provided $Fh(p) = F^2 h(p) = 0$, $F^3 h(p) \neq 0$, and the set $\{\nabla h(p), \nabla(Fh)(p), \nabla(F^2 h)(p)\}$ is linearly independent.

If a point of the phase space which is moving on an orbit of $Z = (F, G)$ falls onto $\mathcal{M}^1$ then it crosses $\mathcal{M}^1$ to another part of the space. In $\mathcal{M}^2$ and $\mathcal{M}^3$, the definition of the local orbit is given by the Filippov convention [5]. We consider the vector field Z^S which is the linear convex combination of F and G tangent to $\mathcal{M}$, that is

$$(3) \qquad \dot{x} = Z^S(x) = \frac{[\nabla h(x) F(x)] G(x) - [\nabla h(x) G(x)] F(x)}{\nabla h(x)[F(x) - G(x)]}.$$

We call Z^S the *sliding vector field* associated to the Filippov system (2), independently on whether it is defined in the sliding or escaping regions (see Figure 1). Solutions of $Z = (F, G)$ through points of $\mathcal{M}^2 \cup \mathcal{M}^3$ follow the orbit of Z^S.
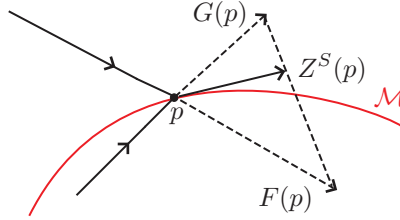

FIGURE 1. The sliding vector field Z^S.

The singularities of a Filippov vector field (2) are

- $p \in \mathcal{M}_\pm$ such that p is an equilibrium of F or G, that is, $F(p) = 0$ or $G(p) = 0$, respectively;
- $p \in \mathcal{M}^2 \cup \mathcal{M}^3$ such that p is an equilibrium of Z^S, that is, $Z^S(p) = 0$;
- $p \in \partial\mathcal{M}^1 \cup \partial\mathcal{M}^2 \cup \partial\mathcal{M}^3$, that is, the tangency points ($Fh(p) = 0$ or $Gh(p) = 0$).

Fenichel (see [**4, 9**]) proved that, in smooth dynamical systems, any structure which persists under regular perturbation also persists under singular perturbation. In [**2**] we extended this theory for the sliding vector field associated to the reduced problem of (1), that is, we study how the sliding mode in Filippov systems is affected by singular perturbations. In this paper we consider again systems of the form (1). Now we analyze the effect of singular perturbations at the tangency points.

For each $\varepsilon \geq 0$ we denote by $\mathcal{M}_\varepsilon$ the set $\mathcal{M}_\varepsilon = \{h(x, y, \varepsilon) = 0\}$. Note that $\mathcal{M}_\varepsilon$ is the switching manifold of (1). The set $\mathcal{S}_0 := \{H(x, y, 0) = 0\}$ is called the *critical manifold* of the singular perturbation problem (1). For $\varepsilon > 0$, we also use the notation $\mathcal{N}_\varepsilon$ to denote the set $\mathcal{N}_\varepsilon = \{H(x, y, \varepsilon) = 0\}$. Here we are supposing that $\mathcal{M}_0$ and $\mathcal{S}_0$ are in a general position, i.e., $\nabla h(p)$ and $\nabla H(p)$ are linearly independent for any $p \in \mathcal{M}_0 \cap \mathcal{S}_0$. Figure 2 illustrates when the manifolds $\mathcal{M}_0$ and $\mathcal{S}_0$ are in a general position and their ε-perturbations $\mathcal{M}_\varepsilon$ and $\mathcal{N}_\varepsilon$. Throughout this article we will assume that the equation $H(x, y, \varepsilon) = 0$ can be solved by $y = f_\varepsilon(x)$, for all $\varepsilon \geq 0$.
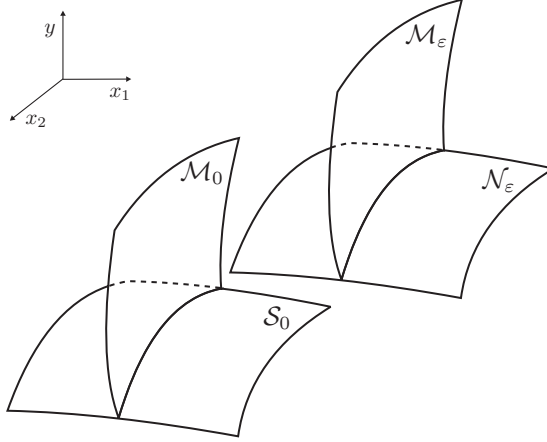
Figure 2. Manifolds $\mathcal{M}_0$ and $\mathcal{S}_0$ in a general position and their ε-perturbations $\mathcal{M}_\varepsilon$ and $\mathcal{N}_\varepsilon$.

For $\varepsilon = 0$ in (1) we have the so-called *reduced problem*

$$(4) \qquad \dot{x} = \begin{cases} \widetilde{F}(x) & \text{if } \widetilde{h}(x) \leq 0, \\ \widetilde{G}(x) & \text{if } \widetilde{h}(x) \geq 0, \end{cases} \qquad 0 = \widetilde{H}(x),$$

where $\widetilde{F}(x) = F(x, f_0(x), 0)$, $\widetilde{G}(x) = G(x, f_0(x), 0)$, $\widetilde{h}(x) = h(x, f_0(x), 0)$ and $\widetilde{H}(x) = H(x, f_0(x), 0)$. The reduced problem (4) is a dynamical system defined on the manifold $\mathcal{S}_0$. For $\varepsilon \neq 0$ we can write the systems of the form (1) in the general form given in (2), namely

$$(5) \qquad (\dot{x}, \dot{y}) = \begin{cases} \overline{F}(x, y, \varepsilon) & \text{if } h(x, y, \varepsilon) \leq 0, \\ \overline{G}(x, y, \varepsilon) & \text{if } h(x, y, \varepsilon) \geq 0, \end{cases}$$

where $\overline{F}(x, y, \varepsilon) = (F, H/\varepsilon)$ and $\overline{G}(x, y, \varepsilon) = (G, H/\varepsilon)$. We will use the notation $Z_\varepsilon = (\overline{F}, \overline{G})$ to represent the Filippov slow-fast system (5) and Z_0 to represent the reduced problem (4).

Definition 1.3. We say that a system of the form (1) is locally simple at $p = (x_0, y_0, 0) \in \mathbb{R}^{n+1} \times \mathbb{R}$ if one of the following conditions is satisfied:

(i) $\frac{\partial h}{\partial x}(p) \neq 0$ and $h(x_0, y, 0) = 0$, for all y close to y_0; or

(ii) there exists a neighborhood U of (x_0, y_0) in $\mathbb{R}^{n+1}$ such that $\frac{\partial H}{\partial x}(q) = 0$, for all $q \in U \cap \mathcal{M}_\varepsilon$.

Remark 1. Note that for the case (i) of Definition 1.3, we can choose local coordinates (around the point p) such that $h(x, y, \varepsilon) = \overline{h}(x, \varepsilon)$, i.e.,

the function h does not depend on the variable y in a given neighborhood of p.

In what follows we summarize a rough overall description of the main results of the paper. The precise statements are given in Sections 2, 3 and 4. In all of them, we assume that the systems of the form (1) are locally simple at the singularity.

First result (Theorem A): This result concerns singular perturbations of fold-regular singularities (see Definition 2.1) of the reduced problem (4). Assuming that system (4) has a singularity of fold-regular type $\overline{p}$, we state that, for $\varepsilon \neq 0$ sufficiently small, there is a family $\overline{p}_\varepsilon$ of fold-regular singularities for the corresponding system (1) such that $\overline{p}_0 = \overline{p}$. Moreover, the features of $\overline{p}$ are preserved for $\overline{p}_\varepsilon$.

Second result (Theorem B): We study singular perturbations of fold-fold singularities (see Definition 2.2) of the reduced problem (4) with $n \geq 3$. If system (4) has a singularity of fold-fold type $\overline{p}$ then, for $\varepsilon \neq 0$ sufficiently small, there is a family $\overline{p}_\varepsilon$ of fold-fold singularities for the corresponding system (1) such that $\overline{p}_0 = \overline{p}$. Moreover, the features of $\overline{p}$ are preserved for $\overline{p}_\varepsilon$.

Third result (Theorem C): This result concerns singular perturbations of cusp-regular singularities (see Definition 3.1) of system (4) with $n = 2$. We prove that the persistence (or not) of tangency points is determined by the sign of a continuous function depending of ε sufficiently small.

Fourth result (Theorem D): This result concerns singular perturbations of fold-cusp singularities of the reduced problem (4) with $n \geq 4$. Supposing that system (4) has a fold-cusp singularity $\overline{p}$, we prove that, for $\varepsilon \neq 0$ sufficiently small, there is a family $\overline{p}_\varepsilon$ of fold-cusp singularities for the corresponding system (1) such that $\overline{p}_0 = \overline{p}$.

Fifth result (Theorem E): We study singular perturbations of hyperbolic/equilibrium-regular singularities (see Definition 4.1) of (4) with $n = 2$. If system (4) has a singularity of hyperbolic/equilibrium-regular type $\overline{p}$ then there is a neighborhood W of $\overline{p}$ such that system (1), for $\varepsilon \neq 0$ sufficiently small, has a hyperbolic equilibrium $\overline{p}_\varepsilon \in W$. Moreover, if $\overline{p}_\varepsilon \notin \mathcal{M}_\varepsilon$, then there exists a family $\{\overline{q}_\varepsilon\}$ of fold-regular singularities of (1).

The paper is organized as follows. In Section 2 we study the effect of singular perturbations at fold points. We state that fold singularities

are persistent with respect to singular perturbations (see Theorems A and B). In Section 3 we study the effect of singular perturbations at cusp points in dimension two. We show, with a minimal example, that such kind of singularities are not persistent with respect to singular perturbations. In this section we also study the unfolding of cusp singularities (see Theorem C). In Subsection 3.1, fold-cusp singularities are considered (see Theorem D). In Section 4 we study the effect of singular perturbation at hyperbolic points in dimension two (see Theorem E). The proofs of the main results are in Section 5. Final remarks are presented in Section 6.

2. Singular perturbation of fold singularities

In this section we will study the effect of singular perturbations at fold singularities of the reduced problem (4). More precisely, by assuming that the reduced problem (4) has fold singularities, we want to know if they persist for the full system (5), for $\varepsilon \neq 0$ small. We also present an example in dimension two showing that in general fold-fold singularities are not persistent with respect to singular perturbations. On the other hand, generically speaking, they are persistent in dimension three (see Theorem B).

Definition 2.1. We say that $p \in \mathcal{M}$ is a fold-regular singularity of (2) if

(i) $Fh(p) = 0$, $F^2h(p) \neq 0$, and $Gh(p) \neq 0$; or
(ii) $Gh(p) = 0$, $G^2h(p) \neq 0$, and $Fh(p) \neq 0$.

In the first case, we say that the fold-regular singularity $p \in \mathcal{M}$ is visible if $F^2h(p) < 0$ and invisible if $F^2h(p) > 0$. In the second case, it is visible provided $G^2h(p) > 0$ and invisible provided $G^2h(p) < 0$.

Definition 2.2. Let $p \in \mathcal{M}$ be a fold-fold singularity of (2), i.e., both vector fields F and G have a fold or quadratic tangency at the same point $p \in \mathcal{M}$. We distinguish the following cases:

(i) **Elliptic case:** $F^2h(p) > 0$ and $G^2h(p) < 0$ (invisible two-fold).
(ii) **Parabolic case:** $F^2h(p) < 0$ and $G^2h(p) < 0$ (visible fold-invisible fold) or $F^2h(p) > 0$ and $G^2h(p) > 0$ (invisible fold-visible fold).
(iii) **Hyperbolic case:** $F^2h(p) < 0$ and $G^2h(p) > 0$ (visible two-fold).

Remark 2. We define fold-regular and fold-fold singularities for system (4) similarly to what was done in Definitions 2.1 and 2.2, respectively. We only note that system (4) is defined on the manifold $\mathcal{S}_0 = \{H(x, y, 0) = 0\}$.

In the theorem below we will see that fold-regular singularities are robust with respect to singular perturbations.

Theorem A. *Let $Z_\varepsilon(x,y)$ be a C^r family defined by (1), with $r \geq 2$. Consider $\overline{p} = (p, f_0(p), 0) \in S_0 \cap M_0$ a fold-regular singularity of the reduced problem Z_0 and suppose that Z_ε is locally simple at $\overline{p}$. Then there exists $\varepsilon_1 > 0$ such that:*

   (i) *There is a C^{r-1} family $\{\overline{p}_\varepsilon : \varepsilon \in (-\varepsilon_1, \varepsilon_1)\}$ such that $\overline{p}_0 = \overline{p}$ and $\overline{p}_\varepsilon$ is a fold-regular singularity of Z_ε.*
   (ii) *If $\overline{p}$ is a visible (resp. invisible) fold-regular of Z_0 then $\overline{p}_\varepsilon$ is a visible (resp. invisible) fold-regular of Z_ε.*

Figure 3 illustrates the case where the reduced problem Z_0 has a visible fold-regular singularity $\overline{p}$. If Z_ε is locally simple at $\overline{p}$, Theorem A ensures that Z_ε has a visible fold-regular singularity $\overline{p}_\varepsilon$, for all ε sufficiently small. Moreover, $\overline{p}_\varepsilon \to \overline{p}$ when $\varepsilon \to 0$. Theorem A is proved in Section 5. It is illustrated by the example below.
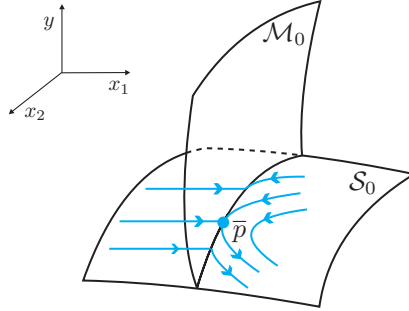

FIGURE 3. A visible fold-regular singularity $\overline{p}$ of the reduced problem Z_0.

Example 1. Consider the singularly perturbed Filippov system

$$(6) \qquad \dot{x} = \begin{cases} (x_2 + y + \varepsilon, 1 + \varepsilon) & \text{if } x_1 - y - \varepsilon \leq 0, \\ (-1 - \varepsilon, -1 - \varepsilon) & \text{if } x_1 - y - \varepsilon \geq 0, \end{cases} \qquad \varepsilon \dot{y} = y - \varepsilon,$$

where $x = (x_1, x_2) \in \mathbb{R}^2$. The functions $h(x, y, \varepsilon)$ and $H(x, y, \varepsilon)$ that define the manifolds M_ε and N_ε are given by $h(x, y, \varepsilon) = x_1 - y - \varepsilon$ and $H(x, y, \varepsilon) = y - \varepsilon$. The critical manifold is $S_0 = \{y = 0\}$. For $\varepsilon = 0$

in (6) we have the reduced problem

$$(7) \qquad \dot{x} = (\dot{x}_1, \dot{x}_2) = \begin{cases} (x_2, 1) & \text{if } x_1 \leq 0, \\ (-1, -1) & \text{if } x_1 \geq 0, \end{cases} \quad y = 0.$$

By denoting system (7) as in (4), we have that

$$\widetilde{F}\widetilde{h} = x_2, \quad \widetilde{F^2}\widetilde{h} = 1, \quad \widetilde{G}\widetilde{h} = -1.$$

Thus, $\overline{p} = (0, 0)$ is an invisible fold-regular singularity of (7). Now, let us write (6) in the general form given by (5), where

$$\overline{F}(x, y, \varepsilon) = \left(x_2 + y + \varepsilon, 1 + \varepsilon, \frac{y}{\varepsilon} - 1 \right)$$

and

$$\overline{G}(x, y, \varepsilon) = \left(-1 - \varepsilon, -1 - \varepsilon, \frac{y}{\varepsilon} - 1 \right).$$

We have that

$$\overline{F}h = x_2 + y - \frac{y}{\varepsilon} + \varepsilon + 1, \quad \overline{F}^2 h = \frac{y}{\varepsilon} - \frac{y}{\varepsilon^2} + \frac{1}{\varepsilon} + \varepsilon, \quad \overline{G}h = -\frac{y}{\varepsilon} - \varepsilon.$$

By solving the system $\overline{F}h(x, f_\varepsilon(x), \varepsilon) = 0$ we obtain the ε-family $\overline{p}_\varepsilon = (2\varepsilon, -2\varepsilon, \varepsilon)$. Since $\overline{F}^2 h(\overline{p}_\varepsilon) = 1 + \varepsilon$ and $\overline{G}h(\overline{p}_\varepsilon) = -1 - \varepsilon$, it follows that $\overline{p}_\varepsilon$ is an invisible fold-regular singularity of (6), for all $\varepsilon \neq 0$ small.

For systems of the form (1) which are not locally simple at $\overline{p}$, we cannot guarantee the validity of the statement (ii) of Theorem A.

Counter-example 1. Consider the Filippov slow-fast system in $\mathbb{R}^3$

$$(8) \quad (\dot{x}_1, \dot{x}_2) = \begin{cases} (x_2, 1) & \text{if } \theta x_1 + (1 - \theta)y \leq 0, \\ (-1, -1) & \text{if } \theta x_1 + (1 - \theta)y \geq 0, \end{cases} \quad \varepsilon \dot{y} = x_1 - y,$$

where θ is a real parameter, $\theta \neq 0$ and 1. For $\varepsilon = 0$ in (8) we have the reduced problem

$$(9) \qquad (\dot{x}_1, \dot{x}_2) = \begin{cases} (x_2, 1) & \text{if } x_1 \leq 0, \\ (-1, -1) & \text{if } x_1 \geq 0, \end{cases} \quad x_1 = y.$$

It is easy to see that $(0, 0)$ is an invisible fold-regular singularity of (9). Note that system (8) is not locally simple at $(0, 0)$.

By denoting system (8) as in (5), for $\varepsilon \neq 0$ we have

$$\overline{F}h = \theta x_2 + \frac{1-\theta}{\varepsilon}(x_1 - y),$$

$$\overline{F}^2 h = \frac{1-\theta}{\varepsilon} x_2 + \theta + \frac{\theta - 1}{\varepsilon^2}(x_1 - y),$$

$$\overline{G}h = -\theta + \frac{1-\theta}{\varepsilon}(x_1 - y).$$

By solving the equation $\overline{F}h(x, f_\varepsilon(x), \varepsilon) = 0$ we see that $\overline{p}_\varepsilon = (0, 0, 0)$ is a tangency point of (8) for all $\varepsilon \neq 0$. Moreover, $\overline{F}^2 h(\overline{p}_\varepsilon) = \theta$ and $\overline{G}h(\overline{p}_\varepsilon) = -\theta$. Thus, for each $\theta < 0$ fixed, $\overline{p}_\varepsilon$ is a visible fold-regular singularity of (8). Therefore, the statement (ii) of Theorem A is not true.

2.1. Singular perturbation of fold-fold singularities. The next example shows that, in dimension two, fold-fold singularities are not persistent with respect to singular perturbations.

Example 2. Consider the following system

$$(10) \qquad \dot{x} = (\dot{x}_1, \dot{x}_2) = \begin{cases} (x_2, 1) & \text{if } x_1 + \varepsilon \leq 0, \\ (x_2 + \varepsilon, -1) & \text{if } x_1 + \varepsilon \geq 0, \end{cases} \qquad \varepsilon \dot{y} = y + 2\varepsilon.$$

The reduced problem of (10) is given by

$$(11) \qquad \dot{x} = (\dot{x}_1, \dot{x}_2) = \begin{cases} (x_2, 1) & \text{if } x_1 \leq 0, \\ (x_2, -1) & \text{if } x_1 \geq 0, \end{cases} \qquad y = 0.$$

It is easy to see that $\overline{p} = (0, 0)$ is a fold-fold singularity of (11). By denoting system (10) as in (5), we have that

$$\overline{F}h = x_2, \quad \overline{F}^2 h = 1, \quad \overline{G}h = x_2 + \varepsilon, \quad \overline{G}^2 h = -1.$$

Thus, for all $\varepsilon \neq 0$, system (10) does not have fold-fold singularities in $\mathcal{M}_\varepsilon = \{x_1 = -\varepsilon\}$. The point $(0, 0, 0)$ was unfolded by families $\overline{p}_\varepsilon = (-\varepsilon, 0, -2\varepsilon)$ and $\overline{q}_\varepsilon = (-\varepsilon, -\varepsilon, -2\varepsilon)$ of fold singularities of the vector fields $\overline{F} = (x_2, 1, y/\varepsilon + \varepsilon)$ and $\overline{G} = (x_2 + \varepsilon, -1, y/\varepsilon + \varepsilon)$, respectively.

On the other hand, in dimension greater than two, fold-fold singularities are persistent with respect to singular perturbations.

Theorem B. *Let $Z_\varepsilon(x, y)$ be a C^r family defined by (1), with $x \in \mathbb{R}^n$, $n \geq 3$ and $r \geq 2$. Consider $\overline{p} = (p, f_0(p), 0) \in \mathcal{S}_0 \cap \mathcal{M}_0$ a fold-fold singularity of the reduced problem Z_0 and suppose that Z_ε is locally simple at $\overline{p}$. Then there exists $\varepsilon_1 > 0$ such that:*

(i) *There is a C^{r-1} family $\{\overline{p}_\varepsilon : \varepsilon \in (-\varepsilon_1, \varepsilon_1)\}$ such that $\overline{p}_0 = \overline{p}$ and $\overline{p}_\varepsilon$ is a fold-fold singularity of Z_ε.*

(ii) *If $\overline{p}$ is either an elliptic or parabolic or hyperbolic fold-fold of Z_0 then $\overline{p}_\varepsilon$ will be either an elliptic or parabolic or hyperbolic fold-fold of Z_ε, respectively.*

Theorem B is proved in Section 5. It is illustrated by the example below.

Example 3. Consider the following system

$$(12) \qquad \dot{x} = \begin{cases} (x_2 + x_3, 1, 1) & \text{if } x_1 + \varepsilon \leq 0, \\ (x_2 - x_3 + \varepsilon, -1, 1) & \text{if } x_1 + \varepsilon \geq 0, \end{cases} \qquad \varepsilon \dot{y} = y + 2\varepsilon,$$

where $x = (x_1, x_2, x_3) \in \mathbb{R}^3$. The manifolds $\mathcal{M}_\varepsilon$ and $\mathcal{N}_\varepsilon$ are given, respectively, by $\mathcal{M}_\varepsilon = \{x_1 = -\varepsilon\}$ and $\mathcal{N}_\varepsilon = \{y = -2\varepsilon\}$. The critical manifold is $\mathcal{S}_0 = \{y = 0\}$. On $\mathcal{S}_0$ we have defined the reduced problem

$$(13) \qquad \dot{x} = (\dot{x}_1, \dot{x}_2, \dot{x}_3) = \begin{cases} (x_2 + x_3, 1, 1) & \text{if } x_1 \leq 0, \\ (x_2 - x_3, -1, 1) & \text{if } x_1 \geq 0, \end{cases} \qquad y = 0.$$

By denoting system (13) as in (4), we have that

$$\widetilde{F}\widetilde{h} = x_2 + x_3, \quad \widetilde{F}^2\widetilde{h} = 2, \quad \widetilde{G}\widetilde{h} = x_2 - x_3, \quad \widetilde{G}^2\widetilde{h} = -2.$$

Thus, $\overline{p} = (0, 0, 0)$ is an elliptic fold-fold singularity of (13). Now, let us write system (12) in the general form given by (5), where

$$\overline{F}(x, y, \varepsilon) = \left(x_2 + x_3, 1, 1, \frac{y}{\varepsilon} + \varepsilon\right)$$

and

$$\overline{G}(x, y, \varepsilon) = \left(x_2 - x_3 + \varepsilon, -1, 1, \frac{y}{\varepsilon} + \varepsilon\right).$$

We have that

$$\overline{F}h = x_2 + x_3, \quad \overline{F}^2 h = 2, \quad \overline{G}h = x_2 - x_3 + \varepsilon, \quad \overline{G}^2 h = -2.$$

By solving the system $\overline{F}h(x, f_\varepsilon(x), \varepsilon) = 0$, $\overline{G}h(x, f_\varepsilon(x), \varepsilon) = 0$ we obtain the ε-family $\overline{p}_\varepsilon = (-\varepsilon, -\frac{\varepsilon}{2}, \frac{\varepsilon}{2}, -2\varepsilon)$. Since $\overline{F}^2 h(\overline{p}_\varepsilon) = 2$ and $\overline{G}^2 h(\overline{p}_\varepsilon) = -2$, it follows that $\overline{p}_\varepsilon$ is an elliptic fold-fold singularity of (12), for all $\varepsilon \neq 0$ small.

For systems of the form (1) which are not locally simple at $\overline{p}$, we cannot guarantee the validity of the statement (ii) of Theorem B.

Counter-example 2. Consider the Filippov slow-fast system in $\mathbb{R}^4$

$$(14) \qquad \dot{x} = \begin{cases} (x_2, 1, 1) & \text{if } 2y - x_1 \leq 0, \\ (x_2 - x_3 + \varepsilon, -1, 1) & \text{if } 2y - x_1 \geq 0, \end{cases} \quad \varepsilon \dot{y} = y - x_1,$$

where $x = (x_1, x_2, x_3) \in \mathbb{R}^3$. For $\varepsilon = 0$ in (14) we have the reduced problem

$$(15) \qquad (\dot{x}_1, \dot{x}_2, \dot{x}_3) = \begin{cases} (x_2, 1, 1) & \text{if } x_1 \leq 0, \\ (x_2 - x_3, -1, 1) & \text{if } x_1 \geq 0, \end{cases} \quad y = x_1.$$

It is easy to see that $\overline{p} = (0, 0, 0)$ is an elliptic fold-fold singularity of (15). Note that system (14) is not locally simple at $\overline{p}$.

By denoting system (14) as in (5), for $\varepsilon \neq 0$ we have

$$\overline{F}h = -x_2 + \frac{2}{\varepsilon}(y - x_1), \qquad \overline{F}^2 h = -\frac{2}{\varepsilon}x_2 - 1 + \frac{2}{\varepsilon^2}(y - x_1),$$

$$\overline{G}h = -x_2 + x_3 - \varepsilon + \frac{2}{\varepsilon}(y - x_1), \quad \overline{G}^2 h = -\frac{2}{\varepsilon}(x_2 - x_3 + \varepsilon) + 2 + \frac{2}{\varepsilon^2}(y - x_1).$$

By solving the system $\overline{F}h(x, f_\varepsilon(x), \varepsilon) = 0$, $\overline{G}h(x, f_\varepsilon(x), \varepsilon) = 0$ we obtain the ε-family $\overline{p}_\varepsilon = (0, 0, \varepsilon, 0)$. Since $\overline{F}^2 h(\overline{p}_\varepsilon) = -1$ and $\overline{G}^2 h(\overline{p}_\varepsilon) = 2$, it follows that $\overline{p}_\varepsilon$ is a hyperbolic fold-fold singularity of (14), for all $\varepsilon \neq 0$ small. Therefore, the statement (ii) of Theorem B is not true.

3. Singular perturbation of cusp singularities

In this section we study the effect of singular perturbations at the singularities of the kind cusp of the reduced problem (4). We restrict our attention to the case $n = 2$, that is, we consider families of systems of the form (1) where $x = (x_1, x_2) \in \mathbb{R}^2$ and $y \in \mathbb{R}$.

Definition 3.1. We say that $p \in \mathcal{M}$ is a cusp-regular singularity of (2) if

(i) $Fh(p) = F^2 h(p) = 0$, $F^3 h(p) \neq 0$ and $Gh(p) \neq 0$; or
(ii) $Gh(p) = G^2 h(p) = 0$, $G^3 h(p) \neq 0$ and $Fh(p) \neq 0$.

Unlike the fold-regular singularities, the cusp-regular ones are not persistent with respect to singular perturbations. In order to prove this statement, consider the following example.

Example 4. Consider the following Filippov slow-fast system in $\mathbb{R}^3$

$$(16) \qquad \dot{x} = (\dot{x}_1, \dot{x}_2) = \begin{cases} (-1, -x_1^2 + \varepsilon) & \text{if } x_2 \leq 0, \\ (1, 1) & \text{if } x_2 \geq 0, \end{cases} \quad \varepsilon \dot{y} = y.$$

The reduced problem of (16) is given by

$$(17) \qquad \dot{x} = (\dot{x}_1, \dot{x}_2) = \begin{cases} (-1, -x_1^2) & \text{if } x_2 \leq 0, \\ (1, 1) & \text{if } x_2 \geq 0, \end{cases} \quad y = 0.$$

It is easy to see that $\overline{p} = (0,0)$ is a cusp-regular singularity of (17). By denoting system (16) as in (5), we have

$$\overline{F}h = -x_1^2 + \varepsilon, \quad \overline{F}^2 h = 2x_1, \quad \overline{F}^3 h = -2, \quad \overline{G}h = 1.$$

Thus, for all $\varepsilon \neq 0$, the vector field $\overline{F}$ does not have cusp singularities in $\mathcal{M}_\varepsilon = \{x_2 = 0\}$. For $\varepsilon > 0$, the cusp-regular point $(0,0)$ is unfolded by families $\overline{p}_\varepsilon = (\sqrt{\varepsilon}, 0, 0)$ and $\overline{q}_\varepsilon = (-\sqrt{\varepsilon}, 0, 0)$ of fold-regular singularities of (16). If $\varepsilon < 0$, system (16) does not have tangency points on $\mathcal{M}_\varepsilon$.

We have the following result.

Theorem C. *Let $Z_\varepsilon(x, y)$ be a C^r family defined by (1), with $x \in \mathbb{R}^2$ and $r \geq 3$. Consider $\overline{p} = (p, f_0(p), 0) \in \mathcal{S}_0 \cap \mathcal{M}_0$ a cusp-regular singularity of the reduced problem Z_0 with $\widetilde{F}(p)\widetilde{G}(p) \neq 0$ and suppose that Z_ε is locally simple at $\overline{p}$. Then, there exist $\varepsilon_0 > 0$, a neighborhood W of $\overline{p}$, and a C^{r-1} function $g \colon (-\varepsilon_0, \varepsilon_0) \to \mathbb{R}$, such that:*

  (i) *$g(\varepsilon) = 0$ if and only if Z_ε has a cusp-regular singularity at the point $\overline{p}_\varepsilon$.*
  (ii) *If $g(\varepsilon) \neq 0$, there are two possibilities: If $g(\varepsilon) > 0$, then Z_ε does not have tangency points on $\mathcal{M}_\varepsilon$. If $g(\varepsilon) < 0$, then $\overline{p}$ is unfolded by two distinct points $\overline{p}_\varepsilon$ and $\overline{q}_\varepsilon$ which are fold-regular singularities of Z_ε.*

Theorem C is proved in Section 5. Note that we can choose $g(\varepsilon) = -\varepsilon$ in Example 4.

3.1. Singular perturbation of fold-cusp singularities. It is not difficult to show that fold-cusp singularities are not persistent with respect to singular perturbations in dimensions two and three. On the other hand, in dimension four they are persistent. In fact,

$$\dot{x} = (\dot{x}_1, \dot{x}_2) = \begin{cases} (1, x_1) & \text{if } x_2 \leq 0, \\ (-1, -x_1^2 + \varepsilon) & \text{if } x_2 \geq 0, \end{cases} \quad \varepsilon \dot{y} = y,$$

and

$$\dot{x} = (\dot{x}_1, \dot{x}_2, \dot{x}_3) = \begin{cases} (1, x_1 - x_3, -1) & \text{if } x_2 \leq 0, \\ (-1, -x_1^2 + \varepsilon, 1) & \text{if } x_2 \geq 0, \end{cases} \quad \varepsilon \dot{y} = y,$$

provide counter-examples for $n = 2$ and $n = 3$ respectively.

Theorem D. *Let $Z_\varepsilon(x,y)$ be a C^r family defined by (1), with $x \in \mathbb{R}^n$, $n \geq 4$ and $r \geq 3$. Consider $\overline{p} = (p, f_0(p), 0) \in \mathcal{S}_0 \cap \mathcal{M}_0$ a fold-cusp singularity of the reduced problem Z_0 and suppose that Z_ε is locally simple at $\overline{p}$. Then there are $\varepsilon_1 > 0$ and a C^{r-1} family $\{\overline{p}_\varepsilon : \varepsilon \in (-\varepsilon_1, \varepsilon_1)\}$ such that $\overline{p}_0 = \overline{p}$ and $\overline{p}_\varepsilon$ is a fold-cusp singularity of Z_ε.*

Theorem D is proved in Section 5. It is illustrated by the example below.

Example 5. Consider the following Filippov slow-fast system in $\mathbb{R}^5$

$$(18) \quad \dot{x} = \begin{cases} (1, x_1 - x_3, -1 - \varepsilon, 1 + \varepsilon) & \text{if } x_2 + \varepsilon \leq 0, \\ (-1, -x_1^2 - x_4 + \varepsilon, 1, 1) & \text{if } x_2 + \varepsilon \geq 0, \end{cases} \quad \varepsilon \dot{y} = y + 2\varepsilon,$$

where $x = (x_1, x_2, x_3, x_4) \in \mathbb{R}^4$. The reduced problem associated to (18) is given by

$$(19) \quad \dot{x} = \begin{cases} (1, x_1 - x_3, -1, 1) & \text{if } x_2 \leq 0, \\ (-1, -x_1^2 - x_4, 1, 1) & \text{if } x_2 \geq 0, \end{cases} \quad y = 0.$$

By denoting system (19) as in (4), we have

$$\widetilde{F}\widetilde{h} = x_1 - x_3, \quad \widetilde{F}^2\widetilde{h} = 2, \quad \widetilde{G}\widetilde{h} = -x_1^2 - x_4, \quad \widetilde{G}^2\widetilde{h} = 2x_1 - 1, \quad \widetilde{G}^3\widetilde{h} = -2.$$

Thus, $\overline{p} = (1/2, 0, 1/2, -1/4, 0)$ is a fold-cusp singularity of (19). Now, by denoting system (18) as in (5), we have

$$\overline{F}h = x_1 - x_3, \qquad \overline{F}^2h = 2 + \varepsilon,$$
$$\overline{G}h = -x_1^2 - x_4 + \varepsilon, \quad \overline{G}^2h = 2x_1 - 1, \quad \overline{G}^3h = -2.$$

By solving the system

$$\overline{F}h(x, f_\varepsilon(x), \varepsilon) = 0,$$
$$\overline{G}h(x, f_\varepsilon(x), \varepsilon) = 0,$$
$$\overline{G}^2h(x, f_\varepsilon(x), \varepsilon) = 0,$$

we obtain the ε-family $\overline{p}_\varepsilon = (1/2, -\varepsilon, 1/2, \varepsilon - 1/4, -2\varepsilon)$. Since $\overline{F}^2h(\overline{p}_\varepsilon) = 2 + \varepsilon$ and $\overline{G}^3h(\overline{p}_\varepsilon) = -2$, it follows that $\overline{p}_\varepsilon$ is a fold-cusp singularity of (18) for all $\varepsilon \neq 0$ small.

4. Singular perturbation of hyperbolic singularities

In this section we study the effect of singular perturbations at hyperbolic singularities of the reduced problem (4). We restrict our attention to the case $n = 2$.

Definition 4.1. We say that $p \in \mathcal{M}$ is a hyperbolic/equilibrium-regular singularity of (2) provided $F(p) = 0$, $Gh(p) \neq 0$, and $DF(p)$ has eigenvalues λ_1 and λ_2 with nonzero real parts. Or, $G(p) = 0$, $Fh(p) \neq 0$ and $DG(p)$ has eigenvalues λ_1 and λ_2 with nonzero real parts.

Remark 3. We define hyperbolic/equilibrium-regular singularity for system (4) similarly to what was done in Definition 4.1. We note that system (4) is defined on the manifold $\mathcal{S}_0 = \{H(x, y, 0) = 0\}$.

We have the following result.

Theorem E. *Let $Z_\varepsilon(x, y)$ be a C^r family defined by (1), with $x \in \mathbb{R}^2$ and $r > 2$. Consider $\overline{p} = (p, f_0(p), 0) \in \mathcal{S}_0 \cap \mathcal{M}_0$ a hyperbolic/equilibrium-regular singularity of the reduced problem Z_0. Suppose that Z_ε is locally simple at $\overline{p}$ and that the eigenspaces of Z_0 are transverse to $\mathcal{M}_0$. Then, there exist $\varepsilon_0 > 0$, a neighborhood W of $\overline{p}$ such that:*

(i) *Each $Z_\varepsilon = (\overline{F}, \overline{G})$, with $\varepsilon \in (-\varepsilon_0, \varepsilon_0)$, has a hyperbolic equilibrium point $\overline{p}_\varepsilon \in W$ such that $\overline{p}_0 = \overline{p}$. Moreover, the hyperbolic equilibrium $\overline{p}_\varepsilon$ is of the same kind as $\overline{p}$.*

(ii) *If $\overline{p}_\varepsilon \notin \mathcal{M}_\varepsilon \cap W$, then there exists a family $\{\overline{q}_\varepsilon : \varepsilon \in (-\varepsilon_0, \varepsilon_0)\}$ of fold-regular singularities of Z_ε such that $\overline{q}_0 = \overline{p}$.*

Theorem E is proved in Section 5. It is illustrated by the example below.

Example 6. Consider the following Filippov slow-fast system in $\mathbb{R}^3$

$$(20) \qquad \dot{x} = \begin{cases} (0, 1) & \text{if } x_1 + x_2 \leq 0, \\ (x_1, -x_2 + \varepsilon) & \text{if } x_1 + x_2 \geq 0, \end{cases} \quad \varepsilon \dot{y} = y,$$

where $x = (x_1, x_2) \in \mathbb{R}^2$. For $\varepsilon = 0$ in (20) we have the reduced problem

$$(21) \qquad \dot{x} = \begin{cases} (0, 1) & \text{if } x_1 + x_2 \leq 0, \\ (x_1, -x_2) & \text{if } x_1 + x_2 \geq 0, \end{cases} \quad y = 0.$$

It is easy to see that $\overline{p} = (0, 0)$ is a saddle-regular singularity of (21). For all $\varepsilon \neq 0$, the point $\overline{p}_\varepsilon = (0, \varepsilon, 0)$ is a saddle for the vector field $\overline{F} = (x_1, -x_2 + \varepsilon, y/\varepsilon)$, and note that $\overline{p}_\varepsilon \notin \mathcal{M}_\varepsilon = \{x_1 + x_2 = 0\}$. Thus, system (20) does not have saddle-regular singularities on the switching manifold $\mathcal{M}_\varepsilon$. For $\varepsilon \neq 0$, there exists the family $\overline{q}_\varepsilon = (-\varepsilon/2, \varepsilon/2, 0)$ of fold-regular singularities of (20). For $\varepsilon > 0$, the point $\overline{q}_\varepsilon$ is an invisible fold for the vector field $\overline{F} = (x_1, -x_2 + \varepsilon, y/\varepsilon)$. If $\varepsilon < 0$, then $\overline{q}_\varepsilon$ is a visible fold for $\overline{F}$ (see Figure 4).
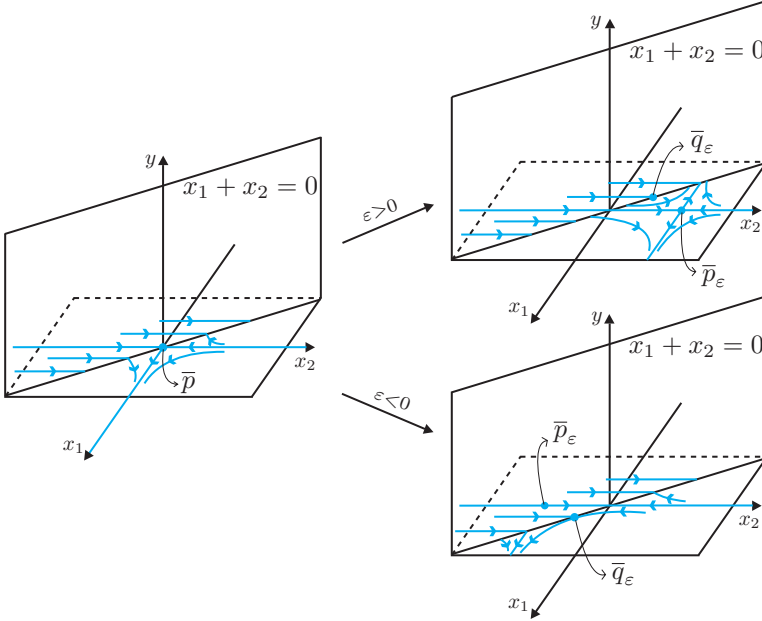
FIGURE 4. Unfolding of the saddle-regular singularity $\overline{p}$ of the reduced problem (21) of Example 6.

5. Proofs of Theorems A, B, C, D, and E

In this section we prove Theorems A, B, C, D, and E. In all these proofs we will use the following result.

Lemma 5.1. *Suppose that system* (1) *is locally simple at* $\overline{p} = (p, f_0(p), 0) \in \mathcal{S}_0 \cap \mathcal{M}_0$. *Then, for all* $i \in \mathbb{N}$,

$$\widetilde{F}^i \widetilde{h}(p) = F^i h(p, f_0(p), 0) \quad and \quad \widetilde{G}^i \widetilde{h}(p) = G^i h(p, f_0(p), 0).$$

Proof: Suppose that $i = 1$. Since $\widetilde{F}(x) = F(x, f_0(x), 0)$ and $\widetilde{h}(x) = h(x, f_0(x), 0)$, it follows by the Chain's rule that

$$(22) \qquad \widetilde{F}\widetilde{h}(p) = \frac{\partial \widetilde{h}}{\partial x}(p) \cdot F(\overline{p}) = \left[\frac{\partial h}{\partial x} \cdot F + \frac{\partial h}{\partial y} \left(\nabla f_0 \cdot F \right) \right] (\overline{p}).$$

Since system (1) is locally simple at $\overline{p}$, one of the following conditions is satisfied: $\frac{\partial h}{\partial x}(\overline{p}) \neq 0$ and $h(p, y, 0) = 0$, or $\frac{\partial H}{\partial x}(\overline{p}) = 0$. In the first case, by Remark 1, it follows that $\frac{\partial h}{\partial y}(\overline{p}) = 0$. The second case implies $\nabla f_0(p) = 0$. In both cases, the relation (22) summarizes to $\widetilde{F}\widetilde{h}(p) = Fh(p, f_0(p), 0)$.

For $i = 2$ we have

$$\widetilde{F}^2\widetilde{h}(p) = \widetilde{F}(p)[\widetilde{F}\widetilde{h}(p)] = F(p, f_0(p), 0)[Fh(p, f_0(p), 0)] = F^2h(p, f_0(p), 0).$$

By induction we can prove that $\widetilde{F}^i\widetilde{h}(p) = F^ih(p, f_0(p), 0)$, for all $i \in \mathbb{N}$.

The second relation $\widetilde{G}^i\widetilde{h}(p) = G^ih(p, f_0(p), 0)$ follows of similar way. $\square$

First we prove Theorem A.

Proof of Theorem A: We suppose that $\overline{p} = (p, f_0(p), 0)$ is a fold for the vector field $\widetilde{F}$ and regular for $\widetilde{G}$. The opposite case is similar. The fold-regular point $\overline{p}$ satisfies the following conditions

$$\widetilde{F}\widetilde{h}(p) = 0, \quad \widetilde{F}^2\widetilde{h}(p) \neq 0, \quad \text{and} \quad \widetilde{G}\widetilde{h}(p) \neq 0.$$

Since Z_ε is locally simple at $\overline{p}$, by Lemma 5.1, these conditions are equivalent to

$$Fh(\overline{p}) = 0, \quad F^2h(\overline{p}) \neq 0, \quad \text{and} \quad Gh(\overline{p}) \neq 0.$$

On the other hand, for each $\varepsilon \neq 0$ small, the vector field $\overline{F}(x, y, \varepsilon) = (F, H/\varepsilon)$ has a tangency at a point q if, and only if, such point satisfies the equation $\overline{F}h(x, y, \varepsilon) = 0$, that is equivalent to

$$(23) \qquad Fh(x, y, \varepsilon) + \frac{\partial h}{\partial y}\frac{H}{\varepsilon}(x, y, \varepsilon) = 0.$$

In order to obtain a family $\overline{p}_\varepsilon$ of tangency points such that $\overline{p}_0 = \overline{p}$ we need to solve the equation (23) restricted to the manifold $y = f_\varepsilon(x)$. Since $H(x, f_\varepsilon(x), \varepsilon) = 0$, it is enough to solve the following equation

$$Fh(x, f_\varepsilon(x), \varepsilon) = 0.$$

We have $F^2h(p, f_0(p), 0) \neq 0$. But, $F^2h = F(Fh) = \frac{\partial(Fh)}{\partial x} \cdot F$. So, in particular, $\frac{\partial(Fh)}{\partial x}(p, f_0(p), 0) \neq 0$. By the Implicit Function Theorem, for each $\varepsilon \neq 0$ sufficiently small there exists a unique $x = x(\varepsilon)$ such that $Fh(x(\varepsilon), f_\varepsilon(x(\varepsilon)), \varepsilon) = 0$. Take $\overline{p}_\varepsilon = (x(\varepsilon), f_\varepsilon(x(\varepsilon)), \varepsilon)$. Thus, $\overline{p}_\varepsilon$ is a family of tangency points of Z_ε such that $\overline{p}_0 = \overline{p}$.

For a fold-regular singularity of Z_ε, $\overline{p}_\varepsilon$, we need to prove that

$$\left[F \cdot \frac{\partial}{\partial x}\left(\frac{\partial h}{\partial x} \cdot F + \frac{\partial h}{\partial y}\frac{H}{\varepsilon}\right) + \frac{H}{\varepsilon}\frac{\partial}{\partial y}\left(\frac{\partial h}{\partial x} \cdot F + \frac{\partial h}{\partial y}\frac{H}{\varepsilon}\right)\right](\overline{p}_\varepsilon) \neq 0$$

and

$$\left(\frac{\partial h}{\partial x} \cdot G + \frac{\partial h}{\partial y}\frac{H}{\varepsilon}\right)(\overline{p}_\varepsilon) \neq 0.$$

Similarly as above since we are restricted to the manifold $y = f_\varepsilon(x)$ and $H(x, f_\varepsilon(x), \varepsilon) = 0$, we obtain

$$F^2 h(\overline{p}_\varepsilon) \neq 0 \quad \text{and} \quad Gh(\overline{p}_\varepsilon) \neq 0.$$

Since $\overline{p} = (p, f_0(p), 0)$ is a fold-regular singularity of Z_0 we have that $\widetilde{F^2 \widetilde{h}}(p) = F^2 h(p, f_0(p), 0) \neq 0$ and $\widetilde{Gh}(p) = Gh(p, f_0(p), 0) \neq 0$. From the continuity of the functions $F^2 h$ and Gh, it follows that $F^2 h(q) \neq 0$ and $Gh(q) \neq 0$ for all q in a given neighborhood of $\overline{p}$. In particular, for each $\varepsilon \neq 0$ sufficiently small, $F^2 h(\overline{p}_\varepsilon) \neq 0$ and $Gh(\overline{p}_\varepsilon) \neq 0$, i.e., $\overline{p}_\varepsilon$ is a fold-regular singularity of Z_ε.

In order to prove the statement (ii), suppose that the fold-regular $\overline{p}$ is visible. So, $\widetilde{F^2 \widetilde{h}}(p) = F^2 h(p, f_0(p), 0) < 0$. Again, by continuity, we can conclude that $F^2 h(\overline{p}_\varepsilon) < 0$, for each $\varepsilon \neq 0$ sufficiently small, i.e., $\overline{p}_\varepsilon$ is a visible fold-regular singularity of Z_ε. Similarly, if $\overline{p}$ is an invisible fold-regular singularity of Z_0, then $\overline{p}_\varepsilon$ will also be an invisible fold-regular singularity of Z_ε. $\qquad \square$

Now we prove Theorem B.

Proof of Theorem B: The fold-fold singularity $\overline{p}$ of Z_0 satisfies the following conditions

$$\widetilde{F \widetilde{h}}(p) = \widetilde{G \widetilde{h}}(p) = 0, \quad \widetilde{F^2 \widetilde{h}}(p) \neq 0, \quad \text{and} \quad \widetilde{G^2 \widetilde{h}}(p) \neq 0.$$

Since Z_ε is locally simple at $\overline{p}$, by Lemma 5.1, these conditions are equivalent to

$$Fh(\overline{p}) = Gh(\overline{p}) = 0, \quad F^2 h(\overline{p}) \neq 0, \quad \text{and} \quad G^2 h(\overline{p}) \neq 0.$$

In order to obtain a family $\overline{p}_\varepsilon$ of tangency points of $\overline{F}$ and $\overline{G}$ such that $\overline{p}_0 = \overline{p}$ we need to solve the system

$$Fh(x, y, \varepsilon) + \frac{\partial h}{\partial y} \frac{H}{\varepsilon}(x, y, \varepsilon) = 0,$$

$$Gh(x, y, \varepsilon) + \frac{\partial h}{\partial y} \frac{H}{\varepsilon}(x, y, \varepsilon) = 0,$$

restricted to the manifold $y = f_\varepsilon(x)$. Since $H(x, f_\varepsilon(x), \varepsilon) = 0$, it is enough to solve the following system

$$Fh(x, f_\varepsilon(x), \varepsilon) = 0,$$

$$Gh(x, f_\varepsilon(x), \varepsilon) = 0.$$

Let Σ_1, Σ_2, Σ_1^ε, and Σ_2^ε be the n-dimensional manifolds in $\mathbb{R}^{n+1}$ given by $\{Fh(x, f_0(x), 0) = 0\}$, $\{Gh(x, f_0(x), 0) = 0\}$, $\{Fh(x, f_\varepsilon(x), \varepsilon) = 0\}$,

and $\{Gh(x, f_\varepsilon(x), \varepsilon) = 0\}$, respectively. The fact that $\overline{p}$ is a tangency point of the vector fields $\widetilde{F}$ and $\widetilde{G}$ means that the manifolds $\mathcal{S}_0$, $\mathcal{M}_0$, Σ_1, and Σ_2 intersect transversally at $\overline{p}$. Since transversal intersection is a stable property, small perturbations of the manifolds $\mathcal{S}_0$, $\mathcal{M}_0$, Σ_1, and Σ_2 still produce a transversal intersection. So we can conclude that the manifolds $\mathcal{N}_\varepsilon$, $\mathcal{M}_\varepsilon$, Σ_1^ε, and Σ_2^ε intersect transversally for all $\varepsilon \neq 0$ sufficiently small. Moreover, $\dim(\mathcal{N}_\varepsilon \cap \mathcal{M}_\varepsilon \cap \Sigma_1^\varepsilon \cap \Sigma_2^\varepsilon) = n - 3$. This means that, for each $\varepsilon \neq 0$ sufficiently small, there exists a point $\overline{p}_\varepsilon$ such that $\overline{p}_0 = \overline{p}$ and $\overline{p}_\varepsilon$ is a tangency point for the vector fields $\overline{F} = (F, H/\varepsilon)$ and $\overline{G} = (G, H/\varepsilon)$. Note that when $n = 3$, the tangency point $\overline{p}_\varepsilon$ is isolated but if $n > 3$ it is not.

In order to prove that $\overline{p}_\varepsilon$ is a fold-fold singularity of Z_ε we need to prove that $F^2h(\overline{p}_\varepsilon) \neq 0$ and $G^2h(\overline{p}_\varepsilon) \neq 0$. Since $\overline{p} = (p, f_0(p), 0)$ is a fold-fold singularity of Z_0 we have that $\widetilde{F}^2\widetilde{h}(p) = F^2h(p, f_0(p), 0) \neq 0$ and $\widetilde{G}^2\widetilde{h}(p) = G^2h(p, f_0(p), 0) \neq 0$. From the continuity of the functions F^2h and G^2h, it follows that $F^2h(q) \neq 0$ and $G^2h(q) \neq 0$ for all q in a given neighborhood of $\overline{p}$. In particular, for each $\varepsilon \neq 0$ sufficiently small, $F^2h(\overline{p}_\varepsilon) \neq 0$ and $G^2h(\overline{p}_\varepsilon) \neq 0$, i.e., $\overline{p}_\varepsilon$ is a fold-fold singularity of Z_ε.

In order to prove the statement (ii), suppose that the fold-fold $\overline{p}$ is elliptic. So, $\widetilde{F}^2\widetilde{h}(p) = F^2h(p, f_0(p), 0) > 0$ and $\widetilde{G}^2\widetilde{h}(p) = G^2h(p, f_0(p), 0) < 0$. Again, by continuity, we can conclude that $F^2h(\overline{p}_\varepsilon) > 0$ and $G^2h(\overline{p}_\varepsilon) < 0$, for each $\varepsilon \neq 0$ sufficiently small, i.e., $\overline{p}_\varepsilon$ is an elliptic fold-fold singularity of Z_ε. Similarly, if $\overline{p}$ is a parabolic (resp. hyperbolic) fold-fold of Z_0, then $\overline{p}_\varepsilon$ will be a parabolic (resp. hyperbolic) fold-fold of Z_ε. $\square$

Now we prove Theorem C.

Proof of Theorem C: We suppose that $\overline{p} = (p, f_0(p), 0)$ is a cusp for the vector field $\widetilde{F}$ and regular for $\widetilde{G}$, i.e, $\widetilde{F}\widetilde{h}(p) = \widetilde{F}^2\widetilde{h}(p) = 0$, $\widetilde{F}^3\widetilde{h}(p) \neq 0$, and $\widetilde{G}\widetilde{h}(p) \neq 0$. The opposite case is similar. Since $\widetilde{F}(p) \neq 0$ and $\widetilde{G}(p) \neq 0$ we can choose $\varepsilon_1 > 0$ and a neighborhood W of $\overline{p}$ in $\mathbb{R}^3$ such that $F(q) \neq 0$ and $G(q) \neq 0$, for all $q \in W$ and all $Z_\varepsilon = [(F, H/\varepsilon), (G, H/\varepsilon)]$ with $\varepsilon \in (-\varepsilon_1, \varepsilon_1)$. In order to prove statements (i) and (ii) of Theorem C, we restrict our attention on the manifold $\mathcal{N}_\varepsilon = \{H(x, y, \varepsilon) = 0\}$. We define $K \colon (-\varepsilon_1, \varepsilon_1) \times \mathbb{R} \to \mathbb{R}$ by

$$K(\varepsilon, \alpha) = \Big[F(r_\varepsilon(\alpha)) \wedge r_\varepsilon'(\alpha)\Big]\Big[G(r_\varepsilon(\alpha)) \wedge r_\varepsilon'(\alpha)\Big],$$

where $r_\varepsilon \colon \mathbb{R} \to \mathcal{M}_\varepsilon$ is an imbedding such that $r_0(0) = \overline{p}$, and "$\wedge$" is the product $(a, b) \wedge (c, d) = ad - bc$, where $(a, b), (c, d) \in \mathbb{R}^2$.

Calling

$$(I) = \frac{d}{d\alpha}[F(r_\varepsilon(\alpha))] \wedge r'_\varepsilon(\alpha) - F(r_\varepsilon(\alpha)) \wedge r''_\varepsilon(\alpha),$$

$$(II) = \frac{d}{d\alpha}[G(r_\varepsilon(\alpha))] \wedge r'_\varepsilon(\alpha) - G(r_\varepsilon(\alpha)) \wedge r''_\varepsilon(\alpha),$$

$$(III) = \frac{d^2}{d\alpha^2}[F(r_\varepsilon(\alpha))] \wedge r'_\varepsilon(\alpha) - 2\frac{d}{d\alpha}[F(r_\varepsilon(\alpha))] \wedge r''_\varepsilon(\alpha) + F(r_\varepsilon(\alpha)) \wedge r'''_\varepsilon(\alpha),$$

$$(IV) = \frac{d^2}{d\alpha^2}[G(r_\varepsilon(\alpha))] \wedge r'_\varepsilon(\alpha) - 2\frac{d}{d\alpha}[G(r_\varepsilon(\alpha))] \wedge r''_\varepsilon(\alpha) + G(r_\varepsilon(\alpha)) \wedge r'''_\varepsilon(\alpha),$$

we have that

$$\frac{\partial K}{\partial \alpha}(\varepsilon, \alpha) = (I)\Big[G(r_\varepsilon(\alpha)) \wedge r'_\varepsilon(\alpha)\Big] + (II)\Big[F(r_\varepsilon(\alpha)) \wedge r'_\varepsilon(\alpha)\Big]$$

and

$$\frac{\partial^2 K}{\partial \alpha^2}(\varepsilon, \alpha) = (III)\Big[G(r_\varepsilon(\alpha)) \wedge r'_\varepsilon(\alpha)\Big] + (IV)\Big[F(r_\varepsilon(\alpha)) \wedge r'_\varepsilon(\alpha)\Big] + 2(I)(II).$$

By a direct calculation we obtain

$$\frac{\partial K}{\partial \alpha}(0,0) = 0 \quad \text{and} \quad \frac{\partial^2 K}{\partial \alpha^2}(0,0) \neq 0.$$

In the above statements we used the fact that $\overline{p}$ is a cusp singularity of $\widetilde{F}$, regular for $\widetilde{G}$, and the hypothesis that Z_ε is locally simple at $\overline{p}$ (Lemma 5.1 states that $F^i h(\overline{p}) = \widetilde{F}^i \widetilde{h}(p)$ and $G^i h(\overline{p}) = \widetilde{G}^i \widetilde{h}(p)$).

By the Implicit Function Theorem, there exist $0 < \varepsilon_0 < \varepsilon_1$, a neighborhood J of 0 in $\mathbb{R}$, and a C^{r-1} function $\xi \colon (-\varepsilon_0, \varepsilon_0) \to J$ such that $\frac{\partial K}{\partial \alpha}(\varepsilon, \xi(\varepsilon)) = 0$ and $\frac{\partial^2 K}{\partial \alpha^2}(\varepsilon, \alpha) \neq 0$, for all $(\varepsilon, \alpha) \in (-\varepsilon_0, \varepsilon_0) \times J$. Assume for simplicity that $\frac{\partial^2 K}{\partial \alpha^2}(0,0) > 0$. The other case is similar. Choose ε_0 and J such that $\frac{\partial^2 K}{\partial \alpha^2}(\varepsilon, \alpha) > 0$, for all $(\varepsilon, \alpha) \in (-\varepsilon_0, \varepsilon_0) \times J$.

For each $\varepsilon \in (-\varepsilon_0, \varepsilon_0)$, consider the function $f_\varepsilon \colon J \to \mathbb{R}$ given by $f_\varepsilon(\alpha) = K(\varepsilon, \alpha)$. Thus, for each $\varepsilon \in (-\varepsilon_0, \varepsilon_0)$, $\xi(\varepsilon)$ is a minimum point of f_ε. Moreover:

(i) If $f_\varepsilon(\xi(\varepsilon)) > 0$ then $f_\varepsilon(\alpha) > 0$, for all $\alpha \in J$. This means that Z_ε is transverse to $\mathcal{M}_\varepsilon$ around $\overline{p}$. Thus, Z_ε does not have tangency points on $\mathcal{M}_\varepsilon$.

(ii) If $f_\varepsilon(\xi(\varepsilon)) = 0$, then $f_\varepsilon(\alpha) = 0$ if, and only if, $\alpha = \xi(\varepsilon)$.

(iii) If $f_\varepsilon(\xi(\varepsilon)) < 0$ then by the Intermediate Value Theorem there exist $\alpha_1, \alpha_2 \in J$, with $\alpha_1 < \xi(\varepsilon) < \alpha_2$ such that $f_\varepsilon(\alpha_1) = f_\varepsilon(\alpha_2) = 0$, however $\frac{\partial K}{\partial \alpha}(\varepsilon, \alpha_i) \neq 0$, $i = 1, 2$. This means that the contact of the vector field $\overline{F}$ with $\mathcal{M}_\varepsilon$ at $r_\varepsilon(\alpha_1)$ and $r_\varepsilon(\alpha_2)$ is quadratic and the vector field $\overline{G}$ is regular at $r_\varepsilon(\alpha_1)$ and $r_\varepsilon(\alpha_2)$. So $\overline{p}$ is unfolded by two distinct points $\{\overline{p}_\varepsilon = r_\varepsilon(\alpha_1)\}$ and $\{\overline{q}_\varepsilon = r_\varepsilon(\alpha_2)\}$ which are fold-regular singularities of Z_ε.

Define $g \colon (-\varepsilon_0, \varepsilon_0) \to \mathbb{R}$ by $g(\varepsilon) = K(\varepsilon, \xi(\varepsilon))$. This completes the proof of Theorem C. $\qquad\blacksquare$

Now we prove Theorem D.

Proof of Theorem D: We suppose that $\overline{p} = (p, f_0(p), 0)$ is a fold singularity for the vector field $\widetilde{F}$ and a cusp singularity for $\widetilde{G}$. The other case is similar. The fold-cusp singularity $\overline{p}$ of Z_0 satisfies the following conditions

$$\widetilde{F}\widetilde{h}(p) = \widetilde{G}\widetilde{h}(p) = \widetilde{G}^2\widetilde{h}(p) = 0, \quad \widetilde{F}^2\widetilde{h}(p) \neq 0, \quad \text{and} \quad \widetilde{G}^3\widetilde{h}(p) \neq 0.$$

Since Z_ε is locally simple at $\overline{p}$, by Lemma 5.1, these conditions are equivalent to

$$Fh(\overline{p}) = Gh(\overline{p}) = G^2h(\overline{p}) = 0, \quad F^2h(\overline{p}) \neq 0, \quad \text{and} \quad G^3h(\overline{p}) \neq 0.$$

Let Σ_1, Σ_2, Σ_3, Σ_1^ε, Σ_2^ε, and Σ_3^ε be the n-dimensional manifolds in $\mathbb{R}^{n+1}$ given by $\{Fh(x, f_0(x), 0) = 0\}$, $\{Gh(x, f_0(x), 0) = 0\}$, $\{G^2h(x, f_0(x), 0) = 0\}$, $\{Fh(x, f_\varepsilon(x), \varepsilon) = 0\}$, $\{Gh(x, f_\varepsilon(x), \varepsilon) = 0\}$, and $\{G^2h(x, f_\varepsilon(x), \varepsilon) = 0\}$, respectively. The fact that $\overline{p}$ is a fold-cusp singularity of Z_0 means that the manifolds $\mathcal{S}_0$, $\mathcal{M}_0$, Σ_1, Σ_2, and Σ_3 intersect transversally at $\overline{p}$ and that $F^2h(\overline{p}) \neq 0$ and $G^3h(\overline{p}) \neq 0$. Since transversal intersection is a stable property, we can conclude that the manifolds $\mathcal{N}_\varepsilon$, $\mathcal{M}_\varepsilon$, Σ_1^ε, Σ_2^ε, and Σ_3^ε intersect transversally for all $\varepsilon \neq 0$ sufficiently small. Moreover, $\dim(\mathcal{N}_\varepsilon \cap \mathcal{M}_\varepsilon \cap \Sigma_1^\varepsilon \cap \Sigma_2^\varepsilon \cap \Sigma_3^\varepsilon) = n - 4$. This means that, for each $\varepsilon \neq 0$ sufficiently small, there exists a point $\overline{p}_\varepsilon \in \mathcal{N}_\varepsilon \cap \mathcal{M}_\varepsilon \cap \Sigma_1^\varepsilon \cap \Sigma_2^\varepsilon \cap \Sigma_3^\varepsilon$. Clearly, $\overline{p}_\varepsilon \to \overline{p}$ when $\varepsilon \to 0$. By the continuity of the functions F^2h and G^3h, we have that $F^2h(\overline{p}_\varepsilon) \neq 0$ and $G^3h(\overline{p}_\varepsilon) \neq 0$, for each $\varepsilon \neq 0$ sufficiently small. Therefore, $\overline{p}_\varepsilon$ is a fold-cusp singularity of Z_ε.

Note that when $n = 4$ the fold-cusp point $\overline{p}_\varepsilon$ is isolated but if $n > 4$ it is not. $\qquad\blacksquare$

We prove below Theorem E.

Proof of Theorem E: In order to prove statements (i) and (ii) of Theorem E, we restrict our attention on the manifold $\mathcal{N}_\varepsilon = \{H(x, y, \varepsilon) = 0\}$.

We suppose that $\overline{p}$ is a hyperbolic equilibrium point for the vector field $\widetilde{F}$ and regular for $\widetilde{G}$. Thus, $\widetilde{F}(p) = F(\overline{p}) = 0$, $D\widetilde{F}(p)$ has eigenvalues λ_1 and λ_2 with nonzero real parts and $\widetilde{G}\widetilde{h}(p) = Gh(\overline{p}) \neq 0$. Let V be a neighborhood of p in $\mathbb{R}^2$ such that p is the unique singularity of $\widetilde{F}_{|_V}$. It is known that there are $\varepsilon_1 > 0$ and a neighborhood W of $\overline{p}$ in $\mathbb{R}^3$ such that each vector field $F(x, y, \varepsilon)$, with $\varepsilon \in (-\varepsilon_1, \varepsilon_1)$, has one unique hyperbolic singularity $\overline{p}_\varepsilon$ in W and this singularity is of the same kind as $\overline{p}$.

First we consider the case where λ_1 and λ_2 are real eigenvalues with $\lambda_1 \neq \lambda_2$. We can choose local coordinates $x = (x_1, x_2)$ around p such that the components of $\widetilde{F}$, $\widetilde{F}_1$, and $\widetilde{F}_2$, satisfy

$$\frac{\partial \widetilde{F}_1}{\partial x_1}(p) = \lambda_1, \quad \frac{\partial \widetilde{F}_2}{\partial x_2}(p) = \lambda_2, \quad \text{and} \quad \frac{\partial \widetilde{F}_1}{\partial x_2}(p) = \frac{\partial \widetilde{F}_2}{\partial x_1}(p) = 0.$$

Define $K \colon (-\varepsilon_1, \varepsilon_1) \times \mathbb{R} \to \mathbb{R}$ by

$$K(\varepsilon, \alpha) = F(r_\varepsilon(\alpha)) \wedge r_\varepsilon'(\alpha),$$

where $r_\varepsilon \colon \mathbb{R} \to \mathcal{M}_\varepsilon$ is an imbedding such that $r_0(0) = \overline{p}$, and "$\wedge$" is the product $(a, b) \wedge (c, d) = ad - bc$, where $(a, b), (c, d) \in \mathbb{R}^2$.

Note that $K(0, 0) = 0$. If $r_\varepsilon(\alpha) = (r_\varepsilon^1(\alpha), r_\varepsilon^2(\alpha))$, by hypothesis we have $(r_0^1)'(0) \neq 0$ and $(r_0^2)'(0) \neq 0$. Thus

$$K(\varepsilon, \alpha) = F_1(r_\varepsilon(\alpha)) \wedge (r_\varepsilon^2)'(\alpha) - F_2(r_\varepsilon(\alpha)) \wedge (r_\varepsilon^1)'(\alpha)$$

and, by direct computation, we obtain

$$\frac{\partial K}{\partial \alpha}(0, 0) = (r_0^1)'(0)(r_0^2)'(0)(\lambda_1 - \lambda_2).$$

Since $\lambda_1 \neq \lambda_2$, then $\frac{\partial K}{\partial \alpha}(0, 0) \neq 0$. By the Implicit Function Theorem, there exist $0 < \varepsilon_0 < \varepsilon_1$, a neighborhood J of 0 in $\mathbb{R}$ and a C^{r-1} function $\xi \colon (-\varepsilon_0, \varepsilon_0) \to J$ such that $K(\varepsilon, \xi(\varepsilon)) = 0$ and $\frac{\partial K}{\partial \alpha}(\varepsilon, \alpha) \neq 0$, for all $(\varepsilon, \alpha) \in (-\varepsilon_0, \varepsilon_0) \times J$.

If $q_\varepsilon := F(r_\varepsilon(\xi(\varepsilon))) \neq 0$, then this vector and $r_\varepsilon'(\xi(\varepsilon))$ are linearly dependent because $K(\varepsilon, \xi(\varepsilon)) = 0$. Define $\overline{q}_\varepsilon =: (q_\varepsilon, f_\varepsilon(q_\varepsilon), \varepsilon)$. This means that $\{\overline{q}_\varepsilon : \varepsilon \in (-\varepsilon_0, \varepsilon_0)\}$ is a family of tangency points of Z_ε. Moreover, the condition $\frac{\partial K}{\partial \alpha}(\varepsilon, \xi(\varepsilon)) \neq 0$ imply that the contact of the vector field $\overline{F}$ with $\mathcal{M}_\varepsilon$ at $\overline{q}_\varepsilon$ is quadratic.

Since $\widetilde{\widetilde{Gh}}(p) = Gh(p, f_0(p), 0) \neq 0$, from the continuity of the function Gh, it follows that $Gh(\overline{q}_\varepsilon) \neq 0$ for all $\varepsilon \neq 0$ sufficiently small, i.e., $\overline{q}_\varepsilon$ is regular for the vector field $\overline{G}$. This completes the proof in this case.

Now consider the case where $\lambda_1 = \alpha + i\alpha_1$ and $\lambda_2 = \beta + i\beta_1$ are complex eigenvalues with $\alpha \neq 0$ and $\beta \neq 0$. Let $x = (x_1, x_2)$ be a system of coordinates around p such that

$$\frac{\partial \widetilde{F}_1}{\partial x_1}(p) = \frac{\partial \widetilde{F}_2}{\partial x_2}(p) = \alpha \quad \text{and} \quad \frac{\partial \widetilde{F}_1}{\partial x_2}(p) = -\frac{\partial \widetilde{F}_2}{\partial x_1}(p) = \beta.$$

Then, this case follows in a similar way as the first part. $\qquad\square$

6. Closing remarks

Non-smooth singularly perturbed systems were studied in this paper. We investigated the effect of singular perturbations at tangency points. We proved that singularities of the type fold are robust with respect to singular perturbations. On the other hand, cusp singularities are not robust with respect to singular perturbations. We also study the unfolding of cusp singularities and hyperbolic equilibria. In short we proved that:

- for any $n \geq 2$, fold-regular singularities are persistent;
- for $n = 2$, cusp-regular singularities are not persistent;
- for $n = 2$, fold-fold singularities are not persistent but for $n \geq 3$ they are;
- for $n = 2, 3$, fold-cusp singularities are not persistent but for $n \geq 4$ they are;
- for $n = 2$, hyperbolic/equilibrium-regular singularities are not persistent.

Some open questions still remain. In [**8**], Sieber and Kowalczyk studied the robustness of periodic motion with sliding. They considered periodic orbits with an infinitesimally small sliding segment, that is, close to a grazing-sliding bifurcation, and proved that the local return map around the grazing periodic orbit develops a discontinuity if the condition on the existence of an attracting sliding region is violated. The question now is: If a sliding mode is persistent, can we develop a Fenichel's theory for this case?

Another open problem, similar to the one described in the previous paragraph, is the study of the robustness of the periodic motion with sewing with respect to singular perturbations.

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