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LYAPUNOV EXPONENT AND ALMOST SURE ASYMPTOTIC STABILITY OF A STOCHASTIC SIRS MODEL

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Abstract: Epidemiological models with bilinear incidence rate usually have an asymptotically stable trivial equilibrium corresponding to the disease-free state, or an asymptotically stable nontrivial equilibrium (*i.e.* interior equilibrium) corresponding to the endemic state. In this paper, we consider an epidemiological model, which is a SIRS (susceptible-infected-removed-susceptible) model influenced by random perturbations. We prove that the solutions of the system are positive for all positive initial conditions and that the solutions are global, that is, there is no finite explosion time. We present necessary and sufficient condition for the almost sure asymptotic stability of the steady state of the stochastic system.

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1. Introduction

In the understanding of different scenarios for disease transmissions and behavior of epidemics, many models in the literature represent dynamics of diseases by systems of ordinary differential equations. The dynamic behaviors of the SIRS models have been investigated by several authors. In the 1920s, a Kermack–McKendrick epidemic SIRS (susceptible-infected-removed-susceptible) model [**9**] was proposed, in which the total population is assumed to be constant and there are infectives $I(t)$, which can pass on the disease to susceptibles $S(t)$, and the remaining members $R(t)$ which have been infected and have become unable to transmit the disease to others. Since then, many people have studied the SIRS disease model (acquired immunity is permanent or acquired immunity is temporary) with different variations in its incidence rate, at which susceptibles become infectives, see [**11, 12, 18**].

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The deterministic SIRS model exhibiting loss of immunity is the following

$$
\begin{aligned}
S'(t) &= -\beta S(t)I(t) - \mu S(t) + \gamma R(t) + \mu, \\
I'(t) &= \beta S(t)I(t) - (\lambda + \mu)I(t), \\
R'(t) &= \lambda I(t) - (\mu + \gamma)R(t),
\end{aligned}
$$

where $S(t)$ is the number of members of the population susceptible to the disease, $I(t)$ is the number of infective members and $R(t)$ is the number of members who have been removed from the possibility of infection through full immunity. The population considered has a constant size N which is normalized to 1, that is $S(t) + I(t) + R(t) = 1$ for all $t \geq 0$. We refer to [**11**] for details about this model.

Parameters in the system are as follows: μ represents the birth and death rate, the constant λ represents the recovery rate of infected people, β is the transmission rate and γ is the per capita rate of loss of immunity. Of course, $\mu > 0$, $\lambda > 0$, $\beta > 0$. We shall assume that $\gamma \geq 0$, the case where $\gamma = 0$ corresponds to the SIRS model. It is easy to see that the above system always has a disease-free equilibrium (*i.e.* boundary equilibrium) $E_0 = (1, 0, 0)$. One of the main issues in the study of the behavior of epidemics is the analysis of the steady states of the model and their stabilities.

A stochastic version of the present model (with or without delay) has been considered in [**13**] for the stability of the disease free equilibrium. The stability of the disease-free of a SIRS model has been studied in [**17**]. The stability of the endemic equilibrium for the SIRS model has been studied in [**5**]. The method used in these papers is that of stochastic Lyapunov functions. Sufficient conditions are given for the stabilities of the corresponding equilibria. Most of the studies in the literature concerning stabilities provide only sufficient conditions. One of the results of the present paper is to give necessary and sufficient conditions for the stability of a stochastic perturbation of the model.

The stochastic model considered in [**13, 17**] is the following

$$
\begin{aligned}
dS(t) &= [-\beta S(t)I(t) - \mu S(t) + \gamma R(t) + \mu]\, dt - \sigma S(t)I(t)\, dw(t), \\
dI(t) &= [\beta S(t)I(t) - (\lambda + \mu)I(t)]\, dt + \sigma S(t)I(t)\, dw(t), \\
dR(t) &= [\lambda I(t) - (\mu + \gamma)R(t)]\, dt,
\end{aligned}
$$

where $w(t)$ is a one-dimensional standard Brownian motion defined on a complete probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, P)$. One notices that there is no white noise perturbation in the third equation.

For the case of a stochastic SIRS model, that is when $\gamma = 0$, sufficient conditions ($0 < \beta < \min\{\lambda+\mu-\frac{\sigma^2}{2}, 2\mu\}$) are given in [**17**] for the stability of the disease free equilibrium. In [**13**], the stability is proved under the condition $0 < \beta < \lambda + \mu - \frac{\sigma^2}{2}$ for the stochastic SIRS model. Numerical simulations are also given in [**13, 17**] to support the conjecture that the equilibrium of the model is still stable under the more general condition $0 < \beta < \lambda + \mu + \frac{\sigma^2}{2}$. One of the aims of the present paper is to show this conjecture for systems with more general type of noise perturbations, namely systems of the form

$$
\begin{aligned}
dS(t) &= [-\beta S(t)I(t) - \mu S(t) + \gamma R(t) + \mu]\, dt \\
&\qquad\qquad - [\sigma_1 S(t)I(t) + \sigma_2 S(t)R(t)]\, dw(t), \\
dI(t) &= [\beta S(t)I(t) - (\lambda + \mu)I(t)]\, dt + \sigma_1 S(t)I(t)\, dw(t), \\
dR(t) &= [\lambda I(t) - (\mu + \gamma)R(t)]\, dt + \sigma_2 S(t)R(t)\, dw(t),
\end{aligned}
\tag{1}
$$

where $w(t)$ is as above, and $\sigma_1, \sigma_2 \in \mathbf{R}$. We recover the systems studied in [**13, 17**] with $\sigma_2 = 0$. The white noises of this system depend on two parameters while in the preceding one they depend only on one parameter.

When dealing with a model for population, it is necessary to prove that the solutions are positive and are defined globally for all $t \geq 0$, *i.e.* there is no finite explosion time. This question is not studied in the papers cited above. We shall prove it for system (1), that is

Theorem 1. *Let the initial data* (ξ_1, ξ_2, ξ_3) *be a* $\mathbf{R}^3$*-valued* $\mathcal{F}_0$*-measurable random variable satisfying* $\mathbb{E}\left[\xi_1^2 + \xi_2^2 + \xi_3^2\right] < \infty$ *and* $\xi_1 > 0$, $\xi_2 > 0$, $\xi_3 > 0$ *a.s. Then there is a unique solution* $(S(t,\omega), I(t,\omega), R(t,\omega))$ *of system* (1), *defined for all* $t \geq 0$, *verifying the initial conditions* $S(0,\omega) = \xi_1$, $I(0,\omega) = \xi_2$, $R(0,\omega) = \xi_3$, *and the solution is positive for all* $t \geq 0$ *with probability* 1, *namely* $S(t,\omega) > 0$, $I(t,\omega) > 0$ *and* $R(t,\omega) > 0$ *for all* $t \geq 0$ *a.s.*

To study the stochastic stabilities of the above system, several definitions of stochastic stabilities could be used, where the convergence of a stochastic sequence could be interpreted in different ways (see [**1, 2, 6, 7, 10**]). One of the most important definitions of stochastic stabilities is the almost sure asymptotic stability, for which necessary and sufficient conditions are always determined by the largest Lyapunov exponent (see [**3, 4, 8, 15**]). To the best of the authors' knowledge, there is no result in the study of the almost sure asymptotic stability of the SIRS model. Our result concerning the stability of the trivial equilibrium of system (1) is the following.

Theorem 2. *The trivial solution* $(1,0,0)$ *of system* (1) *is almost surely asymptotically stable if and only if* $0<\beta<\lambda+\mu+\frac{\sigma_1^2}{2}$.

In the following we shall prove Theorem 1 in Section 2 and Theorem 2 in Section 3.

2. Global and positive solutions

We now prove Theorem 1. Since the coefficients of system (1) are locally Lipschitz continuous, for any given initial data (ξ_1,ξ_2,ξ_3), which is a $\mathbf{R}^3$-valued $\mathcal{F}_0$-measurable random variable satisfying

$$\mathbb{E}\left[\xi_1^2+\xi_2^2+\xi_3^2\right]<\infty,$$

there is a unique maximal local solution $(S(t,\omega),I(t,\omega)),R(t,\omega))$ on $t\in[0,\tau_e(\omega))$, that verifies the initial conditions $S(0,\omega)=\xi_1$, $I(0,\omega)=\xi_2$, $R(0,\omega)=\xi_3$.

From now on we assume that $\xi_1>0$, $\xi_2>0$, $\xi_3>0$ a.s. Firstly, we show that $S(t)$, $I(t)$ and $R(t)$ are positive for all $t\in[0,\tau_e(\omega))$ a.s. Define the stopping time

$$t_+=\sup\left\{t\in(0,\tau_e): S|_{[0,t]}>0,\ I(t)|_{[0,t]}>0 \text{ and } R|_{[0,t]}>0\right\}.$$

We need to show that $t_+=\tau_e$ a.s.

We assume to the contrary that $P\{t_+<\tau_e\}>0$. Itô's formula shows that, for almost all $\omega\in\{t_+<\tau_e\}$ and all $t\in[0,t_+)$,

$$\begin{aligned}
&\ln S(t)+\ln I(t)+\ln R(t)-\ln\xi_1-\ln\xi_2-\ln\xi_3\\
&\quad=\int_0^t\left[-\beta I(s)-(\lambda+\gamma+3\mu)+\frac{\gamma R(s)+\mu}{S(s)}+\beta S(s)+\frac{\lambda I(s)}{R(s)}\right]ds\\
&\qquad-\frac{1}{2}\int_0^t\left[(\sigma_1 I(s)+\sigma_2 R(s))^2+(\sigma_1^2+\sigma_2^2)S(s)^2\right]ds\\
&\qquad+\int_0^t\left[\sigma_1 S(s)+\sigma_2 S(s)-\sigma_1 I(s)-\sigma_2 R(s)\right]dw(s)\\
&\quad\geq\int_0^t\Big[-\beta I(s)-(\lambda+\gamma+3\mu)\Big]ds\\
&\qquad-\int_0^t\frac{1}{2}\Big[\Big(\sigma_1 I(s)+\sigma_2 R(s))^2+(\sigma_1^2+\sigma_2^2)S(s)^2\Big)\Big]ds\\
&\qquad+\int_0^t\Big[\sigma_1 S(s)+\sigma_2 S(s)-\sigma_1 I(s)-\sigma_2 R(s)\Big]dw(s).
\end{aligned}$$

It is easy to see that, for almost all ω in $\{t_+ < \tau_e\}$, $S(t), I(t)$ and $R(t)$ are positive on $[0, t_+)$ and $S(t_+)I(t_+)R(t_+) = 0$, hence

$$\lim_{t \nearrow t_+} [\ln S(t) + \ln I(t) + \ln R(t)] = -\infty.$$

By the above inequality, one has

$$(2)\quad -\infty \geq \int_0^{t_+} \Big[\sigma_1 S(s) + \sigma_2 S(s) - \sigma_1 I(s) - \sigma_2 R(s)\Big]\, dw(s) \\ - \int_0^{t_+} \Big(\beta I(s) + \lambda + \gamma + 3\mu + \frac{1}{2}((\sigma_1 I(s) + \sigma_2 R(s))^2 + (\sigma_1^2 + \sigma_2^2) S(s)^2)\Big)\, ds$$

which is a contradiction since the right hand side of the above inequality is finite, so we must therefore have $t_+ = \tau_e$ a.s.

Now we prove that $\tau_e = \infty$.

For each integer k such that $k \geq \xi_1 + \xi_2 + \xi_3$ a.s., define the stopping time

$$\tau_k = \sup\left\{t \in [0, \tau_e) : (S + I + R)|_{[0,t]} \leq k\right\}.$$

Clearly, τ_k is increasing as $k \to +\infty$. Set $\tau_\infty = \lim_{k \to +\infty} \tau_k$, whence $\tau_\infty \leq \tau_e$ a.s. We claim that $P\{\tau_\infty \neq \tau_e\} = 0$. To see this, we assume to the contrary that $P\{\tau_\infty < \tau_e\} > 0$. Itô's formula shows that, for almost all $\omega \in \{\tau_\infty < \tau_e\}$,

$$e^{\mu \tau_k}(S(\tau_k) + I(\tau_k) + R(\tau_k)) = \xi_1 + \xi_2 + \xi_3 + e^{\mu \tau_k} - 1.$$

It follows from $\tau_\infty < \tau_e$ that $S(\tau_k) + I(\tau_k) + R(\tau_k) = k$ a.s. Letting $k \to +\infty$ leads to

$$+\infty = \xi_1 + \xi_2 + \xi_3 + e^{\mu \tau_\infty} - 1,$$

which is a contradiction since the right hand side of the above inequality is finite for almost all $\omega \in \{\tau_\infty < \tau_e\}$, so we must therefore have $\tau_\infty = \tau_e$ a.s.

Next we prove that $P\{\tau_e < \infty, \tau_\infty < \infty\} = 0$.

Let $T > 0$ be arbitrary. We have by Itô's formula that

$$1_{\{\tau_e < \infty\}} e^{\mu(\tau_k \wedge T)} [S(\tau_k \wedge T) + I(\tau_k \wedge T) + R(\tau_k \wedge T)] \\ = 1_{\{\tau_e < \infty\}} \Big(\xi_1 + \xi_2 + \xi_3\Big) + 1_{\{\tau_e < \infty\}} (e^{\mu(\tau_k \wedge T)} - 1),$$

where $1_{\{\cdot\}}$ means the indicator function of the corresponding set. Taking expectations, we obtain

$$(3)\quad \mathbb{E}\Big[1_{\{\tau_e < \infty\}} e^{\mu(\tau_k \wedge T)} (S(\tau_k \wedge T) + I(\tau_k \wedge T) + R(\tau_k \wedge T))\Big] \\ \leq \mathbb{E}[\xi_1 + \xi_2 + \xi_3] + e^{\mu T}.$$

Obviously,

$$\begin{aligned}&1_{\{\tau_e<\infty\}}e^{\mu(\tau_k\wedge T)}\left[S(\tau_k\wedge T)+I(\tau_k\wedge T)+R(\tau_k\wedge T)\right]\\&\qquad\geq 1_{\{\tau_e<\infty,\,\tau_k\leq T\}}\left[S(\tau_k)+I(\tau_k)+R(\tau_k)\right]=1_{\{\tau_e<\infty,\,\tau_k\leq T\}}k.\end{aligned}$$

Substituting these into (3) gives

$$kP\{\tau_e<\infty,\tau_k\leq T\}\leq \mathbb{E}\left[\xi_1+\xi_2+\xi_3\right]+e^{\mu T}.$$

Letting $k\to+\infty$ leads to $\lim\limits_{k\to\infty} P\{\tau_e<\infty,\tau_k\leq T\}=0$ and hence

$$P\{\tau_e<\infty,\tau_\infty\leq T\}=0.$$

Since $T>0$ is arbitrary, we then have $P\{\tau_e<\infty,\tau_\infty<\infty\}=0$.

Now by using the relation

$$\begin{aligned}\{\tau_e<\infty\}&=\{\tau_e<\infty,\,\tau_\infty=\infty\}\cup\{\tau_e<\infty,\,\tau_\infty<\infty\}\\&\subseteq\{\tau_\infty\neq\tau_e\}\cup\{\tau_e<\infty,\,\tau_\infty<\infty\},\end{aligned}$$

we obtain $P\{\tau_e<\infty\}=0$. Hence $P\{\tau_e=\infty\}=1$. This completes the proof of Theorem 1. □

3. Proof of Theorem 2

We first make changes of variables so that the origin will represent the disease free equilibrium. Under the transformation

$$u_1=S-1,\quad u_2=I,\quad u_3=R, \tag{4}$$

system (1) has the following form

$$\begin{aligned}du_1(t)&=\left[-\beta(u_1(t)+1)u_2(t)-\mu u_1(t)+\gamma u_3(t)\right]dt\\&\qquad\qquad-(u_1(t)+1)(\sigma_1u_2(t)+\sigma_2u_3(t))\,dw(t),\\du_2(t)&=\left[\beta(u_1(t)+1)-\lambda-\mu\right]u_2(t)\,dt+\sigma_1(u_1(t)+1)u_2(t)\,dw(t),\\du_3(t)&=\left[\lambda u_2(t)-(\mu+\gamma)u_3(t)\right]dt+\sigma_2(u_1(t)+1)u_3(t)\,dw(t).\end{aligned}$$

According to the Oseledec Multiplicative Ergodic Theorem, a necessary and sufficient condition for the almost sure asymptotic stability of the trivial solution of the system is that the largest Lyapunov exponent of the linearized system is negative (see [**8, 16**]). We consider the corresponding linearized system

$$\begin{aligned}du_1(t)&=\left[-\beta u_2(t)-\mu u_1(t)+\gamma u_3(t)\right]dt-(\sigma_1u_2(t)+\sigma_2u_3(t))\,dw(t),\\du_2(t)&=(\beta-\lambda-\mu)u_2(t)\,dt+\sigma_1u_2(t)\,dw(t),\\du_3(t)&=\left[\lambda u_2(t)-(\mu+\gamma)u_3(t)\right]dt+\sigma_2u_3(t)\,dw(t).\end{aligned}\tag{5}$$

We denote by $u(t,u_0)$ the unique solution of the system verifying the initial condition $u(0)=u_0=(u_{10},u_{20},u_{30})$. We shall prove from now on: The largest Lyapunov exponent of system (5) is negative if and only if $0 < \beta < \lambda + \mu + \frac{\sigma_1^2}{2}$. More precisely, we prove that, for any $u_0 \in \mathbf{R}^n$, the solution $u(t, u_0)$ of system (5) satisfies the following

$$\limsup_{t\to+\infty} \frac{1}{t} \log \|u(t,u_0)\| = \max\left\{\beta - \left(\lambda + \mu + \frac{\sigma_1^2}{2}\right), -\gamma - \mu - \frac{\sigma_2^2}{2}, -\mu\right\}.$$

The second equation in (5) is a scalar linear stochastic differential equation. So one can solve it explicitly. Denoting $a = \lambda + \mu + \frac{\sigma_1^2}{2} - \beta$, we then have $a > 0$, if and only if $0 < \beta < \lambda + \mu + \frac{\sigma_1^2}{2}$, and

$$u_2(t) = u_2(t, u_{20}) = e^{-at+\sigma_1 w(t)} u_{20}.$$

It is clear that

$$\limsup_{t\to+\infty} \frac{1}{t} \log |u_2(t, u_{20})| = -a.$$

Substituting $u_2(t)$ in the first and third equations in system (5), we obtain, by denoting $U(t) = \binom{u_1(t)}{u_3(t)}$, the following system

$$dU(t) = \big(FU(t) + f(t)\big)\, dt + \big(GU(t) + g(t)\big)\, dw(t),$$

where

$$F = \begin{pmatrix} -\mu & \gamma \\ 0 & -\mu-\gamma \end{pmatrix}, \quad f(t) = \begin{pmatrix} -\beta u_2(t) \\ \lambda u_2(t) \end{pmatrix},$$
$$G = \begin{pmatrix} 0 & -\sigma_2 \\ 0 & \sigma_2 \end{pmatrix}, \qquad g(t) = \begin{pmatrix} -\sigma_1 u_2(t) \\ 0 \end{pmatrix}.$$

We use the variation of constants method to solve the above system (see for example [**14**]). The corresponding homogeneous system is

$$dU(t) = FU(t)\, dt + GU(t)\, dw(t).$$

Since $FG = GF$, one can solve it to obtain a fundamental matrix solution $\Phi(t)$ which is given by

$$\Phi(t) = e^{(F-\frac{1}{2}G^2)t+Gw(t)} = e^{Ht} e^{Gw(t)},$$

where

$$H = \begin{pmatrix} -\mu & \gamma + \frac{1}{2}\sigma_2^2 \\ 0 & -\mu-\gamma-\frac{1}{2}\sigma_2^2 \end{pmatrix}.$$

And consequently, since $Gg(t) = \binom{0}{0}$, the solution of system (5) is

$$U = e^{Ht+Gw(t)}\left[U(0) + \int_0^t e^{-Hs-Gw(s)} f(s)\,ds + \int_0^t e^{-Hs-Gw(s)} g(s)\,dw(s)\right].$$

Let

$$P = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}.$$

Then

$$P^{-1}HP = \begin{pmatrix} -\mu & 0 \\ 0 & -\mu-\gamma-\frac{1}{2}\sigma_2^2 \end{pmatrix}.$$

Therefore, with $a' = \mu+\gamma+\frac{1}{2}\sigma_2^2 > 0$, one has

$$e^{Ht} = P\begin{pmatrix} e^{-\mu t} & 0 \\ 0 & e^{-a't} \end{pmatrix}P^{-1} = \begin{pmatrix} e^{-\mu t} & e^{-\mu t} - e^{-a't} \\ 0 & e^{-a't} \end{pmatrix}.$$

Similarly

$$e^{Gw(t)} = \begin{pmatrix} 1 & 1-e^{\sigma_2 w(t)} \\ 0 & e^{\sigma_2 w(t)} \end{pmatrix}.$$

Hence

$$e^{Ht+Gw(t)} = \begin{pmatrix} e^{-\mu t} & e^{-\mu t} - e^{-a't+\sigma_2 w(t)} \\ 0 & e^{-a't+\sigma_2 w(t)} \end{pmatrix}.$$

Therefore

$$\int_0^t e^{-Hs-Gw(s)} f(s)\,ds = \begin{pmatrix} \int_0^t ((\lambda-\beta)e^{(\mu-a)s+\sigma_1 w(s)} - \lambda e^{(a'-a)s+(\sigma_1-\sigma_2)w(s)})\,ds \\ \lambda\int_0^t e^{(a'-a)s+(\sigma_1-\sigma_2)w(s)}\,ds \end{pmatrix} u_{20}$$

and

$$\int_0^t e^{-Hs}e^{-Gw(s)} g(s)\,dw(s) = \begin{pmatrix} -\sigma_1\int_0^t e^{(\mu-a)s+\sigma_1 w(s)}\,dw(s) \\ 0 \end{pmatrix} u_{20}.$$

Therefore

$$e^{Ht+Gw(t)}\int_0^t e^{-Hs-Gw(s)}f(s)\,ds$$

$$=\begin{pmatrix}(\lambda-\beta)e^{-\mu t}\int_0^t e^{(\mu-a)s+\sigma_1 w(s)}ds-\lambda e^{-a't+\sigma_2 w(t)}\int_0^t e^{(a'-a)s+(\sigma_1-\sigma_2)w(s)})\,ds\\ \lambda e^{-a't+\sigma_2 w(t)}\int_0^t e^{(a'-a)s+(\sigma_1-\sigma_2)w(s)}\,ds\end{pmatrix}u_{20},$$

and

$$e^{Ht+Gw(t)}\int_0^t e^{-Hs-Gw(s)}g(s)\,dw(s)$$

$$=\begin{pmatrix}-\sigma_1 e^{-\mu t}\int_0^t e^{(\mu-a)s+\sigma_1 w(s)}\,dw(s)\\ 0\end{pmatrix}u_{20}.$$

We obtain finally

$$\begin{aligned}u_1(t)&=u_{10}e^{-\mu t}+u_{30}(e^{-\mu t}-e^{-a't+\sigma_2 w(t)})\\&\quad+u_{20}(\lambda-\beta)e^{-\mu t}\int_0^t e^{(\mu-a)s+\sigma_1 w(s)}\,ds\\&\quad-u_{20}\lambda e^{-a't+\sigma_2 w(t)}\int_0^t e^{(a'-a)s+(\sigma_1-\sigma_2)w(s)}\,ds\\&\quad-u_{20}\sigma_1 e^{-\mu t}\int_0^t e^{(\mu-a)s+\sigma_1 w(s)}\,dw(s)\end{aligned}$$

and

$$u_3(t)=e^{-a't+\sigma_2 w(t)}\left(u_{30}+u_{20}\lambda\int_0^t e^{(a'-a)s+(\sigma_1-\sigma_2)w(s)}\,ds\right).$$

By Itô's formula one has

$$\begin{aligned}e^{(\mu-a)t+\sigma_1 w(t)}-1=\int_0^t\left(\mu-a+\frac{1}{2}\sigma_1^2\right)e^{(\mu-a)s+\sigma_1 w(s)}\,ds\\+\sigma_1\int_0^t e^{(\mu-a)s+\sigma_1 w(s)}\,dw(s).\end{aligned}$$

Then

$$\begin{aligned}\sigma_1\int_0^t e^{(\mu-a)s+\sigma_1 w(s)}\,dw(s)=e^{(\mu-a)t+\sigma_1 w(t)}\\-1-\left(\mu-a+\frac{1}{2}\sigma_1^2\right)\int_0^t e^{(\mu-a)s+\sigma_1 w(s)}\,ds.\end{aligned}$$

Since $w(t)$ is a standard Wiener process, one has $|w(t)| \le c\sqrt{2t|\log(|\log t|)|}$ a.s. where c is a constant. Let $\mu' = \mu$ or a', and $\sigma = \sigma_1$ or $\sigma_1 - \sigma_2$.

If $a = \mu'$, then for $t > e$,

$$\begin{aligned}\int_0^t e^{(\mu'-a)s+\sigma w(s)}\,ds = \int_0^t e^{\sigma w(s)}\,ds &= \int_0^e e^{\sigma w(s)}\,ds + \int_e^t e^{\sigma w(s)}\,ds \\ &\le \int_0^e e^{\sigma w(s)}\,ds + \int_e^t e^{|\sigma|c\sqrt{2s\log\log s}}\,ds \\ &\le \int_0^e e^{\sigma w(s)}\,ds + (t-e)e^{|\sigma|c\sqrt{2t\log\log t}} \\ &\le \int_0^e e^{\sigma w(s)}\,ds + te^{|\sigma|c\sqrt{2t\log\log t}}.\end{aligned}$$

If $a \neq \mu'$, then for $t > e$,

$$\begin{aligned}\int_0^t e^{(\mu'-a)s+\sigma w(s)}\,ds &= \int_0^e e^{(\mu'-a)s+\sigma w(s)}\,ds + \int_e^t e^{(\mu'-a)s+\sigma w(s)}\,ds \\ &\le \int_0^e e^{(\mu'-a)s+\sigma w(s)}\,ds + \int_e^t e^{(\mu'-a)s+|\sigma|c\sqrt{2s\log\log s}}\,ds \\ &\le \int_0^e e^{(\mu'-a)s+\sigma w(s)}\,ds + \int_e^t e^{(\mu'-a)s+|\sigma|c\sqrt{2t\log\log t}}\,ds \\ &= \int_0^e e^{(\mu'-a)s+\sigma w(s)}\,ds \\ &\qquad + \frac{e^{(\mu'-a)t} - e^{(\mu'-a)e}}{\mu'-a} e^{|\sigma_1|c\sqrt{2t\log\log t}} \\ &\le \int_0^e e^{(\mu'-a)s+\sigma w(s)}\,ds + \frac{e^{(\mu'-a)t}}{|\mu'-a|} e^{|\sigma|c\sqrt{2t\log\log t}} \\ &\qquad + \frac{e^{(\mu'-a)e}}{|\mu'-a|} e^{|\sigma|c\sqrt{2t\log\log t}}.\end{aligned}$$

One then has for $t \geq e$,

$$
\begin{aligned}
|u_1(t)| \leq & C_1 e^{-\mu t} + C_2 e^{-a' t + \sigma_2 w(t)} \\
& + C_3 \left[t e^{-\mu t} + e^{-at} + e^{-\mu t} \right] e^{|\sigma_1| c \sqrt{2t \log\log t}} \\
& + C_4 \left[t e^{-a' t + \sigma_2 w(t)} + e^{-at + \sigma_2 w(t)} + e^{-a' t + \sigma_2 w(t)} \right] e^{|\sigma_1 - \sigma_2| c \sqrt{2t \log\log t}} \\
& + C_5 e^{-at + \sigma_1 w(t)},
\end{aligned}
$$

where the C_j are positive constants. Therefore

$$
\limsup_{t \to +\infty} \frac{1}{t} \log |u_1(t)| = \max\{-\mu, -a', -a\} < 0.
$$

The assertion for $u_3(t)$ can be proved similarly. Therefore the Lyapunov exponent of the system is $\Lambda = \max\{-\mu, -a', -a\}$ which is negative if and only if $a = \lambda + \mu + \frac{\sigma_1^2}{2} - \beta > 0$. This completes the proof of Theorem 2. □

4. Conclusional discussions

Mathematically, $\sigma_1^2/2$ can be regarded as the intensity of the environmental stochastic perturbation on the transmission rate of the disease. We see that, for $\sigma_1 = \sigma_2 = 0$, *i.e.*, there is no environmental stochastic perturbation for the transmission rate, $\beta < \beta_0 \triangleq \lambda + \mu$ guarantees the disappearance of the disease, which agrees well with the classical results. Taken the environment noise into account, the introduction of the noise in the deterministic SIRS model increase the deterministic stability threshold β_0 of the disease-free equilibrium to $\hat{\beta}_0 \triangleq \lambda + \mu + \frac{\sigma_1^2}{2}$, under which the disease-free equilibrium is almost sure asymptotic stable such that the disease cannot establish itself and it will disappear finally leaving all the population susceptible.

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