Publ. Mat. (2014), 209–219
Proceedings of *New Trends in Dynamical Systems.* Salou, 2012.
DOI: 10.5565/PUBLMAT_Extra14_12

FROM ABEL EQUATIONS TO JACOBIAN CONJECTURE

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Dedicated to Jaume Llibre for his 60th birthday

Abstract: This is a survey relating several subjects bridging algebraic differential equations and their integrability, pertubative approach based on algebraic moments, the Jacobian conjecture and optimal transport of measures defined on the complement of an arrangement of hyperplanes. This text was written in relation with Jaume Llibre's 60th birthday and it touches several fields in which Jaume gave important contributions. It also contains some results wich have not yet been published before.

2010 Mathematics Subject Classification: Primary: 34C05, 34A34, 34C14.

Key words: Algebraic moments, Jacobian conjecture, Hamilton–Jacobi equation.

1. Introduction

Among the "New trends in Dynamical Systems", we focus on results that have been recently obtained in the field of algebraic differential equations and that look to be promising trends for future research. This field has a long tradition which begins with N. Abel, S. Lie and R. Liouville. Their contributions are at the origin of many modern fields like integrability of dynamical systems and encompasses both fundamental mathematics and applications to physics and mechanics. We like to underline herein the perturbation theory of integrable algebraic dynamical systems and its link with other subjects of algebraic geometry including the Jacobian conjecture. The Jacobian conjecture deals with a polynomial map $F=(F_1,\dots,F_n)\colon k^n\to k^n$, k is any algebraically closed field of characteristic zero. Assume that the Jacobian of F: $\mathrm{Jac}_{ij} F := \frac{\partial F_i}{\partial x_j}$ is so that $J(F)=\det \mathrm{Jac}(F)=1$. Then the (local analytic) inverse of F is a polynomial map. It was first formulated by Keller in 1939 for the case of polynomial with integer coefficients. Keller checked the conjecture in the birational case [**24**]. An important progress was achieved by

The author expresses his warmest thanks to the organizers of the conference "New Trends in Dynamical Systems" in Salou, from October 1–October 5, 2012.

Abhyankar and Gurjar who gave an expansion formula for the inverse of the map F ([**2**], [**4**], and [**30**]). This expansion has been studied in terms of trees theory and displays interesting connections with the general approach to perturbative theory of dynamical systems developped by G. Gallavotti. In relation with perturbation theory of Abel equations the algebraic moment approach has been at the origin of many articles. Recently the algebraic moment approach was extended to any dimensions [**22**] and revealed several links with the last deep results obtained around the Jacobian conjecture [**20**]. This article which could not be reasonably exhaustive in such a broad area intends to survey some aspects. It includes some new results and open questions.

2. The Abel equation

One of the simplest setting to investigate integrability of dynamical systems by algebraic methods is the case of 1-dimensional non-autonomous systems:

$$\frac{dy}{dx} = f(x, y). \tag{1}$$

In particular, this equation

$$\frac{dy}{dx} = p(x)y^3 + q(x)y^2 \tag{2}$$

is named Abel equation because it was first studied by the famous Norwegian Mathematician Heindrick Abel. It is not so easy to spot a published reference of a related work by Abel. Indeed in the "Œuvres complètes" [**1**, tome II, paragraphs 14 and 15] refers to Abel work (in french). Paragraph 14 is devoted to the Riccati equation written in the differential form notation:

$$dy - [p + qy + ry^2]\, dx = 0. \tag{3}$$

It contains several results on the integrability of Riccati equations which are now well-known. In contrast, paragraph 15, which seems developped by Abel as a natural extension of the previous one has not been often revisited. It deals with the differential equation:

$$(y + s)dy - [p + qy + ry^2]\, dx = 0. \tag{4}$$

This equation can be reduced to

$$ydy - [p + qy]\, dx = 0, \tag{5}$$

which is the same (as far as integrability is concerned!) as the equation (2) (cf. replace y by $1/y$).

2.1. An algebraic perturbation setting. In the articles [**8**] and [**9**], we introduced an algebraic perturbation setting:

$$\frac{dy}{dx} = p(x)y^3 + \epsilon q(x)y^2, \tag{6}$$

where $p(x)$, $q(x)$ are (real or complex) polynomials and $x \in [0,1]$. We computed the first derivative relatively to the pertubation parameter ϵ of the solution $y = y(y_0, \epsilon, x)$ of the equation which equates y_0 at initial time $x = 0$:

$$\frac{dy(y_0, \epsilon, 1)}{d\epsilon}\Big|_{\epsilon=0} = \Sigma_{k=1}^{+\infty} m_k y_0^{k+1}, \tag{7}$$

where

$$m_k = \int_0^1 P(x)^k q(x)\, dx \tag{8}$$

are so-called algebraic moments associated to the polynomial $P = \int_0^x p(t)\, dt$ and the polynomial measure $\mu = q(x)\, dx$. Note that this setting extends immediately to several other algebraic perturbation settings including trigonometric polynomials or more generally complex foliation defined by two complex polynomials $p(x)$, $q(x)$ and integration of the complex differential equation along a fixed path in the x-plane (see for instance [**23**]). We posed then the "algebraic moment problem" in relation with Hilbert's 16th problem and Pugh's problem for Abel equation and developed our viewpoint in a series of articles (cf. [**10**], [**11**], [**12**], and [**13**]). This algebraic moment problem reads: what can be said of a polynomial P and a polynomial measure $\mu = q(x)\, dx$ such that

$$m_k = \int_0^1 P(x)^k q(x)\, dx = 0, \quad k \geq 1. \tag{9}$$

In contrast with the usual case of classical moment theory where measure μ is positive, this does not imply that $P = 0$ and we discovered in this setting the relevance of composition of polynomials. In this direction an important progress was given by C. Christopher ([**16**] and [**17**]). The whole problem was latter fully solved by Pakovich and Muzychuk [**27**]. There has been indeed many contributions related to this question and among the most recent contributions, we mention [**18**]. In this paragraph, we like to put emphasis on the eventual deeper link with the classical moment problem. Classically the moment problem is indeed the problem to reconstruct a measure from its moments. In this regard, we should also consider the following reconstruction problem for algebraic perturbation theory of Abel equations (that we set here as an open question):

Reconstruction problem for Abel equations.

Given a (real or complex) polynomial P and the integrable Abel equation $\frac{dy}{dx} = p(x)y^3$ is it possible to choose the perturbation $\epsilon q(x)$ so that the first order perturbation function $\frac{dy(y_0,\epsilon,1)}{d\epsilon}|_{\epsilon=0} = \Sigma_{k=1}^{+\infty} m_k y_0^{k+1}$ is prescribed in advance? This is, of course equivalent to finding a polynomial measure $q(x)\,dx$ so that the moments $m_k = \int_0^1 P(x)^k q(x)\,dx$ are imposed.

The problem of reconstruction of measure from moments data relates to the problem of algebraic sampling discussed by Y. Yomdin *et al.* in recent articles ([**5**] and [**28**]).

2.2. A natural extension to any dimensions. In the article [**22**], we considered natural extensions of the previous algebraic moment problems to any dimensions. Given two (complex or real) polynomial P, q in n real variables, let Ω be a compact domain in $\mathbb{R}^n$, what can be said of P and q if (moment vanishing problem)

$$m_k(P,q) = \int_\Omega P^k(x)q(x)\,d\mu(x) = 0, \quad k = 1,\dots \tag{10}$$

We analyzed several special cases (complex atomic measure μ, μ concentrated on an algebraic curve, complex measures on $\mathbb{S}^1$). Indeed it seems natural to extend this problem to any measure μ with exponentially decreasing density with $\Omega = \mathbb{R}^n$.

3. The Jacobian conjecture

The Jacobian conjecture deals with a polynomial map

$$F = (F_1,\dots,F_n)\colon k^n \to k^n,$$

k is any algebraically closed field of characteristic zero. By Lefschetz principle we can assume $k = \mathbb{C}$. Assume that the Jacobian of F: $\mathrm{Jac}_{ij} F := \frac{\partial F_i}{\partial x_j}$ is so that $J(F) = \det \mathrm{Jac}(F) = 1$. Then the (local analytic) inverse of F should be a polynomial map. For more history and known results on the Jacobian conjecture, see [**4**], [**19**], and the references therein. There has been recently many interesting developments about this conjecture and we shall focuss in this article on a few of them to motivate the interest of specialists of differential equations. In particular, the problem was recently reduced to the case of symmetric jacobians ([**7**] and [**26**]):

$$F\colon x \mapsto x - \nabla P, \tag{11}$$

where P denotes a polynomial with terms of degree strictly larger than 2. It is possible to show, for instance by using the Abhyankar–Gurjar formula ([**30**], [**2**], and [**4**]) that the inverse of a map of this type is also a map of this type:

$$G\colon w \mapsto z, \quad z_i = w_i + \frac{\partial Q}{\partial w_i}. \tag{12}$$

W. Zhao proved [**29**] the following:

Theorem 1. *Let t be a small parameter, consider the deformation $F_t(z) = z - t\nabla P$. The inverse map of $z \mapsto F_t(z)$ can be written*

$$G_t(z) = z + t\nabla Q_t(z), \tag{13}$$

where $Q_t(z)$ is the unique solution of the Cauchy problem for the Hamilton-Jacobi equation

$$\begin{aligned} \frac{\partial Q_t(z)}{\partial t} &= \frac{1}{2}\langle \nabla Q_t, \nabla Q_t\rangle, \\ Q_{t=0}(z) &= P(z). \end{aligned} \tag{14}$$

This theorem is important as it brings new analytic methods in the algebraic Jacobian problem.

More can be assumed on P such as that the Hessian of P is nilpotent (recall that we are concerned with complex polynomials). Using this reduction Zhao proved that the Jacobian conjecture is equivalent to the following

Conjecture 1 (Vanishing Conjecture)**.** *Let $\Delta = \Sigma_{i=1}^n \frac{\partial^2}{\partial z_i^2}$ be the (complex) Laplace operator. For any homogeneous polynomial (of degree 4) P if $\Delta^m P^m = 0$ for all $m \geq 1$, then $\Delta^m P^{m+1} = 0$, for all $m \gg 0$.*

Inspired by a conjecture of O. Mathieu [**25**], equivalent to the Jacobian conjecture, Zhao formulated the Image Conjecture upon noticing the resemblance of Mathieu conjecture with his own Vanishing Conjecture. Along this way, the notion of a Mathieu subspace was introduced by W. Zhao in [**29**].

Definition 1. Let k be a field and A be a commutative k-algebra. A subspace M of A is said to be a Mathieu space if:

For all $f \in A$ such that $f^m \in M$ for all $m \geq 1$, then for all $g \in A$, $f^k g \in M$ for $k \gg 0$.

Conjecture 2 (Image Conjecture)**.** *Let k be a field and let A be a k-algebra and let $B = A[z_1, \dots, z_n]$ be the ring of polynomials in n variables on A. For $a_1, \dots, a_n \in A$ a regular sequence the image of the A-linear map from B^n to B defined by $D = (\delta_{z_1} - a_1, \dots, \delta_{z_n} - a_n)$ is a Mathieu subspace in B.*

If the Image Conjecture is true then necessarily the following should be true.

Conjecture 3. *Let L be the linear $\mathbb{C}$-mapping of $\mathbb{C}[U_1, \dots, U_n] \to \mathbb{C}$ such that $L(U^\alpha) = \alpha!$, then the kernel of L is a Mathieu space.*

This conjecture is a special case of the Conjecture A discussed in [**22**] where the measure μ is of exponential decay.

In fact van den Essen, Wright and Zhao [**20**] dub the so-called Factorial conjecture:

Conjecture 4. *If $f \in \mathbb{C}[U_1, \dots, U_n]$ is such that $L(f^m) = 0$ for all $m > 1$ then $f = 0$.*

This conjecture is even more optimistic than the preceding ones and is presented as a good step to help understanding all the subject. This last conjecture is typically a vanishing algebraic moment problem. Indeed let $D_n = \{x = (x_1, \dots, x_n),\, 0 \le x_i \le +\infty\}$ be the positive n-tant in $\mathbb{R}^n$. The above conjecture writes:

Conjecture 5. *Let $f \in \mathbb{C}[x_1, \dots, x_n]$ assume that for all $k \ge 1$*

$$\int_{D^n} f^k \exp\Big(-(x_1 + \cdots + x_n)\Big)\, dx_1 \wedge \cdots \wedge dx_n = 0, \tag{15}$$

then $f = 0$.

Note that this conjecture is obviously true if $f \in \mathbb{R}[x_1, \dots, x_n]$ as shown by taking even values of k.

4. Measures defined on the complement of an arrangement of hyperplanes

Let A be a finite arrangement of hyperplanes in the n-dimensional complex affine space $\mathbb{C}^n$. Let $N(A)$ be the union of hyperplanes of A in $\mathbb{C}^n$ and $M(A)$ be its complement. It is further assumed that A is real, meaning that the defining function of every hyperplane $f_h(z) = u_{h,0} + \sum_{i=1}^n u_{h,i} z_i$ has real coefficients. We consider the family of functions

$$V = \frac{1}{2} \sum_{i=1}^{n} \left(x_i - \sum_{h \in A} t \frac{\lambda_h u_{h,i}}{f_h(x)} \right)^2, \quad t \in [0,1],\ \lambda_h > 0, \tag{16}$$

and the associated family of measures defined by densities:

$$\rho_t = \left(\frac{\beta}{2\pi}\right)^{n/2} \exp(-\beta V). \tag{17}$$

In the case when the arrangement of hyperplanes is given by the walls of the Weyl Chambers of a root system, the function V represents the potential of the generalized Calogero Hamiltonian system (where all masses must be equal). In that case, it was possible to prove that all the measures $\rho_t\,dx$ are indeed probability measure because the system is associated to a symplectic action of the torus (cf. [**21**] and [**14**]).

In [**3**], Aomoto and Forrester proved that, in general, for any finite arrangement of hyperplanes, we have

$$\int_{M(A)} \rho_t\,dx = 1. \tag{18}$$

Although the Jacobian conjecture is for complex polynomials, Theorem 1 above makes perfectly sense in the real. With the help of this Theorem 1, it is possible to give a new interpretation of the Aomoto–Forrester result.

Theorem 2. *The mapping*

$$F_t\colon x \mapsto w, \quad w_i = x_i - t\frac{\partial P}{\partial x_i}, \quad P(x) = \log[\Pi_{h\in A}\langle f_h(x)\rangle^{\lambda_h}] \tag{19}$$

displays a local inverse for t small, solution of the Hamilton–Jacobi equation. This local inverse admits an analytic prolongation to $t \in [0,1]$ on each connected components Δ_j, $j = 1,\ldots,k$. This inverse mapping of F_t achieves an optimal transport of $\rho_0\,dx = (\frac{\beta}{2\pi})^{n/2}\exp(-\frac{1}{2}\beta|w|^2)\,dx$ to $\rho_1\,dx = (\frac{\beta}{2\pi})^{n/2}\exp(-\frac{1}{2}\beta|x-\nabla P|^2)\,dx$:

$$\rho_1(T(w))\,\mathrm{Det}(DT(w)) = \rho_0(w), \tag{20}$$

$$T_j(w) = x_j = w_j + t\frac{\partial Q_t}{\partial w_j}, \tag{21}$$

where $\Sigma_j \frac{1}{2}w_j^2 + tQ_t$ is convex and Q_t solves Hamilton–Jacobi equation

$$\begin{aligned} \frac{\partial Q_t(z)}{\partial t} &= \frac{1}{2}\langle\nabla Q_t, \nabla Q_t\rangle, \\ Q_{t=0}(z) &= P(z). \end{aligned} \tag{22}$$

See for instance [**6**] for optimal transport of measures.

Proof: Consider the Aomoto–Forrester mapping

$$F\colon x \mapsto w, \quad w_i = x_i - \sum_{h\in A} \frac{\lambda_h u_{h,i}}{f_h(x)}. \tag{23}$$

A connected component Δ_j, $j = 1, \dots, k$ of $M(A) \cap \mathbb{R}^n$ is called a chamber. Aomoto and Forrester proved that the mapping F defines a bijection of Δ_j on $\mathbb{R}^n$. Let us denote $T_j\colon \mathbb{R}^n \to \Delta_j$ the inverses of the mapping F. Given a $w \in \mathbb{R}^n$, there exists a unique $x \in \Delta_j$ so that $w = F(x)$, $x = T_j(w)$ and

$$\begin{aligned}\int_{M(A)} \rho_1 &= \Sigma_{j=1}^k \int_{\Delta_j} \exp(-\beta V)\, d^n x \\ &= \Sigma_{j=1}^k \int_{\mathbb{R}^n} \exp(-\beta \Sigma_i w_i^2)\, \mathrm{Jac}(T_j(w))\, dw.\end{aligned} \tag{24}$$

Aomoto and Forrester proved (by a convexity argument) that for all $w \in \mathbb{R}^n$, $\mathrm{Jac}(T_j(w)) > 0$. Furthermore they showed using Griffiths-Harris residue techniques that

$$\Sigma_{j=1}^k \mathrm{Jac}(T_j(w)) = 1. \tag{25}$$

This implies that

$$\int_{M(A)} \rho_1\, dx = \int_{\mathbb{R}^n} \exp(-\beta \Sigma_i w_i^2)[\Sigma_{j=1}^k \mathrm{Jac}(T_j(w))]\, dw = 1. \quad \square \tag{26}$$

Several consequences of Theorem 1 have been analyzed from the viewpoint of perturbation analysis of Hamiltonian systems and symplectic geometry in [**15**].

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