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OPTIMAL QUASI-METRICS IN A GIVEN POINTWISE EQUIVALENCE CLASS DO NOT ALWAYS EXIST

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Abstract: In this paper we provide an answer to a question found in [**3**], namely when given a quasi-metric ρ, if one examines all quasi-metrics which are pointwise equivalent to ρ, does there exist one which is most like an ultrametric (or, equivalently, exhibits an optimal amount of Hölder regularity)? The answer, in general, is negative, which we demonstrate by constructing a suitable Rolewicz–Orlicz space.

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1. Introduction

Quasi-normed spaces such as certain Lebesgue spaces, Lorentz spaces, Hardy spaces, Besov spaces, and Triebel–Lizorkin spaces (to name a few), arise naturally in analysis. In general, given a vector space X and a norm $\|\cdot\|\colon X\to[0,\infty]$, if $\|\cdot\|'$ is pointwise equivalent to $\|\cdot\|$ (in the sense that vectors have comparable sizes when measured in $\|\cdot\|$ and $\|\cdot\|'$), the most we can conclude is that $\|\cdot\|'$ is merely a quasi-norm; heuristically speaking, the collection of norms on a vector space is not stable under pointwise equivalence, while the collection of quasi-norms is. Furthermore, an operator from one quasi-normed space to another is bounded with respect to one quasi-norm if and only if it is bounded to some (hence, any) pointwise equivalent quasi-norm, and also pointwise equivalent quasi-norms induce the same topologies.

A natural question arises: given a quasi-norm $\|\cdot\|$ on a vector space X, if one looks at the collection of all pointwise equivalent quasi-norms, is there a best quasi-norm to work with? To answer this we shall adopt a more general point of view and consider this question formulated in the

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larger category of quasi-metric spaces (since every quasi-norm may be viewed as a quasi-metric). In this context, the notion of "best" is made precise in (2.5). In loose terms, quasi-metrics are the most rudimentary tools for measuring distances, possibly failing to be continuous and their associated balls failing to be open in the topology they induce, while metrics, which may be thought of as nicer versions of quasi-metrics, do not exhibit such pathological properties, and finally ultrametrics are better still, enjoying a number of highly specialized properties (such as every triangle being isosceles and every point in a ball being its center). With this mindset, the issue we address is, given a quasi-metric and examining all quasi-metrics which are pointwise equivalent (i.e., comparable in size) to it, whether or not there exists one which is most ultrametric-like.

This aspect is particularly relevant for the type of analysis one may carry out on a quasi-metric space since, as we indicate in Remark 4.2, it may be rephrased in the following form: given a quasi-metric, among all others that are pointwise equivalent to it, is there one that exhibits the most amount of Hölder regularity?

We shall see that, in general, this optimization problem does not necessarily have a global minimizer (a phenomenon akin to a lack of compactness). We establish this following these steps: first we present certain regularization results (processes associating to given quasi-metrics other quantitatively better quasi-metrics), then introduce the Rolewicz modulus of concavity of a topological vector space and relate this modulus of concavity to certain optimal constants regarding the geometry of quasi-metric spaces, both of which are infima. After this we develop the notion of Rolewicz–Orlicz spaces (which are topological vector spaces) and construct one whose modulus of concavity is not attained. When brought back to the setting of quasi-metrics, this proves our main result. We conclude by identifying a scenario containing additional constraints in which the aforementioned optimization problem happens to have a global minimizer.

2. Requisite notions

First we record some basic definitions. Given a set $\mathscr{X}$, call two functions $f, g\colon \mathscr{X} \to [0,\infty]$ *pointwise equivalent*, written $f \approx g$, if there exists $C \in [1,\infty)$ such that $C^{-1}f(x) \leq g(x) \leq Cf(x)$ for all $x \in \mathscr{X}$.

Definition 2.1. Let X be an ambient set (tacitly assumed to have cardinality at least 2) and denote by $\operatorname{diag}(X)$ the diagonal in Cartesian product $X \times X$.

(1) Call $\rho\colon X\times X\to[0,\infty]$ a *quasi-metric* on X if $\rho^{-1}(\{0\})=\mathrm{diag}(X)$ and

$$(2.1)\qquad \begin{aligned} C_\rho &:= \sup_{\substack{x,y,z\in X\\ \text{not all equal}}} \frac{\rho(x,y)}{\max\{\rho(x,z),\rho(z,y)\}}<\infty,\\ \widetilde{C}_\rho &:= \sup_{\substack{x,y\in X\\ x\neq y}} \frac{\rho(y,x)}{\rho(x,y)}<\infty. \end{aligned}$$

In such a scenario, call the pair (X,ρ) a *quasi-metric space.*

(2) Given a quasi-metric ρ on X, denote by $[\rho]$ the collection of all quasi-metrics on X which are pointwise equivalent to ρ, and call it a *quasi-metric structure* on X. Also, given a quasi-metric structure $\mathbf{q}$ on X, call the pair $(X,\mathbf{q})$ a *quasi-metric structure space.*

A few remarks are in order. First, $\rho^{-1}(\{0\})=\mathrm{diag}(X)$ is the typical nondegeneracy condition. Also, by design, $C_\rho,\widetilde{C}_\rho\geq 1$, hence ultimately $C_\rho,\widetilde{C}_\rho\in[1,\infty)$. That for each $x,y\in X$ there holds $\rho(y,x)\leq\widetilde{C}_\rho\,\rho(x,y)$ expresses the fact ρ is *quasi-symmetric.* Call ρ *symmetric* if $\widetilde{C}_\rho=1$. That C_ρ is finite reflects the fact that ρ satisfies a *quasi-ultrametric condition*, i.e.,

$$(2.2)\qquad \rho(x,y)\leq C_\rho\max\big\{\rho(x,z),\rho(z,y)\big\},\quad \forall\,x,y,z\in X.$$

Call ρ an *ultrametric* when $C_\rho=1$ and $\widetilde{C}_\rho=1$. If ρ is a metric then we have $C_\rho\in[1,2]$ and $\widetilde{C}_\rho=1$. Ultrametrics exhibit highly specialized characteristics which metrics, in general, do not, and metrics satisfy some very useful properties which generic quasi-metrics may fail to enjoy. For example, any metric (hence any ultrametric) is continuous with respect to the topology it induces, while not all quasi-metrics are, where the topology τ_ρ induced by a quasi-metric ρ on X is given by

$$(2.3)\qquad \mathscr{O}\in\tau_\rho \overset{\text{def}}{\Longleftrightarrow} \mathscr{O}\subseteq X \text{ and } \forall\,x\in\mathscr{O}\;\exists\,r>0 \text{ such that } B_\rho(x,r):=\{y\in X:\rho(x,y)<r\}\subseteq\mathscr{O}.$$

Let us mention that if ρ is a quasi-metric on X and $\varrho\colon X\times X\to[0,\infty]$ is such that there exists $\gamma\in(0,\infty)$ for which ϱ^γ is pointwise equivalent to ρ, then ϱ is also a quasi-metric on X, and $\tau_\rho=\tau_\varrho$. Thus, if $(X,\mathbf{q})$ is a quasi-metric structure space, defining $\tau_{\mathbf{q}}:=\tau_\rho$ for some (hence, any) $\rho\in\mathbf{q}$ is unambiguous.

Related to the fact that quasi-metrics may fail to be continuous with respect to their canonical topologies, given an abstract quasi-metric ρ, the above ρ-balls $B_\rho(x,r)$ may not be open (in τ_ρ), while balls induced by genuine metrics are always open sets. In other words, heuristically speaking, the smaller C_ρ is, the better properties we expect ρ to exhibit.

Note the quasi-ultrametric condition (2.2) implies the existence of a constant $C \in [1,\infty)$ (usually different from the constant C_ρ appearing in (2.2)) with the property that

$$\rho(x,y) \le C\big(\rho(x,z)+\rho(z,y)\big), \quad \forall\, x,y,z \in X. \tag{2.4}$$

We shall refer to (2.4) as the *quasi-triangle inequality.* Conversely, if ρ satisfies the quasi-triangle inequality (2.4) then the quasi-ultrametric condition (2.2) holds (again, with a typically different constant).

One final remark concerns the stability of the class of quasi-metrics under the operation of taking powers. Specifically, if ρ is a quasi-metric on X and $\alpha \in (0,\infty)$, then ρ^α is also a quasi-metric on X. If the latter actually happens to satisfy a genuine triangle inequality (i.e., (2.4) with $C=1$) we shall say that the original ρ is α*-subadditive.* Corresponding to $\alpha=\infty$, call ρ ∞*-subadditive* if $C_\rho = 1$. Note if ρ is symmetric and α-subadditive for some $\alpha \in (0,\infty]$ then ρ^β is a genuine metric for any finite $\beta \in (0,\alpha]$.

Now for the formulation of the main question addressed in this article: keeping in mind the thinking that the smaller C_ρ is, the nicer properties we expect ρ to exhibit, given a quasi-metric structure $\mathbf{q}$ on some set X, does there exist $\rho \in \mathbf{q}$ which has the smallest optimal constant in the quasi-ultrametric condition (2.2)? In precise terms,

$$\text{is the infimum } C_{\mathbf{q}} := \inf\big\{C_\rho : \rho \in \mathbf{q}\big\} \text{ actually attained?} \tag{2.5}$$

3. Main result

The answer to the question posed at the end of Section 2 is contained in the following theorem.

Theorem 3.1. *There exists a quasi-metric structure space* $(X,\mathbf{q})$ *so that the infimum* $C_{\mathbf{q}}$ *is not attained. Furthermore,* X *may be taken to be a vector space which, when equipped with* $\mathbf{q}$*, is separable, complete, locally bounded, and* $\mathbf{q}$ *is finite.*

For example, let L *be the collection of equivalence classes of complex-valued, Lebesgue measurable functions defined on* $[0,1]$*. Also, fix* $p_0 \in (0,1]$*, define* $\|\cdot\| \colon L \to [0,\infty]$ *by*

$$\|u\| := \inf\left\{\lambda \in (0,\infty) : \int_0^1 \frac{|u(x)/\lambda|^{p_0}}{\ln(|u(x)/\lambda|^{p_0}+e)}\,dx \le \lambda\right\}, \quad \forall\, u \in L, \tag{3.1}$$

and set $X := \{u \in L : \|u\| < \infty\}$*. Then* X *is a vector space,* $\|\cdot\|$ *is a quasi*-F*-norm on* X*, and for each* $p \in (0, p_0)$ *there exists a* p*-homogeneous norm on* X*, call it* $\|\cdot\|_p$*, which induces the same topology on* X *as* $\|\cdot\|$*. Furthermore, if* $\rho_p\colon X \times X \to [0, \infty)$ *is defined by* $\rho_p(u, v) := \|u - v\|_p$ *for all* $u, v \in X$*, then* ρ_p *is a quasi-metric (in fact, a translation invariant,* p*-homogeneous, genuine metric) on* X *such that* $C_{\rho_p} \in (2^{p/p_0}, 2]$ *and, with* $\mathbf{q}_p := [\rho_p]$*, the quasi-metric structure space* $(X, \mathbf{q}_p)$ *has all the attributes mentioned in the first part of the statement. In particular,* $C_{\mathbf{q}_p} = 2^{p/p_0}$ *but* $C_\varrho > 2^{p/p_0}$ *for every* $\varrho \in \mathbf{q}_p$*.*

It is understood that a quasi-metric structure space $(X, \mathbf{q})$ is called separable or complete whenever the topological space $(X, \tau_{\mathbf{q}})$ is so. Similarly for locally bounded, where a topological vector space is called locally bounded provided there exists a topologically bounded neighborhood of the origin. By saying $\mathbf{q}$ is finite it is understood that some (hence, any) $\rho \in \mathbf{q}$ takes values in $[0, \infty)$. For other pieces of terminology the reader is referred to the body of the paper.

The topological space $(X, \tau_{\mathbf{q}})$, with X and $\mathbf{q}$ in the last part of Theorem 3.1, is a particular case of a Rolewicz–Orlicz space. A more inclusive point of view is adopted in the proof of Theorem 3.1 (presented in the last part of Section 7), where certain Rolewicz–Orlicz spaces of a more general nature are considered.

4. Regularization results

Here we discuss a number of regularization results, or procedures which associate to an abstract quasi-metric a similar (i.e., pointwise equivalence) yet quantitatively better quasi-metric. The first such result, correcting quasi-symmetry and the quasi-triangle inequality, may be found in [**3**, Theorem 3.46, p. 144]. It may be regarded as the sharp form of an earlier result, with similar aims, from [**2**].

Theorem 4.1. *Let* (X, ρ) *be a quasi-metric space. Define the function* $\rho_{\max}\colon X \times X \to [0, \infty]$ *by*

$$\rho_{\max}(x, y) := \max\big\{\rho(x, y), \rho(y, x)\big\}, \quad \forall\, x, y \in X. \tag{4.1}$$

Then the function $\rho_{\max}$ *is a symmetric quasi-metric on* X *which satisfies* $\rho \leq \rho_{\max} \leq \widetilde{C}_\rho\, \rho$ *on* $X \times X$ *and* $C_{\rho_{\max}} \leq C_\rho$*.*

Given $\alpha \in \big(0, (\log_2 C_\rho)^{-1}\big]$*, define the* α*-subadditive regularization* $\rho_\alpha\colon X\times X\to[0,\infty]$ *of* ρ *by*

$$(4.2)\quad \rho_\alpha(x,y):=\inf\Bigg\{\Big(\sum_{i=1}^{N}\rho(\xi_i,\xi_{i+1})^\alpha\Big)^{\frac{1}{\alpha}} : N\in\mathbb{N} \text{ and } \xi_1,\dots,\xi_{N+1}\in X, \text{ such that } \xi_1=x \text{ and } \xi_{N+1}=y\Bigg\},\quad \forall\, x,y\in X,$$

if α *is finite and, corresponding to* $\alpha=\infty$ *(which occurs precisely when* $C_\rho=1$*), take* $\rho_\infty(x,y):=\rho(x,y)$*.*

Then ρ_α *is a quasi-metric on* X *which satisfies* $(C_\rho)^{-2}\rho\le\rho_\alpha\le\rho$ *on* $X\times X$ *(hence,* $\rho_\alpha\approx\rho$*) as well as* $C_{\rho_\alpha}\le C_\rho\le 2^{1/\alpha}$*. Also,* ρ_α *is* β*-subadditive for each* $\beta\in(0,\alpha]$*, and* $\rho=\rho_\alpha$ *if and only if* ρ *is* α*-subadditive.*

Finally, define $\rho_\#\colon X\times X\to[0,\infty]$ *by* $\rho_\#:=(\rho_{\max})_\alpha$ *with* α *taken to be precisely* $(\log_2 C_\rho)^{-1}$*. Then* $\rho_\#$ *is a symmetric quasi-metric on* X *which is* β*-subadditive for each* $\beta\in(0,\alpha]$*; hence* $(\rho_\#)^\beta$ *is a metric on* X *for each finite* $\beta\in(0,\alpha]$*. Furthermore* $C_{\rho_\#}\le C_\rho$ *and*

$$(4.3)\qquad (C_\rho)^{-2}\rho(x,y)\le\rho_\#(x,y)\le\widetilde{C}_\rho\,\rho(x,y),\quad \forall\, x,y\in X.$$

In particular, the topology induced by the distance $(\rho_\#)^\beta$ *on* X *is precisely* τ_ρ*, thus the topology induced by any quasi-metric is metrizable.*

Moreover, for each finite exponent $\beta\in\big(0,(\log_2 C_\rho)^{-1}\big]$*, the function* $\rho_\#$ *satisfies the following local Hölder-type regularity condition of order* β *in both variables simultaneously:*

$$(4.4)\quad \big|\rho_\#(x,y)-\rho_\#(w,z)\big| \le \tfrac{1}{\beta}\max\big\{\rho_\#(x,y)^{1-\beta},\rho_\#(w,z)^{1-\beta}\big\}\big(\rho_\#(x,w)^\beta+\rho_\#(y,z)^\beta\big),$$

for all $x,y,w,z\in X$ *such that* $\min\{\rho(x,y),\rho(w,z)\}<\infty$*, and if* $\beta\ge1$*, it is assumed that* $x\neq y, w\neq z$*. In particular, in the case* $x=w$*, formula* (4.4) *becomes*

$$(4.5)\quad \big|\rho_\#(x,y)-\rho_\#(x,z)\big|\le\tfrac{1}{\beta}\max\big\{\rho_\#(x,y)^{1-\beta},\rho_\#(x,z)^{1-\beta}\big\}\big[\rho_\#(y,z)\big]^\beta,$$

for all $x,y,z\in X$ *such that* $\min\{\rho(x,y),\rho(x,z)\}<\infty$ *where, if* $\beta\ge1$*, it is assumed that* $x\notin\{y,z\}$*.*

Finally, the Hölder-type results from (4.4)–(4.5) *are sharp in the sense that they may fail if* $\beta>(\log_2 C_\rho)^{-1}$*.*

A few comments are in order. The first ties up the issue of whether the infimum defining $C_{\mathbf{q}}$ in (2.5) is attained to the existence of a quasi-distance $\rho\in\mathbf{q}$ that exhibits an optimal amount of Hölder regularity.

Remark 4.2. As seen from the last part of Theorem 4.1 the expression $\big(\log_2 C_\rho\big)^{-1}$ constitutes an optimal upper bound for the Hölder regularity exponent of a quasi-distance ρ on X. Note that the smaller C_ρ the larger $\big(\log_2 C_\rho\big)^{-1}$. In light of this observation, the issue whether the infimum $C_{\mathbf{q}} := \inf\big\{C_\rho : \rho \in \mathbf{q}\big\}$ is actually attained becomes equivalent to asking whether in a quasi-metric structure space $(X, \mathbf{q})$ there is a quasi-metric $\rho \in \mathbf{q}$ which exhibits the most amount of Hölder regularity. This aspect is particularly relevant for the type of analysis that may be carried out on $(X, \mathbf{q})$. For example, a theory of Hardy spaces $H^p(X, \mathbf{q})$ for an optimal range of p's that takes into account this maximal amount of Hölder regularity that the environment $(X, \mathbf{q})$ may sustain has been recently developed in [**1**].

Our next remark provides an example of a setting where the question asked in (2.5) has a positive answer.

Remark 4.3. Fix $\gamma \in (0, \infty)$ and denote by $\mathbf{q}_\gamma$ the pointwise equivalence class of the quasi-distance $|\cdot - \cdot|^\gamma$ in $\mathbb{R}^n$. Then $\inf\big\{C_\rho : \rho \in \mathbf{q}_\gamma\big\}$ is actually attained. Indeed, from the first formula in (2.1) one readily obtains $C_{|\cdot - \cdot|^\gamma} = 2^\gamma$, and we claim that $C_\rho \geq 2^\gamma$ for every $\rho \in \mathbf{q}_\gamma$. In turn, the claim is justified via reasoning by contradiction. Specifically, apply the fact that every function defined on an open connected subset of the Euclidean space satisfying a Hölder condition with exponent > 1 is necessarily constant, to the function $\rho_{\#}(x, \cdot)$ (with x arbitrarily fixed in $\mathbb{R}^n$) in any Euclidean ball whose closure is contained in $\mathbb{R}^n \setminus \{x\}$. In light of the Hölder-type condition formulated in (4.5) from Theorem 4.1, this yields a contradiction whenever $\beta \in \big(0, (\log_2 C_\rho)^{-1}\big)$ is such that $\beta > \gamma^{-1}$.

Another thing to note is that while ρ may fail to be continuous with respect to τ_ρ and ρ-balls may not be open in τ_ρ, the α-subadditive regularization ρ_α of ρ does not exhibit these pathologies; see below.

Remark 4.4. Given a quasi-metric space (X, ρ) and $\alpha \in \big(0, (\log_2 C_\rho)^{-1}\big]$, there holds $B_{\rho_\alpha}(x, r)$ is open in τ_ρ for all $x \in X, r \in (0, \infty)$. Indeed, pick $y \in B_{\rho_\alpha}(x, r)$, thus $\rho_\alpha(x, y) < r$. Fix some finite $\beta \in (0, \alpha]$ and take $R := \big(r^\beta - \rho_\alpha(x, y)^\beta\big)^{1/\beta} > 0$. Then for each $z \in B_{\rho_\alpha}(y, R)$ estimate, using the β-subadditivity of ρ_α,

$$\rho_\alpha(x, z) \leq \big(\rho_\alpha(x, y)^\beta + \rho_\alpha(y, z)^\beta\big)^{1/\beta} < \big(\rho_\alpha(x, y)^\beta + R^\beta\big)^{1/\beta} = r. \tag{4.6}$$

This shows $B_{\rho_\alpha}(y, R) \subseteq B_{\rho_\alpha}(x, r)$, hence $B_{\rho_\alpha}(x, r)$ is open in $\tau_\rho = \tau_{\rho_\alpha}$. Also, while $B_\rho(x, r)$ may not be an open set containing x, it is always a neighborhood of x. In particular, $\{B_\rho(x,1/n)\}_{n\in\mathbb{N}}$ is a countable fundamental system of neighborhoods of x.

We next present a regularization procedure of a quasi-metric ρ with respect to a semigroup of transformations of the ambient set X. Given a mapping $T\colon X\to X$, call ρ *T-invariant* provided

$$\rho\big(T(x),T(y)\big)=\rho(x,y),\quad\forall\, x,y\in X. \tag{4.7}$$

Theorem 4.5. *Let (X,ρ) be a quasi-metric space. Assume that $\mathscr{F}$ is a semigroup of maps from X into itself (with respect to the operation of composition) that contains the identity operator on X, and assume that $\sigma\colon\mathscr{F}\to(0,\infty)$ is a multiplicative map, i.e., one has $\sigma(S\circ T)=\sigma(S)\sigma(T)$ for every $S,T\in\mathscr{F}$. Finally, suppose that the quasi-metric ρ satisfies*

$$C_\rho^{\mathscr{F}}:=\sup_{x,y\in X,\,T\in\mathscr{F}}\frac{\rho\big(T(x),T(y)\big)}{\sigma(T)\rho(x,y)}<\infty. \tag{4.8}$$

Define $\rho^{\mathscr{F}}: X\times X\to[0,\infty]$ by setting

$$\rho^{\mathscr{F}}(x,y):=\sup\big\{\sigma(T)^{-1}\rho\big(T(x),T(y)\big):T\in\mathscr{F}\big\}\ \text{for each } x,y\in X. \tag{4.9}$$

Then $C_\rho^{\mathscr{F}}\in[1,\infty)$ and $\rho^{\mathscr{F}}$ satisfies the following properties:

1. *$\rho^{\mathscr{F}}\big(T(x),T(y)\big)\le\sigma(T)\rho^{\mathscr{F}}(x,y)$, for each $x,y\in X$ and each $T\in\mathscr{F}$;*
2. *$\rho\le\rho^{\mathscr{F}}\le C_\rho^{\mathscr{F}}\rho$, so $\rho^{\mathscr{F}}\approx\rho$, hence, in particular, $\rho^{\mathscr{F}}$ is a quasi-metric on X;*
3. *the optimal constants in the quasi-ultrametric condition satisfy $C_{\rho^{\mathscr{F}}}\le C_\rho$;*
4. *if ρ is symmetric then so is $\rho^{\mathscr{F}}$;*
5. *if ρ is α-subadditive for some $\alpha\in(0,\infty]$ then so is $\rho^{\mathscr{F}}$;*
6. *if ρ is T-invariant for some transformation $T\colon X\to X$ which commutes with every mapping in $\mathscr{F}$ (i.e., $T\circ S=S\circ T$ for each $S\in\mathscr{F}$) then $\rho^{\mathscr{F}}$ is also T-invariant;*
7. *if the map $T\in\mathscr{F}$ is bijective and has the property that $T^{-1}\in\mathscr{F}$, then $\rho^{\mathscr{F}}\big(T(x),T(y)\big)=\sigma(T)\rho^{\mathscr{F}}(x,y)$ for each $x,y\in X$. In particular, $\rho^{\mathscr{F}}$ is T-invariant for each invertible T in the semigroup $\mathscr{F}$ satisfying $\sigma(T)=1$.*

We make a quick comment before presenting the proof of this theorem.

Remark 4.6. In the context of Theorem 4.9, if the semigroup $\mathscr{F}$ is actually a group then $C_\rho^{\mathscr{F}}=1$ if and only if $\rho\big(T(x),T(y)\big)=\sigma(T)\rho(x,y)$ for each $x,y\in X$ and $T\in\mathscr{F}$. Thus, if in addition we also have that σ is constant (hence necessarily identically 1), then $C_\rho^{\mathscr{F}}=1$ if and only if ρ is T-invariant for each $T\in\mathscr{F}$.

Proof of Theorem 4.5: The multiplicative property of the map σ implies $\sigma(I) = \sigma(I)^2$, forcing $\sigma(I) = 1$. In turn, given that $I \in \mathscr{F}$, this entails $C_\rho^{\mathscr{F}} \geq 1$, hence $C_\rho^{\mathscr{F}} \in [1, \infty)$. For (1), given any $x, y \in X$ and $T \in \mathscr{F}$,

$$\begin{aligned} \rho^{\mathscr{F}}\big(T(x), T(y)\big) &= \sup\big\{\sigma(S)^{-1}\rho\big(S(T(x)), S(T(y))\big) : S \in \mathscr{F}\big\} \\ &\leq \sup\big\{\sigma(R)^{-1}\sigma(T)\rho\big(R(x), R(y)\big) : R \in \mathscr{F}\big\} \\ &= \sigma(T)\rho^{\mathscr{F}}(x, y), \end{aligned} \tag{4.10}$$

where the inequality above uses the fact that for each $S, T \in \mathscr{F}$ we have $R := S \circ T \in \mathscr{F}$ and $\sigma(R) = \sigma(S)\sigma(T)$.

For (2), the fact that $\sigma(I) = 1$ implies $\rho^{\mathscr{F}} \geq \rho$ on $X \times X$, as the supremum in the definition of $\rho^{\mathscr{F}}$ is taken over the set $\mathscr{F}$ which contains I. For the opposite inequality, for each $x, y \in X$ we write

$$\begin{aligned} \rho^{\mathscr{F}}(x, y) &= \sup\big\{\sigma(T)^{-1}\rho\big(T(x), T(y)\big) : T \in \mathscr{F}\big\} \\ &\leq \sup\big\{C_\rho^{\mathscr{F}}\rho(x, y) : T \in \mathscr{F}\big\} = C_\rho^{\mathscr{F}}\rho(x, y). \end{aligned} \tag{4.11}$$

To see $C_{\rho^{\mathscr{F}}} \leq C_\rho$, hence treat (3), fix $x, y, z \in X$ and write

$$\begin{aligned} \rho^{\mathscr{F}}(x, y) &= \sup\big\{\sigma(T)^{-1}\rho\big(T(x), T(y)\big) : T \in \mathscr{F}\big\} \\ &\leq \sup\Big\{\sigma(T)^{-1}C_\rho \max\big\{\rho\big(T(x), T(z)\big), \rho\big(T(z), T(y)\big)\big\} : T \in \mathscr{F}\Big\} \\ &= C_\rho \max\{\rho^{\mathscr{F}}(x, z), \rho^{\mathscr{F}}(z, y)\}, \end{aligned} \tag{4.12}$$

as sup and max commute. Then as $C_{\rho^{\mathscr{F}}}$ is the optimal constant in this estimate, the desired conclusion follows. That symmetry is hereditary is clear from definitions; this takes care of (4). Regarding (5), assuming that ρ is α-subadditive for some $\alpha \in (0, \infty)$, for each $x, y, z \in X$ we may estimate

$$\begin{aligned} \rho^{\mathscr{F}}(x, y) &= \sup_{T \in \mathscr{F}} \sigma(T)^{-1}\rho\big(T(x), T(y)\big) \\ &\leq \sup_{T \in \mathscr{F}} \sigma(T)^{-1}\Big\{\rho\big(T(x), T(z)\big)^\alpha + \rho\big(T(z), T(y)\big)^\alpha\Big\}^{1/\alpha} \\ &\leq \Big\{\sup_{T \in \mathscr{F}} \sigma(T)^{-1}\rho\big(T(x), T(z)\big)^\alpha + \sup_{T \in \mathscr{F}} \sigma(T)^{-1}\rho\big(T(z), T(y)\big)^\alpha\Big\}^{1/\alpha} \\ &= \Big(\rho^{\mathscr{F}}(x, z)^\alpha + \rho^{\mathscr{F}}(z, y)^\alpha\Big)^{1/\alpha}, \end{aligned} \tag{4.13}$$

implying that $\rho^{\mathscr{F}}$ is also α-subadditive. The case $\alpha = \infty$ is similar. Moving on to (6), assume that ρ is T-invariant for some $T\colon X \to X$ commuting with every mapping in $\mathscr{F}$. Then, bearing in mind that we have $T \circ S = S \circ T$ for each $S \in \mathscr{F}$, for each $x, y \in X$ we may compute

$$\begin{aligned} \rho^{\mathscr{F}}\big(T(x),T(y)\big) &= \sup\big\{\sigma(S)^{-1}\rho\big(S(T(x)),S(T(y))\big)\colon S\in\mathscr{F}\big\} \\ &= \sup\big\{\sigma(S)^{-1}\rho\big(T(S(x)),T(S(y))\big)\colon S\in\mathscr{F}\big\} \qquad (4.14) \\ &= \sup\big\{\sigma(S)^{-1}\rho\big(S(x),S(y)\big)\colon S\in\mathscr{F}\big\} = \rho^{\mathscr{F}}(x,y), \end{aligned}$$

which goes to show that $\rho^{\mathscr{F}}$ is T-invariant as well. Finally, as far as (7) is concerned, suppose that $\rho^{\mathscr{F}}$ is T-invariant for some bijective transformation $T \in \mathscr{F}$ such that $T^{-1} \in \mathscr{F}$. Then $\sigma(T^{-1}) = \sigma(T)^{-1}$ since $1 = \sigma(I) = \sigma\big(T \circ T^{-1}\big) = \sigma(T)\sigma(T^{-1})$. Writing the inequality established in (1) for T^{-1} in place of T and for $T(x)$, $T(y)$ in place of x, y then gives $\rho^{\mathscr{F}}(x,y) \le \sigma(T)^{-1}\rho^{\mathscr{F}}\big(T(x),T(y)\big)$ for each $x, y \in X$. In concert with (1), this shows that actually $\rho^{\mathscr{F}}\big(T(x),T(y)\big) = \sigma(T)\rho^{\mathscr{F}}(x,y)$ for each $x, y \in X$, as desired. □

Now we shift our attention to quasi-metrics on vector spaces (always tacitly assumed to be complex).

Definition 4.7. Let X be a vector space and let ρ be a quasi-metric on X.

(1) Call (X, ρ) a *quasi-metric linear space* provided (X, τ_ρ) is a topological vector space. In this case $(X, [\rho])$ is said to be a *quasi-metric linear structure space.*

(2) Say ρ is *quasi-translation invariant* provided

$$\overline{C}_\rho := \sup_{\substack{x,y,z\in X \\ x\neq y}} \frac{\rho(x+z, y+z)}{\rho(x,y)} < \infty. \qquad (4.15)$$

Call ρ *translation invariant* provided $\overline{C}_\rho = 1$.

(3) Given $p \in (0, \infty)$, say ρ is *lower quasi-p-homogeneous* if

$$C^{*,\ell}_{\rho,p} := \sup_{\substack{x,y\in X,\lambda\in\mathbb{C} \\ x\neq y,\, 0<|\lambda|\le 1}} \frac{\rho(\lambda x, \lambda y)}{|\lambda|^p \rho(x,y)} < \infty. \qquad (4.16)$$

If $p = 1$ abbreviate $C^{*,\ell}_{\rho,p} := C^{*,\ell}_{\rho}$ and say ρ is *lower quasi-homogeneous.* If the finiteness condition (4.16) holds with the supremum taken over nonzero scalars λ call ρ *quasi-p-homogeneous* and label the corresponding supremum as $C^{*}_{\rho,p}$. Similarly for C^{*}_{ρ}, quasi-homogeneous, p-homogeneous, and homogeneous.

Much like C_ρ and $\widetilde{C}_\rho$, the constants $\overline{C}_\rho$ and $C^{*,\ell}_{\rho,p}$ belong to $[1,\infty)$ and are designed to be the optimal constants in certain inequalities, namely

$$\rho(x+z,y+z) \leq \overline{C}_\rho\, \rho(x,y) \quad \text{and} \quad \rho(\lambda x,\lambda y) \leq C^{*,\ell}_{\rho,p}|\lambda|^p \rho(x,y), \tag{4.17}$$

with $x,y,z \in X$ and $\lambda \in \mathbb{C}$ such that $|\lambda| \leq 1$. Note that if $\overline{C}_\rho = 1$ then the first inequality holds with equality, but if $C^{*,\ell}_{\rho,p} = 1$ we do not necessarily have equality in the second (though this does hold if ρ satisfies the stronger condition of p-homogeneity). Further, observe that ρ is (lower) quasi-p-homogeneous if and only if $\rho^{1/p}$ is (lower) quasi-homogeneous. We shall now employ Theorem 4.5 in order to correct the quasi-translation invariance without straying too far from the original quasi-metric.

Proposition 4.8. *Let (X,ρ) be a quasi-metric linear space so that ρ is quasi-translation invariant. Define the function $\overline{\rho}\colon X\times X \to [0,\infty]$ by setting*

$$\overline{\rho}(x,y) := \sup\big\{\rho(x+z,y+z) : z \in X\big\} \text{ for each } x,y \in X. \tag{4.18}$$

Then $\overline{\rho}$ satisfies the following properties:

1. *$\overline{\rho}$ is a genuinely translation invariant quasi-metric;*
2. *$\rho \leq \overline{\rho} \leq (\overline{C}_\rho)^{-1}\rho$ on $X\times X$, thus $\overline{\rho}$ is pointwise equivalent to ρ;*
3. *the optimal constant in the quasi-ultrametric condition of $\overline{\rho}$ is at most that of ρ, i.e., $C_{\overline{\rho}} \leq C_\rho$;*
4. *if ρ is α-subadditive for some $\alpha \in (0,\infty]$ then so is $\overline{\rho}$, and if ρ is symmetric then so is $\overline{\rho}$.*

Proof: This is a direct consequence of Theorem 4.5 used here for the family $\mathscr{F} := \{T_x\}_{x\in X}$ where $T_x\colon X\to X$ is given by $T_x(y) := x+y$ for each $y \in X$, and where $\sigma\colon \mathscr{F} \to (0,\infty)$ is identically equal to 1. □

The next proposition deals with correcting the lower quasi-p-homogeneity property.

Proposition 4.9. *Suppose (X,ρ) is a quasi-metric linear space such that ρ is lower quasi-p-homogeneous for some $p \in (0,\infty)$. Define the function $\rho^{*,\ell}\colon X\times X \to [0,\infty]$ by setting*

$$\rho^{*,\ell}(x,y) := \sup\big\{|\lambda|^{-p}\rho(\lambda x,\lambda y) : 0<|\lambda|\leq 1\big\} \text{ for each } x,y \in X. \tag{4.19}$$

Then $\rho^{*,\ell}$ satisfies the following properties:

1. *$\rho^{*,\ell}$ is a genuinely lower p-homogeneous quasi-metric on X, and $C_{\rho^{*,\ell}} \leq C_\rho$;*

(2) $\rho^{*,\ell}$ *is pointwise equivalent to the quasi-metric* ρ *in the precise sense that* $\rho \leq \rho^{*,\ell} \leq C^{*,\ell}_{\rho,p}\rho$;

(3) *if* ρ *is* α*-subadditive for some* $\alpha \in (0,\infty]$ *then so is* $\rho^{*,\ell}$;

(4) *if* ρ *is translation invariant then so is* $\rho^{*,\ell}$*, and if* ρ *is symmetric then so is* $\rho^{*,\ell}$.

Proof: Apply Theorem 4.5 with $\mathscr{F} := \{T_\lambda : \lambda \in \mathbb{C},\ 0 < |\lambda| \leq 1\}$ where $T_\lambda \colon X \to X$ is given by $T_\lambda(x) := \lambda x$ for each $x \in X$, and where the map $\sigma \colon \mathscr{F} \to (0,\infty)$ acts according to $\sigma(T_\lambda) := |\lambda|^p$ for each $T_\lambda \in \mathscr{F}$. □

An analogous result may be formulated when ρ is assumed to be quasi-p-homogeneous rather than lower quasi-p-homogeneous, in which case we would define $\rho^*(x,y) := \sup\big\{|\lambda|^{-p}\rho(\lambda x, \lambda y) : \lambda \in \mathbb{C}\setminus\{0\}\big\}$ for all $x, y \in X$. Then results (1)–(4) above hold by omitting the word "lower" and the superscript "ℓ."

5. Relating $C_{\mathbf{q}}$ to $\mathrm{ind}_R(X, \tau_{\mathbf{q}})$

We first record some concepts pertaining to topological vector spaces, which are always tacitly assumed to be Hausdorff. Taking after Rolewicz in [**4**], we make the the following definition.

Definition 5.1. (1) Let X be a vector space and let U be a subset of X. Then the (Rolewicz) *modulus of concavity* of U, denoted $c(U)$, is defined by

$$c(U) := \inf\big\{s \in [1,\infty) : U + U \subseteq sU\big\}, \tag{5.1}$$

convening that $\inf \emptyset := \infty$. Call U *pseudoconvex* if it has a finite modulus concavity, i.e., if $c(U) < \infty$.

(2) Let (X,τ) be a topological vector space. Define the *Rolewicz modulus of concavity of* (X,τ) as

$$\mathrm{ind}_R(X,\tau) := \inf\big\{c(U) : \emptyset \neq U \subseteq X,\ U \text{ open, balanced, and topologically bounded}\big\}. \tag{5.2}$$

Recall a subset U of a vector space is *balanced* provided $sU \subseteq U$ for every $s \in \mathbb{C}$ satisfying $|s| \leq 1$, and a subset U of a topological vector space is called *topologically bounded* if for every neighborhood of the origin N there exists a scalar s so that $U \subseteq sN$. Also note $c(\lambda U) = c(U)$ for all $\lambda \in \mathbb{C} \setminus \{0\}$.

Given a quasi-metric linear structure space $(X, \mathbf{q})$, a setting in which both $\mathrm{ind}_R(X,\tau_{\mathbf{q}})$ and $C_{\mathbf{q}}$ are defined, our goal is to relate $\mathrm{ind}_R(X,\tau_{\mathbf{q}})$ to $C_{\mathbf{q}} := \inf\big\{C_\rho : \rho \in \mathbf{q}\big\}$. Most importantly, we seek to show that if the infimum in $C_{\mathbf{q}}$ is attained then so is the infimum in $\mathrm{ind}_R(X, \tau_{\mathbf{q}})$, granted proper conditions on $\mathbf{q}$; see Theorem 5.6 below. We need several preliminary results in order to accomplish this task: properties of Minkowski functionals, which allow us to pass from certain sets (namely open, topologically bounded, balanced sets) to quasi-norms, then negotiating passing from a quasi-norm to a quasi-metric, and finally a couple of lemmas detailing properties of certain balls.

Recall that, given a topological vector space X and a neighborhood of the origin $U \subseteq X$, the *Minkowski functional* (or *Minkowski gauge function*) of U, denoted $\|\cdot\|_U \colon X \to [0, \infty]$, is defined as

$$\|x\|_U := \inf\big\{\lambda \in (0, \infty) : x/\lambda \in U\big\}, \quad \forall\, x \in X, \tag{5.3}$$

where the infimum of the empty set is taken to be ∞. Note any topologically bounded neighborhood of the origin is pseudoconvex. For a proof of the next proposition see [**3**, (4.391)–(4.394), p. 222].

Proposition 5.2. *Given topological vector space (X, τ) and a nonempty, open, balanced, topologically bounded set $U \subseteq X$, the Minkowski gauge function $\|\cdot\|_U$ associated with U is nondegenerate (in the sense that $\|x\|_U = 0$ if and only if $x = 0$), is finite (in the sense that $\|\cdot\|_U$ does not take the value ∞), is homogeneous (in the sense that one has $\|\lambda x\|_U = |\lambda| \cdot \|x\|_U$ for all $\lambda \in \mathbb{C}$ and $x \in X$), and satisfies the quasi-ultrametric-like condition $\|x + y\|_U \leq c(U) \max\{\|x\|_U, \|y\|_U\}$ for all $x, y \in X$.*

Thus the notation $\|\cdot\|_U$ employed in (5.3) is not merely suggestive: if U is a nonempty, open, balanced, topologically bounded set then $\|\cdot\|_U$ is indeed a quasi-norm. Below we state and prove a lemma which gives us a quasi-metric from any given Minkowski functional. Recall that a subset U of a quasi-metric space (X, ρ) is said to be *ρ-bounded* provided $\mathrm{diam}_\rho(U) := \sup\{\rho(x, y) : x, y \in U\} < \infty$.

Lemma 5.3. *Let (X, ρ) be a quasi-metric linear space such that ρ is both quasi-translation invariant and quasi-p-homogeneous for some exponent $p \in (0, \infty)$. Then any nonempty, open, ρ-bounded, and balanced set $U \subseteq X$ induces a symmetric, finite, translation invariant, homogeneous quasi-metric ρ_U on X via*

$$\rho_U(x, y) := \|x - y\|_U \text{ for every } x, y \in X, \tag{5.4}$$

where $\|\cdot\|_U$ is as in (5.3). Furthermore, $\rho_U \approx \rho^{1/p}$ (hence $\tau_{\rho_U} = \tau_\rho$) and $C_{\rho_U} \leq c(U)$.

Proof: Observe that ρ_U is translation invariant by design, while its homogeneity and finiteness are inherited from $\|\cdot\|_U$'s. Note also that these imply ρ_U is symmetric. We claim that there exists a constant $C \in [1,\infty)$ so that $C^{-1}\rho_U \leq \rho^{1/p} \leq C\rho_U$ on $X \times X$ (and this will force ρ_U to be a quasi-metric). To see a constant multiple of ρ_U dominates $\rho^{1/p}$, set $k := \mathrm{diam}_\rho(U)$. Then $k \in (0,\infty)$ since U is ρ-bounded and does not consist of only the zero vector. Fix $x, y \in X$ along with some arbitrary $\varepsilon > 0$. By definition of ρ_U, we have

$$\frac{y-x}{\rho_U(y,x)+\varepsilon} \in U, \text{ hence } \rho\left(0, \frac{y-x}{\rho_U(y,x)+\varepsilon}\right) \leq k. \tag{5.5}$$

Calling upon the quasi-p-homogeneity and quasi-translation invariance of ρ, we may estimate

$$k \geq \rho\left(0, \frac{y-x}{\rho_U(y,x)+\varepsilon}\right) \geq \frac{\rho(0,y-x)}{C^*_{\rho,p}\big(\rho_U(y,x)+\varepsilon\big)^p}. \tag{5.6}$$

Since ρ_U is symmetric and $\varepsilon > 0$ was arbitrary, this ultimately forces $\rho^{1/p} \leq \big[kC^*_{\rho,p}\overline{C}_\rho\big]^{1/p}\rho_U$ on $X \times X$.

To show a constant multiple of $\rho^{1/p}$ dominates ρ_U, fix $x, y \in X$ distinct. Since $\{B_\rho(0,1/n)\}_{n\in\mathbb{N}}$ is a fundamental system of neighborhood of the origin (cf. Remark 4.4) and U is a neighborhood of the origin, there exists $n \in \mathbb{N}$ so that $B_\rho(0,1/n) \subseteq U$. Due to this and quasi-$p$-homogeneity of ρ it follows that

$$\frac{y-x}{\big[2nC^*_{\rho,p}\,\rho(0,y-x)\big]^{1/p}} \in B_\rho(0,1/n) \subseteq U. \tag{5.7}$$

From this, keeping in mind that $1 \geq \|z\|_U$ for all $z \in U$, the definition of ρ_U, and that ρ_U is translation invariant and homogeneous, we obtain

$$\begin{aligned} 1 &\geq \left\| \frac{y-x}{\big[2nC^*_{\rho,p}\,\rho(0,y-x)\big]^{1/p}} \right\|_U = \rho_U\left(0, \frac{y-x}{\big[2nC^*_{\rho,p}\,\rho(0,y-x)\big]^{1/p}}\right) \\ &= \frac{\rho_U(x,y)}{\big[2nC^*_{\rho,p}\,\rho(0,y-x)\big]^{1/p}}. \end{aligned} \tag{5.8}$$

In turn, this further implies that $\rho_U \leq \big[2nC^*_{\rho,p}\overline{C}_\rho\big]^{1/p}\rho^{1/p}$ on $X \times X$. At this point, there remains to show that $C_{\rho_U} \leq c(U)$. In this regard, by the last part in Proposition 5.2, for any $x, y, z \in X$ we have

$$\|x-y\|_U = \|x-z+z-y\|_U \leq c(U)\max\big\{\|x-z\|_U, \|z-y\|_U\big\}. \tag{5.9}$$

Hence, $\rho_U(x,y) \leq c(U)\max\big\{\rho_U(x,z), \rho_U(z,y)\big\}$ for all $x, y, z \in X$, so $C_{\rho_U} \leq c(U)$ by (2.1). $\square$

The final two auxiliary results needed for relating $C_{\mathbf{q}}$ to $\mathrm{ind}_R(X,\tau_{\mathbf{q}})$ in a proper setting are contained in Lemmas 5.4–5.5. The reader is reminded that the regularization $\rho^{*,\ell}$ of a quasi-metric ρ is defined in (4.19).

Lemma 5.4. *Suppose (X,τ) is a topological vector space such that $\tau=\tau_{\mathbf{q}}$ for some quasi-metric structure $\mathbf{q}$ on X and, further, that there exists $\rho\in\mathbf{q}$ which is lower quasi-p-homogeneous for some $p\in(0,\infty)$. Then any ρ-bounded subset of X is topologically bounded. Moreover, for any $r\in(0,\infty)$, the ball $B_{\rho^{*,\ell}}(0,r)$ is a topologically bounded, balanced neighborhood of the origin in (X,τ).*

Proof: From (4.16) it readily follows that for every given $R,r\in(0,\infty)$ there holds

$$c\geq\max\Big\{1,\big(C^{*,\ell}_{\rho,p}\,R/r\big)^{1/p}\Big\}\Longrightarrow B_\rho(0,R)\subseteq cB_\rho(0,r). \tag{5.10}$$

Based on this and the last part of Remark 4.4, the first conclusion in the statement of the lemma follows.

To establish the second conclusion, fix an arbitrary $r\in(0,\infty)$. From the last part of Remark 4.4 it follows that $B_{\rho^{*,\ell}}(0,r)$ is a neighborhood of the origin. In addition, if $|s|\leq 1$ and $x\in B_{\rho^{*,\ell}}(0,r)$ then, since $\rho^{*,\ell}(0,sx)\leq|s|^p\rho^{*,\ell}(0,x)<r$, we have $sx\in B_{\rho^{*,\ell}}(0,r)$. This goes to show that $B_{\rho^{*,\ell}}(0,r)$ is also balanced. Finally, given that $B_{\rho^{*,\ell}}(0,r)$ is ρ-bounded (thanks to $\rho\approx\rho^{*,\ell}$), the first part of the proof gives that this set is topologically bounded as well. □

Lemma 5.5. *Suppose (X,τ) is a topological vector space such that $\tau=\tau_{\mathbf{q}}$ for some quasi-metric structure $\mathbf{q}$ on X and, further, that there exists $\rho\in\mathbf{q}$ which is both quasi-translation invariant and lower quasi-p-homogeneous for some $p\in(0,\infty)$. Then a subset of X is topologically bounded if and only if it is ρ-bounded.*

Proof: Define $\|\cdot\|\colon X\to[0,\infty]$ by setting $\|x\|:=\rho(0,x)$ for each $x\in X$. Then with $c_0:=C^{*,\ell}_{\rho,p}\in[1,\infty)$ and $c_1:=(\overline{C}_\rho)^2C_\rho\in[1,\infty)$, we have

$$\|\lambda x\|\leq c_0\|x\|,\qquad \forall\,x\in X\text{ and }\forall\,\lambda\in\mathbb{C}\text{ such that }|\lambda|\leq 1, \tag{5.11}$$

$$\|x+y\|\leq c_1\big(\|x\|+\|y\|\big),\quad \forall\,x,y\in X. \tag{5.12}$$

Indeed, (5.11) is clear from (4.16), while (5.12) may be justified by estimating

$$\begin{aligned}\|x+y\|=\rho(0,x+y)&\leq\overline{C}_\rho\,\rho(-x,y)\leq\overline{C}_\rho C_\rho\max\big\{\rho(-x,0),\rho(0,y)\big\}\\&\leq\overline{C}_\rho C_\rho\max\big\{\overline{C}_\rho\,\rho(0,x),\rho(0,y)\big\}\leq(\overline{C}_\rho)^2C_\rho\big(\|x\|+\|y\|\big),\end{aligned} \tag{5.13}$$

by (4.17) and (2.1). Granted (5.11)–(5.12), we may invoke [**3**, Lemma 3.40, p. 133] in order to conclude that

$$\|\lambda x\| \leq c_0 c_1\big(c_0 + 4c_1^2|\lambda|^{1+\log_2 c_1}\big)\|x\|, \quad \forall\, \lambda \in \mathbb{C} \text{ and } \forall\, x \in X. \tag{5.14}$$

Bearing in mind the definition of $\|\cdot\|$, this ultimately implies

$$\lambda B_\rho(0,r) \subseteq B_\rho\big(0, c_{\rho,\lambda} r\big), \quad \forall\, \lambda \in \mathbb{C}, \tag{5.15}$$

where we have abbreviated $c_{\rho,\lambda} \in [1,\infty)$ as

$$\begin{aligned} c_{\rho,\lambda} &:= c_0 c_1\big(c_0 + 4c_1^2|\lambda|^{1+\log_2 c_1}\big) \\ &= (\overline{C}_\rho)^2 C_\rho C^{*,\ell}_{\rho,p}\Big(C^{*,\ell}_{\rho,p} + 4(\overline{C}_\rho)^4 (C_\rho)^2 |\lambda|^{1+\log_2((\overline{C}_\rho)^2 C_\rho)}\Big). \end{aligned} \tag{5.16}$$

At this stage, the fact that any topologically bounded subset of X is ρ-bounded follows from (5.15)–(5.16), while the converse statement has been already established in Lemma 5.4. □

Now we are ready to prove the aforementioned theorem, relating $C_{\mathbf{q}}$ and $\mathrm{ind}_R(X,\tau)$; this is of independent interest. Recall that for a quasi-metric structure space $(X,\mathbf{q})$ we have set $C_{\mathbf{q}} := \inf\{C_\rho : \rho \in \mathbf{q}\}$ and that the Rolewicz modulus of concavity $\mathrm{ind}_R(X,\tau)$ of a topological vector space (X,τ) is defined as in (5.2). Finally, the topology $\tau_{\mathbf{q}}$ induced by a quasi-metric structure $\mathbf{q}$ is given immediately after (2.3).

Theorem 5.6. *Let $(X,\mathbf{q})$ be a quasi-metric linear structure space for which there exists $\rho \in \mathbf{q}$ that is both quasi-translation invariant and lower quasi-p-homogeneous for some $p \in (0,\infty)$. Then $C_{\mathbf{q}} \geq \big[\mathrm{ind}_R(X,\tau_{\mathbf{q}})\big]^p$.*

Moreover, if the lower quasi-p-homogeneity condition on ρ is strengthened to quasi-p-homogeneity then

$$C_{\mathbf{q}} = \big[\mathrm{ind}_R(X,\tau_{\mathbf{q}})\big]^p, \tag{5.17}$$

and the infimum in $C_{\mathbf{q}}$ is attained if and only if the infimum in $\mathrm{ind}_R(X,\tau_{\mathbf{q}})$ is attained.

The fact that, under the condition specified above, formula (5.17) holds is remarkable since $C_{\mathbf{q}}$ is a purely quasi-metric entity, while $\mathrm{ind}_R(X,\tau_{\mathbf{q}})$ pertains to the topological vector space structure of the ambient.

Proof of Theorem 5.6: Unraveling definitions, the first claim in the statement follows as soon as we prove

$$\inf\{C_\rho : \rho \in \mathbf{q}\} \geq \inf\{c(U)^p : \emptyset \neq U \subseteq X,\ U \text{ open}, \text{ topologically bounded, and balanced}\}. \tag{5.18}$$

To this end, fix $\rho \in \mathbf{q}$ lower quasi-p-homogeneous and quasi-translation invariant. Consider the quasi-metric $\rho' := \big(\overline{(\rho_\alpha)}\big)^{*,\ell}$ where the α-regularization is as in Theorem 4.1, the bar-regularization is as in Proposition 4.8, and the $(\cdot)^{*,\ell}$-regularization is as in Proposition 4.9. By the aforementioned theorem and propositions it follows that $C_\rho \geq C_{\rho'}$ and that ρ' is α-subadditive, translation invariant, and lower p-homogeneous. Moreover, $\rho \approx \rho'$ which makes $C_{\rho'}$ a contender in the infimum in the left side of (5.18).

Define $U_\rho := B_{\rho'}(0,1)$. By Lemma 5.4 it follows that U_ρ is a topologically bounded, balanced neighborhood of the origin, and by Remark 4.4 (and the α-subadditivity of ρ') U_ρ is also open. We wish to prove that $c(U_\rho)^p \leq C_{\rho'}$. Fix $x, y \in U_\rho$ arbitrary with the aim of showing $x + y \in (C_{\rho'})^{1/p} U_\rho$; this suffices as $c(U_\rho)$ is the infimum over all such constants $c > 0$ satisfying $x + y \in c\, U_\rho$ for all $x, y \in U_\rho$. Making use of lower p-homogeneity, the fact that $C_{\rho'} \in [1, \infty)$, and translation invariance, we estimate

$$
\begin{aligned}
\rho'\left(0, \frac{x+y}{(C_{\rho'})^{1/p}}\right) &\leq \frac{1}{C_{\rho'}}\, \rho'(0, x+y) = \frac{1}{C_{\rho'}}\, \rho'(-y, x) \\
&\leq \frac{1}{C_{\rho'}} \cdot C_{\rho'} \max\Big\{\rho'(-y, 0), \rho'(0, x)\Big\} \qquad (5.19) \\
&= \max\big\{\rho'(0, y), \rho'(0, x)\big\} < 1,
\end{aligned}
$$

as desired. Hence, $C_\rho \geq C_{\rho'} \geq c(U_\rho)^p$, and this readily yields (5.18), i.e., $C_{\mathbf{q}} \geq \big[\mathrm{ind}_R(X, \tau_{\mathbf{q}})\big]^p$.

Moving on, strengthen the lower quasi-p-homogeneity condition on ρ to quasi-p-homogeneity, with the goal of proving the opposite inequality in (5.18). Specifically, fix $U \subseteq X$ such that U is nonempty, open, topologically bounded, and balanced. Bringing in Lemma 5.3 (whose applicability is ensured by Lemma 5.5 and current hypotheses, since $U \subseteq X$ is topologically bounded) gives $(\rho_U)^p \approx \rho$ and $C_{\rho_U} \leq c(U)$. Hence, $C_{(\rho_U)^p} = \big(C_{\rho_U}\big)^p \leq c(U)^p$, thus in the current case (5.18) also holds with the inequality sign reversed.

The above reasoning also makes possible to show that, under the latter (stronger) hypotheses on ρ, the infimum in $C_{\mathbf{q}}$ is attained if and only if the infimum in $\mathrm{ind}_R(X, \tau_{\mathbf{q}})$ is attained. Indeed, in one direction, suppose $\rho \in \mathbf{q}$ is such that $C_\rho = C_{\mathbf{q}}$. Then $c(U_\rho)^p \leq C_\rho = \big[\mathrm{ind}_R(X, \tau_{\mathbf{q}})\big]^p$ and, given the nature of the infimum in $\mathrm{ind}_R(X, \tau_{\mathbf{q}})$, it follows that $c(U_\rho) = \mathrm{ind}_R(X, \tau_{\mathbf{q}})$. In the opposite direction, assume $U \subseteq X$ is a nonempty, open, topologically bounded, and balanced set, with the property that

$c(U) = \mathrm{ind}_R(X, \tau_{\mathbf{q}})$. Then $C_{(\rho_U)^p} \leq c(U)^p = \big[\mathrm{ind}_R(X, \tau_{\mathbf{q}})\big]^p = C_{\mathbf{q}}$, and since $(\rho_U)^p \in \mathbf{q}$ the definition of $C_{\mathbf{q}}$ forces $C_{(\rho_U)^p} = C_{\mathbf{q}}$. □

6. Rolewicz–Orlicz spaces

Theorem 5.6 shifts the focus to finding a topological vector space (X,τ) with the topology induced by a suitable quasi-metric and for which the infimum $\mathrm{ind}_R(X, \tau)$ is not attained. We shall eventually identify such a specimen in the category of Rolewicz–Orlicz spaces. As a preamble, we discuss quasi-modulars, a notion that refines a concept found in [**4**].

Definition 6.1. Let X be a vector space. A function $m\colon X \to [0,\infty]$ is called a *quasi-modular* (on X) provided there exist $k_0, k_1 \in [1,\infty)$ such that the following conditions are satisfied:

(1) m is nondegenerate in the sense that $m(x) = 0 \iff x = 0$, for each $x \in X$;

(2) $m(\lambda x) \leq k_0\, m(x)$ for all $x \in X$ and $\lambda \in \mathbb{C}$ such that $|\lambda| = 1$;

(3) $m(\lambda x + (1-\lambda)y) \leq k_1(m(x) + m(y))$ for all $x \in X$ and $\lambda \in [0,1]$;

(4) $m(\lambda_n x) \to 0$ if the sequence $\{\lambda_n\}_{n\in\mathbb{N}} \subseteq \mathbb{C}$ converges to zero and $x \in X$ is such that $m(x) < \infty$.

Call m a *modular* if k_0 and k_1 may be taken to be 1. If further m satisfies the property

(5) $m(\lambda x_n) \to 0$ whenever $\lambda \in \mathbb{C}$ and $\{x_n\}_{n\in\mathbb{N}} \subseteq X$ is such that $m(x_n) \to 0$ as $n \to \infty$,

call m a *metrizing quasi-modular*. Again, if 1 is an admissible value for k_0 and k_1 and m satisfies (5), call m a *metrizing modular*. Lastly, given a quasi-modular m on X define

$$X_m := \big\{x \in X : \text{there exists } C \in (0,\infty) \text{ so that } m(Cx) < \infty\big\}. \tag{6.1}$$

The above axioms imply a number of things, including

$$k_0^{-1} m(\lambda x) \leq m(|\lambda| x) \leq k_0 m(\lambda x) \quad \text{for all } \lambda \in \mathbb{C} \text{ and } x \in X, \tag{6.2}$$

$$m(\lambda x) \leq k_0 k_1 m(x) \qquad \text{whenever } x \in X \text{ and } |\lambda| \leq 1. \tag{6.3}$$

A special feature of metrizing quasi-modulars is that they give rise to quasi-F-norms in the precise sense described in Theorem 6.2 below. Following [**3**, (3.426)–(3.429), p. 135], we shall call a function $\|\cdot\|\colon X \to [0,\infty]$ a *quasi*-F-*norm* on a vector space X provided $\|\cdot\|$ is nondegenerate, quasi-subadditive, *quasi-subhomogeneous* (i.e., $\|\lambda x\| \leq C\|x\|$ for all $x \in X$ and $\lambda \in \mathbb{C}$ such that $|\lambda| \leq 1$, where C is some fixed, positive constant), and $\|\lambda_n x\| \to 0$ for each fixed $x \in X$ if $\lambda_n \to 0$ and $\|x\| < \infty$.

Note lower quasi-p-homogeneity implies quasi-subhomogeneity for any $p \in (0,\infty)$. Also, agree to remove the prefix "quasi" if $\|\cdot\|$ is actually subadditive and *subhomogeneous* (the latter meaning $\|\lambda x\| \leq \|x\|$ for all $x \in X$ and all $\lambda \in \mathbb{C}$ with $|\lambda| \leq 1$). Note the F in F-norm is for Fréchet.

In the next theorem we reference metrizing quasi-modulars being *topologically equivalent* to quasi-F-norms, meaning they induce the same topology. The topology τ_m induced by a quasi-modular m on a vector space X is defined as

$$(6.4)\quad \mathscr{O} \in \tau_m \overset{\text{def}}{\Longleftrightarrow} \mathscr{O} \subseteq X \text{ and } \forall\, x \in \mathscr{O}\ \exists\, r > 0 \text{ such that } \{y \in X : m(x-y) < r\} \subseteq \mathscr{O},$$

with a similar definition for the topology induced by a quasi-F-norm.

Theorem 6.2. *Let X be a vector space and $m\colon X \to [0,\infty]$ be a metrizing quasi-modular. Then X_m is a vector space and there exists a quasi-*F*-norm $\|\cdot\|_m\colon X_m \to [0,\infty)$ so that for every sequence $\{x_n\}_{n\in\mathbb{N}} \subseteq X_m$ one has $\|x_n\|_m \to 0$ if and only if $m(x_n) \to 0$. Hence, m is topologically equivalent to a quasi-*F*-norm on X_m and, ultimately, to an* F*-norm on X_m.*

Proof: Define $\|\cdot\|_m\colon X \to [0,\infty]$ by setting (with the convention that $\inf\emptyset := \infty$)

$$(6.5)\quad \|x\|_m := \inf\big\{|c| : c \in \mathbb{C}\setminus\{0\} \text{ satisfying } m(x/c) \leq |c|\big\}, \quad \forall\, x \in X.$$

First, we propose to show that

$$(6.6)\quad X_m = \big\{x \in X : \text{there exists } C \in (0,\infty) \text{ so that } \|Cx\|_m < \infty\big\}.$$

To prove the left-to-right inclusion in (6.6), fix an arbitrary $x \in X_m$. By definition there exists $C_1 \in (0,\infty)$ such that $m(C_1 x) < \infty$. We need $C_2 \in (0,\infty)$ such that $\|C_2 x\|_m < \infty$ or, equivalently, such that there exists $c \in \mathbb{C}\setminus\{0\}$ for which $m(C_2 x/c) \leq |c|$. However, this latter inequality is satisfied if we choose $c := m(C_1 x) + 1 \in [1,\infty) \subseteq \mathbb{C}\setminus\{0\}$ and $C_2 := C_1\big(m(C_1 x)+1\big) \in (0,\infty)$.

To establish the right-to-left inclusion in (6.6), fix some x belonging to the right side of (6.6). Unraveling definitions, we are guaranteed some $C_1 \in (0,\infty)$ and some $c \in \mathbb{C}\setminus\{0\}$ satisfying $m(C_1 x/c) \leq |c|$. Invoking (6.2) with $\lambda := C_1/c \in \mathbb{C}$ yields $m(C_1 x/|c|) \leq k_0 m(C_1 x/c) \leq k_0|c| < \infty$. Thus, $m(C_2 x) < \infty$ if $C_2 := C_1/|c| \in (0,\infty)$, which shows that $x \in X_m$. This concludes the proof of (6.6).

Second, regarding quasi-subhomogeneity, for each $x \in X$ and each scalar $\lambda \in \mathbb{C}$ satisfying $|\lambda| \leq 1$ we have

$$
\begin{aligned}
\|\lambda x\|_m &= \inf\big\{|c| \colon c \in \mathbb{C} \setminus \{0\} \text{ such that } m(\lambda x/c) \leq |c|\big\} \\
&= \inf\left\{|k_0 k_1 c| \colon c\in\mathbb{C} \setminus \{0\} \text{ such that } m\Big(\frac{\lambda}{k_0 k_1} x/c\Big) \leq |k_0 k_1 c|\right\} \qquad (6.7)\\
&\leq k_0 k_1 \inf\big\{|c| \colon c\in\mathbb{C} \setminus \{0\} \text{ such that } m(x/c) \leq |c|\big\} = k_0 k_1 \|x\|_m,
\end{aligned}
$$

where in going from the first line to the second we replaced $c \in \mathbb{C} \setminus \{0\}$ with $k_0 k_1 c \in \mathbb{C} \setminus \{0\}$, and the inequality comes from taking the infimum over a possibly smaller set, as seen with the help of (6.3). This establishes the quasi-subhomogeneity of $\|\cdot\|_m$ on X.

Third, to prove the finiteness of $\|\cdot\|_m$ on X_m we make use of (6.6). Specifically, given $x \in X_m$ this implies that there exists $C \in (0, \infty)$ such that $\|Cx\|_m < \infty$. If $C \in [1, \infty)$, we may invoke (6.7) with $\lambda := C^{-1}$ in order to write $\|x\|_m = \big\|C^{-1}(Cx)\big\|_m \leq k_0 k_1 \|Cx\|_m < \infty$, as wanted. If, on the other hand, $C \in (0, 1)$, we proceed as follows. The finiteness condition $\|Cx\|_m < \infty$ entails the existence of some $c_1 \in \mathbb{C} \setminus \{0\}$ such that $m(Cx/c_1) \leq |c_1|$. As such, taking $c := c_1/C \in \mathbb{C} \setminus \{0\}$ yields $m(x/c) = m(Cx/c_1) \leq |c_1| \leq |c|$ given that $C \in (0, 1)$. Hence, we have $\|x\|_m \leq |c| < \infty$ in this case as well. Let us note that, as a simple consequence of (6.6) and the finiteness of $\|\cdot\|_m$ on X_m we have the description

$$
X_m = \big\{x \in X : \|x\|_m < \infty\big\}. \qquad (6.8)
$$

Fourth, to show that $\|\cdot\|_m$ is nondegenerate on X, assume that $x \in X$ is such that $\|x\|_m = 0$. By definition, this implies the existence of a sequence $\{c_j\}_{j\in\mathbb{N}} \subseteq \mathbb{C} \setminus \{0\}$ such that $\lim_{j\to\infty} c_j = 0$ and $m(x/c_j) \leq |c_j|$ for each $j \in \mathbb{N}$. Given that by (6.3) when the number j is large we have $m(x) = m(c_j \cdot x/c_j) \leq k_0 k_1 m(x/c_j) \leq k_0 k_1 |c_j|$, we deduce that necessarily $m(x) = 0$. Hence, $x = 0$ since m is nondegenerate.

Fifth, for quasi-subadditivity of $\|\cdot\|_m$ on X_m, fix $x_1, x_2 \in X_m$ and $\varepsilon > 0$. By the definition of $\|\cdot\|_m$, it follows that there exist $c_1, c_2 \in \mathbb{C}\setminus\{0\}$ such that $|c_i| \leq \|x_i\|_m + \varepsilon$ and $m(x_i/c_i) \leq |c_i|$ for $i = 1, 2$. Abbreviate

$k := k_0 k_1 \in [1, \infty)$. Then by the third axiom of Definition 6.1 and (6.2),

$$
\begin{aligned}
m\left(\frac{k(x_1+x_2)}{k(|c_1|+|c_2|)}\right) &= m\left(\frac{x_1+x_2}{|c_1|+|c_2|}\right) \\
&= m\left(\frac{|c_1|}{|c_1|+|c_2|}\cdot\frac{x_1}{|c_1|}+\frac{|c_2|}{|c_1|+|c_2|}\cdot\frac{x_2}{|c_2|}\right) \\
&\leq k_1\left[m\Big(\frac{x_1}{|c_1|}\Big)+m\Big(\frac{x_2}{|c_2|}\Big)\right] \\
&\leq k_0 k_1\left[m\Big(\frac{x_1}{c_1}\Big)+m\Big(\frac{x_2}{c_2}\Big)\right] \leq k\big(|c_1|+|c_2|\big),
\end{aligned}
\tag{6.9}
$$

which forces $\|k(x_1+x_2)\|_m \leq k\big(|c_1|+|c_2|\big) \leq k\big(\|x_1\|_m + \|x_2\|_m + 2\varepsilon\big)$. As this holds for all $\varepsilon > 0$, we get $\|k(x_1+x_2)\|_m \leq k\big(\|x_1\|_m + \|x_2\|_m\big)$ for each $x_1, x_2 \in X_m$. To proceed, observe next from (6.6) that for each $x_1, x_2 \in X_m$ we have $x_1/k, x_2/k \in X_m$. Writing the last inequality for x_1/k, x_2/k in place of x_1, x_2 then shows that for each $x_1, x_2 \in X_m$ we have $\|x_1+x_2\|_m \leq k\big(\|x_1/k\|_m + \|x_2/k\|_m\big) \leq k^2\big(\|x_1\|_m + \|x_2\|_m\big)$, where the last inequality uses (6.3). This establishes the quasi-subadditivity of $\|\cdot\|_m$ on X_m.

Sixth, we shall show that X_m is a vector space. That X_m is stable under addition is a consequence of the quasi-subadditivity of $\|\cdot\|_m$ on X_m and (6.8). From (6.6) we also see that X_m is stable under multiplication by positive scalars, while from (6.7) and (6.8) it follows that X_m is stable under multiplication by scalars $\lambda \in \mathbb{C}$ satisfying $|\lambda| \leq 1$. Combining these, the desired conclusion readily follows.

Going further, pick a sequence $\{\lambda_n\}_{n\in\mathbb{N}} \subseteq \mathbb{C}$ convergent to zero, along with some $x \in X_m$. The goal is to prove that $\|\lambda_n x\|_m \to 0$ as $n \to \infty$. Since $x \in X_m$ there exists $c \in \mathbb{C} \setminus \{0\}$ such that $m(x/c) \leq |c|$. In particular, $m(x/c) < \infty$ and for each fixed $\varepsilon > 0$, axiom (4) in Definition 6.1 then gives $m\big(\lambda_n x/(\varepsilon c)\big) \to 0$ as $n \to \infty$. As such, $m\big(\lambda_n x/(\varepsilon c)\big) \leq |\varepsilon c|$ for n large, which goes to show that $\|\lambda_n x\|_m \leq \varepsilon|c|$ from a rank on, depending on ε. Ultimately, this shows that $\|\lambda_n x\|_m \to 0$ as $n \to \infty$, so the final property needed for $\|\cdot\|_m$ to be a quasi-F-norm on X_m holds.

There remains to prove that for any sequence $\{x_n\}_{n\in\mathbb{N}} \subseteq X_m$ one has $\|x_n\|_m \to 0$ if and only if $m(x_n) \to 0$. First suppose $\{x_n\}_{n\in\mathbb{N}}$ is such that $\|x_n\|_m$ converges to zero. By definition of $\|\cdot\|_m$, for each $n \in \mathbb{N}$ there exists $c_n \in \mathbb{C} \setminus \{0\}$ such that $m(x_n/c_n) \leq |c_n|$ and $c_n \to 0$ as $n \to \infty$. Making use of (6.3) we may write, for sufficiently large n (in particular, we need $|c_n| \leq 1$),

$$m(x_n) = m\left(c_n \frac{x_n}{c_n}\right) \leq k_0 k_1 m(x_n/c_n) \leq k_0 k_1 |c_n|, \tag{6.10}$$

implying $m(x_n) \to 0$ as $n \to \infty$.

Conversely, suppose $m(x_n) \to 0$ as $n \to \infty$ and fix $\varepsilon > 0$ arbitrary. As m is a metrizing quasi-modular it follows that $m(x_n/\varepsilon)$ also converges to 0. Thus, for sufficiently large n we have $m(x_n/\varepsilon) \leq \varepsilon$. This shows $\|x_n\|_m \to 0$ as $n \to \infty$. Lastly, that m is topologically equivalent to a genuine F-norm follows from every quasi-F-norm being topologically equivalent to an F-norm. This is a consequence of Theorem 4.1 used in an appropriate setting (see [**3**, Theorem 3.41, p. 136] for details). □

We now state a lemma which ties up with the definition of Rolewicz–Orlicz spaces. Given $(\mathscr{X}, \mathfrak{M}, \mu)$ a measure space, denote the collection of (equivalence classes of) $\mathfrak{M}$-measurable, complex-valued functions defined on $\mathscr{X}$ by $L(\mathscr{X}, \mathfrak{M}, \mu)$. We will be interested in functions obeying a slow growth condition; namely, $f\colon [0,\infty) \to [0,\infty)$ is said to satisfy the Δ_2-*condition* provided there exists $k \in (0,\infty)$ such that

$$f(2x) \leq k f(x), \quad \forall\, x \in [0,\infty). \tag{6.11}$$

Lemma 6.3. *Let $(\mathscr{X}, \mathfrak{M}, \mu)$ be a measure space and suppose the function N: $[0,\infty) \to [0,\infty)$ satisfies the Δ_2-condition, is continuous, nondecreasing, and vanishes only at zero. Define $m_N\colon L(\mathscr{X}, \mathfrak{M}, \mu) \to [0,\infty]$ by setting*

$$m_N(u) := \int_{\mathscr{X}} (N \circ |u|)\, d\mu \quad \textit{for each} \quad u \in L(\mathscr{X}, \mathfrak{M}, \mu). \tag{6.12}$$

Then m_N is a metrizing modular.

Proof: We must show m_N satisfies conditions (1)–(5) in Definition 6.1. Throughout, $u, v \in L(\mathscr{X}, \mathfrak{M}, \mu)$.

For (1), as the function N vanishes only at 0 and is nondecreasing, $\int_{\mathscr{X}} (N \circ |u|)\, d\mu = 0$ if and only if $|u| = 0$ μ-a.e., and so u vanishes μ-almost everywhere. Next, condition (2) from Definition 6.1 is obvious in the present context. To see (3), set $\mathscr{X}_0 := \big\{x \in \mathscr{X} : |u(x)| \geq |v(x)|\big\}$,

$\mathscr{X}_1 := \{x \in \mathscr{X} : |u(x)| < |v(x)|\}$, and fix $\lambda \in [0,1]$. Then

$$
\begin{aligned}
m_N\big(\lambda u + (1-\lambda)v\big) &= \int_{\mathscr{X}} \big(N \circ (|\lambda u + (1-\lambda)v|)\big)\, d\mu \\
&\le \int_{\mathscr{X}_0 \cup \mathscr{X}_1} \big(N \circ (\lambda|u| + (1-\lambda)|v|)\big)\, d\mu \\
&\le \int_{\mathscr{X}_0} \big(N \circ (\lambda|u| + (1-\lambda)|u|)\big)\, d\mu \\
&\quad + \int_{\mathscr{X}_1} \big(N \circ (\lambda|v| + (1-\lambda)|v|)\big)\, d\mu \\
&\le \int_{\mathscr{X}} (N \circ |u|)\, d\mu + \int_{\mathscr{X}} (N \circ |v|)\, d\mu \\
&= m_N(u) + m_N(v),
\end{aligned} \tag{6.13}
$$

as desired. For (4), suppose $\{\lambda_n\}_{n\in\mathbb{N}}$ is a sequence of scalars such that $\lambda_n \to 0$ as $n \to \infty$, and assume $m_N(u) < \infty$. Without loss of generality suppose $|\lambda_n| \le 1$ for $n \in \mathbb{N}$. Since N is nondecreasing we have

$$
m_N(\lambda_n u) = \int_{\mathscr{X}} (N \circ |\lambda_n u|)\, d\mu \le \int_{\mathscr{X}} (N \circ |u|)\, d\mu < \infty. \tag{6.14}
$$

Noting due to continuity of N that $N(|\lambda_n u(x)|) \to N(0) = 0$ for all $x \in \mathscr{X}$, we may invoke Lebesgue's dominated convergence theorem and conclude $m_N(\lambda_n u) \to 0$ as $n \to \infty$.

Finally, regarding (5), suppose $\{u_n\}_{n\in\mathbb{N}} \subset L(\mathscr{X}, \mathfrak{M}, \mu)$ is a sequence such that $m_N(u_n) \to 0$ as $n \to \infty$, and fix a scalar λ. Since N satisfies the Δ_2-condition and is nondecreasing, there exists some $k > 0$ so that, for all $n \in \mathbb{N}$, $N \circ |\lambda u_n| \le k(N \circ |u_n|)$ on $\mathscr{X}$, hence $m_N(\lambda u_n) \le k\, m_N(u_n)$, ultimately showing $m_N(\lambda u_n) \to 0$ as $n \to \infty$, and concluding the proof. □

We now define a topological space based on the above lemma. The reader is advised to recall (6.1).

Definition 6.4. Let $(\mathscr{X}, \mathfrak{M}, \mu)$ be a measure space and suppose $N\colon [0,\infty) \to [0,\infty)$ is continuous, nondecreasing, vanishes only at zero, and satisfies the Δ_2-condition. With m_N as in (6.12) and X abbreviating $L(\mathscr{X}, \mathfrak{M}, \mu)$, call the pair (X_{m_N}, τ_{m_N}) a *Rolewicz–Orlicz space*, written $L_N^R(\mathscr{X}, \mathfrak{M}, \mu)$.

At its heart, a Rolewicz–Orlicz space is a topological space. However, one could alternatively, and quite naturally, define Rolewicz–Orlicz spaces to be F-normed spaces instead, using Theorem 6.2. Indeed, the F-norm which is topologically equivalent to m_N used in showing τ_{m_N} is given by an F-norm was displayed in (6.5). In particular, (6.5) becomes, after replacing the generic m with m_N from (6.12),

$$\|u\|_{m_N} = \inf\left\{\lambda \in (0,\infty) : \int_{\mathscr{X}} \left(N \circ \left|\frac{u}{\lambda}\right|\right)\, d\mu \leq \lambda\right\}. \tag{6.15}$$

Our next proposition elaborates on the relationship between certain types of F-norms and local p-convexity. For a quick definition, a topological vector space (X,τ) is said to be *locally p-convex*, for some $p \in (0,\infty)$, if it has a fundamental system of neighborhoods of the origin $\{U_n\}_{n\in\mathbb{N}}$ satisfying $c(U_n) \leq 2^{1/p}$ for all $n \in \mathbb{N}$. Also, given a vector space X and $p \in (0,\infty)$, call $\|\cdot\|\colon X \to [0,\infty]$ a *lower p-homogeneous norm* if $\|\cdot\|$ satisfies the usual nondegeneracy and triangle inequality axioms but rather than homogeneity we have $\|\lambda x\| = |\lambda|^p\|x\|$ for all $x \in X$ and $\lambda \in \mathbb{C}$ such that $|\lambda| \leq 1$. Instead, call $\|\cdot\|$ a *p-homogeneous norm* if the above condition holds for all $\lambda \in \mathbb{C}$.

Proposition 6.5. *Suppose (X,τ) is a topological vector space and let $p \in (0,\infty)$ be such that there exists a lower p-homogeneous norm $\|\cdot\|\colon X \to [0,\infty]$ for which $\tau = \tau_{\|\cdot\|}$. Then if $B_n := \{x \in X : \|x\| < 1/n\}$ for $n \in \mathbb{N}$, it follows that $\{B_n\}_{n\in\mathbb{N}}$ is a fundamental system of neighborhoods of the origin, with the property that B_n is open, balanced, topologically bounded, and $c(B_n) \leq 2^{1/p}$, for each $n \in \mathbb{N}$. In particular, (X,τ) is locally p-convex.*

Proof: That each B_n is open and $\{B_n\}_{n\in\mathbb{N}}$ is a fundamental system of neighborhoods of 0 follows from Remark 4.4 upon viewing $\|\cdot\|$ as a quasi-metric on X via $\rho\colon X \times X \to [0,\infty]$, $\rho(x,y) := \|x-y\|$ for all $x,y \in X$. In a similar manner, Lemma 5.4 implies that each B_n is topologically bounded and balanced. Regarding their moduli of concavity, fix $n \in \mathbb{N}$ and $x,y \in B_n$, and so by definition $\|x\|,\|y\| < 1/n$. As $2^{-1/p} \in (0,1)$, a scalar for which we may use lower p-homogeneity of $\|\cdot\|$, estimate

$$\left\|2^{-1/p}(x+y)\right\| \leq \tfrac{1}{2}\|x+y\| \leq \tfrac{1}{2}\big(\|x\|+\|y\|\big) < 1/n. \tag{6.16}$$

Hence $x+y \in 2^{1/p}B_n$ and, as $c(B_n)$ is the infimum over such constants, it follows that $c(B_n) \leq 2^{1/p}$. As this happens for all $n \in \mathbb{N}$, (X,τ) is locally p-convex. □

Remark 6.6. Regarding the converse direction in Proposition 6.5, we wish to note that if (X,τ) is a topological vector space that is locally p-convex for some $p \in (0,\infty)$ then there exists a p-homogeneous norm $\|\cdot\|\colon X \to [0,\infty)$ with the property that $\tau = \tau_{\|\cdot\|}$. This follows from a version of the Aoki–Rolewicz theorem which may be found in [**3**, Theorem 1.4, p. 5].

Call a measure space $(X, \mathfrak{M}, \mu)$ *separable* provided there exists $\{A_n\}_{n\in\mathbb{N}} \subseteq \mathfrak{M}$ such that for any set $B \in \mathfrak{M}$ of finite measure and for any $\varepsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that $\mu(B \setminus A_{n_0}) + \mu(A_{n_0} \setminus B) < \varepsilon$. In this case call μ *separable*. And now we record a couple of results from [**4**] regarding Rolewicz–Orlicz spaces.

Proposition 6.7. *Let $(\mathscr{X}, \mathfrak{M}, \mu)$ be a measure space and fix a function $N\colon [0,\infty) \to [0,\infty)$ which is continuous, nondecreasing, satisfies the Δ_2-condition, and which vanishes only at zero. In this setting, consider the Rolewicz–Orlicz space $L_N^R(\mathscr{X}, \mathfrak{M}, \mu)$. Also define $n\colon (0,\infty) \to [0,1]$ by*

$$n(x) := \inf\left\{\lambda > 0 : N(\lambda x) \geq \frac{N(x)}{2}\right\}, \quad \forall\, x \in (0,\infty). \tag{6.17}$$

Then the following are valid:

(1) *$L_N^R(\mathscr{X}, \mathfrak{M}, \mu)$ is complete;*
(2) *$L_N^R(\mathscr{X}, \mathfrak{M}, \mu)$ is separable if and only if μ is separable;*
(3) *if $\inf\limits_{0<x<\infty} n(x) > 0$ then $L_N^R(\mathscr{X}, \mathfrak{M}, \mu)$ is locally bounded.*

Proof: These are, respectively, Propositions 1.5.1 and 1.6.4, and Theorem 3.3.1 of [**4**]. □

Next are two important results from [**4**]. Given $f, g\colon [0,\infty) \to [0,\infty)$, write $f \overset{\infty}{\approx} g$ if $f \approx g$ for sufficiently large inputs. Also, call a measure space *purely atomic* if every measurable set contains an *atom*, where an atom is a set which has positive measure but no proper subset of it has positive measure.

Proposition 6.8. (1) *Let $(\mathscr{X}, \mathfrak{M}, \mu)$ be a measure space such that μ is not purely atomic and fix $p \in (0,\infty)$. Further let $N\colon [0,\infty) \to [0,\infty)$ be a continuous, nondecreasing function satisfying the Δ_2-condition and which vanishes only at zero. If*

$$\liminf_{x\to\infty} \frac{N(x)}{x^p} = 0 \tag{6.18}$$

then the Rolewicz–Orlicz space $L_N^R(\mathscr{X}, \mathfrak{M}, \mu)$ is not locally p-convex.

As a corollary, in this context the topology of $L_N^R(\mathscr{X}, \mathfrak{M}, \mu)$ cannot be given by a p-homogeneous norm.

(2) *Let $(\mathscr{X}, \mathfrak{M}, \mu)$ be a finite measure space. Fix $p \in (0,1]$ and let $N\colon [0,\infty) \to [0,\infty)$ be a continuous, nondecreasing function which vanishes only at zero, and satisfies the Δ_2-condition.*

If there exists a convex function $M\colon [0,\infty) \to [0,\infty)$ such that $M(x^p) \overset{\infty}{\approx} N(x)$ then the topology of the Rolewicz–Orlicz space $L^R_N(\mathscr{X},\mathfrak{M},\mu)$ is given by a p-homogeneous norm.

As a corollary, in this setting $L^R_N(\mathscr{X},\mathfrak{M},\mu)$ is locally p-convex.

Proof: The above two main claims are Theorem 3.4.4 and Proposition 3.4.2 of [**4**], respectively, while the corollary statements follow from Proposition 6.5. □

We wish to point out here that, while (6.15) illustrates that every Rolewicz–Orlicz space may be thought of as an F-normed space, the second part of Proposition 6.8 takes this a step further. In particular, it gives a sufficient condition for Rolewicz–Orlicz spaces to be thought of as p-homogeneous normed spaces for $p \in (0,1]$. The latter is a stronger condition than the former, as every p-homogeneous norm is an F-norm but not every F-norm is a p-homogeneous norm for some $p \in (0,\infty)$, let alone $p \in (0,1]$.

7. Construction of the key quasi-metric structure space

Here we construct our desired Rolewicz–Orlicz space. To set the stage, first we prove a lemma elaborating on the properties of a key function in the construction of this example.

Lemma 7.1. *Fix a number $p_0 \in (0,\infty)$ and consider the function $N\colon [0,\infty) \to [0,\infty)$ defined by*

$$N(x) := \frac{x^{p_0}}{\ln(x^{p_0}+e)}, \quad \forall\, x \in [0,\infty). \tag{7.1}$$

Then N enjoys the following properties:

1. *N vanishes only at zero and $\lim_{x\to\infty} N(x) = \infty$;*
2. *$N \in C^0\big([0,\infty)\big) \cap C^\infty\big((0,\infty)\big)$;*
3. *N is strictly increasing on $[0,\infty)$;*
4. *N satisfies the Δ_2-condition with constant 2^{p_0};*
5. *for $p \in (0,\infty)$ one has $\lim\limits_{x\to\infty} N(x)/x^p = 0 \iff p \geq p_0$;*
6. *for $q \in (0,\infty)$, the function $N(x^q)$ is convex for large x (dependent upon q) $\iff q > 1/p_0$;*
7. *if $n\colon (0,\infty) \to [0,1]$ is associated with the current N as in (6.17) then $\inf\limits_{0<x<\infty} n(x) \geq 4^{-1/p_0}$.*

Proof: (1) and (2) are clear by design. For (3), the first derivative of N is

$$N'(x) = \frac{p_0\, x^{p_0-1}\big((x^{p_0}+e)\ln(x^{p_0}+e) - x^{p_0}\big)}{(x^{p_0}+e)\big(\ln(x^{p_0}+e)\big)^2}, \quad \forall\, x \in (0,\infty). \tag{7.2}$$

Since $\ln(x^{p_0}+e) > 1$ for $x > 0$ we have $(x^{p_0}+e)\ln(x^{p_0}+e) - x^{p_0} > e > 0$ for $x > 0$, ultimately showing N is strictly increasing on $[0,\infty)$. Regarding (4), observe that for each $x \in [0,\infty)$,

$$\begin{aligned} N(2x) \leq 2^{p_0} N(x) &\iff \frac{(2x)^{p_0}}{\ln((2x)^{p_0}+e)} \leq 2^{p_0} \frac{x^{p_0}}{\ln(x^{p_0}+e)} \\ &\iff \ln(x^{p_0}+e) \leq \ln\big((2x)^{p_0}+e\big), \end{aligned} \tag{7.3}$$

with the last inequality obviously true. Moving on, as for (5), fix a number $p \in (0,\infty)$. Then

$$\lim_{x\to\infty} \frac{N(x)}{x^p} = \lim_{x\to\infty} \frac{x^{p_0-p}}{\ln(x^{p_0}+e)}, \tag{7.4}$$

and if $p_0 - p \leq 0$ then the limit is 0 and otherwise the limit is ∞. For (6), fix $q \in (0,\infty)$ and, to ease the exposition, define $f(x) := N(x^q)$ and relabel $a := qp_0 \in (0,\infty)$. Then

$$f(x) := \frac{x^a}{\ln(x^a+e)}, \quad \forall\, x \in (0,\infty). \tag{7.5}$$

After a somewhat lengthy but straightforward differentiation, we see that the second derivative of f is

$$\begin{aligned} f''(x) = a x^{a-2}\big(2a x^{2a} &+ (a-1)(x^a+e)^2(\ln(x^a+e))^2 \\ &- x^a(3ae + (2a-1)x^a - e)\ln(x^a+e)\big)\big/ \\ &\qquad \big((x^a+e)^2(\ln(x^a+e))^3\big). \end{aligned} \tag{7.6}$$

Note that $ax^{a-2} > 0$ since $a, x \in (0,\infty)$. If $a \in (1,\infty)$ (which is equivalent to $q > 1/p_0$) we wish to show that, for x large,

$$\begin{aligned} 2a x^{2a} &+ (a-1)(x^a+e)^2\big(\ln(x^a+e)\big)^2 \\ &> x^a\big(3ae + (2a-1)x^a - e\big)\ln(x^a+e). \end{aligned} \tag{7.7}$$

For large x, upon expanding, the dominant contributor on the left side will be $(a-1)x^{2a}\big(\ln(x^a+e)\big)^2$, while the dominant term on the right side is $(2a-1)x^{2a}\ln(x^a+e)$. Since $a > 1$ and $\ln(x^a+e)$ increases without bound, it follows that (7.7) is valid for large x. Hence, $f''(x) > 0$ for x large in this case.

If however $a \in (0,1]$ (which is the same as saying $q \leq 1/p_0$) then a similar analysis based on the behavior of the dominant terms leads to the conclusion that $f''(x)$ is negative for large x.

Finally, to establish (7), for each fixed number $x \in (0,\infty)$ introduce $I(x) := \{\lambda > 0 : N(\lambda x) \geq N(x)/2\}$. Since N is nondecreasing and nonnegative, $I(x)$ is either $\big(n(x),\infty\big)$ or $\big[n(x),\infty\big)$. We claim that

$$4^{-1/p_0} \notin I(x) \quad \text{for every} \quad x \in (0,\infty). \tag{7.8}$$

Indeed, if $x \in (0,\infty)$ is such that $4^{-1/p_0} \in I(x)$ then we necessarily have $N\big(4^{-1/p_0}x\big) \geq N(x)/2$, which is further equivalent to

$$\begin{aligned}\frac{\big(4^{-1/p_0}x\big)^{p_0}}{\ln\big((4^{-1/p_0}x)^{p_0}+e\big)} &\geq \frac{1}{2}\frac{x^{p_0}}{\ln(x^{p_0}+e)} \\ &\iff \frac{1}{2}\ln(x^{p_0}+e) \geq \ln\big(x^{p_0}/4+e\big).\end{aligned} \tag{7.9}$$

Introducing $t := x^{p_0} \in (0,\infty)$ this further becomes equivalent to

$$\begin{aligned}\ln\sqrt{t+e} \geq \ln\big(t/4+e\big) &\iff t+e \geq \big(t/4+e\big)^2 \\ &\iff t^2/16+(e/2-1)t+(e^2-e) \leq 0,\end{aligned} \tag{7.10}$$

with the last inequality an impossibility since the discriminant of the above quadratic expression is $\Delta = (e/2-1)^2-(e^2-e)/4 = 1-3e/4 < 0$. This concludes the proof of (7.8). In turn, from (7.8) and the format of $I(x)$ we deduce that necessarily $n(x) \geq 4^{-1/p_0}$ for each $x \in (0,\infty)$, as wanted. □

And here is the key Rolewicz–Orlicz space.

Theorem 7.2. *For each $p_0 \in (0,1]$ there exists a complete, separable, locally bounded, topological vector space (X,τ) whose topology τ can be given by a p-homogeneous norm for $p \in (0,p_0)$ (hence (X,τ) is locally p-convex for such p) but is not locally p_0-convex (hence τ cannot be given by a p_0-homogeneous norm).*

Proof: The idea is to construct a Rolewicz–Orlicz space and then invoke the two parts of Proposition 6.8.

Fix $p_0 \in (0,1]$ and let $(\mathscr{X},\mathfrak{M},\mu)$ be a measure space so that μ is separable, not purely atomic, and $\mu(\mathscr{X}) < \infty$. For example, $[0,1]$ equipped with the Lebesgue measurable sets and the Lebesgue measure suffices. Take $N\colon [0,\infty) \to [0,\infty)$ to be the function from Lemma 7.1. Then, in light of (1)–(4) of Lemma 7.1, N satisfies all the properties needed to construct the Rolewicz–Orlicz space $L^R_N(\mathscr{X},\mathfrak{M},\mu)$ as in Definition 6.4. By part (7) in Lemma 7.1 and part (3) of Proposition 6.7 it follows that

$L^R_N(\mathscr{X}, \mathfrak{M}, \mu)$ is locally bounded. Parts (1) and (2) of Proposition 6.7 also give that $L^R_N(\mathscr{X}, \mathfrak{M}, \mu)$ is complete and separable.

Next we show that for every $p \in (0, p_0)$ the topology of $L^R_N(\mathscr{X}, \mathfrak{M}, \mu)$ can be given by a p-homogeneous norm. In hopes of calling upon (2) of Proposition 6.8, we need that for every $p \in (0, p_0)$ there exists a convex function $M_p\colon [0, \infty) \to [0, \infty)$ such that $M_p(x^p) \overset{\infty}{\approx} N(x)$. First, $M_p(x^p) \overset{\infty}{\approx} N(x)$ is equivalent to $M_p(x) \overset{\infty}{\approx} N(x^{1/p})$. That is, calling $q := 1/p$, we wish to show for every $q \in (1/p_0, \infty)$ there exists a convex function that is pointwise equivalent with $N(x^q)$ for large inputs. Fix such a q. Indeed, something stronger happens, namely by (6) of Lemma 7.1 the function that $N(x^q)$ itself is convex for large x dependent upon q. Bearing this in mind, fix $x_0 \geq 0$ so that $N(x^q)$ is convex for $x^q \geq x_0$, then define $M_p(x) := N(x^q)$ for $x \geq x_0$, and for $x \in [0, x_0]$ take $M_p(x)$ to be the linear function whose graph is the line segment which connects the origin to $(x_0, N(x_0^q))$. Invoking part (2) of Proposition 6.8 we then conclude the topology of $L^R_N(\mathscr{X}, \mathfrak{M}, \mu)$ can be given by a p-homogeneous norm, as desired.

Finally, property (5) in Lemma 7.1 permits us to invoke part (1) of Proposition 6.8, whose conclusion is that the Rolewicz–Orlicz space $L^R_N(X, \mathfrak{M}, \mu)$ is not locally p_0-convex. In concert with Proposition 6.5 this further shows that the topology of $L^R_N(\mathscr{X}, \mathfrak{M}, \mu)$ cannot be given by a p_0-homogeneous norm. □

It should be noted that Proposition 3.4.8 on p. 115 of [**4**] is similar to the above Theorem 7.2, its beginning stating "There is a locally bounded space X such that the topology in X can be determined by a p-homogeneous norm for p, $0 < p < p_0$ but cannot be determined by a p_0-homogeneous norm", but a closer inspection shows that the argument sketched there simply does not work. More specifically, S. Rolewicz claims for any $h\colon [0, \infty) \to [0, \infty)$ which is positive, decreasing, continuous, convex, and so that $h(0) < p_0$ and $h(x) \to 0$ as $x \to \infty$, the function $N\colon [0, \infty) \to [0, \infty)$ given by

$$N(x) := x^{p_0 - h(x)}, \quad \forall\, x \in [0, \infty), \tag{7.11}$$

works in lieu of our N in Theorem 7.2, at least for the purposes of invoking both parts of Proposition 6.8. However, for example, the function $h(x) := p_0/(x + 2)$, $x \in [0, \infty)$, satisfies the aforementioned properties yet (6.18) does not hold for N as in (7.11) when $p = p_0$ (as the limit inferior in question is 1 when $p = p_0$).

Corollary 7.3. *For every $p_0 \in (0, 1]$ there exists a locally bounded, complete, separable topological vector space (X, τ) so that* $\mathrm{ind}_R(X, \tau) = 2^{1/p_0}$

but for every open, topologically bounded, balanced set $U \subseteq X$ there holds $c(U) > 2^{1/p_0}$. In particular, the infimum defining $\mathrm{ind}_R(X,\tau)$ *(as in* (5.2)*) is not attained.*

Proof: Fix $p_0 \in (0,1]$ and consider the Rolewicz–Orlicz space $L_N^R(\mathscr{X},\mathfrak{M},\mu)$ from the proof of Theorem 7.2, which we denote (X,τ). Pick $p \in (0,p_0)$, and suppose $\|\cdot\|_p$ is the p-homogeneous norm such that $\tau = \tau_{\|\cdot\|_p}$ guaranteed by Theorem 7.2. From Proposition 6.5 (with r in place of $1/n$) it follows that, for any $r > 0$, $B_{\|\cdot\|_p}(0,r) := \{x \in X : \|x\|_p < r\}$ is open, topologically bounded, balanced, and $c\big(B_{\|\cdot\|_p}(0,r)\big) \leq 2^{1/p}$.

Since, by Theorem 7.2, the space (X,τ) is not locally p_0-convex, by definition there is no fundamental system of neighborhoods of the origin $\{U_n\}_{n\in\mathbb{N}}$ satisfying $c(U_n) \leq 2^{1/p_0}$ for each $n \in \mathbb{N}$. In fact, something stronger happens, namely every open, topologically bounded, balanced set U satisfies $c(U) > 2^{1/p_0}$. For otherwise, suppose U is open, topologically bounded, balanced, and such that $c(U) \leq 2^{1/p_0}$. Then defining $U_n := n^{-1}U$ for all $n \in \mathbb{N}$, it follows that $c(U_n) = c(U) \leq 2^{1/p_0}$ for all natural numbers n, and $\{U_n\}_{n\in\mathbb{N}}$ is a fundamental system of neighborhoods. That $c(U_n) = c(U)$ was noted after Definition 5.1, while the second property may be justified as follows. Fix an arbitrary neighborhood $V \subseteq X$ of the origin. Since U is topologically bounded, there exists $\lambda \in (0,\infty)$ so that $U \subseteq \lambda V$. Choose $n \in \mathbb{N}$ so that $n \geq \lambda$. Since U is balanced so is $U_n = n^{-1}U$, thus $\lambda U_n \subseteq nU_n = U \subseteq \lambda V$, or $U_n \subseteq V$, as desired.

Combining the above two paragraphs we see that the infimum in the definition of $\mathrm{ind}_R(X,\tau)$ is $2^{1/p_0}$, and further that it is not attained. $\square$

Lastly, we are now ready to present the

Proof of Theorem 3.1: Pick $p_0 \in (0,1]$ and consider the Rolewicz–Orlicz space $L_N^R(\mathscr{X},\mathfrak{M},\mu)$ as in the proofs of Theorem 7.2 and Corollary 7.3, which we will denote (X,τ). Then (X,τ) is a topological vector space which is locally bounded, complete, separable, and such that the infimum defining $\mathrm{ind}_R(X,\tau)$ is not attained. Fix some $p \in (0,p_0)$. By Theorem 7.2 there exists a p-homogeneous norm $\|\cdot\|_p$ on X such that $\tau = \tau_{\|\cdot\|_p}$. We then define $\rho\colon X \times X \to [0,\infty)$ by $\rho(x,y) := \|x-y\|_p$ for all $x,y \in X$. By design, (X,ρ) is a quasi-metric linear space so that ρ is translation invariant and p-homogeneous. Granted these, we are in a position to apply Theorem 5.6, from which, labeling $\mathbf{q} := [\rho]$ (implying $\tau = \tau_{\mathbf{q}}$), we conclude that the infimum defining $C_{\mathbf{q}}$ is not attained. $\square$

In closing, we note that while the infimum in $C_{\mathbf{q}}$ is not necessarily attained, a truncated version of it is, granted proper constraints. To make this precise, given a quasi-metric space (X, ρ), for $\lambda \in [1, \infty)$ define

$$[\rho]_\lambda := \big\{\rho' \colon X \times X \to [0, \infty] : \lambda^{-1}\rho \leq \rho' \leq \lambda\rho \text{ on } X \times X\big\}. \tag{7.12}$$

The following result has been established in [**3**, Theorem 4.64, p. 219], using Ascoli's compactness theorem and the metrization result presented in Theorem 4.1.

Theorem 7.4. *Suppose that (X, ρ) is a quasi-metric space with the property that (X, τ_ρ) is a separable topological space. In this setting, define $\alpha := (\log_2 C_\rho)^{-1} \in (0, \infty]$ and consider ρ_α as in Theorem 4.1. Then*

$$\inf\big\{C_{\rho'} : \rho' \in [\rho_\alpha]_\lambda\big\} \text{ is attained, for every } \lambda \in [1, \infty). \tag{7.13}$$

In other words, as long as ρ induces a separable topology and the focus is on quasi-metrics which do not differ too drastically from ρ_α (i.e., stay within prescribed multiplicative bounds), then we are guaranteed a best (in the sense of most ultrametric-like) pointwise equivalent quasi-metric, in the same class.

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