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INTEGRAL RESTRICTION FOR BILINEAR OPERATORS

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Abstract: We consider the integral domain restriction operator T_Ω for certain bilinear operator T. We obtain that if (s,p_1,p_2) satisfies $\frac{1}{p_1}+\frac{1}{p_2}\geq\frac{2}{\min\{1,s\}}$ and $\|T\|_{L^{p_1}\times L^{p_2}\to L^s}<\infty$, then $\|T_\Omega\|_{L^{p_1}\times L^{p_2}\to L^s}<\infty$. For some special domain Ω, this property holds for triplets (s,p_1,p_2) satisfying $\frac{1}{p_1}+\frac{1}{p_2}>\frac{1}{\min\{1,s\}}$.

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1. Introduction

Let T be a bilinear operator defined by

$$T(f_1,f_2)(x)=\int_{\mathbb{R}^2}K(x,y_1,y_2)f_1(y_1)f_2(y_2)\,dy_1\,dy_2. \tag{1.1}$$

We consider its integral domain restriction operator (IDRO)

$$T_\Omega(f_1,f_2)(x)=\int_{\Omega}K(x,y_1,y_2)f_1(y_1)f_2(y_2)\,dy_1\,dy_2, \tag{1.2}$$

where Ω is an open set in $\mathbb{R}^2$. We will show that T_Ω inherits the $L^{p_1}\times L^{p_2}\to L^s$ boundedness from the operator T, if (s,p_1,p_2) satisfies $\frac{1}{p_1}+\frac{1}{p_2}\geq\frac{2}{\min\{1,s\}}$. For special domains Ω, we can extend the range of (s,p_1,p_2) to be $\frac{1}{p_1}+\frac{1}{p_2}>\frac{1}{\min\{1,s\}}$.

In order to study the stability of absolutely continuous spectrum for certain one-dimensional Schrödinger operator, in [**1**] Christ and Kiselev considered linear operators

$$(K_if)(\lambda)=\int_{\mathbb{R}^+}k_i(\lambda,x)f(x)\,dx,\quad i=1,2,\dots,n, \tag{1.3}$$

where the functions $k_i(\lambda, x)$, $i = 1, 2 \ldots, n$ are defined on $I \times \mathbb{R}^+$ and I is a measurable set of $\mathbb{R}$, and the multilinear operator

$$(1.4)\quad T_n(f_1, \ldots, f_n)(\lambda) = \int_{\mathbb{R}^n} \prod_{j=1}^{n} f_j(x_j) k_j(x_j, \lambda) \prod_{\alpha \in A} \chi_{\mathbb{R}^+}(x_{i_\alpha} - x_{i'_\alpha})\, dx,$$

where A is any set of ordered pairs $\alpha = (i_\alpha, i'_\alpha)$, with $1 \le i_\alpha, i'_\alpha \le n$. Christ and Kiselev proved that if, for some $p \in [1, 2)$ and $q > p$

$$(1.5)\qquad \|K_i f\|_{L^q(I, d\lambda)} \le C_i \|f\|_{L^p(\mathbb{R}^+, dx)}, \quad i = 1, 2, \ldots, n,$$

for all functions $f \in L^p(\mathbb{R}^+)$ with compact supports, then

$$(1.6)\qquad \|T_n(f_1, f_2, \ldots, f_n)\|_{L^{s_n}} \le C_n \prod_{i=1}^{n} \|f_i\|_{L^p},$$

for all $f_i \in L^p(\mathbb{R}^+, dx)$ with $s_n^{-1} = nq^{-1}$. Especially, when $n = 2$ and $\mathcal{A} = \{(1, 2)\}$, we see

$$(1.7)\qquad T_2(f_1, f_2)(\lambda) = \int_{x_1 \ge x_2} f_1(x_1) f_2(x_2) k_1(x_1, \lambda) k_2(x_2, \lambda)\, dx.$$

Then Christ–Kiselev's result says that if for some $1 \le p < 2$ and $q > p$

$$(1.8)\qquad \|K_i f\|_{L^q} \le C_i \|f\|_{L^p}, \quad i = 1, 2,$$

then

$$(1.9)\qquad \|T_2(f_1, f_2)\|_{L^{q/2}(\mathbb{R})} \le C \|f_1\|_{L^p(\mathbb{R})} \|f_2\|_{L^p(\mathbb{R})},$$

where C depends on the constants C_1 and C_2.

A natural question raised is that, in the above restriction domain inequality, can we replace the domain $\{(x_1, x_2),\, x_1 \ge x_2\}$ by any measurable set? More precisely, for a bilinear operator

$$(1.10)\qquad T(f_1, f_2)(x) = \int_{\mathbb{R}^2} K(x, y_1, y_2) f(y_1) f(y_2)\, dy_1\, dy_2$$

we will study its IDRO

$$(1.11)\qquad T_\Omega(f_1, f_2)(x) = \int_{\Omega} K(x, y_1, y_2) f(y_1) f(y_2)\, dy_1\, dy_2$$

for any measurable set $\Omega \subset \mathbb{R}^2$. Unlike the linear operator

$$(1.12)\qquad L(f)(x) = \int_{\mathbb{R}^2} K(x, y_1, y_2) f(y_1, y_2)\, dy_1\, dy_2,$$

$f(y_1, y_2) = f_1(y_1) f_2(y_2)$ is separable in the bilinear operator $T(f_1, f_2)$. This definition allows that T_Ω can inherit the $L^{p_1} \times L^{p_2} \to L^s$ boundedness from the operator T for some triplets (s, p_1, p_2).

We have the following theorem.

Theorem 1.1. *Let T be the bilinear operator with kernel $K(x, y_1, y_2)$, that satisfies*

$$\|T(f_1, f_2)\|_{L^s(\mathbb{R})} \le C_1 \|f_1\|_{L^{p_1}(\mathbb{R})} \|f_2\|_{L^{p_2}(\mathbb{R})}, \tag{1.13}$$

for some constant C_1 and both $f_i \in L^{p_i}$, $i = 1, 2$, where (s, p_1, p_2) satisfies $\frac{1}{p_1} + \frac{1}{p_2} \ge \frac{2}{\min\{1,s\}}$ and $0 < s, p_1, p_2 < \infty$. Then for any open subset $\Omega \subseteq \mathbb{R}^2$ and

$$T_\Omega(f_1, f_2)(x) = \int_\Omega K(x, y_1, y_2) f_1(y_1) f_2(y_2)\, dy_1\, dy_2, \tag{1.14}$$

we have

$$\|T_\Omega(f_1, f_2)\|_{L^s(\mathbb{R})} \le C \|f_1\|_{L^{p_1}(\mathbb{R})} \|f_2\|_{L^{p_2}(\mathbb{R})}, \tag{1.15}$$

where C only depends on C_1.

For a special domain, (1.15) is also true for (s, p_1, p_2) which satisfies $\frac{1}{p_1} + \frac{1}{p_2} > \frac{1}{\min\{1,s\}}$ and $0 < s, p_1, p_2 < \infty$.

Let Ψ be a family of sets Ω_x, $x \in \mathbb{R}$ which satisfies the following conditions:

$$\emptyset \in \Psi \quad \text{and} \quad \mathbb{R} \in \Psi, \tag{1.16}$$

$$\text{for any } x < y, \quad \Omega_x \subseteq \Omega_y. \tag{1.17}$$

We set $\Theta = \{\Psi : \Psi \text{ satisfies (1.16) and (1.17)}\}$, and set $\Sigma_1(\Psi) = \{(x, y) \in \mathbb{R}^2 : y \in \Omega_x^c,\, \Omega_x \in \Psi\}$, $\Sigma_2(\Psi) = \{(x, y) \in \mathbb{R}^2 : x \in \Omega_y^c,\, \Omega_x \in \Psi\}$. Let $\mathcal{A}$ be the algebra generated by some finite subset of $\cup_{\Psi\in\Theta}\{\Sigma_1(\Psi), \Sigma_2(\Psi) : \Psi = \{\Omega_x\}_{x\in\mathbb{R}}\}$, then for any $\Sigma \in \mathcal{A}$ there exists n such that Σ can be written as finite unions and complements of sets in

$$\big\{\Sigma_1(\Psi^1), \Sigma_2(\Psi^1), \ldots, \Sigma_1(\Psi^n), \Sigma_2(\Psi^n)\big\},$$

with $\Psi^k \in \Theta$ for $k = 1, 2, \ldots, n$.

For the bilinear operator T, we consider its restriction operator

$$T_\Sigma(f_1, f_2)(x) = \int_\Sigma K(x, y_1, y_2) f_1(y_1) f_2(y_2)\, dy_1\, dy_2. \tag{1.18}$$

Theorem 1.2. *Suppose T is the bilinear operator defined in Theorem 1.1 and satisfies (1.13) for (s, p_1, p_2) satisfying $\frac{1}{p_1} + \frac{1}{p_2} > \frac{1}{\min\{1,s\}}$, $0 < s, p_1, p_2 < \infty$. Then for $\Sigma \in \mathcal{A}$, there exists a constant $C > 0$ dependent only on the domain and C_1 in (1.15), for any function $f_i \in L^{p_i}(\mathbb{R})$, $i = 1, 2$,*

$$\|T_\Sigma(f_1, f_2))\|_{L^s} \le C \|f_1\|_{L^{p_1}} \|f_2\|_{L^{p_2}}. \tag{1.19}$$

We give some notations used in this paper. Let $\mathscr{S} := \mathscr{S}(\mathbb{R}^n)$ be the Schwartz space. If $\Omega \subset \mathbb{R}^n$, then $|\Omega|$ denotes the Lebesgue measure of Ω and $\#\Omega$ denotes the cardinality of Ω. For $\Omega_1, \Omega_2 \subset \mathbb{R}^n$, notation $\Omega_1 \subset_{\text{a.e.}} \Omega_2$ means that there exists a 0 measure set A such that $\Omega_1 \setminus A \subset \Omega_2$. We define $N(f) = \{x \in \mathbb{R}^n : f(x) \neq 0\}$ for any function f defined on $\mathbb{R}^n$.

This paper is organized as follows. In Section 2, we give the proof for Theorem 1.1, by using dividing integral domains into rectangles. In Section 3, we prove Theorem 1.2. And in Section 4, we extend the results to high dimension and give corresponding results on special domains.

2. Proof of Theorem 1.1

In this section we give the proof of Theorem 1.1 by using inductive argument to divide the integral domains into domains on which the integral is controlled.

Since T_Ω is bilinear, we may assume without loss of generality throughout the proof that $\|f_1\|_{L^{p_1}}^{p_1} = \|f_2\|_{L^{p_2}}^{p_2} = 1/2$ and $f_i \in \mathscr{S}(\mathbb{R})$, $i = 1, 2$. Let

$$f(x) = |f_1(x)|^{p_1} + |f_2(x)|^{p_2}.$$

Since

$$T_\Omega(f_1, f_2)(x) = \int_{\Omega \cap (N(f) \times N(f))} K(x, y_1, y_2) f_1(y_1) f_2(y_2)\, dy_1\, dy_2. \tag{2.1}$$

If $|\Omega \cap (N(f) \times N(f))| = 0$, then $T_\Omega(f_1, f_2) = 0$. So we should consider the the nontrivial case $|\Omega \cap (N(f) \times N(f))| \neq 0$ and $\Omega \neq \mathbb{R}^2$. To estimate $T_\Omega(f_1, f_2)$, we need to decompose the support of the function f into dyadic pieces and show that there exists a constant C, such that $\|T_\Omega(f_1, f_2)\|_{L^s} \leq C$.

We consider a partition of $\Omega \cap (N(f) \times N(f))$ into the dyadic pieces by the following inductive method. In this part, we denote $E(m, i, j) = E(m, i) \times E(m, j)$ for simplicity.

Dividing process. Denote $E(0, 1) = \mathbb{R}$.

Step 1: Find the smallest $x_{1,1} \in \overline{N(f)}$, such that

$$\|f\chi_{E(1,1)}\|_{L^1} = \|f\chi_{E(1,2)}\|_{L^1} = \frac{1}{2}. \tag{2.2}$$

Here $E(1, 1)$ and $E(1, 2)$ are defined as follows: If $x_{1,1}$ is an inner point of $N(f)$, then we define $E(1, 1) = (-\infty, x_{1,1})$ and $E(1, 2) = [x_{1,1}, \infty)$; if $x_{1,1}$ is at the boundary, we define $E(1, 1) = (-\infty, x_{1,1})$ and $E(1, 2) = (x_{1,1}, \infty)$. We remark here since we let $f_i \in \mathscr{S}(\mathbb{R})$, $i = 1, 2$, $x_{1,1}$ exists and is unique. So we divide $E(0, 1)$ into 2 intervals $E(1, 1)$, $E(1, 2)$.

Step m: For each $E(m-1,i)$, we divide it into 2 intervals $E(m,2i-1)$, $E(m,2i)$ such that

$$\|f\chi_{E(m,2i-1)}\|_{L^1} = \|f\chi_{E(m,2i)}\|_{L^1} = \frac{1}{2^m} \tag{2.3}$$

for $1 \le i \le 2^{m-1}$ respectively.

Selecting process.

Step 1: Denote

$$T_1 = \{E(1,i,j) : i,j = 1,2\}.$$

We drop the domains in T_1 which intersect $\Omega \cap (N(f) \times N(f))$ with 0 measure and let D_1 be the set of dropped domains in Step 1. So the *dropped* family in Step 1

$$D_1 = \big\{E(1,i,j) \in T_1 : |E(1,i,j) \cap \Omega \cap (N(f) \times N(f))| = 0,\quad i,j = 1,2\big\}. \tag{2.4}$$

Then in $T_1 \setminus D_1$ we select the domains which are contained a.e. in $\Omega \cap (N(f) \times N(f))$, and let S_1 be the set of selected domains in Step 1. So the *selected* family in Step 1

$$S_1 = \big\{E(1,i,j) \in T_1 \setminus D_1 : E(1,i,j) \subset_{\text{a.e.}} \Omega \cap (N(f) \times N(f)),\ i,j = 1,2\big\}.$$

Denote the remaining domains in Step 1 by R_1, and we have $R_1 = T_1 \setminus (S_1 \cup D_1)$.

Step m: Denote

$$T_m = \big\{E(m,2i-1,2j-1), E(m,2i,2j-1), E(m,2i-1,2j), E(m,2i,2j) : \text{for any } i,j = 1,2,\dots,2^{m-1},\ \text{such that } E(m-1,i,j) \in R_{m-1}\big\}.$$

Also we drop the domains in T_m which intersect $\Omega \cap (N(f) \times N(f))$ with 0 measure and let D_m be the set of dropped domains in Step m. So the *dropped* family in Step m is

$$D_m = \big\{E(m,i,j) \in T_m : |E(m,i,j) \cap (\Omega \cap (N(f) \times N(f)))| = 0,\quad i,j = 1,2,3,\dots,2^m\big\}. \tag{2.5}$$

Then in $T_m \setminus D_m$ we select the domains which are almost everywhere contained in $\Omega \cap (N(f) \times N(f))$, and let S_m be the set of selected domains in Step m. So the *selected* family in Step m is

$$S_m = \big\{E(m,i,j) \in T_m \setminus D_m : E(m,i,j) \subset_{\text{a.e.}} \Omega \cap (N(f) \times N(f)),\quad i,j = 1,2,3,\dots,2^m\big\}.$$

Denote the remaining domains in Step m by R_m, we have $R_m = T_m \setminus (S_m \cup D_m)$.

Next, we show that there is a 0 measure set $A \subset \cup_{m=1}^{\infty} \cup_{E(m,i,j)\in S_m} E(m,i,j)$, such that

$$\Omega \cap (N(f) \times N(f)) = \bigcup_{m=1}^{\infty} \bigcup_{E(m,i,j)\in S_m} E(m,i,j) \setminus A$$

and

$$A \cap \Omega \cap (N(f) \times N(f)) = \emptyset.$$

According to our selecting process, there is a 0 measure set $A(m,i,j) \subset E(m,i,j)$ and $A(m,i,j) \cap \Omega \cap (N(f) \times N(f)) = \emptyset$ such that $E(m,i,j) \setminus A(m,i,j) \subset \Omega \cap (N(f) \times N(f))$. We take $A = \cup_{m=1}^{\infty} \cup_{E(m,i,j)\in S_m} A(m,i,j)$, then we have

$$\Omega \cap (N(f) \times N(f)) \supset \bigcup_{m=1}^{\infty} \bigcup_{E(m,i,j)\in S_m} E(m,i,j) \setminus A$$

and

$$A \cap \Omega \cap (N(f) \times N(f)) = \emptyset.$$

Now we show $\Omega \cap (N(f) \times N(f)) \subset \cup_{m=1}^{\infty} \cup_{E(m,i,j)\in S_m} E(m,i,j) \setminus A$. Since $N(f)$ is open, we have $\Omega \cap (N(f) \times N(f))$ is an open set. For all $x = (x_1, x_2) \in \Omega \cap (N(f) \times N(f))$, there exists a $\delta > 0$, such that

$$(x_1 - \delta, x_1 + \delta) \times (x_2 - \delta, x_2 + \delta) \subset \Omega \cap (N(f) \times N(f)).$$

And there exists $\varepsilon > 0$, $\|f\chi_{(x_1-\delta,x_1+\delta)}\|_{L^1} > \varepsilon$, $\|f\chi_{(x_2-\delta,x_2+\delta)}\|_{L^1} > \varepsilon$. By the process of dividing, for the above $\delta > 0$, there exist N_1, N_2, for all $n > N_1$ there exists $E(n, i_n)$ such that

$$x_1 \in E(n, i_n) \subset (x_1 - \delta, x_1 + \delta),$$

and for all $n > N_2$ there exists $E(n, j_n)$ such that

$$x_2 \in E(n, j_n) \subset (x_2 - \delta, x_2 + \delta).$$

So for all $n > \max\{N_1, N_2\}$,

$$\begin{aligned} x = (x_1, x_2) \in E(n, i_n, j_n) \subset (x_1 - \delta, x_1 + \delta) &\times (x_2 - \delta, x_2 + \delta) \\ &\subset \Omega \cap (N(f) \times N(f)). \end{aligned}$$

Remark 2.1. For $E(n,i,j)$ where $i,j \in \{1,2,3,\ldots,2^n\}$, there exists a unique $E(k, i_k, j_k)$ such that $E(n,i,j) \subset E(k, i_k, j_k)$ for all $1 \le k \le n, k \in \mathbb{Z}$.

Proof: By the processes above, the existence is obvious. If there exist $E(k,i_k,j_k)$, $E(k,\widetilde{i}_k,\widetilde{j}_k)$ such that $E(n,i,j)\subset E(k,i_k,j_k), E(k,\widetilde{i}_k,\widetilde{j}_k)$. Then $E(n,i,j)\subset E(k,i_k,j_k)\cap E(k,\widetilde{i}_k,\widetilde{j}_k)\neq\emptyset$, contradictory. □

Then we choose any $n>\max\{N_1,N_2\}$ and claim the fact that there exist $E(k,i_k,j_k)\supset E(n,i_n,j_n)$, for some $k=1,2,\dots,n$, such that $E(k,i_k,j_k)\in S_k$. If not, for $|E(1,i_1,j_1)\cap\Omega\cap(N(f)\times N(f))|\neq 0$, so $E(1,i_1,j_1)\notin D_1$ and $E(1,i_1,j_1)$ must be in R_1. Then we get $E(2,i_2,j_2)\in T_2$, by the same argument, we get $E(2,i_2,j_2)\in R_2$. Step by step, finally we have $E(n,i_n,j_n)\in R_n$, this is contradictory, since $E(n,i_n,j_n)\subset\Omega\cap(N(f)\times N(f))$.

According to our dividing and selecting, we get

$$\begin{gathered}\#S_m+\#D_m+\#R_m=\#T_m=4\#R_{m-1},\\ \#T_1=4,\\ \#S_m\geq 0,\quad \#D_m\geq 0,\quad \#R_m\geq 0.\end{gathered}\tag{2.6}$$

Since $\#R_m\leq 4\#R_{m-1}$, $\#R_1\leq 4$, we get $\#R_m\leq 4^m$, then we have

$$\sum_{k=1}^{m}\frac{1}{4^k}(\#S_k+\#D_k)=1-\frac{1}{4^m}\#R_m\leq 2.\tag{2.7}$$

Then we have the estimate for $T_\Omega(f_1,f_2)$:

$$|T_\Omega(f_1,f_2)|\leq\sum_{m=1}^{\infty}\sum_{E(m,i,j)\in S_m}|T_\Omega(f_1\chi_{E(m,i)},f_2\chi_{E(m,j)})|.\tag{2.8}$$

We now consider two cases: $s\leq 1$ and $s>1$. Suppose first that $0<s\leq 1$ and $\frac{1}{p_1}+\frac{1}{p_2}\geq\frac{2}{s}$. We use the fact $\left(\sum_{i=1}^\infty|A_i|\right)^s\leq\sum_{i=1}^\infty|A_i|^s$, where $A_i\in\mathbb{R}$. By our assumption we have $\|f_i\chi_{E_{m,l}}\|_{L^{p_i}}^{p_i}\leq\|f\chi_{E_{m,l}}\|_{L^1}=2^{-m}$, $i=1,2$. Then by (1.13) and (2.7), we have

$$\begin{aligned}\|T_\Omega(f_1,f_2)\|_{L^s}^s&\leq C_1\sum_{m=1}^{\infty}\sum_{E(m,i,j)\in S_m}\|f_1\chi_{E(m,i)}\|_{L^{p_1}}^s\|f_2\chi_{E(m,j)}\|_{L^{p_2}}^s\\ &\leq C_1\sum_{m=1}^{\infty}\#S_m2^{-\frac{ms}{p_1}-\frac{ms}{p_2}}\\ &\leq C_1\sum_{m=1}^{\infty}\frac{1}{4^m}(\#S_m+\#D_m)4^m2^{-\frac{ms}{p_1}-\frac{ms}{p_2}}\\ &\leq C_2\big(\sup_{m\geq1}2^{m(2-s(\frac{1}{p_1}+\frac{1}{p_2}))}\big)\sum_{m=1}^{\infty}\frac{1}{4^m}(\#S_m+\#D_m)\leq C_3.\end{aligned}$$

If $s > 1$, by our assumption we have $\frac{1}{p_1} + \frac{1}{p_2} \geq 2$. Using Minkowski inequality on L^s norm, we have:

$$\begin{aligned}
\|T_\Omega(f_1, f_2)\|_{L^s} &\leq C_4 \sum_{m=1}^{\infty} \sum_{E(m,i,j)\in S_m} \|f_1\chi_{E(m,i)}\|_{L^{p_1}} \|f_2\chi_{E(m,j)}\|_{L^{p_2}} \\
&\leq C_4 \sum_{m=1}^{\infty} \#S_m 2^{-\frac{m}{p_1}-\frac{m}{p_2}} \\
&\leq C_4 \sum_{m=1}^{\infty} \frac{1}{4^m}(\#S_m + \#D_m)4^m 2^{-\frac{m}{p_1}-\frac{m}{p_2}} \\
&\leq C_4 \sup_{m\geq 1}\big(2^{m(2-(\frac{1}{p_1}+\frac{1}{p_2}))}\big) \sum_{m=1}^{\infty} \frac{1}{4^m}(\#S_m + \#D_m) \leq C_5.
\end{aligned}$$

So, for both cases, there exists a constant C, such that $\|T_\Omega(f_1,f_2)\|_{L^s} \leq C$. This completes the proof.

3. Proof of Theorem 1.2

In this section we give the proof of Theorem 1.2. To prove Theorem 1.2, we only need to show it is true for $\Sigma = \Sigma_1(\Psi)$ with $\Psi = \{\Omega_x\}_{x\in\mathbb{R}} \in \Theta$.

First, we prove the theorem under an additional assumption that $|\Omega_x|$ (the Lebesgue measure of Ω_x) depends continuously on x. We set $a = \sup_{\Omega_x\in\Psi}\{x : \Omega_x = \emptyset\}$, $b = \inf_{\Omega_x\in\Psi}\{x : \Omega_x = \mathbb{R}\}$. Here a may be $-\infty$ and b may be $+\infty$. Then

$$\begin{aligned}
T_{\Sigma_1}(f_1, f_2)(x) &= \int_{\mathbb{R}} \int_{x_2\in\Omega^c_{x_1}} K(x, x_1, x_2) f_1(x_1) f_2(x_2)\, dx_2\, dx_1 \\
&= \int_a^b \int_{x_2\in\Omega^c_{x_1}} K(x, x_1, x_2) f_1(x_1) f_2(x_2)\, dx_2\, dx_1 \\
&\quad + \int_{-\infty}^a \int_{\mathbb{R}} K(x, x_1, x_2) f_1(x_1) f_2(x_2)\, dx_2\, dx_1 \\
&= I_1 + I_2.
\end{aligned}$$

For the boundedness of I_2, by the boundedness of T, we have

$$\|I_2\| \leq \|T(f_1\chi_{(-\infty,a)}, f_2)\|_{L^s} \leq C_1 \|f_1\|_{L^{p_1}} \|f_2\|_{L^{p_2}}. \tag{3.1}$$

Next we consider the boundedness of I_1. Since T_{Σ_1} is bilinear, we may assume without loss of generality throughout the proof that $\|f_1\|^{p_1}_{L^{p_1}} =$

$\|f_2\|_{L^{p_2}}^{p_2} = 1/2$. Let

$$f(x) = |f_1(x)|^{p_1} + |f_2(x)|^{p_2}.$$

To estimate I_1 we need to decompose the support of the function f into dyadic pieces and show that there exists a constant C, such that $\|I_1\|_{L^s} \le C$. We consider a partition of $\mathbb{R}$ into the dyadic pieces in the following way:

$$E_{m,j} = \Omega_{x_{m,j+1}} \setminus \Omega_{x_{m,j}},$$

where

$$x_{m,j} = \inf\left\{t : \int_{\Omega_t} |f(x)|\, dx = 2^{-m} j\right\}.$$

The value $x_{m,j}$ is well-defined for $j = 1, 2, 3, \ldots, 2^m$ and $m \in Z^+$ because $|\Omega_x|$ depends continuously on x. We denote by $M \subset \mathbb{R}^2$ the set

$$M = \{(x_1, x_2) \in \mathbb{R}^2 : x_2 \in \Omega_{x_1}^c\}.$$

Then we can claim that

$$M \cap (\mathrm{supp}(f_1) \times \mathrm{supp}(f_2)) = \bigcup_{m=1}^{\infty} \bigcup_{\substack{l=1 \\ l \text{ odd}}}^{2^m} (E_{m,l} \times E_{m,l+1})$$

(the proof is similar to the proof of Lemma 4.2 in [**1**]). We begin to estimate I_1:

$$|I_1| \le \sum_{m=1}^{\infty} \sum_{\substack{l=1 \\ l \text{ odd}}}^{2^m} |T_{\Sigma_1}(f_1 \chi_{E_{m,l}}, f_2 \chi_{E_{m,l+1}})|.$$

We consider two different cases: $s \le 1$ and $s > 1$. Suppose first that $0 < s \le 1$. We use the fact $\left(\sum_{i=1}^{\infty} |A_i|\right)^s \le \sum_{i=1}^{\infty} |A_i|^s$. By our assumption we have $\|f_i \chi_{E_{m,l}}\|_{L^{p_i}}^{p_i} \le \|f \chi_{E_{m,l}}\|_{L^1} = 2^{-m}$, $i = 1, 2$. Then, by (1.13),

$$\begin{aligned}
\|I_1\|_{L^s}^s &\le C_1 \sum_{m=1}^{\infty} \sum_{\substack{l=1 \\ l \text{ odd}}}^{2^m} \|f_1 \chi_{E_{m,l}}\|_{L^{p_1}}^s \|f_2 \chi_{E_{m,l+1}}\|_{L^{p_2}}^s \\
&\le C_1 \sum_{m=1}^{\infty} \sum_{\substack{l=1 \\ l \text{ odd}}}^{2^m} 2^{-\frac{ms}{p_1} - \frac{ms}{p_2}} \\
&\le C_2 \sum_{m=1}^{\infty} 2^{m(1 - s(\frac{1}{p_1} + \frac{1}{p_2}))} \le C_3.
\end{aligned}$$

If $s > 1$, by our assumption we have $\frac{1}{p_1} + \frac{1}{p_2} > 1$. Using Minkowski inequality on L^s norm, we have:

$$\begin{aligned}
\|I_1\|_{L^s} &\le C_4 \sum_{m=1}^{\infty} \sum_{\substack{l=1 \\ l \text{ odd}}}^{2^m} \|f_1\chi_{E_{m,l}}\|_{L^{p_1}} \|f_2\chi_{E_{m,l+1}}\|_{L^{p_2}} \\
&\le C_4 \sum_{m=1}^{\infty} \sum_{\substack{l=1 \\ l \text{ odd}}}^{2^m} 2^{-\frac{m}{p_1}} 2^{-\frac{m}{p_2}} \\
&\le C_5 \sum_{m=1}^{\infty} 2^{m(1-\frac{1}{p_1}-\frac{1}{p_2})} \le C_6.
\end{aligned}$$

So, in either case, there exists a constant C, such that $\|I_1\|_{L^s} \le C$. This completes the proof under the assumption that $|\Omega_x|$ is a continuous function of x.

Then we consider the general case. We use the same method that Kiselev used in [**3**] when dealing with the a.e. convergence of integral operators.

We consider the function $|\Omega_x|$. We say there is a jump for Ω_x at x if $\lim_{x\to x_{0_+}} |\Omega_x| \ne |\Omega_{x_0}|$ or $\lim_{x\to x_{0_-}} |\Omega_x| \ne |\Omega_{x_0}|$. We denote the value of jump by $h_\pm$:

$$h_+(x) = |\Omega_{x+0} \setminus \Omega_x|, \quad h_-(x) = |\Omega_x \setminus \Omega_{x-0}|. \tag{3.2}$$

Since $|\Omega_x|$ is monotone, according to our assumption $\Omega_x \subset \Omega_y$ for $x < y$. The set of values of $h_\pm$ where any jump may occur is at most countable. Let $\{x_n\}_{n=1}^{\infty}$ be the sequence of these points. If for some x, both h_+ and h_- are nonzero, let $x_n = x_{n+1} = x$, for some n. We set $\Delta_{x_n} = \Omega_{x_n+0} \setminus \Omega_{x_n}$, if $|\Omega_{x_n+0} \setminus \Omega_x| \ne 0$, and $\Delta_{x_n} = \Omega_{x_n} \setminus \Omega_{x_n-0}$ otherwise, and set the new measure space:

$$\Theta = \Theta_0 \cup \left(\bigcup_m \Theta_m \right)$$

in which $\Theta_0 = \big(\mathbb{R} \setminus \cup_n \Delta_{x_n}\big) \times \{0\}$ and $\Theta_m = \Delta_{x_m} \times [0,1]$. While on Θ_m the measure μ equals the product measure $dx \times d\nu$ ($d\nu$ being a Lebesgue measure on $[0,1]$) and on Θ_0, $d\mu = dx \times d\delta(0)$.

Then we construct the new bilinear operator:

$$\widetilde{T}(\widetilde{f}_1, \widetilde{f}_2)(x) = \int_{\Theta^2} \widetilde{K}(x, \mathrm{y}_1, \mathrm{y}_2) \widetilde{f}_1(\mathrm{y}_1) \widetilde{f}_2(\mathrm{y}_2)\, d\mu(\mathrm{y}_1, \mathrm{y}_2), \tag{3.3}$$

where $\widetilde{K}(x, \mathrm{y}_1, \mathrm{y}_2)$ is a new kernel defined on $\mathbb{R} \times \Theta \times \Theta$, which is equal to $K(x, x_1, x_2)$ for all x, when $(\mathrm{y}_1, \mathrm{y}_2) = (x_1, 0, x_2, 0) \in \Theta_0 \times \Theta_0$. If

$(\mathrm{y}_1, \mathrm{y}_2) = (x_1, y_1, x_2, y_2) \in \Theta_{x_n} \times \Theta_{x_m}$, which is $(y_1, y_2) \in [0,1] \times [0,1]$ and $(x_1, x_2) \in \Delta_n \times \Delta_m$ for some n and m, then $\widetilde{K}(x, \mathrm{y}_1, \mathrm{y}_2) = K(x, x_1, x_2)$ for all x. If $(\mathrm{y}_1, \mathrm{y}_2) = (x_1, 0, x_2, y_2) \in \Theta_0 \times \Theta_n$, which is $(x_1, x_2) \in \big(\mathbb{R} \backslash \cup_n \Delta_{x_n}\big) \times \Delta_{x_n}$ for some n, then $\widetilde{K}(x, \mathrm{y}_1, \mathrm{y}_2) = K(x, x_1, x_2)$ for all x. If $(\mathrm{y}_1, \mathrm{y}_2) = (x_1, y_1, x_2, 0) \in \Theta_n \times \Theta_0$, which is $(x_1, x_2) \in \Delta_{x_n} \times \big(\mathbb{R} \setminus \cup_n \Delta_{x_n}\big)$ for some n, then $\widetilde{K}(x, \mathrm{y}_1, \mathrm{y}_2) = K(x, x_1, x_2)$ for all x.

Next, define a family $\widetilde{\Omega}_u$ of the extending measurable sets in Θ. We construct $\widetilde{\Omega}_u$ so that $\mu(\widetilde{\Omega}_u) = u$ which means $\mu(\widetilde{\Omega}_u)$ is continuous on u.

Let $x_0(u) = \sup_t \{t : |\Omega_t| \le u\}$. If $x_0(u) \ne x_n$ for any n, we let

$$\widetilde{\Omega}_u = \big((\Omega_{x_0(u)} \times \{0\}) \cap \Theta_0\big) \cup \left(\bigcup_{x_m < x_0(u)} \Theta_m \right).$$

If at $x_0(u)$ we have a jump on the left, we let

$$\widetilde{\Omega}_u = \big((\Omega_{x_0(u)-0} \times \{0\}) \cap \Theta_0\big) \cup \left(\bigcup_{x_m < x_0(u)} \Theta_m \right) \cup \left(\Delta_{x_n} \times \left[0, \frac{u - |\Omega_{x_0(u)-0}|}{|\Omega_{x_0(u)}|}\right] \right).$$

If at $x_0(u)$ we have a jump on the right, we let

$$\widetilde{\Omega}_u = \big((\Omega_{x_0(u)} \times \{0\}) \cap \Theta_0\big) \cup \left(\bigcup_{x_m < x_0(u)} \Theta_m \right) \cup \left(\Delta_{x_n} \times \left[0, \frac{u - |\Omega_{x_0(u)}|}{|\Omega_{x_0(u)}|}\right] \right).$$

We can give the following claim without proof (details can be found in [**2**]):

(1) Let $\widetilde{\Psi}$ be a family of sets $\widetilde{\Omega}_u$, then $\emptyset \in \widetilde{\Psi}$, $\Theta \in \widetilde{\Psi}$, and $\widetilde{\Omega}_{u_1} \subseteq \widetilde{\Omega}_{u_2}$ for any $u_1 < u_2$.

(2) If we let $u(x) = \sup_{t \le x}\{|\Omega_t|\}$, and for given $f_i \in L^{p_i}(\mathbb{R})$, let $\widetilde{f}_i(\widetilde{x}) = f_i(x)$ when $(x, 0) \in \Theta_0$ and $\widetilde{f}_i(\widetilde{x}) = f_i(x)$ in which $\widetilde{x} = (x, y)$ for any $y \in [0,1]$ if $x \in \Delta_{x_n}$ for some n, then we have

$$\iint_{u(y_2) \in \widetilde{\Omega}^c_{u(y_1)}} \widetilde{K}(x, \widetilde{y}_1, \widetilde{y}_2) \widetilde{f}_1(\widetilde{y}_1) \widetilde{f}_2(\widetilde{y}_2)\, d\mu(\widetilde{y}_1, \widetilde{y}_2) = T_{\Sigma_1}(f_1, f_2).$$

(3) There exists a constant C, such that

$$\|\widetilde{T}(\widetilde{f}_1, \widetilde{f}_2)\|_{L^s(\mathbb{R})} \le C_1 \|\widetilde{f}_1\|_{L^{p_1}(\Theta, d\mu)} \|\widetilde{f}_2\|_{L^{p_2}(\Theta, d\mu)}.$$

Hence, we have shown the bound (1.19) in general case.

4. Some corollaries and remarks

In this section, we will show some corollaries and remarks.

Remark 4.1. For a multilinear operator

$$T(f_1, f_2, \dots, f_n)(x) = \int_{\mathbb{R}^n} K(x, y_1, y_2, \dots, y_n) f_1(y_1) f_2(y_2) \cdots f_n(y_n)\, dy_1\, dy_2 \cdots dy_n, \tag{4.1}$$

we assume that it satisfies

$$\|T(f_1, f_2, \dots, f_n)\|_{L^s} \le C \prod_{k=1}^{n} \|f_k\|_{L^{p_k}}, \tag{4.2}$$

where (s, p_k), $k = 1, 2, \dots, n$, satisfy:

$$0 < s,\ p_k < \infty,\ \sum_{k=1}^{n} \frac{1}{p_k} \ge n \max\left\{\frac{1}{s}, 1\right\}. \tag{4.3}$$

Let $\Omega \in \mathbb{R}^n$ be any open set, and

$$T_\Omega(f_1, f_2, \dots, f_n)(x) = \int_{\Omega} K(x, y_1, y_2, \dots, y_n) f_1(y_1) f_2(y_2) \cdots f_n(y_n)\, dy_1\, dy_2 \cdots dy_n, \tag{4.4}$$

we have

$$\|T_\Omega(f_1, f_2, \dots, f_n)\|_{L^s} \le C_1 \prod_{k=1}^{n} \|f_k\|_{L^{p_k}}, \tag{4.5}$$

where C_1 depends only on the $\|T\|_{L^{p_1} \times L^{p_2} \times \cdots \times L^{p_k} \to L^s}$ norm.

We can see that the special case in [**1**] ($\{(x, y),\ x \ge y\}$) satisfies the condition required for the region in Theorem 1.2 even though it looks complicated. Also, there are some other special cases, such as convex sets.

Remark 4.2. For $k = 1, 2$, let $f_k \in L^{p_k}(I_k)$ and I_k be measurable sets in $\mathbb{R}$. Assume that (s, p_1, p_2) satisfies $\frac{1}{p_1} + \frac{1}{p_2} > \frac{1}{\min\{1,s\}}$ and $0 < s, p_1, p_2 < \infty$. If the whole space $\mathbb{R} \in \Psi$ in the first condition (1.16) is replaced by $I_1, I_2 \in \Psi$, then (1.15) and (1.19) also holds.

Corollary 4.3. *Let T be as in Theorem 1.2, (s, p_1, p_2) satisfies $\frac{1}{p_1}+\frac{1}{p_2} > \frac{1}{\min\{1,s\}}$ and $0 < s, p_1, p_2 < \infty$. Then for any function $f_i \in L^{p_i}(\mathbb{R})$, $i = 1, 2$,*

$$\|T_B(f_1, f_2)\|_{L^s} \le C \prod_{i=1}^{2} \|f_i\|_{L^{p_i}}, \tag{4.6}$$

where T_B is given by

$$T_B(f_1, f_2)(x) = \int_{B(0,1)} K(x, y_1, y_2) f_1(y_1) f_2(y_2)\, dy. \tag{4.7}$$

The constant C depends only on the constants in the norm bound (1.13) *for the operator T.*

Proof: We set $g_k = f_k \chi_{[-1,1]}$, $i = 1, 2$. We divide the ball $B(0, 1) = \{(y_1, y_2) : y_1^2 + y_2^2 < 1\}$ into two parts $B_+ = \big\{(y_1, y_2) : y_1 \in [0, 1]$, $y_2 \in (-\sqrt{1-y_1^2}, \sqrt{1-y_1^2})\big\}$ and $B_- = \big\{(y_1, y_2) : y_1 \in [-1, 0]$, $y_2 \in (-\sqrt{1-y_1^2}, \sqrt{1-y_1^2})\big\}$.

Let $\Phi_+ = \{\Omega_x\}_{x\in[0,1]}$ with $\Omega_x = (-1, -\sqrt{1-x^2}] \cup [\sqrt{1-x^2}, 1)$ for $x \in [0, 1]$. Then $\emptyset = \Omega_0 \in \Phi_+$, $(-1, 1) = \Omega_1 \in \Phi_+$ and for $x < y$, $\Omega_x \subset \Omega_y$.

Let $\Phi_- = \{\widetilde{\Omega}_x\}_{x\in[-1,0]}$ with $\widetilde{\Omega}_x = (-\sqrt{1-x^2}, \sqrt{1-x^2})$. Then $\emptyset = \widetilde{\Omega}_{-1} \in \Phi_-$, $(-1, 1) = \widetilde{\Omega}_0 \in \Phi_-$ and for $x < y$, $\widetilde{\Omega}_x \subset \widetilde{\Omega}_y$. Since

$$\begin{aligned}
T_B(f_1, f_2)(x) &= \int_{B(0,1)} K(x, y_1, y_2) f_1(y_1) f_2(y_2)\, dy \\
&= \int_{B_+} K(x, y_1, y_2) f_1(y_1) f_2(y_2)\, dy \\
&\quad + \int_{B_-} K(x, y_1, y_2) f_1(y_1) f_2(y_2)\, dy \\
&= \int_{y_2 \in \Omega^c_{y_1}, y_1 \in [0,1]} K(x, y_1, y_2) g_1(y_1) g_2(y_2)\, dy \\
&\quad + \int_{-1}^{1} \int_{-1}^{0} K(x, y_1, y_2) g_1(y_1) g_2(y_2)\, dy_1\, dy_2 \\
&\quad - \int_{y_2 \in \widetilde{\Omega}^c_{y_1}, y_1 \in [-1,0]} K(x, y_1, y_2) g_1(y_1) g_2(y_2)\, dy.
\end{aligned}$$

So by using Theorem 1.2 and Remark 4.2, we get (4.6). $\square$

The proof gives us a method of dealing with some area whose boundary consists of finite monotonic functions and lines parallel to coordinate axes.

Corollary 4.4. *Let the bilinear operator T with kernel $K(x, y_1, y_2)$ be as in Theorem 1.2 with (s, p_1, p_2) satisfying $\frac{1}{p_1} + \frac{1}{p_2} > \frac{1}{\min\{1,s\}}$ and $0 < s, p_1, p_2 < \infty$. For a convex bounded measurable set $\mathcal{K} \subseteq \mathbb{R}^2$, $T_{\mathcal{K}}$ is given by*

$$T_{\mathcal{K}}(f_1, f_2)(x) = \int_{\mathcal{K}} K(x, y_1, y_2) f_1(y_1) f_2(y_2)\, dy. \tag{4.8}$$

Then we have

$$\|T_{\mathcal{K}}(f_1, f_2)\|_{L^s} \le C \prod_{i=1}^{2} \|f_i\|_{L^{p_i}}. \tag{4.9}$$

Finally we make some remarks of extending our theorem to high dimensional cases $n \ge 1$. The extension of Theorem 1.2 is involved. The main difficulty is how to define an order for sets Ω_x and Ω_y where $x, y \in \mathbb{R}^n$. To this end, we should give an order for points in higher dimension. A lot of orders can be given. In this paper, we just show two typical cases which can be proved in the same method.

We consider the bounded bilinear operator $T\colon L^{p_1} \times L^{p_2} \to L^s$ given by

$$T(f_1, f_2)(\lambda) = \int K(\lambda, x_1, x_2) f_1(x_1) f_2(x_2)\, dx, \tag{4.10}$$

and assume

$$\|T(f_1, f_2)\|_{L^s} \le C_1 \|f_1\|_{L^{p_1}} \|f_2\|_{L^{p_2}}. \tag{4.11}$$

Let Ψ be a family of sets Ω_x and $\kappa_1(x) = |x|$, $\kappa_2(x) = \sum_{i=1}^n x_i$, which satisfies the following conditions:

(1) $\emptyset \in \Psi$ and $\mathbb{R}^n \in \Psi$;

(2) for any $\kappa_1(x) < \kappa_1(y)$ or $\kappa_2(x) < \kappa_2(y)$, $\Omega_x \subseteq \Omega_y$.

Set $\Theta = \{\Psi,\ \Psi \text{ satisfies (1) and (2)}\}$, $\Sigma_1(\Psi) = \{(x, y) \in \mathbb{R}^2 : y \in \Omega_x^c\}$, and $\Sigma_2(\Psi) = \{(x, y) \in \mathbb{R}^2 : x \in \Omega_y^c\}$. Let Σ be any set in $\mathcal{A}$, where $\mathcal{A}$ is the algebra generated by some finite subset of $\cup_{\Psi\in\Theta}\{\Sigma_1(\Psi), \Sigma_2(\Psi)\}$. Consider the bilinear operator T_2 given by

$$T_2(f_1, f_2)(\lambda) = \int_{\Sigma} K(\lambda, x_1, x_2) f_1(x_1) f_2(x_2)\, dx_1\, dx_2. \tag{4.12}$$

Remark 4.5. For any function $f_i \in L^{p_i}(\mathbb{R}^n)$, $i = 1, 2$, and (s, p_1, p_2) satisfying $\frac{1}{p_1} + \frac{1}{p_2} > \frac{1}{\min\{1,s\}}$ and $0 < s, p_1, p_2 < \infty$, if (4.11) holds, then

$$\|T_2(f_1, f_2)\|_{L^s} \leq C \prod_{i=1}^{2} \|f_i\|_{L^{p_i}}.$$

The constant C depends only on the constant C_1 in (4.11) and the domain.

At last we remark that the relationship of (p_1, p_2, s) in Theorem 1.2 is optimal in some sense.

Remark 4.6. Here we refer to Muscalu et al. [**4**]. Let

$$T(f_1, f_2)(x) = \int_{\mathbb{R}^2} e^{-2\pi i(x_2 - x_1)x} f_1(x_1) f_2(x_2)\, dx_1\, dx_2 = \widehat{f_1}(-x)\widehat{f_2}(x).$$

Here $\widehat{f}$ is the Fourier transform of f. Let $\Sigma = \{(x_1, x_2) \in \mathbb{R}^2,\ x_1 < x_2\}$ and

$$T_\Sigma(f_1, f_2)(x) = \int_{x_1 < x_2} e^{-2i(x_2 - x_1)x} f_1(x_1) f_2(x_2)\, dx_1\, dx_2.$$

Actually one may find that if we choose $f_1 = f_2$, then $T_\Sigma(f_1, f_1)(x)$ is essentially $P(|\widehat{f_1}|^2)(x)$. Here $\widehat{P(f)} = \chi_{(-\infty,0]}\widehat{f} = \frac{1}{2}(I - iH)(|\widehat{f_1}|^2)$ and H is the Hilbert transform. Thus although we have

$$\|T(f_1, f_2)\|_{L^1} \leq C\|\widehat{f_1}\|_{L^2}\|\widehat{f_2}\|_{L^2} \leq C\|f_1\|_{L^2}\|f_2\|_{L^2},$$

it fails that

$$\|T_\Sigma(f_1, f_2)\|_{L^1} \leq C\|f_1\|_{L^2}\|f_2\|_{L^2}.$$

Since the L^1 boundedness of Hilbert transform fails.

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