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GLOBAL WELL-POSEDNESS AND SYMMETRIES FOR DISSIPATIVE ACTIVE SCALAR EQUATIONS WITH POSITIVE-ORDER COUPLINGS

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Abstract: We consider a family of dissipative active scalar equations outside the L^2-space. This was introduced in [**7**] and its velocity fields are coupled with the active scalar via a class of multiplier operators which morally behave as derivatives of positive order. We prove global well-posedness and time-decay of solutions, without smallness assumptions, for initial data belonging to the critical Lebesgue space $L^{\frac{n}{2\gamma-\beta}}(\mathbb{R}^n)$ which is a class larger than that of the above reference. Symmetry properties of solutions are investigated depending on the symmetry of initial data and coupling operators.

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1. Introduction

We are concerned with the initial value problem (IVP) for a family of dissipative active scalar equation, which reads as

$$\begin{cases} \dfrac{\partial\theta}{\partial t} + \kappa(-\Delta)^\gamma\theta + u\cdot\nabla_x\theta = 0, & x\in\mathbb{R}^n,\ t>0,\\ \theta(x,0)=\theta_0(x), & x\in\mathbb{R}^n, \end{cases} \tag{1.1}$$

where $n\geq 2$, $\kappa\geq 0$, and $\gamma>0$. The fractional laplacian operator $(-\Delta)^\gamma$ is defined by

$$[\widehat{(-\Delta)^\gamma f}](\xi) = |\xi|^{2\gamma}\widehat{f}(\xi),$$

where $\widehat{f}=\int_{\mathbb{R}^n} e^{-ix\cdot\xi} f(\xi)\,d\xi$ stands for the Fourier transform of f. The velocity field u is determined by the active scalar θ by means of the multiplier operators

$$u = P[\theta] = (\widetilde{P}_1[\theta],\dots,\widetilde{P}_n[\theta]), \tag{1.2}$$

such that $\nabla \cdot u = 0$, and

$$u_j = \widetilde{P}_j[\theta] = \sum_{i=1}^{n} a_{ij}\mathcal{R}_i\Lambda^{-1}P_i[\theta], \quad \text{for } 1 \le j \le n, \tag{1.3}$$

where $\Lambda = (-\Delta)^{\frac{1}{2}}$, $\mathcal{R}_i = -\partial_i(-\Delta)^{-\frac{1}{2}}$ is the i-th Riesz transform, a_{ij}'s are constant and

$$\widehat{P_i[\theta]}(\xi) = P_i(\xi)\widehat{\theta}(\xi). \tag{1.4}$$

Denoting $I = \sqrt{-1}$, it follows that

$$\widehat{\widetilde{P}_j[\theta]}(\xi) = \widetilde{P}_j(\xi)\widehat{\theta}(\xi) \text{ with } \widetilde{P}_j(\xi) = \sum_{i=1}^{n} a_{ij}\frac{\xi_i I}{|\xi|^2}P_i(\xi),$$

and the vector field u can be expressed in Fourier variables in the shorter form

$$\widehat{u} = \widehat{P[\theta]} = P(\xi)\widehat{\theta}(\xi) \text{ where } P(\xi) = (\widetilde{P}_1(\xi), \ldots, \widetilde{P}_n(\xi)). \tag{1.5}$$

Throughout this manuscript the symbol $P_i(\xi)$ in (1.4) is assumed to belong to $C^{[\frac{n}{2}]+1}(\mathbb{R}^n\backslash\{0\})$ with

$$\left|\frac{\partial^\alpha P_i}{\partial \xi^\alpha}(\xi)\right| \le C|\xi|^{\beta-|\alpha|}, \tag{1.6}$$

for all $\alpha \in (\mathbb{N}\cup\{0\})^n$, $|\alpha| \le [\frac{n}{2}]+1$, and $\xi \neq 0$, where $\beta \ge 0$. The brackets $[\cdot]$ stand for the greatest integer function. In particular, for $\alpha = 0$ it follows from (1.5) and (1.6) that

$$|\widehat{u}(\xi)| \le C|\xi|^{\beta-1}|\widehat{\theta}(\xi)|, \quad \text{for all } \xi \neq 0. \tag{1.7}$$

Concerning the criticality of (1.1)–(1.3), there is an interplay between the field u and fractional viscosity $(-\Delta)^\gamma$ expressed by means of three basic cases: sub-critical $\beta < 2\gamma$, critical $\beta = 2\gamma$, and super-critical $\beta > 2\gamma$.

We could consider an arbitrary $\kappa > 0$, nevertheless $\kappa = 1$ is assumed for the sake of simplicity. The IVP (1.1)–(1.3) can be converted into the integral equation

$$\theta(t) = G_\gamma(t)\theta_0 + B(\theta,\theta)(t), \tag{1.8}$$

where

$$B(\theta,\varphi)(t) = -\int_0^t G_\gamma(t-s)(\nabla_x \cdot (P[\theta]\varphi))(s)\,ds \tag{1.9}$$

and $G_\gamma(t)$ is the convolution operator with kernel given in Fourier variables by $\hat{g}_\gamma(\xi,t) = e^{-t|\xi|^{2\gamma}}$. Solutions of (1.8) are called mild ones for (1.1)–(1.3).

Assuming that P_i's are homogeneous functions of degree β, we have formally that

$$\theta_\lambda = \lambda^{2\gamma-\beta}\theta(\lambda x, \lambda^{2\gamma}t)$$

verifies (1.1)–(1.3), for all $\lambda > 0$, provided that θ does so. It follows that

$$\theta \to \theta_\lambda = \lambda^{2\gamma-\beta}\theta(\lambda x, \lambda^{2\gamma}t), \quad \text{for } \lambda > 0, \tag{1.10}$$

is the scaling map for (1.1)–(1.3). Also, making $t \to 0^+$ in (1.10), one obtains the scaling for the initial data

$$\theta_0 \to \lambda^{2\gamma-\beta}\theta_0(\lambda x). \tag{1.11}$$

In view of (1.6), even when P_i is not homogeneous, we can consider (1.10) as an intrinsic scaling for (1.1)–(1.3) in the sense that it is useful to identify threshold indexes for functional settings and properties of solutions. One of our aims is to provide a global well-posedness result for (1.1)–(1.3) in a scaling invariant framework outside the L^2-space.

Active scalar equations like (1.1)–(1.3) arise in a large number of physical models in fluid mechanics and atmospheric science. Examples of those are 2D surface quasi-geostrophic equation (SQG) $u = \nabla^\perp((-\Delta)^{-1/2}\theta)$ $(\beta = 1)$, Burgers equation $u = \theta$ $(\beta = 1)$, and 2D vorticity equation $u = \nabla^\perp(-\Delta)^{-1}\theta$ $(\beta = 0)$. SQG is a famous model with a lot of papers concerning existence, uniqueness, regularity and asymptotic behavior of solutions in the inviscid case $\kappa = 0$, or in the subcritical $(1/2 < \gamma < 1)$, critical $(\gamma = 1/2)$, and supercritical $(\gamma \in (0, 1/2))$ ranges. Without making a complete list, we would like to mention [**1, 3, 9, 11, 12, 13, 16, 19, 21, 24, 25, 26, 33, 35**] and their references. In the case $u = \theta$, see [**25**] and [**17**] for results on blow-up, global existence, and regularity of solutions. One dimensional active scalar models have also attracted the attention of many authors, see e.g. [**4, 14, 15, 29**] where the reader can find global existence, finite-time singularity, and asymptotic behavior results with velocity coupled via singular integral operators that are zero-order multiplier ones.

In the case of SQG, notice that u can be written by using Riesz transform as

$$u = (-\mathcal{R}_2\theta, \mathcal{R}_1\theta) \tag{1.12}$$

and then the velocity is coupled to the active scalar via zero-order multiplier operators. The model (1.1)–(1.3) was introduced in [**5, 7**] and covers positive-order couplings when $\beta > 1$ (see (1.7)). In this range, the operator $P[\cdot]$ behaves “morally” like a positive derivative of $(\beta - 1)$-order and produces more difficulties in comparison with SQG ($\beta = 1$, zero-order) and $\beta < 1$ (negative-order).

The paper [**5**] dealt mainly with the inviscid case $\kappa = 0$, while [**7**] with the dissipative one $\kappa > 0$. This last work is our main motivation since we also focus in the dissipative model. The authors of [**7**] showed existence of global solutions in $L^\infty((0,\infty);Y)$ for (1.1)–(1.3) where $Y = L^1 \cap L^\infty \cap B^{s,M}_{q,\infty}$ with $s > 1$ and $2 \leq q \leq \infty$. The index $M = \{M_j\}_{j\geq -1}$ is a sequence and the space $B^{s,M}_{q,\infty}$ is an extension of the classical Besov space $B^s_{q,\infty}$ where the $B^{s,M}_{q,\infty}$-norm increases according to the growth of M. The results of [**7**] consider couplings $P[\cdot]$ in (1.2) such that $P_i \in C^\infty(\mathbb{R}^n\backslash\{0\})$, P_i is radially symmetric, $P_i = P_i(|\xi|)$ is nondecreasing with $|\xi|$, and a technical growth hypothesis involving $P_i(\xi)$ and the sequence M. Applying their results to the special case

$$u = \nabla^\perp(\Lambda^{\beta-2}\theta) = \Lambda^{\beta-1}(-\mathcal{R}_2\theta, \mathcal{R}_1\theta) \tag{1.13}$$

with $n = 2$, $0 \leq \beta < 2\gamma < 1$ (within the sub-critical range), $\kappa > 0$, and $M_j = j+1$, they obtained well-posedness of solutions with initial data in $L^1 \cap L^\infty \cap B^{s,M}_{q,\infty}$. Roughly speaking, the technique employed in [**7**] for constructing solutions relies on a successive approximation scheme together *a priori* estimates involving Besov norms. The field (1.13) corresponds to the modified SQG that interpolates 2D vorticity equation and SQG by varying the parameter β from 0 to 1. This model has been studied for instance in [**5, 10, 25, 30, 31, 32**] where one can find existence and regularity results with data in Sobolev spaces H^m with $m \geq 0$. The conditions $\kappa > 0$, $\beta \in [0,1]$, and $\beta = 2\gamma$ were assumed in [**10, 25, 30, 32**]; $\kappa > 0$ and $1 \leq \beta < 2\gamma < 2$ in [**31**]; and $\kappa = 0$ and $\beta \in [1,2]$ in [**5**]. In this last work, local well-posedness of $H^m(\mathbb{R}^2)$-solutions was proved for (1.1)–(1.13) with $m \geq 4$. We also would like to mention the work [**18**] where the authors showed results of self-similar solutions in Fourier–Besov–Morrey spaces for a wide class of symbols (1.4).

In this paper we prove the global well-posedness of (1.1)–(1.3) in the Lebesgue space $L^{\frac{n}{2\gamma-\beta}}(\mathbb{R}^n)$ without smallness conditions (see Theorem 3.1). This is the unique L^r-space whose norm is invariant by the scaling (1.11), that is, $L^{\frac{n}{2\gamma-\beta}}$ is the critical one in the scale of Lebesgue spaces. We can consider initial data outside the L^2-framework and, due to the inclusion $L^1 \cap L^\infty \subset L^{\frac{n}{2\gamma-\beta}}$, our initial data class is larger than that of [**7**]. In comparison with [**7**], some new symbols $P_i(\xi)$ are considered here (e.g. non-radially symmetric ones). Even for a singular initial data $\theta_0 \in L^{\frac{n}{2\gamma-\beta}}(\mathbb{R}^n)$, the global solution $\theta \in BC([0,\infty); L^{\frac{n}{2\gamma-\beta}}(\mathbb{R}^n))$ is instantaneously C^∞-smoothed out and verifies (1.1)–(1.3) classically, for all $t > 0$. Here we focus in the range $\beta \geq 1$ and consider the sub-critical case $\beta < 2\gamma$. More precisely, we assume

$$1 \le 2\beta - 1 < 2\gamma < \min\left\{ \frac{2}{3}(n+\beta+1), (n+1) \right\}. \tag{1.14}$$

The range $0 \le \beta < 2\gamma$ with $\beta < 1$ also can be treated with an adaptation on the proofs (see Remark 3.3).

Also, we show some decay properties in L^q-norms (see Theorem 3.1). Precisely, for $\frac{n}{2\gamma-\beta} \le q \le \infty$ and $\theta_0 \in L^{\frac{n}{2\gamma-\beta}}(\mathbb{R}^n)$, we obtain the time-polynomial decay

$$\|\theta(\cdot,t)\|_{L^q} \le Ct^{-(\frac{2\gamma-\beta}{2\gamma} - \frac{n}{2\gamma q})}, \quad \text{for all } t > 0. \tag{1.15}$$

Assuming further $\theta_0 \in L^{\frac{n}{2\gamma-\beta}}(\mathbb{R}^n) \cap L^1(\mathbb{R}^n)$, the solution θ belongs to $BC([0,\infty); L^1(\mathbb{R}^n))$ and the estimate (1.15) is improved to

$$\|\theta(\cdot,t)\|_{L^q} \le Ct^{-(\frac{2\gamma-\beta}{2\gamma} - \frac{n}{2\gamma q}) - (\frac{n+\beta}{2\gamma} - 1)}, \quad \text{for all } t > 0, \tag{1.16}$$

where $1 \le q \le \infty$. Notice that the decay in (1.16) is faster than those of (1.15) due to the condition $2\gamma < \frac{2}{3}(n+\beta+1) < n+\beta$.

In view of the L^p-L^q estimate (2.5) for the semigroup $G_\gamma(t)$, it is not expected that (1.15) holds true for $q < \frac{n}{2\gamma-\beta}$ and an arbitrary $\theta_0 \in L^{\frac{n}{2\gamma-\beta}}(\mathbb{R}^n)$. Thus $\theta(\cdot,t)$ may not be a L^2-solution when $2 < \frac{n}{2\gamma-\beta}$ although $\theta(\cdot,t) \in C^\infty(\mathbb{R}^n)$, for all $t > 0$. Even in the subcritical case, this fact seems to prevent an adaptation from previous techniques based on L^2-frameworks (see e.g. the famous papers [**1, 26**]) in order to obtain global well-posedness of $L^{\frac{n}{2\gamma-\beta}}(\mathbb{R}^n)$-solutions. Roughly speaking, the approach employed here relies on time-weighted Kato type norms, scaling arguments, and arguments of the type parabolic De Giorgi–Nash–Moser. These ingredients also were used in [**3**] in order to analyze SQG ($\beta = 1$). However, due to the coupling between θ and u being via a positive-order operator, the model (1.1)–(1.3) requires more involved arguments and further care in comparison with SQG. For instance, since $P[\cdot]$ is not continuous from L^{p_1} to L^{p_2} when $\beta > 1$, we need to employ an auxiliary time-weighted Kato-type norm based on homogeneous Sobolev spaces $\dot{H}^s_q$ with $q > \frac{n}{2\gamma-\beta}$ in order to control the nonlinear term in (1.1)–(1.3). So, different from SQG, Sobolev norms play here a crucial role for the local existence and extension of solutions with data in Lebesgue spaces (see e.g. (5.5) and (5.27)–(5.29), respectively). Let us also mention that there is no maximum principle for $\dot{H}^s_q$-norms when $s > 0$; and consequently there is a lack of global-in-time control on these norms (see (3.2)).

In view of the structure of (1.1), it is natural to wonder about symmetry properties of solutions under symmetry conditions for the symbols $P_i(\xi)$ and initial data θ_0. In Theorem 3.4, we show that the global solution given in Theorem 3.1 is radially symmetric, for all $t > 0$, provided that θ_0 and $\operatorname{div}_\xi(P(\xi))$ present this same property. Moreover, results on odd and even symmetry of solutions are obtained under parity conditions for θ_0 and P_i's. In Remark 3.5, we also comment about conditions for solutions to be non-symmetric.

Let us also comment on *log*-type couplings which are interesting ones covered by (1.1)–(1.3). Ohkitani [**34**] has presented numerical evidences that, even with $\kappa = 0$, (1.1) with $n = 2$ and

$$u = \nabla^\perp(\log(I - \Delta))^\chi\theta, \quad \chi > 0, \tag{1.17}$$

may be globally well-posed. The authors of [**5**] have proved local well-posedness of $H^4(\mathbb{R}^2)$-solutions for (1.1)–(1.17) with $\kappa > 0$. As pointed in [**5**], the field (1.17) is of order higher (at least logarithmically) than derivatives of order 1 and in particular than (1.12). Another examples are

$$P_i(\xi) = |\xi|^\sigma(\log(1 + |\xi|^2))^\chi, \qquad \chi \geq 0, \tag{1.18}$$

$$P_i(\xi) = |\xi|^\sigma(\log(1 + \log(1 + |\xi|^2)))^\chi, \quad \chi \geq 0, \tag{1.19}$$

which are of order higher than (1.17) when $\sigma > 2$. These couplings are also treated in [**7**] with $\sigma = \beta$ and $\chi \geq 0$. When $\sigma = 0$ and $n = 2$, (1.18) and (1.19) correspond to *log* and *log-log* Navier–Stokes which are intermediate models between 2D vorticity equation and SQG. See [**6**] for further details and global existence results in the case $\kappa = 0$, $0 \leq \chi \leq 1$ and data $\theta_0 \in L^1 \cap L^\infty \cap B^s_{q,\infty}$, where $B^s_{q,\infty}$ stands for an inhomogeneous Besov space with $s > 1$ and $q > 2$. An interest in *log*-type couplings has also arisen in connection with other fluid mechanics models (see [**8**]).

Finally, we remark that our results cover the couplings (1.13), (1.18), and (1.19). The condition (1.6) is clearly satisfied by (1.13), and if $\beta \in [1, 2]$ and $2\beta - 1 < 2\gamma < \min\{2 + \frac{2\beta}{3}, 3\}$ then (1.14) holds true. Also, (1.18)–(1.19) with $\chi > 0$ verify (1.6) with $\beta = \sigma + \varepsilon$ and small $\varepsilon > 0$, and we have (1.14) when $\frac{1}{2} < \gamma < \frac{4}{3}$ and $0 < \varepsilon < \gamma - \frac{1}{2}$. The cases (1.18) and (1.19) with $\chi = 0$ are similar to (1.13).

This manuscript is organized as follows. In the next section we recall some estimates in $L^q(\mathbb{R}^n)$ and Sobolev homogeneous spaces for Fourier multiplier operators and the semigroup $\{G_\gamma(t)\}_{t\geq 0}$. Our results are stated in Section 3 in two theorems, namely Theorems 3.1 and 3.4. Estimates for the bilinear operator (1.9) are obtained in Section 4. Local well-posedness and some properties of solutions are proved in Subsection 5.1. The proofs of Theorems 3.1 and 3.4 are performed in Subsections 5.2 and 5.3, respectively.

2. Preliminaries

In this section we recall some estimates for the fundamental solution of the linear part of (1.1) in $L^p(\mathbb{R}^n)$ and $\dot{H}^s_p(\mathbb{R}^n)$, whose norms will be denoted by $\|\cdot\|_p$ and $\|\cdot\|_{\dot{H}^s_p}$, respectively.

We remember that given $s \in \mathbb{R}$ and $1 < p < \infty$, the homogeneous Sobolev space $\dot{H}^s_p(\mathbb{R}^n)$ is the space of all $u \in \mathcal{S}'/\mathcal{P}$ such that $(-\Delta)^{\frac{s}{2}} u \in L^p(\mathbb{R}^n)$. In other words, $\dot{H}^s_p = (-\Delta)^{-\frac{s}{2}} L^p$ and it is a Banach space with norm

$$\|u\|_{\dot{H}^s_p} = \|(-\Delta)^{\frac{s}{2}} u\|_p.$$

For $s \in \mathbb{R}$, $1 < p_1, p_2, p < \infty$, and $\alpha \in (0,1)$ such that $\frac{1}{p} = \frac{\alpha}{p_1} + \frac{1-\alpha}{p_2}$, we have the interpolation property

$$\dot{H}^s_p = (\dot{H}^s_{p_1}, \dot{H}^s_{p_2})_{\alpha,p} \text{ with } \|u\|_{\dot{H}^s_p} \leq \|u\|^{\alpha}_{\dot{H}^s_{p_1}} \|u\|^{1-\alpha}_{\dot{H}^s_{p_2}}. \tag{2.1}$$

The following Sobolev type embedding holds true

$$\dot{H}^{s_2}_{p_2}(\mathbb{R}^n) \subset \dot{H}^{s_1}_{p_1}(\mathbb{R}^n), \tag{2.2}$$

for $1 < p_2 \leq p_1 < \infty$ and $s_1 - \frac{n}{p_1} = s_2 - \frac{n}{p_2}$. The reader is refereed to [**20**, Chapter 6] for further details on these spaces.

The next lemma gives estimates for certain multiplier operators acting in $\dot{H}^s_p(\mathbb{R}^n)$ (see e.g. [**27**]).

Lemma 2.1. *Let $m, s \in \mathbb{R}$, $1 < p < \infty$, and $F(\xi) \in C^{[\frac{n}{2}]+1}(\mathbb{R}^n \backslash \{0\})$, where $[\cdot]$ stands for the greatest integer function. Assume that there is $L > 0$ such that*

$$\left| \frac{\partial^\alpha F}{\partial \xi^\alpha}(\xi) \right| \leq L|\xi|^{m-|\alpha|}, \tag{2.3}$$

for all $\alpha \in (\mathbb{N} \cup \{0\})^n$, $|\alpha| \leq [\frac{n}{2}] + 1$, and $\xi \neq 0$. Then the multiplier operator $F(D)$ on $\mathcal{S}'/\mathcal{P}$ is bounded from $\dot{H}^s_p$ to $\dot{H}^{s-m}_p$. Moreover, the following estimate holds true

$$\|F(D)f\|_{\dot{H}^{s-m}_p} \leq C\|f\|_{\dot{H}^s_p}, \tag{2.4}$$

where $C > 0$ is independent of f.

The next lemma gives estimates for $\{G_\gamma(t)\}_{t \geq 0}$ on spaces $L^p(\mathbb{R}^n)$ and $\dot{H}^s_p(\mathbb{R}^n)$.

Lemma 2.2. *Let $n \geq 2$, $0 < \gamma < \infty$, $1 \leq p \leq q \leq \infty$, and $k \in (\mathbb{N} \cup \{0\})^n$. Then*

$$\|\nabla_x^k G_\gamma(t) f\|_q \leq C\, t^{-\frac{|k|}{2\gamma} - \frac{n}{2\gamma}(\frac{1}{p} - \frac{1}{q})} \|f\|_p, \tag{2.5}$$

for all $f \in L^p(\mathbb{R}^n)$. Now, let $s_1 \leq s_2$, $s_i \in \mathbb{R}$, and $1 < p_1 \leq p_2 < \infty$. There is a constant $C > 0$ such that

$$\|G_\gamma(t) f\|_{\dot{H}^{s_2}_{p_2}} \leq C t^{-\frac{(s_2 - s_1)}{2\gamma} - \frac{n}{2\gamma}(\frac{1}{p_1} - \frac{1}{p_2})} \|f\|_{\dot{H}^{s_1}_{p_1}}, \tag{2.6}$$

for all $f \in \dot{H}^{s_1}_{p_1}$. Moreover, let $f \in L^{\frac{n}{2\gamma - \beta}}(\mathbb{R}^n)$ with $1 \leq \beta < 2\gamma \leq n + \beta$ and let $\frac{n}{2\gamma - \beta} \leq q < \infty$ with $q \neq \frac{n}{2\gamma - \beta}$ if $\beta = 1$. Then

$$\sup_{0<t<T} t^{\eta_q} \|G_\gamma(t) f\|_{\dot{H}^{\beta-1}_q} \leq C \|f\|_{\frac{n}{2\gamma-\beta}} \text{ and } \lim_{t \to 0^+} t^{\eta_q} \|G_\gamma(t)\theta_0\|_{\dot{H}^{\beta-1}_q} = 0, \tag{2.7}$$

where $\eta_q = \frac{2\gamma - 1}{2\gamma} - \frac{n}{2\gamma q}$ and C is independent of f and $0 < T \leq \infty$. The inequality in (2.7) *still holds true in the case $q = \frac{n}{2\gamma - \beta}$ and $\beta = 1$.*

Proof: The estimate (2.5) is well-known (see e.g. [**3**] for a proof). Also, (2.5) still holds true by replacing ∇_x^k by $(-\Delta)^{\frac{|k|}{2}}$. In view of the latter comment and $(-\Delta)^{\frac{s_2}{2}} = (-\Delta)^{\frac{s_2 - s_1}{2}} (-\Delta)^{\frac{s_1}{2}}$, we obtain (2.6) from (2.5) because $G_\gamma(t)$ commutates with $(-\Delta)^{\frac{s_1}{2}}$. The estimate in (2.7) comes from (2.6) with $p_2 = q$, $s_2 = \beta - 1$, $p_1 = \frac{n}{2\gamma - \beta}$, and $s_1 = 0$. Due to (2.6), it is easy to see that the limit in (2.7) holds true for every $\theta_0 \in L^{\frac{n}{2\gamma-\beta}} \cap \dot{H}^{\beta-1}_q$. This fact together with $\overline{L^{\frac{n}{2\gamma-\beta}} \cap \dot{H}^{\beta-1}_q}^{\|\cdot\|_{\frac{n}{2\gamma-\beta}}} = L^{\frac{n}{2\gamma-\beta}}$ and the estimate in (2.7) yield the limit in (2.7), for every $\theta_0 \in L^{\frac{n}{2\gamma-\beta}}(\mathbb{R}^n)$. □

3. Results

This section is devoted to the statements of the results whose proofs will be performed in Section 5.

Theorem 3.1 (Global solutions)**.** *Assume the condition* (1.14) *and let $\eta_q = \frac{2\gamma-1}{2\gamma} - \frac{n}{2\gamma q}$ and $\tilde{\eta}_q = \frac{2\gamma-\beta}{2\gamma} - \frac{n}{2\gamma q}$. If $\theta_0 \in L^{\frac{n}{2\gamma-\beta}}(\mathbb{R}^n)$ then there is a unique global solution $\theta \in BC([0,\infty); L^{\frac{n}{2\gamma-\beta}}(\mathbb{R}^n))$ for* (1.1)*–*(1.3) *such that*

$$t^{\tilde{\eta}_q} \theta \in BC((0,\infty), L^q(\mathbb{R}^n)), \quad \text{for all } \frac{n}{2\gamma - \beta} < q \leq \infty, \tag{3.1}$$

$$t^{\eta_q} \theta \in C((0,\infty); \dot{H}^{\beta-1}_q(\mathbb{R}^n)), \quad \text{for all } \frac{n}{2\gamma - \beta} \leq q < \infty, \tag{3.2}$$

where, as $t \to 0^+$, the limits of $t^{\eta_q}\theta$ in (3.2) *(except the case $(\beta, q) = (1, \frac{n}{2\gamma-\beta})$) and $t^{\tilde{\eta}_q}\theta$ in* (3.1) *are zero. Moreover, if $\theta_0 \in L^1(\mathbb{R}^n) \cap$*

$L^{\frac{n}{2\gamma-\beta}}(\mathbb{R}^n)$ *and* $1 < q \leq \infty$, *then* $\theta \in BC([0,\infty); L^1(\mathbb{R}^n) \cap L^{\frac{n}{2\gamma-\beta}}(\mathbb{R}^n))$ *and*

$$t^{\tilde{\eta}_q + \frac{n+\beta}{2\gamma} - 1}\theta \in BC((0,\infty); L^q(\mathbb{R}^n)). \tag{3.3}$$

Remark 3.2 (Continuous dependence on initial data). The proof of Theorem 3.1 also gives that the solution θ depends continuously on the data θ_0 in finite time intervals $[0,T]$. More precisely, if $\theta_{k,0} \to \theta_0$ in $L^{\frac{n}{2\gamma-\beta}}(\mathbb{R}^n)$ then $\theta_k \to \theta$ in $C([0,T]; L^{\frac{n}{2\gamma-\beta}}(\mathbb{R}^n))$, for all $0 < T < \infty$, where θ_k is the solution with initial data $\theta_{k,0}$.

Remark 3.3. One can treat the range $0 \leq \beta < 1$ by modifying the estimates of Section 4 (particularly (4.1) and (4.4)). For that matter, one should replace $\sup_{0<t<T} t^{\eta_l}\|\theta(\cdot,t)\|_{\dot{H}_l^{\beta-1}}$ by $\sup_{0<t<T} t^{\tilde{\eta}_l}\|\theta(\cdot,t)\|_l$ into those estimates ($l = q, r$). In fact, due to Hardy–Littlewood–Sobolev inequality, this case is easier to handling than $\beta > 1$ and it is not necessary to use norms based on homogeneous Sobolev spaces in order to prove existence of solutions.

Before proceeding, we recall the concept of even and odd symmetry. A function h is said to be even (resp. odd) when $h(x) = h(-x)$ (resp. $h(x) = -h(-x)$).

Theorem 3.4 (Symmetry). *Under the hypotheses of Theorem 3.1.*

(i) *The solution* $\theta(x,t)$ *is odd (resp. even) for all* $t > 0$, *provided that* θ_0 *and* P_i*'s are odd (resp. even).*

(ii) *Let* $P(\xi)$ *be as in* (1.5). *If* θ_0 *and* $\mathrm{div}_\xi(P(\xi))$ *are radially symmetric then* $\theta(x,t)$ *is radially symmetric for all* $t > 0$.

Remark 3.5. Adapting the arguments in the proof of Theorem 3.4, one also can prove the following non-symmetry results: if θ_0 is not odd (resp. not even) and P_i's are odd (resp. even) then $\theta(x,t)$ is not odd (resp. not even). Also, if θ_0 is nonradial and $\mathrm{div}_\xi(P(\xi))$ is radial, then $\theta(x,t)$ is not radially symmetric.

4. Bilinear estimates

This part of the article is devoted to estimates for the bilinear term (1.9).

Lemma 4.1. *Let* $0 < T \leq \infty$, $n \geq 2$, $1 \leq \beta < 2\gamma < \infty$, *and let* $1 < q < \infty$ *be such that* $\frac{\beta-1}{n} < \frac{1}{q} < \frac{2\gamma-1}{n}$. *Denote* $\eta_q = \frac{2\gamma-1}{2\gamma} - \frac{n}{2\gamma q}$ *and* $\tilde{\eta}_q = \frac{2\gamma-\beta}{2\gamma} - \frac{n}{2\gamma q}$.

(i) *If* $\frac{2\gamma-(\beta+1)}{n}-\frac{1}{q}<\frac{1}{r}\leq\frac{1}{q'}$ *and* $q'\leq p\leq\infty$ *then there are positive constants* K_1, K_2, K_3, *independent of* θ, ϕ, *and* T, *such that*

$$\sup_{0<t<T} t^{\tilde{\eta}_r}\|B(\theta,\phi)\|_r\leq K_1\sup_{0<t<T} t^{\eta_q}\|\theta\|_{\dot{H}_q^{\beta-1}}\sup_{0<t<T} t^{\tilde{\eta}_r}\|\phi\|_r, \tag{4.1}$$

$$\sup_{0<t<T}\|B(\theta,\phi)\|_p\leq K_2\sup_{0<t<T} t^{\eta_q}\|\theta\|_{\dot{H}_q^{\beta-1}}\sup_{0<t<T}\|\phi\|_p, \tag{4.2}$$

$$\sup_{0<t<T}\|B(\theta,\phi)\|_1\leq K_3T^{\frac{2\gamma-1}{2\gamma}-\eta_q}\sup_{0<t<T} t^{\eta_q}\|\theta\|_{\dot{H}_q^{\beta-1}}\sup_{0<t<T}\|\phi\|_{q'}. \tag{4.3}$$

(ii) *If* $\frac{2\gamma-2}{n}-\frac{1}{q}<\frac{1}{r}<\frac{n+\beta-1}{n}-\frac{1}{q}$ *then*

$$\sup_{0<t<T} t^{\eta_r}\|B(\theta,\phi)\|_{\dot{H}_r^{\beta-1}}\leq K_4\sup_{0<t<T} t^{\eta_r}\|\theta\|_{\dot{H}_r^{\beta-1}}\sup_{0<t<T} t^{\eta_q}\|\phi\|_{\dot{H}_q^{\beta-1}}, \tag{4.4}$$

where $K_4>0$ *is a constant independent of* θ, ϕ, *and* T.

Proof: Part (i). Let $p_1=p$ and $p_2=r$. Notice that $\frac{p_iq}{p_i+q}\geq 1$ because $p,r\geq q'$. Using Lemma 2.2 and Hölder inequality, we estimate

$$\begin{aligned}\|B(\theta,\phi)\|_{p_i}&\leq\int_0^t\|\nabla_xG_\gamma(t-s)(P[\theta]\phi)(s)\|_{p_i}\,ds\\&\leq C\int_0^t(t-s)^{-\frac{1}{2\gamma}-\frac{n}{2\gamma q}}\|(P[\theta]\phi)(s)\|_{\frac{p_iq}{p_i+q}}\,ds\\&\leq C\int_0^t(t-s)^{-\frac{1}{2\gamma}-\frac{n}{2\gamma q}}\|P[\theta(s)]\|_q\|\phi(s)\|_{p_i}\,ds\\&\leq C\int_0^t(t-s)^{-\frac{1}{2\gamma}-\frac{n}{2\gamma q}}\|\theta(s)\|_{\dot{H}_q^{\beta-1}}\|\phi(s)\|_{p_i}\,ds,\end{aligned} \tag{4.5}$$

where $i=1,2$ and in the third line we have used Lemma 2.1 in order to infer

$$\|P[\theta]\|_q\leq C\|\theta\|_{\dot{H}_q^{\beta-1}}.$$

Therefore

$$\|B(\theta,\phi)\|_p\leq C\,I_1(t)\sup_{0<t<T}\|\phi(t)\|_p\sup_{0<t<T} t^{\eta_q}\|\theta(t)\|_{\dot{H}_q^{\beta-1}}, \tag{4.6}$$

$$\|B(\theta,\phi)\|_r\leq C\,I_2(t)\sup_{0<t<T} t^{\tilde{\eta}_r}\|\phi(t)\|_r\sup_{0<t<T} t^{\eta_q}\|\theta(t)\|_{\dot{H}_q^{\beta-1}}, \tag{4.7}$$

where the integrals $I_1(t)$ and $I_2(t)$ can be computed as

$$
\begin{aligned}
I_1(t) &= \int_0^t (t-s)^{-\frac{1}{2\gamma}-\frac{n}{2\gamma q}} s^{-\eta_q}\, ds \\
&= \int_0^1 (1-s)^{\eta_q-1} s^{-\eta_q}\, ds = C < \infty,
\end{aligned} \tag{4.8}
$$

$$
\begin{aligned}
I_2(t) &= \int_0^t (t-s)^{-\frac{1}{2\gamma}-\frac{n}{2\gamma q}} s^{-\eta_q-\tilde{\eta}_r}\, ds \\
&= t^{\eta_q-1-\eta_q-\tilde{\eta}_r+1} \int_0^1 (1-s)^{\eta_q-1} s^{-\eta_q-\tilde{\eta}_r}\, ds = C\, t^{-\tilde{\eta}_r}.
\end{aligned} \tag{4.9}
$$

The estimates (4.1) and (4.2) follow from (4.7) with (4.9), and (4.6) with (4.8), respectively.

Moreover, we have that

$$
\begin{aligned}
\|B(\theta,\phi)(t)\|_1 &\le \int_0^t \|\nabla_x G_\gamma(t-s)(P[\theta]\phi)(s)\|_1\, ds \\
&\le C \int_0^t (t-s)^{-\frac{1}{2\gamma}} \|P[\theta]\|_q \|\phi\|_{q'}\, ds \\
&\le C \int_0^t (t-s)^{-\frac{1}{2\gamma}} \|\theta\|_{\dot{H}_q^{\beta-1}} \|\phi\|_{q'}\, ds \\
&\le C \int_0^t (t-s)^{-\frac{1}{2\gamma}} s^{-\eta_q}\, ds \sup_{0<t<T} t^{\eta_q} \|\theta\|_{\dot{H}_q^{\beta-1}} \sup_{0<t<T} \|\phi\|_{q'} \\
&\le C T^{1-\frac{1}{2\gamma}-\eta_q} \sup_{0<t<T} t^{\eta_q} \|\theta\|_{\dot{H}_q^{\beta-1}} \sup_{0<t<T} \|\phi\|_{q'},
\end{aligned}
$$

which gives (4.3).

Part (ii). Consider

$$
\frac{1}{h} = \frac{1}{r} + \frac{1}{q} - \frac{\beta-1}{n} \text{ and } \delta = \frac{n}{h} - \frac{n}{r}. \tag{4.10}
$$

Note that $\frac{\beta+\delta}{2\gamma} < 1$ because $\frac{1}{q} < \frac{2\gamma-1}{n}$. We employ the continuous inclusion $\dot{H}_h^{\beta-1+\delta} \subset \dot{H}_r^{\beta-1}$, (2.6) and afterwards (2.4) to obtain

$$\begin{aligned}
\|B(\theta,\phi)\|_{\dot{H}_r^{\beta-1}} &\leq \int_0^t \|G_\gamma(t-s)[\nabla_x\cdot(P[\theta]\phi)(s)]\|_{\dot{H}_r^{\beta-1}}\, ds \\
&\leq \int_0^t \|G_\gamma(t-s)[\nabla_x\cdot(P[\theta]\phi)(s)]\|_{\dot{H}_h^{\beta-1+\delta}}\, ds \\
&\leq \int_0^t (t-s)^{-\frac{\beta+\delta}{2\gamma}} \|\nabla_x\cdot(P[\theta]\phi)(s)]\|_{\dot{H}_h^{-1}}\, ds \\
&\leq \int_0^t (t-s)^{-\frac{\beta+\delta}{2\gamma}} \|P[\theta]\phi(s)\|_h\, ds. \qquad (4.11)
\end{aligned}$$

In view of (4.10), we can choose $1 < l < \infty$ in such a way that $l > q$, $\frac{1}{h} = \frac{1}{r} + \frac{1}{l}$, and $\frac{1}{l} = \frac{1}{q} - \frac{\beta-1}{n}$. Then, Hölder inequality, (2.4), and Sobolev embedding (2.2) imply that

$$\begin{aligned}
\|P[\theta]\phi\|_h &\leq \|P[\theta]\|_r \|\phi\|_l \\
&\leq \|\theta\|_{\dot{H}_r^{\beta-1}} \|\phi\|_{\dot{H}_q^{\beta-1}}.
\end{aligned} \qquad (4.12)$$

Inserting (4.12) into (4.11), we get

$$\begin{aligned}
\|B(\theta,\phi)\|_{\dot{H}_r^{\beta-1}} &\leq C\int_0^t (t-s)^{-\frac{\beta+\delta}{2\gamma}} \|\theta\|_{\dot{H}_r^{\beta-1}} \|\phi\|_{\dot{H}_q^{\beta-1}}\, ds \\
&\leq C\int_0^t (t-s)^{-\frac{\beta+\delta}{2\gamma}} s^{-\eta_r-\eta_q}\, ds \\
&\quad \times \left(\sup_{0<t<T} t^{\eta_r} \|\theta(t)\|_{\dot{H}_r^{\beta-1}} \sup_{0<t<T} t^{\eta_q} \|\phi(t)\|_{\dot{H}_q^{\beta-1}} \right) \\
&\leq t^{-\frac{\beta+\delta}{2\gamma}-\eta_r-\eta_q+1} \int_0^1 (1-s)^{-\frac{\beta+\delta}{2\gamma}} s^{-\eta_r-\eta_q}\, ds \\
&\quad \times \left(\sup_{0<t<T} t^{\eta_r} \|\theta(t)\|_{\dot{H}_r^{\beta-1}} \sup_{0<t<T} t^{\eta_q} \|\phi(t)\|_{\dot{H}_q^{\beta-1}} \right) \\
&\leq Ct^{-\eta_r} \left(\sup_{0<t<T} t^{\eta_r} \|\theta(t)\|_{\dot{H}_r^{\beta-1}} \sup_{0<t<T} t^{\eta_q} \|\phi(t)\|_{\dot{H}_q^{\beta-1}} \right),
\end{aligned}$$

which is equivalent to (4.4). □

5. Proofs

5.1. Local in time solutions. We start by recalling an abstract lemma in Banach spaces which is useful in order to avoid extensive fixed point computations (see e.g. [**28**, Theorem 9]).

Lemma 5.1. *Let X be a Banach space with norm $\|\cdot\|_X$, and $B\colon X\times X\to X$ be a continuous bilinear map, i.e., there exists $K>0$ such that*

$$\|B(x_1,x_2)\|_X \le K\|x_1\|_X\|x_2\|_X,$$

for all $x_1,x_2\in X$. Given $0<\varepsilon<\frac{1}{4K}$ and $y\in X$ such that $\|y\|_X\le\varepsilon$, there exists a solution $x\in X$ for the equation $x=y+B(x,x)$ such that $\|x\|_X\le 2\varepsilon$. The solution x is unique in the closed ball $\{x\in X : \|x\|_X\le 2\varepsilon\}$. Moreover, the solution depends continuously on y in the following sense: If $\|\tilde y\|_X\le\varepsilon$, $\tilde x=\tilde y+B(\tilde x,\tilde x)$, and $\|\tilde x\|_X\le 2\varepsilon$, then

$$\|x-\tilde x\|_X \le \frac{1}{1-4K\varepsilon}\|y-\tilde y\|_X.$$

Remark 5.2 (Picard sequence)*.* The solution given by Lemma 5.1 can be obtained as the limit in X of the Picard sequence $\{x_k\}_{k\in\mathbb{N}}$ where $x_1=y$ and $x_{k+1}=y+B(x_k,x_k)$, for all $k\in\mathbb{N}$. Moreover, $\|x_k\|_X\le 2\varepsilon$ for all $k\in\mathbb{N}$.

The following proposition shows that (1.1)–(1.3) is locally in time well-posed for $L^{\frac{n}{2\gamma-\beta}}$-data.

Proposition 5.3 (Local in time solutions)**.** *Assume* (1.14) *and let q be such that*

$$\max\left\{\frac{\beta-1}{n},\frac{\gamma-1}{n}\right\}<\frac{1}{q}<\min\left\{\frac{2\gamma-\beta}{n},\frac{n+\beta-2\gamma}{n},\frac{n+\beta-1}{2n}\right\}. \tag{5.1}$$

If $\theta_0\in L^{\frac{n}{2\gamma-\beta}}(\mathbb{R}^n)$ then there exists $T>0$ such that (1.1)*–*(1.3) *has a unique mild solution θ in the class*

$$t^{\eta_q}\theta\in BC((0,T);\dot H_q^{\beta-1}(\mathbb{R}^n)) \text{ and } \lim_{t\to 0^+} t^{\eta_q}\|\theta\|_{\dot H_q^{\beta-1}}=0. \tag{5.2}$$

Moreover, $\theta\in BC([0,T);L^{\frac{n}{2\gamma-\beta}}(\mathbb{R}^n))$ and $t^{\frac{\beta-1}{2\gamma}}\theta\in BC((0,T);\dot H^{\beta-1}_{\frac{n}{2\gamma-\beta}}(\mathbb{R}^n))$. If $\beta>1$ then the limit of $t^{\frac{\beta-1}{2\gamma}}\theta$ in $\dot H^{\beta-1}_{\frac{n}{2\gamma-\beta}}$ is zero, as $t\to 0^+$.

Proof: Step 1. For $T>0$, let us define the Banach space

$$\mathcal{E}_T=\left\{\theta \text{ measurable}; t^{\eta_q}\theta\in BC((0,T);\dot H_q^{\beta-1}(\mathbb{R}^n))\right\}$$

with norm given by

$$\|\theta\|_{\mathcal{E}_T}=\sup_{0<t<T} t^{\eta_q}\|\theta(\cdot,t)\|_{\dot H_q^{\beta-1}}. \tag{5.3}$$

Due to (2.7) in Lemma 2.2 and $\theta_0 \in L^{\frac{n}{2\gamma-\beta}}(\mathbb{R}^n)$, for any $\varepsilon > 0$, there exists a $T > 0$ such that

$$\sup_{0<t<T} t^{\eta_q} \|G_\gamma(t)\theta_0\|_{\dot{H}_q^{\beta-1}} \le \varepsilon. \tag{5.4}$$

Take $0 < \varepsilon < \frac{1}{4K_4}$ where K_4 is as in (4.4) with $r = q$. In view of (5.4) and (4.4), we can apply Lemma 5.1 in $\mathcal{E}_T$ to obtain a unique solution $\theta(x,t)$ for (1.8) such that

$$\sup_{0<t<T} t^{\eta_q} \|\theta\|_{\dot{H}_q^{\beta-1}} \le 2\varepsilon. \tag{5.5}$$

Using (2.7) and an induction argument, one can shows that every element θ_k of the Picard sequence

$$\theta_1(x,t) = G_\gamma(t)\theta_0(x), \tag{5.6}$$

$$\theta_{k+1}(x,t) = \theta_1(x,t) + B(\theta_k, \theta_k), \quad k \in \mathbb{N}, \tag{5.7}$$

satisfies $\lim_{t\to 0^+} t^{\eta_q} \|\theta_k\|_{\dot{H}_q^{\beta-1}} = 0$. Then the second property in (5.2) follows from the fact that the fixed point θ is the limit in $\mathcal{E}_T$ of $\{\theta_k\}_{k\in\mathbb{N}}$ (see Remark 5.2). Further details are left to the reader.

Step 2. In what follows we show that $\theta \in BC([0,T); L^{\frac{n}{2\gamma-\beta}}(\mathbb{R}^n))$. We have that the recursive sequence (5.6)–(5.7) satisfies (see Remark 5.2)

$$\sup_{0<t<T} t^{\eta_q} \|\theta_k\|_{\dot{H}_q^{\beta-1}} \le 2\varepsilon, \quad \text{for all } k \in \mathbb{N}. \tag{5.8}$$

Using Lemma 2.2, (4.2) with $p = \frac{n}{2\gamma-\beta}$, and (5.8), we get

$$\sup_{0<t<T} \|\theta_1(t)\|_{\frac{n}{2\gamma-\beta}} \le C\|\theta_0\|_{\frac{n}{2\gamma-\beta}},$$

and

$$\begin{aligned} &\sup_{0<t<T} \|\theta_{k+1}(t)\|_{\frac{n}{2\gamma-\beta}} \\ &\qquad \le C\|\theta_0\|_{\frac{n}{2\gamma-\beta}} + K_2 \sup_{0<t<T} t^{\eta_q}\|\theta_k(t)\|_{\dot{H}_q^{\beta-1}} \sup_{0<t<T} \|\theta_k(t)\|_{\frac{n}{2\gamma-\beta}} \\ &\qquad \le C\|\theta_0\|_{\frac{n}{2\gamma-\beta}} + 2\varepsilon K_2 \sup_{0<t<T} \|\theta_k(t)\|_{\frac{n}{2\gamma-\beta}}, \quad \text{for all } k \in \mathbb{N}. \end{aligned} \tag{5.9}$$

By reducing $T > 0$ in (5.4) if necessary, we can consider $0 < \varepsilon < \min\{\frac{1}{4K_4}, \frac{1}{2K_2}\}$. Since $2K_2\varepsilon < 1$, an induction argument in (5.9) shows that $\{\theta_k\}_{k\in\mathbb{N}}$ is uniformly bounded in $L^\infty((0,T); L^{\frac{n}{2\gamma-\beta}}(\mathbb{R}^n))$ and then there exists a subsequence of $\{\theta_k\}_{k\in\mathbb{N}}$ (denoted in the same way) that converges toward $\tilde{\theta}$ weak-$*$ in that space and consequently in $\mathcal{D}'(\mathbb{R}^n \times [0,T))$. Because $\theta_k \to \theta$ in $\mathcal{E}_T$, which implies convergence in $\mathcal{D}'(\mathbb{R}^n \times [0,T))$, the uniqueness of the limit in the sense of distributions yields $\theta =$

$\tilde{\theta} \in L^{\infty}((0,T); L^{\frac{n}{2\gamma-\beta}}(\mathbb{R}^n))$. The time-continuity of θ follows from standard arguments by using that θ verifies (1.8), $\theta \in L^{\infty}((0,T); L^{\frac{n}{2\gamma-\beta}}(\mathbb{R}^n)) \cap \mathcal{E}_T$, and the second property in (5.2) (see e.g. [**22, 23**]).

Step 3. Here we deal with the statement about $t^{\frac{\beta-1}{2\gamma}}\theta$ in $\dot{H}^{\beta-1}_{\frac{n}{2\gamma-\beta}}(\mathbb{R}^n)$. Noting that $\eta_{\frac{n}{2\gamma-\beta}} = \frac{\beta-1}{2\gamma}$, a direct application of Lemma 2.2 gives

$$\sup_{0<t<T} t^{\frac{\beta-1}{2\gamma}} \|\theta_1(t)\|_{\dot{H}^{\beta-1}_{\frac{n}{2\gamma-\beta}}} \leq C\|\theta_0\|_{\frac{n}{2\gamma-\beta}}. \tag{5.10}$$

Next, we use (5.10), (4.4) with $r = \frac{n}{2\gamma-\beta}$, and (5.8) to obtain

$$\begin{aligned} &\sup_{0<t<T} t^{\frac{\beta-1}{2\gamma}} \|\theta_{k+1}(t)\|_{\dot{H}^{\beta-1}_{\frac{n}{2\gamma-\beta}}} \\ &\leq C\|\theta_0\|_{\frac{n}{2\gamma-\beta}} + K_4 \sup_{0<t<T} t^{\frac{\beta-1}{2\gamma}} \|\theta_k(t)\|_{\dot{H}^{\beta-1}_{\frac{n}{2\gamma-\beta}}} \sup_{0<t<T} t^{\eta_q}\|\theta_k(t)\|_{\dot{H}^{\beta-1}_q} \\ &\leq C\|\theta_0\|_{\frac{n}{2\gamma-\beta}} + 2\varepsilon K_4 \sup_{0<t<T} t^{\frac{\beta-1}{2\gamma}} \|\theta_k(t)\|_{\dot{H}^{\beta-1}_{\frac{n}{2\gamma-\beta}}}, \quad \text{for all } k \in \mathbb{N}, \end{aligned} \tag{5.11}$$

where K_4 is the constant in (4.4) when $r = \frac{n}{2\gamma-\beta}$ and $q > r$ is as in (5.1). Now, the remainder of the proof follows similarly to Step 2 by using (5.11) instead of (5.9). $\square$

In the next proposition we investigate the L^1-persistence of the solutions obtained in Proposition 5.3.

Proposition 5.4. *Under the hypotheses of Proposition 5.3, there exists $T > 0$ such that the solution θ belongs to $BC([0,T); L^1(\mathbb{R}^n))$ provided that $\theta_0 \in L^1(\mathbb{R}^n) \cap L^{\frac{n}{2\gamma-\beta}}(\mathbb{R}^n)$.*

Proof: Let q be such that $1 < q' < \frac{n}{2\gamma-\beta}$. From interpolation, we have that $\theta_0 \in L^{q'}(\mathbb{R}^n)$. Employing (5.8) and the estimate (4.2) with $p = q'$, we get

$$\sup_{0<t<T} \|\theta_1(t)\|_{q'} \leq C\|\theta_0\|_{q'},$$

and

$$\begin{aligned} \sup_{0<t<T} \|\theta_{k+1}(t)\|_{q'} &\leq C\|\theta_0\|_{q'} + K_2 \sup_{0<t<T} t^{\eta_q}\|\theta_k(t)\|_{\dot{H}^{\beta-1}_q} \sup_{0<t<T} \|\theta_k(t)\|_{q'} \\ &\leq C\|\theta_0\|_{q'} + 2K_2\varepsilon \sup_{0<t<T} \|\theta_k(t)\|_{q'}, \quad \text{for all } k \in \mathbb{N}. \end{aligned}$$

Again reducing $T>0$ if necessary, we can consider $2K_2\varepsilon<1$ and proceed similarly to the end of the proof of Proposition 5.3 to obtain that

$$\theta \in BC([0,T);L^{q'}(\mathbb{R}^n)). \tag{5.12}$$

Now we use (2.5), (4.3), (5.2), and (5.12) to estimate

$$\begin{aligned}\sup_{0<t<T}\|\theta(t)\|_1 &\leq C\|\theta_0\|_1+\sup_{0<t<T}\|B(\theta,\theta)\|_1\\ &\leq C\|\theta_0\|_1+K_3T^{1-\frac{1}{2\gamma}-\eta_q}\sup_{0<t<T}t^{\eta_q}\|\theta\|_{\dot{H}^{\beta-1}_q}\sup_{0<t<T}\|\theta\|_{q'}<\infty,\end{aligned}$$

as required. □

The existence time T in Propositions 5.3 and 5.4 may depend on index q. In the next proposition we show that indeed one can take a same small time $T>0$ for all q.

Proposition 5.5. *Under the hypotheses of Proposition 5.3. Let θ be the solution given by Proposition 5.3 with data $\theta_0\in L^{\frac{n}{2\gamma-\beta}}(\mathbb{R}^n)$. There is a $T>0$ such that*

$$t^{\eta_r}\theta\in BC([0,T);\dot{H}^{\beta-1}_r(\mathbb{R}^n)),\quad \textit{for all } \frac{n}{2\gamma-\beta}\leq r<\infty, \tag{5.13}$$

$$t^{\tilde{\eta}_r}\theta\in BC([0,T);L^r(\mathbb{R}^n)),\qquad \textit{for all } \frac{n}{2\gamma-\beta}\leq r<\infty, \tag{5.14}$$

where the values at $t=0$ of $t^{\eta_r}\theta$ in (5.13) (except for $(\beta,r)=(1,\frac{n}{2\gamma-\beta})$) and of $t^{\tilde{\eta}_r}\theta$ in (5.14) (except for $r=\frac{n}{2\gamma-\beta}$)) are zero.

Proof: Let q be fixed as in Proposition 5.3. Given $\frac{n}{2\gamma-\beta}<r<\infty$ verifying $\frac{2\gamma-2}{n}-\frac{1}{q}<\frac{1}{r}<\frac{n+\beta-1}{n}-\frac{1}{q}$, we can use (4.4) instead of (4.1) and proceed just like as in Step 2 of the proof of Proposition 5.3 to obtain (reducing $T>0$ if necessary)

$$t^{\eta_r}\theta\in BC([0,T);\dot{H}^{\beta-1}_r(\mathbb{R}^n)). \tag{5.15}$$

Now let $\frac{2\gamma-2}{n}-\frac{1}{q}<\frac{1}{r}<\frac{2\gamma-1}{n}-\frac{1}{q}$, and consider $r<\tilde{r}<\infty$. Taking $\frac{1}{h}=\frac{1}{q}+\frac{1}{z}=\frac{1}{q}+\frac{1}{r}-\frac{\beta-1}{n}$ and $\delta=\frac{n}{h}-\frac{n}{\tilde{r}}$, it follows that $\delta>0$, $\frac{\beta+\delta}{2\gamma}<1$, and $\eta_q + \eta_r < 1$. Then, we can estimate

$$
\begin{aligned}
\|\theta\|_{\dot{H}^{\beta-1}_{\tilde{r}}} &\leq Ct^{-\eta_{\tilde{r}}}\|\theta_0\|_{\frac{n}{2\gamma-\beta}} \\
&\quad + \int_0^t \|G_\gamma(t-s)\nabla_x \cdot (P[\theta]\theta)(s)\|_{\dot{H}^{\beta-1}_{\tilde{r}}}\, ds \\
&\leq Ct^{-\eta_{\tilde{r}}}\|\theta_0\|_{\frac{n}{2\gamma-\beta}} \\
&\quad + C\int_0^t \|G_\gamma(t-s)\nabla_x \cdot (P[\theta]\theta)(s)\|_{\dot{H}^{\beta-1+\delta}_{h}}\, ds \\
&\leq Ct^{-\eta_{\tilde{r}}}\|\theta_0\|_{\frac{n}{2\gamma-\beta}} \\
&\quad + C\int_0^t (t-s)^{-\frac{\beta+\delta}{2\gamma}}\|\nabla_x\cdot (P[\theta]\theta)(s)\|_{\dot{H}^{-1}_{h}},
\end{aligned} \tag{5.16}
$$

where above we have used Sobolev embedding and afterwards (2.6).

Now we employ (2.4), Hölder inequality, and Sobolev embedding in order to estimate

$$
\begin{aligned}
&\text{R.H.S. of (5.16)} \\
&\leq Ct^{-\eta_{\tilde{r}}}\|\theta_0\|_{\frac{n}{2\gamma-\beta}} + C\int_0^t (t-s)^{-\frac{\beta+\delta}{2\gamma}}\|P[\theta]\theta(s)\|_h\, ds \\
&\leq Ct^{-\eta_{\tilde{r}}}\|\theta_0\|_{\frac{n}{2\gamma-\beta}} + C\int_0^t (t-s)^{-\frac{\beta+\delta}{2\gamma}}\|P[\theta]\|_q\|\theta(s)\|_z\, ds \\
&\leq Ct^{-\eta_{\tilde{r}}}\|\theta_0\|_{\frac{n}{2\gamma-\beta}} + C\int_0^t (t-s)^{-\frac{\beta+\delta}{2\gamma}}\|\theta\|_{\dot{H}^{\beta-1}_{q}}\|\theta\|_{\dot{H}^{\beta-1}_{r}}\, ds \\
&\leq Ct^{-\eta_{\tilde{r}}}\|\theta_0\|_{\frac{n}{2\gamma-\beta}} + C\int_0^t (t-s)^{-\frac{\beta+\delta}{2\gamma}} s^{-\eta_q-\eta_r}\, ds \\
&\qquad \times \Big(\sup_{0<t<T} t^{\eta_q}\|\theta\|_{\dot{H}^{\beta-1}_{q}}\Big)\Big(\sup_{0<t<T} t^{\eta_r}\|\theta\|_{\dot{H}^{\beta-1}_{r}}\Big) \\
&\leq Ct^{-\eta_{\tilde{r}}}\|\theta_0\|_{\frac{n}{2\gamma-\beta}} + Ct^{-\eta_{\tilde{r}}}\int_0^1 (1-s)^{-\frac{\beta+\delta}{2\gamma}} s^{-\eta_q-\eta_r}\, ds \\
&\leq Ct^{-\eta_{\tilde{r}}},
\end{aligned} \tag{5.17}
$$

and then we arrive at (5.15) with $\tilde{r}$ in place of r, and with the same existence time $T > 0$. In Proposition 5.3, we already have proved that

$$
t^{\frac{\beta-1}{2\gamma}}\theta \in BC([0,T); \dot{H}^{\beta-1}_{\frac{n}{2\gamma-\beta}}(\mathbb{R}^n)), \tag{5.18}
$$

which is (5.15) with $r = \frac{n}{2\gamma-\beta}$. In view of (2.1), we can interpolate (5.15) (with $\tilde{r}$ in place of r) and (5.18) in order to obtain that (5.15) holds true for every $r = l$ such that $\frac{n}{2\gamma-\beta} \leq l \leq \tilde{r}$ (and the same $T > 0$). Since $\tilde{r} > r$ is arbitrary, we get (5.15) with $r = l$ (and the same $T > 0$), for all $\frac{n}{2\gamma-\beta} \leq l < \infty$, which gives (5.13).

The proof of (5.14) can be performed in a similar way by using (4.1) instead of (4.4). □

5.2. Proof of Theorem 3.1.

Step 1: Local smoothness and maximum principle. Since we are assuming (1.14), Proposition 5.3 assures the existence of a local-in-time mild solution $\theta(x,t)$. In fact, this solution is C^∞-smooth for any $t > 0$ belonging to the existence interval $(0,T)$. This smooth effect holds for several parabolic equations in several frameworks, like e.g. L^p, weak-L^p, Morrey, Besov–Morrey, when mild solutions are constructed by using time-weighted norms of Kato type (see [**22**]). Precisely, adapting arguments from [**22**] (see also [**2**]), one can obtain that the solution verifies

$$\partial_t^m \nabla_x^k \theta(x,t) \in C((0,T); L^{\frac{n}{2\gamma-\beta}}(\mathbb{R}^n) \cap L^q(\mathbb{R}^n)), \tag{5.19}$$

for all $\frac{n}{2\gamma-\beta} < q < \infty$, $m \in \{0\} \cup \mathbb{N}$, and multi-index $k \in (\{0\} \cup \mathbb{N})^n$, where $T > 0$ is the existence time given in Proposition 5.5. In particular, it follows that $\theta(x,t) \in C^\infty(\mathbb{R}^n \times (0,T))$ and $\theta(t) \in L^\infty(\mathbb{R}^n)$ with

$$\|\theta(t)\|_\infty \leq C\|\theta(t)\|^{\alpha}_{\frac{n^2}{2\gamma-\beta}} \|\nabla_x \theta(t)\|^{1-\alpha}_{\frac{n^2}{2\gamma-\beta}}, \tag{5.20}$$

for all $0 < t < T$, where $\alpha = \frac{n+\beta-2\gamma}{n}$. If further $\theta_0 \in L^1(\mathbb{R}^n) \cap L^{\frac{n}{2\gamma-\beta}}(\mathbb{R}^n)$ then q in (5.19) can be taken in the range $1 < q < \infty$.

Due to (5.19) we have that θ verifies (1.1)–(1.3) in the classical sense and $\partial_t^m \nabla_x^k \theta(x,t) \to 0$ when $|x| \to \infty$, for all $0 < t < T$. In view of $\nabla \cdot P[\theta] = 0$, we can integrate by parts to obtain

$$\begin{aligned} \frac{\partial}{\partial t}\|\theta(t)\|_p^p &= p\int_{\mathbb{R}^n} \theta(t)^{p-1} \frac{\partial}{\partial t}\theta(t)\, dx \\ &= p\int_{\mathbb{R}^n} \theta(t)^{p-1}(-(-\Delta)^\gamma \theta - \nabla_x \cdot (P[\theta]\theta))\, dx \\ &= -p\int_{\mathbb{R}^n} \theta(t)^{p-1}(-\Delta)^\gamma \theta\, dx \leq -\int_{\mathbb{R}^n} \left|(-\Delta)^{\frac{\gamma}{2}}(\theta^{\frac{p}{2}})\right|^2 dx, \end{aligned} \tag{5.21}$$

for all $t \in (0,T)$. The last inequality in (5.21) can be found in [**9, 13**] (see also [**21**]).

In view of the estimate (5.21), we have that L^p-norms of $\theta(t)$ are non-increasing in $(0,T)$. If $\theta_0 \in L^{\frac{n}{2\gamma-\beta}}(\mathbb{R}^n)$ and $\theta_0 \in L^1(\mathbb{R}^n) \cap L^{\frac{n}{2\gamma-\beta}}(\mathbb{R}^n)$, we obtain respectively

$$\|\theta(t)\|_{\frac{n}{2\gamma-\beta}} \le \|\theta(t_0)\|_{\frac{n}{2\gamma-\beta}} \text{ and } \|\theta(t)\|_1 \le \|\theta(t_0)\|_1, \tag{5.22}$$

for $0 < t_0 \le t < T$.

Making $t_0 \to 0^+$ in (5.22), it follows that the solution $\theta(x,t)$ satisfies

$$\|\theta(t)\|_{\frac{n}{2\gamma-\beta}} \le \|\theta_0\|_{\frac{n}{2\gamma-\beta}} \text{ and } \|\theta(t)\|_1 \le \|\theta_0\|_1, \tag{5.23}$$

for all $t \in (0,T)$, when $\theta_0 \in L^{\frac{n}{2\gamma-\beta}}(\mathbb{R}^n)$ and $\theta_0 \in L^1(\mathbb{R}^n) \cap L^{\frac{n}{2\gamma-\beta}}(\mathbb{R}^n)$, respectively.

Step 2: Extension of the local solution. We start by making the following observation: if $\frac{n}{2\gamma-\beta} < q < \infty$ and $\theta_0 \in L^q(\mathbb{R}^n)$ then

$$t^{\eta_q}\|G_\gamma(t)\theta_0\|_{\dot{H}_q^{\beta-1}} \le Ct^{\tilde{\eta}_q}\|\theta_0\|_q \to 0 \text{ when } t \to 0^+. \tag{5.24}$$

Therefore, for $\theta_0 \in L^{\frac{n}{2\gamma-\beta}}(\mathbb{R}^n) \cap L^{q_0}(\mathbb{R}^n)$ with q_0 as in (5.1), the existence time $T > 0$ obtained in Proposition 5.3 can be taken depending on the norm $\|\theta_0\|_{q_0}$. Indeed it can be chosen as

$$T = \left(\frac{\varepsilon}{C\|\theta_0\|_{q_0}}\right)^{\frac{1}{\tilde{\eta}_{q_0}}}, \tag{5.25}$$

where $0 < \varepsilon < \frac{1}{4K_4}$ and $C > 0$ is as in (5.24).

Now let $\theta_0 \in L^{\frac{n}{2\gamma-\beta}}(\mathbb{R}^n)$. From Propositions 5.3 and 5.5, there exists $T_0 > 0$ and a unique mild solution for (1.1)–(1.3) verifying (5.13)–(5.14) in $[0,T_0)$ such that

$$\sup_{0<t<T_0} \|\theta(t)\|_{\frac{n}{2\gamma-\beta}} \le \|\theta_0\|_{\frac{n}{2\gamma-\beta}}, \tag{5.26}$$

where (5.26) comes from (5.23).

Let us now denote

$$T^* = \sup\left\{\widetilde{T} > 0; \theta \text{ verifies (5.13)–(5.14) and (5.26) in } [0,\widetilde{T})\right\}. \tag{5.27}$$

We desire to prove that $T^* = \infty$. Suppose by contradiction that $T^* < \infty$, and let $a = \theta(T^* - \delta)$ where $0 < \delta < T^*$ will be chosen later. Property (5.14) gives that $a \in L^{\frac{n}{2\gamma-\beta}}(\mathbb{R}^n) \cap L^q(\mathbb{R}^n)$, for all $\frac{n}{2\gamma-\beta} < q < \infty$. Moreover, if $\delta < \frac{T^*}{2}$ we get $\|\theta(T^* - \delta)\|_q \le \|\theta(\frac{T^*}{2})\|_q$. Therefore, taking $\delta < T^*/2$ and a as initial data, we have that given $0 < \varepsilon < \frac{1}{4K_4}$

there exist $T_1 > 0$ and a unique mild solution $\tilde{\theta}$ for (1.1)–(1.3) satisfying (5.13)–(5.14) and (5.26) in $I = [T^* - \delta, T_1 + T^* - \delta)$, that is,

$$t^{\eta_q}\theta \in BC(I; \dot{H}_q^{\beta-1}) \text{ and } t^{\tilde{\eta}_q}\theta \in BC(I; L^q), \tag{5.28}$$

for all $\frac{n}{2\gamma-\beta} \leq q < \infty$. From uniqueness part of Proposition 5.3, it follows that $\theta = \tilde{\theta}$ in $[T^* - \delta, T^*)$. In view of (5.25), we can choose $T_1 = \min\{(\frac{\varepsilon}{C\|\theta(\frac{T^*}{2})\|_q})^{\frac{1}{\tilde{\eta}_q}}, T^*\}$. Taking $0 < \delta < \min\{\frac{T^*}{2}, T_1\}$ and $T_2 = T_1 + T^* - \delta$, we have that $T^* < T_2$ and get a solution

$$t^{\eta_q}\theta \in BC([0, \widetilde{T}); \dot{H}_q^{\beta-1}) \text{ and } t^{\tilde{\eta}_q}\theta \in BC([0, \widetilde{T}); L^q), \tag{5.29}$$

for all $\frac{n}{2\gamma-\beta} \leq q < \infty$, and for all $0 < \widetilde{T} < T_2$, which contradicts the maximality of T^* in (5.27). Therefore $T^* = \infty$ and, in particular, it follows that

$$t^{\eta_q}\theta \in C([0, \infty); \dot{H}_q^{\beta-1}) \text{ and } t^{\tilde{\eta}_q}\theta \in C([0, \infty); L^q), \tag{5.30}$$

for all $\frac{n}{2\gamma-\beta} \leq q < \infty$, where the values at $t = 0^+$ in (5.30) are as in Proposition 5.5.

Step 3: Global L^q-decay of solutions. In view of (5.30), it remains to prove (3.1) and (3.3). We will prove only the part of the statement corresponding to the case $\theta_0 \in L^{\frac{n}{2\gamma-\beta}}(\mathbb{R}^n)$. The estimate (3.3) for $\theta_0 \in L^1(\mathbb{R}^n) \cap L^{\frac{n}{2\gamma-\beta}}(\mathbb{R}^n)$ follows similarly to the first one by using $\|\theta(t)\|_1 \leq \|\theta_0\|_1$ and the sequence $q_k = 2^k$ instead of $\|\theta(t)\|_{\frac{n}{2\gamma-\beta}} \leq \|\theta_0\|_{\frac{n}{2\gamma-\beta}}$ and $q_k = \frac{n}{2\gamma-\beta}2^k$.

Since we have extended the solution θ, it follows that (5.19) and (5.20) hold true for $T = \infty$. Then

$$\|\theta(\cdot, t)\|_\infty < \infty, \quad \text{for all } t > 0. \tag{5.31}$$

Now we proceed as in [**3**] and [**23**]. In view of the Gagliardo–Nirenberg inequality, we have that

$$\|\phi\|_2 \leq C\|\phi\|_1^\alpha \left\|(-\Delta)^{\frac{\gamma}{2}}\phi\right\|_2^{1-\alpha} \text{ with } \alpha = \frac{2\gamma}{n+2\gamma}. \tag{5.32}$$

Taking $\phi = \theta^{\frac{q}{2}}$ in (5.32), it follows that

$$\|\theta\|_q^{q(\frac{n+2\gamma}{n})} \leq C\|\theta\|_{\frac{q}{2}}^{\frac{2q\gamma}{n}} \left\|(-\Delta)^{\frac{\gamma}{2}}(\theta^{\frac{q}{2}})\right\|_2^2. \tag{5.33}$$

Denoting $\psi_q(t) = \|\theta(t)\|_q^q$, we obtain from (5.21) and (5.33) that

$$\frac{\partial}{\partial t}\psi_q \leq -C(\psi_{\frac{q}{2}})^{-\frac{4\gamma}{n}}\psi_q^{\frac{n+2\gamma}{n}}. \tag{5.34}$$

The differential inequality (5.34) can be solved by an induction procedure. In fact, using the first inequality in (5.23) and considering the sequence $q_k = \frac{n}{2\gamma-\beta}2^k$ for $k \geq 0$, we arrive at

$$\psi_{q_k}(t) \leq M_{q_k} t^{-\frac{n}{2}(\frac{2^k-1}{\gamma})}, \tag{5.35}$$

where

$$M_{q_0} = \|\theta_0\|_{\frac{n}{2\gamma-\beta}} \text{ and } M_{q_k} = \left(\frac{n(2^k-1)}{2C\gamma}\right)^{\frac{n}{2\gamma}} M^2_{\frac{q_k}{2}}, \quad \text{for } k \in \mathbb{N}.$$

It follows that

$$\begin{aligned} M_{q_k}^{\frac{1}{q_k}} &= \left(\frac{n(2^k-1)}{2C\gamma}\right)^{\frac{n}{2\gamma q_k}} M_{q_{k-1}}^{\frac{1}{q_{k-1}}} \\ &= \left(\frac{n(2^k-1)}{2C\gamma}\right)^{\frac{n}{2\gamma 2^k q_0}} \left(\frac{n(2^{k-1}-1)}{2C\gamma}\right)^{\frac{n}{2\gamma 2^{k-1} q_0}} M_{q_{k-2}}^{\frac{1}{q_{k-2}}} \\ &\vdots \\ &= \left[\prod_{i=1}^{k}\left(\frac{n(2^i-1)}{2C\gamma}\right)^{\frac{n}{2^i 2\gamma q_0}}\right](M_{q_0})^{\frac{1}{q_0}}, \quad \text{for all } k \in \mathbb{N}, \end{aligned}$$

and then

$$\begin{aligned} \|\theta(t)\|_{2^k q_0} &\leq M_{q_k}^{\frac{1}{q_k}} t^{-\frac{n}{2}(\frac{2^k-1}{\gamma q_k})} \\ &= \left(\prod_{i=1}^{k}\left(\frac{n}{2}\frac{2^i-1}{C\gamma}\right)^{\frac{n}{2^i 2\gamma q_0}}\right)(M_{q_0})^{\frac{1}{q_0}} t^{-\frac{n}{2\gamma q_0}(\frac{2^k-1}{2^k})}, \end{aligned} \tag{5.36}$$

where $q_0 = \frac{n}{2\gamma-\beta}$. In view of (5.31), we can make $k \to \infty$ in (5.36) to obtain

$$\|\theta(t)\|_\infty \leq C\|\theta_0\|_{\frac{n}{2\gamma-\beta}}^{\frac{2\gamma-\beta}{n}} t^{-\frac{2\gamma-\beta}{2\gamma}}, \quad \text{for all } t > 0. \tag{5.37}$$

Interpolating the first inequality in (5.23) with (5.37), the result is

$$\|\theta(t)\|_q \leq C\, t^{-\tilde{\eta}_q}, \text{ for all } t > 0 \text{ and } \frac{n}{2\gamma-\beta} \leq q \leq \infty,$$

as required.

The uniqueness statement follows from the local uniqueness property in Proposition 5.3. □

5.3. Proof of Theorem 3.4.

Part (i). We will prove only the odd part of the statement since the even one follows similarly. Let θ be the solution of Proposition 5.3 with existence time $T>0$. From Step 2 of the proof of Theorem 3.1, θ can be extended by using Proposition 5.3 and solving (1.1)–(1.3) consecutively with initial data $\theta(\frac{T}{2})$, $\theta(\frac{T}{2}+T_1)$, $\theta(\frac{T}{2}+2T_1)$, and so on, where $T_1 = \min\{(\frac{\varepsilon}{C\|\theta(\frac{T}{2})\|_{q_0}})^{\frac{1}{\tilde{\eta}_{q_0}}}, T\}$, $\varepsilon = \frac{1}{8K_4}$, and C as in (5.24). Because of that, it is sufficient to show the following claim: if $\theta_0 \in L^{\frac{n}{2\gamma-\beta}}$ is odd then so is the solution $\theta(x,t)$ given by Proposition 5.3, for all $t \in (0,T)$. In fact, notice that we can use this claim repeatedly to show that the global solution $\theta(x,t)$ is odd, for all $t>0$.

Let $\psi(x,t) = G_\gamma(t)\theta_0$. We have that $\theta_0(-x) = -\theta_0(x)$ is equivalent to

$$-\widehat{\theta_0}(\xi) = [\theta_0(-x)]^\wedge(\xi) = \widehat{\theta_0}(-\xi) \text{ in } \mathcal{S}'(\mathbb{R}^n). \tag{5.38}$$

It follows from (5.38) that

$$\begin{aligned}[\psi(-x,t)]^\wedge(\xi) &= e^{-t|\xi|^{2\gamma}}\widehat{\theta_0}(-\xi)\\ &= -e^{-t|\xi|^{2\gamma}}\widehat{\theta_0}(\xi) = -\widehat{\psi(x,t)}(\xi),\end{aligned}$$

which shows that $G_\gamma(t)\theta_0$ is odd, for each fixed $t>0$. Also, if θ is odd then $\nabla\theta$ is even, because

$$\nabla(\theta(x,t)) = \nabla(-\theta(-x,t)) = (\nabla\theta)(-x,t).$$

Recall that $A = (a_{ij})$ and

$$P(\xi) = (\widetilde{P}_1(\xi), \ldots, \widetilde{P}_n(\xi)),$$

where

$$\widetilde{P}_j(\xi) = \sum_{i=1}^{n} a_{ij}\frac{\xi_i I}{|\xi|^2}P_i(\xi).$$

It follows that

$$\widehat{u}(-\xi) = (\widehat{u_1}(-\xi), \ldots, \widehat{u_n}(-\xi)) = P(-\xi)\widehat{\theta}(-\xi,t)$$

with

$$\begin{aligned}P(-\xi) &= \frac{I}{|\xi|^2}[-\xi_1P_1(-\xi), \ldots, -\xi_nP_n(-\xi)]A\\ &= \frac{I}{|\xi|^2}[\xi_1P_1(\xi), \ldots, \xi_nP_n(\xi)]A = P(\xi),\end{aligned}$$

because P_i's are odd. Therefore $u = P[\theta]$ is odd when θ is odd, and then $(u\cdot\nabla\theta) = (P[\theta]\cdot\nabla\theta)$ is odd too. Hence if θ is odd then so is $B(\theta,\theta)$.

So, employing an induction argument, one can prove that each element θ_k of the Picard sequence (5.6)–(5.7) is odd. Since $\theta_k \to \theta$ in the norm (5.3), then the sequence (5.6)–(5.7) also converges (up to a subsequence) to θ a.e. $x \in \mathbb{R}^n$, for all $t \in (0,T)$. It follows that $\theta(x,t)$ is odd, for each fixed $t \in (0,T)$, because pointwise convergence preserves odd symmetry. This shows the desired claim.

Part (ii). From the same reasons given in Part (i), we need only to prove that the local solution of Proposition 5.3 is radially symmetric whenever θ_0 and $\mathrm{div}_\xi(P(\xi))$ are too. For that matter, we first observe that $G_\gamma(t)\theta_0$ is radial because θ_0 and the kernel $\hat{g}_\gamma(\xi,t) = e^{-|\xi|^{2\gamma}t}$ are radial, for all $t > 0$. Also, for θ radially symmetric, we have that

$$
\begin{aligned}
(u \cdot \nabla\theta) &= \sum_{j=1}^{n} u_j \partial_{x_j}\theta = \frac{\theta'(r)}{r}\sum_{j=1}^{n} u_j x_j \\
&= I\frac{\theta'(r)}{r}\sum_{j=1}^{n}(\partial_{\xi_j}\widehat{u_j})^{\vee} = I\frac{\theta'(r)}{r}\sum_{j=1}^{n}\left(\partial_{\xi_j}\widetilde{P}_j(\xi)\widehat{\theta}\right)^{\vee} \\
&= I\frac{\theta'(r)}{r}\left(\widehat{\theta}(\xi,t)(\mathrm{div}_\xi(P(\xi))\right)^{\vee}.
\end{aligned}
\tag{5.39}
$$

It follows from (5.39) that if θ and $(\mathrm{div}_\xi(P(\xi))$ are radial then so is $(u \cdot \nabla\theta)$. Using that $G_\gamma(t)$ preserves radiality, we obtain that $B(\theta,\theta)$ defined in (1.9) is radially symmetric, for each $t \in (0,T)$, whenever θ is too. Analogously to Part (i), we now can use induction in order to show that each function θ_k defined in (5.6)–(5.7) is also radially symmetric. Since θ_k converges (up to a subsequence) to θ a.e. $x \in \mathbb{R}^n$, for each $t \in (0,T)$, we obtain the required conclusion. □

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