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LOW ENERGY CANONICAL IMMERSIONS INTO HYPERBOLIC MANIFOLDS AND STANDARD SPHERES

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Abstract: We consider critical points of the global L^2-norm of the second fundamental form, and of the mean curvature vector of isometric immersions of compact Riemannian manifolds into a fixed background Riemannian manifold, as functionals over the space of deformations of the immersion. We prove new gap theorems for these functionals into hyperbolic manifolds, and show that the celebrated gap theorem for minimal immersions into the standard sphere can be cast as a theorem about their critical points having constant mean curvature function, and whose second fundamental form is suitably small in relation to it. In this case, the various minimal submanifolds that occur at the pointwise upper bound on the norm of the second fundamental form are realized by manifolds of nonnegative Ricci curvature, and of these, the Einstein ones are distinguished from the others by being those that are immersed on the sphere as critical points of the first of the functionals mentioned.

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1. Introduction

Let n and $\tilde{n}$ be any two integers such that $2 \leq n \leq \tilde{n}-1$. Let (M, g) be a closed Riemannian manifold of dimension n isometrically immersed into a background Riemannian manifold $(\tilde{M}, \tilde{g})$ of dimension $\tilde{n}$. Thus, we have an immersion $f\colon M \to \tilde{M}$ of M into $\tilde{M}$ such that $g = f^*\tilde{g}$. It then follows that small deformations of the metric g on M can be realized similarly by a family $f_t\colon M \to \tilde{M}$ of isometric immersions into $(\tilde{M}, \tilde{g})$ that deform the immersion f, and for each such deformation, we obtain a family $(M, f_t^*\tilde{g})$ of Riemannian metrics on M. If we attach to f a suitable functional, we can then single out metrics g on M of this type that are stationary under any deformation f_t of the immersion f that yields g. This is the general theme of our paper.

We let α and H be the second fundamental form and mean curvature vector of an immersion $f\colon M \to \tilde{M}$, respectively. If $d\mu = d\mu_M$ denotes the Riemannian measure on M, we define

$$\Pi(M) = \Pi_f(M) = \int_M \|\alpha\|^2 \, d\mu, \tag{1}$$

$$\Psi(M) = \Psi_f(M) = \int_M \|H\|^2 \, d\mu, \tag{2}$$

and view them as functionals defined over the space of all isometric immersions of M into $\tilde{M}$. Ψ is generally named the Willmore functional in the literature. Willmore used it to study surfaces in $\mathbb{R}^3$ [**11**]. This functional had been studied earlier by Blaschke [**2**] and Thomsen [**10**] also.

We shall say that a Riemannian manifold (M, g) is *canonically placed* into $(\tilde{M}, \tilde{g})$ if it admits an isometric immersion into the latter that is a critical point of Π. Thus, for any family of metrics $g_t = f_t(\tilde{g})$ in M that deforms g, $g_0 = g$, given by a family of immersions $f_t\colon M \hookrightarrow \tilde{M}$, (M, g) is canonically placed into $(\tilde{M}, \tilde{g})$ if f_0 is a critical point of Π_{f_t}. The use of the functional Ψ instead leads to a related, and alternative, notion of a canonical placing of (M, g) into $(\tilde{M}, \tilde{g})$.

In the case when the background manifold is the space form $S_c = S_c^{\tilde{n}}$ of curvature c, we introduce for consideration a third functional given by

$$\Theta_c(M) = \int_M (n(n-1)c + \|H\|^2) \, d\mu. \tag{3}$$

When $c = 1$, we denote Θ_c simply by Θ. We view Π, Ψ, and Θ as energies of the immersion.

The first goal of our work is the study of the critical points of lowest energy of Π and Ψ when the background manifold is the sphere $\mathbb{S}^{n+p}$ with its standard metric. Our results reinterpret the celebrated gap theorem of Simons [**9**] in these contexts, and show that this result can be derived as one about critical points of Ψ and Π of constant mean curvature function, and whose density is pointwise small.

Let us begin our explanation by recalling the classical gap theorem in a form convenient to our work:

Theorem 1 ([**9**, Theorem 5.3.2, Corollary 5.3.2], [**3**, Main Theorem], [**6**, Corollary 2])**.** *Suppose that $M^n \hookrightarrow \mathbb{S}^{n+p}$ is an isometric minimal immersion. Assume that the pointwise inequality $\|\alpha\|^2 \leq np/(2p-1)$ holds everywhere. Then*

(1) *Either* $\|\alpha\|^2 = 0$, *or*

(2) $\|\alpha\|^2 = np/(2p-1)$ *if, and only if, either* $p = 1$ *and* M^n *is the minimal Clifford torus* $\mathbb{S}^m(\sqrt{m/n}) \times \mathbb{S}^{n-m}(\sqrt{(n-m)/n}) \subset \mathbb{S}^{n+1}$, $1 \leq m < n$, *with* $\|\alpha\|^2 = n$, *or* $n = p = 2$ *and* M *is the real projective plane embedded into* $\mathbb{S}^4$ *by the Veronese map with* $\|\alpha\|^2 = 4/3$.

Thus, among minimal n-manifolds $M \hookrightarrow \mathbb{S}^{n+p}$, or critical points of the volume functional of M, the lowest value that $\|\alpha\|^2$ can achieve is isolated and it is achieved by submanifolds inside equatorial totally geodesic spheres, while its first nonzero value is the constant $pn/(2p-1)$, which is achieved by all the Clifford toruses when $p = 1$, or by a minimal real projective plane of scalar curvature $2/3$ in $\mathbb{S}^4$. We prove here that this gap theorem can be reconstructed as a characterization of compact Riemannian manifolds (M, g) that are isometrically embedded into $\mathbb{S}^{n+p}$ as critical points of the functionals (2) or (3), with constant mean curvature function and whose density is pointwise small. The latter conditions make of the said critical points absolute minimums of the functional (2). The gap theorem then produces two types of such critical points, and those that achieve the upper bound for $\|\alpha\|^2$ carry metrics of nonnegative Ricci tensor. Among them, the ones that are Einstein are distinguished by being, in addition, critical points of (1).

For a smooth immersion $M \hookrightarrow \tilde{M}$, we let ν_H denote a normal vector in the direction of the mean curvature vector H, and denote by A_{ν_H} and ∇^ν the shape operator in the direction of ν_H and covariant derivative of the normal bundle, respectively. Such a choice of ν_H can be made to depend smoothly upon smooth deformations of the immersion. We consider immersions that satisfy the estimates

$$\begin{aligned} -\lambda\|H\|^2 - n &\leq \operatorname{trace} A_{\nu_H}^2 - \|H\|^2 - \|\nabla^\nu \nu_H\|^2 \\ &\leq \|\alpha\|^2 - \|H\|^2 - \|\nabla^\nu \nu_H\|^2 \leq \frac{np}{2p-1} \end{aligned} \tag{4}$$

for some constant λ. Notice that $\|A_{\nu_H}\|^2 = \operatorname{trace} A_{\nu_H}^2$ is bounded above by $\|\alpha\|^2$, and so the second of the inequalities above is always true.

Our first result is the following:

Theorem 2. *Suppose that* (M^n, g) *is a closed Riemannian manifold isometrically immersed into* $\mathbb{S}^{n+p}$ *as a critical point of the functional* Ψ *above, and having constant mean curvature function* $\|H\|$. *Assume that the immersion is such that* (4) *holds for some constant* $\lambda \in [0, 1/2)$. *Then* M *is minimal, and so it is a critical point of the functional* Θ *also,* $0 \leq \|\alpha\|^2 \leq np/(2p-1)$, *and either*

(1) $\|\alpha\|^2 = 0$, *in which case M lies in an equatorial sphere, or*

(2) $\|\alpha\|^2 = np/(2p-1)$, *in which case either $p = 1$ and M^n is the Clifford torus $\mathbb{S}^m(\sqrt{m/n})\times\mathbb{S}^{n-m}(\sqrt{(n-m)/n}) \subset \mathbb{S}^{n+1}$, $1\leq m<n$, with $\|\alpha\|^2 = n$, or $n = p = 2$ and M^2 is the real projective plane embedded into $\mathbb{S}^4$ by the Veronese map with $\|\alpha\|^2 = 4/3$ and scalar curvature $2/3$, all cases of metrics with nonnegative Ricci tensor.*

Let us now consider immersions that satisfy the estimates

$$\begin{aligned} -\lambda\|H\|^2-1 &\leq \mathrm{trace}\left(\frac{1}{\|H\|}A_{\nu_H}\sum A^2_{\nu_j}\right)-\frac{1}{2}\|\alpha\|^2-\|\nabla^\nu \nu_H\|^2-\|H\|^2 \\ &\leq \|\alpha\|^2-\|H\|^2-\|\nabla^\nu\nu_H\|^2 \leq \frac{np}{2p-1} \end{aligned} \tag{5}$$

for some constant λ. Here $\{\nu_j\}$ is a local orthonormal frame of the normal bundle, which can be chosen to be such that $\nu_1 = \nu_H$ and to depend smoothly upon deformations of the immersion. The trace term is homogeneous of degree two, and may be defined by continuity in the limit in the case when $\|H\| = 0$. The details will be given below, where we shall see also that the second of the inequalities above is always true.

Our second result distinguishes further the critical points of Ψ obtained in Theorem 2.

Theorem 3. *Suppose that (M^n, g) is a closed Riemannian manifold isometrically immersed into $\mathbb{S}^{n+p}$ as a critical point of the functional Π above, and having constant mean curvature function. Assume that the immersion is such that (5) holds for some constant $\lambda \in [0,1)$. Then M is minimal, so it is a critical point of Ψ, and of Θ also, $0 \leq \|\alpha\|^2 \leq np/(2p-1)$, and either*

(1) $\|\alpha\|^2 = 0$, *in which case M lies inside an equatorial sphere, or*

(2) $\|\alpha\|^2 = np/(2p-1)$, *in which case either $n = p = 2$ and M is a minimal real projective plane with an Einstein metric embedded into $\mathbb{S}^4$, or $p = 1$, $n = 2m$ and M is the Clifford torus $\mathbb{S}^m(\sqrt{1/2})\times\mathbb{S}^m(\sqrt{1/2}) \subset \mathbb{S}^{n+1}$ with its Einstein product metric.*

The following observation follows easily, but it emphasizes the fact that among these submanifolds, we have some that are critical points of the total scalar curvature functional among metrics in M realized by isometric immersions into the sphere $\mathbb{S}^{n+p}$.

Corollary 4. *Let (M, g) be a closed Riemannian manifold that is canonically placed in $\mathbb{S}^{n+p}$ with constant mean curvature function and satisfying* (5) *for some $\lambda \in [0, 1)$. Then M is a minimal critical point of the total scalar curvature functional under deformations of the isometric immersion, and either $\|\alpha\|^2 = 0$ or $\|\alpha\|^2 = np/(2p-1)$. In the latter case, (M, g) is Einstein and the two possible surface cases in codimension $p = 1$ and $p = 2$ correspond to Einstein manifolds that are associated to different critical values of the total scalar curvature.*

Thus, among Riemannian manifolds of constant mean curvature function, the pointwise estimates (5), $\lambda \in [0, 1)$, for a critical point of the L^2-norm of the second fundamental form implies that this is also a minimal critical point of the L^2-norm of the mean curvature vector, and therefore of the total scalar curvature functional, under deformations of the immersion. Of course, the latter in itself does not imply necessarily that (M, g) is Einstein, as the space of metrics on M that can be realized by isometric embeddings into the sphere does not have to equal the space of all Riemannian metrics on M. But if $\|\alpha\|^2 = np/(2p-1)$, the canonically placed submanifold (M, g) is an Einstein critical point of the total scalar curvature functional among metrics on M that can be realized by isometric immersions into the standard sphere $\mathbb{S}^{n+p}$. The cases in Theorem 1 that are excluded in Theorem 3 correspond to critical points of Ψ that are not critical points of Π.

That the functional Π distinguishes the symmetric minimal Clifford torus in $\mathbb{S}^{2m+1}$ from the others have been observed previously [**8**], cf. [**5**, §3] and [**7**, p. 366]. It has been observed also that, even on the sphere, the functionals Π and Ψ contain critical points that are not minimal submanifolds [**8**].

It is slightly easier to prove Theorem 2 by replacing the role that the functional Ψ plays by that of the functional Θ, and derive the same conclusion. The point is not the use of critical points of Θ versus those of Ψ. Rather, since the curvature of the sphere is positive, if the second fundamental form is pointwise small in relation to the mean curvature vector, the constant mean curvature function condition forces the critical points of these functionals to be the same, and minimal.

The pointwise estimates (4) and (5) that we assume on the immersion quantities ensure that the critical submanifolds (M, g) of Ψ or Π that we consider are not too far off from being totally geodesic, among immersions with constant mean curvature function. Thus, the role that the volume functional plays in the gap theorem can be reinterpreted using the functional Ψ, replacing accordingly the minimality condition by that of having a constant mean curvature function, and the estimates (4). This singles out compact immersions into the sphere that minimize the volume, and so Θ. This type of critical points is further distinguished by selecting those that are canonically placed in $\mathbb{S}^{n+p}$ and satisfy the stricter estimates (5), and among these, it singles out those for which the metric g on M is Einstein. That the critical points of Ψ and Π are this closely interconnected is a consequence of Gauss' identity and the fact that the sectional curvature of the standard metric on the sphere is the (positive) constant 1.

Although we cast Simons' gap theorem in this manner, Theorems 2 and 3 are both consequences of nonlinear versions of the first eigenvalue of the Laplacian of the metric g on M, when (M, g) is a critical point of the functionals in question of constant mean curvature function and small density. There is a natural gap theorem for the functionals Ψ or Π themselves that we now derive, and this is the second goal of our work. This result is somewhat dual to that proven for minimal immersions into spheres, and occur on quotients of space forms of negative curvature instead.

We recall that a closed hyperbolic manifold is of the form $\mathbb{H}^m/\Gamma$ for Γ a torsion-free discrete group of isometries of $\mathbb{H}^m$. We have the following:

Theorem 5. *Let M be a critical point of* (2) *on a hyperbolic compact manifold $\mathbb{H}^{n+p}/\Gamma$. If the pointwise inequality $0 \leq \|\alpha\|^2 - \frac{1}{2}\|H\|^2 - \|\nabla^\nu \nu_H\|^2 \leq n$ holds on M, then either $\|H\|^2 = 0$ and M is minimal, or $\|\alpha\|^2 = \frac{1}{2}\|H\|^2 + n = \|A_{\nu_H}\|^2$ and M is a nonminimal submanifold whose mean curvature vector is a covariantly constant section of its normal bundle.*

Theorem 6. *Let M be a critical point of* (1) *on a hyperbolic compact manifold $\mathbb{H}^{n+p}/\Gamma$. If the pointwise inequality*

$$\|\alpha\|^2 \left(\left(3 - \frac{n}{2}\right) \|\alpha\|^2 - \|H\|^2 \right) \leq (n\|\alpha\|^2 + 2\|H\|^2)$$

holds, then either $\|H\|^2 = 0$ and M is a minimal submanifold, or $n \leq 5$, the equality above holds, $\|\alpha\|^2 = \|A_{\nu_H}\|^2$, and M is a submanifold whose mean curvature vector is a covariantly constant section of its normal bundle.

Notice that neither of these theorems requires a priori the assumption that the mean curvature function be constant.

We carry out our work assuming that $n \geq 2$ in order to see the effect of curvature quantities. However, Theorems 5 and 6 remain true for $n = 1$, case where being minimal and totally geodesic are equivalent notions, and where the two theorems yield the same result. This dimension shows also the special nature of embedding into spheres or parabolic spaces: There are canonically placed circles that are deformations of closed geodesics in the space form $S^{\tilde{n}}_c$, $c \geq 0$, having nonconstant principal curvatures of changing signs.

2. Critical points of the Lagrangians

Consider a closed Riemannian manifold $(\tilde{M}, \tilde{g})$, and let M be a submanifold of $\tilde{M}$. With the metric g induced by $\tilde{g}$, M becomes a Riemannian manifold. We denote by $\nabla^{\tilde{g}}$ and ∇^g the Levi-Civita connections of $\tilde{g}$ and g, respectively, and by α the second fundamental form of the isometric immersion. The dimensions of M and $\tilde{M}$ are n and $\tilde{n}$, respectively.

We have Gauss' identity

$$\nabla^{\tilde{g}}_X Y = \nabla^g_X Y + \alpha(X, Y). \tag{6}$$

If N is a section of the normal bundle $\nu(M)$, the shape operator A_N is defined by

$$A_N X = -\pi_{TM}(\nabla^{\tilde{g}}_X N),$$

where in the right side above, N stands for an extension of the original section to a neighborhood of M. If ∇^ν is the connection on $\nu(M)$ induced by $\nabla^{\tilde{g}}$, we have Weingarten's identity

$$\nabla^{\tilde{g}}_X N = -A_N X + \nabla^\nu_X N. \tag{7}$$

For a detailed development of these and some of the expressions that follow, see [**4**].

Gauss' identity implies Gauss' equation

$$\begin{aligned} g(R^g(X,Y)Z,W) = \tilde{g}(R^{\tilde{g}}(X,Y)Z,W) + \tilde{g}(\alpha(X,W),\alpha(Y,Z)) \\ - \tilde{g}(\alpha(X,Z),\alpha(Y,W)). \end{aligned} \tag{8}$$

Here, R^g stands for the Riemann curvature tensor of the corresponding metric g, and X, Y, Z, and W are vector fields in $\tilde{M}$ tangent to the submanifold M.

Let $\{e_1, \ldots, e_{\tilde{n}}\}$ be an orthonormal frame for $\tilde{g}$ in a tubular neighborhood of M such that $\{e_1, \ldots, e_n\}$ constitutes an orthonormal frame for g on points of M. We denote by H the mean curvature vector, the trace of α. The immersion $M \hookrightarrow \tilde{M}$ is said to be minimal if $H = 0$. By (8), the Ricci tensors r_g and $r_{\tilde g}$ are related to each other by the expression

$$
\begin{aligned}
r_g(X,Y) &= \sum_{i=1}^{n} \tilde g(R^{\tilde g}(e_i,X)Y,e_i) + \tilde g(H,\alpha(X,Y)) \\
&\quad - \sum_{i=1}^{n} \tilde g(\alpha(e_i,X),\alpha(e_i,Y)) \\
&= r_{\tilde g}(X,Y) - \sum_{i=n+1}^{\tilde n} \tilde g(R^{\tilde g}(e_i,X)Y,e_i) + \tilde g(H,\alpha(X,Y)) \\
&\quad - \sum_{i=1}^{n} \tilde g(\alpha(e_i,X),\alpha(e_i,Y)),
\end{aligned} \tag{9}
$$

and the scalar curvatures s_g and $s_{\tilde g}$ by the expression

$$
\begin{aligned}
s_g &= s_{\tilde g} - 2\sum_{i=1}^{n}\sum_{j=n+1}^{\tilde n} K_{\tilde g}(e_i,e_j) - K_{\tilde g}(e_i,e_j) + \tilde g(H,H) - \tilde g(\alpha,\alpha) \\
&= \sum_{i,j\leq n} K_{\tilde g}(e_i,e_j) + \tilde g(H,H) - \tilde g(\alpha,\alpha),
\end{aligned} \tag{10}
$$

where $K_{\tilde g}(e_i,e_j)$ is the $\tilde g$-curvature of the section spanned by the orthonormal vectors e_i and e_j, and $\tilde g(H,H)$ and $\tilde g(\alpha,\alpha)$ are the squared-norms of the mean curvature vector H and the form α, respectively.

The critical submanifolds for the functionals Π and Ψ in (1) and (2) under deformations of the immersion f are described in full generality in [**8**, Theorem 3.10]. We recall the equations they satisfy adapted to the case of interest here. We denote by S^n_c the nth dimensional space form of curvature c. For convenience, we use the standard double index summation convention.

Theorem 7 ([**8**, Theorem 3.10])**.** *Let M be an n-manifold of codimension p isometrically immersed into the space form S^{n+p}_c. Let $\{e_1,\ldots,e_n\}$ be an orthonormal frame of tangent vectors to M, and $\{\nu_1,\ldots,\nu_p\}$ be an orthonormal frame of the normal bundle of the immersion such that $H = h\nu_1$. If $\alpha(e_i,e_j) = h^r_{ij}\nu_r$, then M is a critical point of Π if, and only if,*

$$2\Delta h = 2ch - 2h\|\nabla^\nu_{e_i}\nu_1\|^2 - h\|\alpha\|^2 + 2\,\mathrm{trace}\,A_{\nu_1}A^2_{\nu_k},$$

and for all m in the range $2\leq m\leq p$, we have that

$$0 = 2\langle e_i(h)\nabla^\nu_{e_i}\nu_1,\nu_m\rangle + he_i\langle\nabla^\nu_{e_i}\nu_1,\nu_m\rangle - h\langle\nabla^\nu_{e_i}\nu_1,\nabla^\nu_{e_i}\nu_m\rangle + 2\,\mathrm{trace}\,A_{\nu_m}A^2_{\nu_k}.$$

In addition M is a a critical point of the functional (2) *if, and only if,*

$$2\Delta h = 2cnh - 2h\|\nabla^{\nu}_{e_i}\nu_1\|^2 - h^3 + 2h\,\mathrm{trace}\,A^2_{\nu_1},$$

and for all m in the range $2 \leq m \leq p$, we have that

$$0 = 4e_i(h)\langle\nabla^{\nu}_{e_i}\nu_1, \nu_m\rangle + 2he_i\langle\nabla^{\nu}_{e_i}\nu_1, \nu_m\rangle - 2h\langle\nabla^{\nu}_{e_i}\nu_1, \nabla^{\nu}_{e_i}\nu_m\rangle + 2h\,\mathrm{trace}\,A_{\nu_1}A_{\nu_m}.$$

Finally M is a critical point of the functional (3) *if, and only if, the last $p-1$ equations above hold, and the one before these is replaced by*

$$2\Delta h = (3n - n^2)ch - 2h\|\nabla^{\nu}_{e_i}\nu_1\|^2 - h^3 + 2h\,\mathrm{trace}\,A^2_{\nu_1}.$$

For hypersurfaces in an Einstein background $(\tilde{M}, \tilde{g})$, these critical point equations can be described completely using only the principal curvatures $k_1, \ldots, k_n$. Observe that we have $h = k_1 + \cdots + k_n$ and $\|\alpha\|^2 = k_1^2 + \cdots + k_n^2$, respectively.

Theorem 8 ([**8**, Theorem 3.3, Corollary 3.4])**.** *Let M be a hypersurface in an Einstein manifold $(\tilde{M}, \tilde{g})$. Assume that $k_1, \ldots, k_n$ are the principal curvatures, with associated orthonormal frame of principal directions $e_1, \ldots, e_n$. Let ν be a normal field along M. Then M is a critical point of the functional* (1) *if, and only if,*

$$2\Delta h = 2(k_1K_{\tilde{g}}(e_1, \nu) + \cdots + k_nK_{\tilde{g}}(e_n, \nu)) - h\|\alpha\|^2 + 2(k_1^3 + \cdots + k_n^3),$$

and a critical point of the functional (2) *if, and only if,*

$$2\Delta h = 2h(K_{\tilde{g}}(e_1, \nu) + \cdots + K_{\tilde{g}}(e_n, \nu)) + 2h\|\alpha\|^2 - h^3.$$

In particular, a hypersurface M in S^{n+1}_c is a critical point for the functional (1) *if, and only if, its mean curvature function h satisfies the equation*

$$2\Delta h = 2ch - h\|\alpha\|^2 + 2(k_1^3 + \cdots + k_n^3),$$

while M is a critical point of the functional (2) *if, and only if, its mean curvature function h satisfies the equation*

$$2\Delta h = 2cnh + 2h\|\alpha\|^2 - h^3.$$

3. The Laplacian of the second fundamental form in space forms

The Laplacian of the second fundamental form of the immersion $f\colon M \to \tilde{M}$ was initially computed by Simons [**9**] under the assumption of minimality on $f(M)$. Bérard [**1**] wrote down the general result, though he applied it to immersions in space forms under the assumption that the mean curvature vector of the immersion H was covariantly constant. In this section, we recall this fact in general. Once we show the minimality of the critical points we consider of the functionals (1) and (2), the result can be used in the same manner as before to explain the reinterpretation of the classical gap theorem that we now make.

Given any tensor field Z on M, we set $\nabla^2_{X,Y}Z = (\nabla^g_X \nabla^g_Y - \nabla^g_{\nabla^g_X Y})Z$, an operator of order 2. It is tensorial in X, Y, and its symmetric properties are captured in the relation $(\nabla^2_{X,Y} - \nabla^2_{Y,X})Z = R^g(X,Y)Z$. The Laplacian is given by $\nabla^2 = \sum_{i=1}^n (\nabla^g_{e_i}\nabla^g_{e_i} - \nabla^g_{\nabla^g_{e_i} e_i})$, where $\{e_i\}_{i=1}^n$ is a g-orthonormal frame on M. If we think of the shape operator as the linear mapping $A\colon \nu(M) \to \mathcal{S}(M)$, where $\mathcal{S}(M)$ is the bundle of symmetric bilinear maps on TM, then we obtain an adjoint ${}^{\dagger}A\colon \mathcal{S}(M) \to \nu(M)$ defined by $\langle {}^{\dagger}As, N\rangle = \langle A_N, s\rangle$. Let $\widetilde{A} = {}^{\dagger}A \circ A$ and $\underset{\sim}{A} = \sum \operatorname{ad} A_{\nu_j} \operatorname{ad} A_{\nu_j}$, where $\{\nu_j\}_{j=1}^q$ is an orthonormal basis of $\nu(M)$ at the point. Finally, we have the curvature type operators $\overline{R}'_W$ and $\overline{R}(A)^W$ given by

$$\langle R^{\tilde{g}\prime}{}_W(X), Y\rangle = \sum_i \langle (\nabla^{\tilde{g}}_X R^{\tilde{g}})(e_i, Y)e_i, W\rangle + \langle (\nabla^{\tilde{g}}_{e_i} R^{\tilde{g}})(e_i, X)Y, W\rangle,$$

$$\begin{aligned}\langle R^{\tilde{g}}(A)^W(X), Y\rangle = \sum_i & 2(\langle R^{\tilde{g}}(e_i, Y)\alpha(X, e_i), W\rangle + \langle R^{\tilde{g}}(e_i, X)\alpha(Y, e_i), W\rangle) \\ & - \langle A_N X, R^{\tilde{g}}(e_i, Y)e_i\rangle - \langle A_W(Y), R^{\tilde{g}}(e_i, X)e_i\rangle \\ & + \langle R^{\tilde{g}}(e_i, \alpha(X, Y))e_i, W\rangle - 2\langle A_W e_i, R^{\tilde{g}}(e_i, X)Y\rangle,\end{aligned}$$

where W is a normal vector at the point.

Theorem 9 ([**1**, Theorem 2])**.** *We have that*

$$\begin{aligned}\langle \nabla^2\alpha(X, Y), W\rangle = & -\langle A \circ \widetilde{A}(W)(X), Y\rangle - \langle \underset{\sim}{A} \circ A_W(X), Y\rangle \\ & + \langle R^{\tilde{g}}(A)^W(X), Y\rangle \\ & + \langle R^{\tilde{g}\prime}{}_W(X), Y\rangle + \langle \nabla^2_{X,Y} H, W\rangle \\ & + \langle R^{\tilde{g}}(H, X)Y, W\rangle + \langle A_W Y, A_H X\rangle.\end{aligned} \tag{11}$$

We now assume that $(\tilde{M}, \tilde{g})$ has constant sectional curvature c. Then we have that

$$R^{\tilde{g}}(X, Y)Z = c(\langle Y, Z\rangle X - \langle X, Z\rangle Y), \tag{12}$$

that $R^{\tilde{g}\prime}{}_W = 0$, and that $R^{\tilde{g}}(A)^W$ is given by

$$R^{\tilde{g}}(A)^W = cn\left(A_W - \frac{2}{n}\langle H, W\rangle \mathbb{1}_{TM}\right). \tag{13}$$

The following result parallels [**1**, Corollary 3] but leaves in the term involving the second derivatives on H that appear in (11), as we do not assume that the mean curvature vector is covariantly constant.

Corollary 10. *If $(\tilde{M}, \tilde{g})$ has constant sectional curvature c, then*

$$\nabla^2 \alpha = -(A \circ \widetilde{A} + \underset{\sim}{A} \circ A) + cn \left(A - \frac{1}{n} \langle H, \cdot \rangle 1\!\!1_{TM} \right) + \nabla^2_{\cdot,\cdot} H + A \circ A_H .$$

Proof: We identify the terms in the right side of (11). The first two yield $-(A \circ \widetilde{A} + \underset{\sim}{A} \circ A)$. Now, by (12), we have that $\langle R^{\tilde{g}}(H, e_i)e_j, \alpha(e_i, e_j)\rangle = c\|H\|^2$, and so $R^{\tilde{g}}(H, \cdot)\cdot = c\langle H, \cdot \rangle 1\!\!1_{TM}$. This and (13) show that the third and sixth terms in the right side of (11) yield the next term in the right side of the stated expression for $\nabla^2 \alpha$. The fourth term in the right side of (11) is zero, the next is $\nabla^2_{\cdot,\cdot} H$, and the last is $A \circ A_H$. □

We develop now a basic inequality for the analysis of (1) and (2), still in the case when $(\tilde{M}, \tilde{g})$ has constant sectional curvature c. This inequality parallels and extends that of Simons [**9**, Theorem 5.3.2], and reproduces Simons' result [**9**, Theorem 5.3.2, Corollary 5.3.2] when dealing with critical points of our functionals that are minimal submanifolds. As we will see, this embodies already the fact that the gap phenomenon is one about the lowest critical values of the said functionals.

We recall Simons' inequality [**9**, Lemma 5.3.1]

$$\langle A \circ \widetilde{A} + \underset{\sim}{A} \circ A, A \rangle \leq \left(2 - \frac{1}{p} \right) \|A\|^4, \tag{14}$$

where p is the codimension of M in $\tilde{M}$. Then, by Corollary 10, we obtain that

$$\begin{aligned} 0 &\leq - \int \langle \nabla^2 \alpha, \alpha \rangle \, d\mu \\ &= \int \langle A \circ \widetilde{A} + \underset{\sim}{A} \circ A, A \rangle + \left(2c\|H\|^2 - cn\|\alpha\|^2 \right) \, d\mu \\ &\quad - \int \left(\langle \nabla^2_{e_i,e_j} H, \alpha(e_i, e_j) \rangle + c\|H\|^2 + \langle A_{\alpha(e_i,e_j)} e_j, A_H e_i \rangle \right) d\mu \qquad (15) \\ &\leq \int \left(\left(2 - \frac{1}{p} \right) \|\alpha\|^4 + c\|H\|^2 - cn\|\alpha\|^2 \right) d\mu \\ &\quad - \int \left(\langle \nabla^2_{e_i,e_j} H, \alpha(e_i, e_j) \rangle + \langle A_{\alpha(e_i,e_j)} e_j, A_H e_i \rangle \right), \end{aligned}$$

inequality that when $H \equiv 0$ reduces to the inequality in Simons [**9**, Theorem 5.3.2], the key ingredient in the proof of his gap theorem. Both of its sides involve differential terms of second order that might cancel each other out, and so (15) is inadequate to analyze restrictions on critical points of (1) or (2) that are not necessarily minimal. In the proofs of our main results, we overcome this difficulty by using instead the elliptic equation that the mean curvature function h of these critical point satisfies, Theorem 7, and the corresponding inequality that arises by the nonnegativity of the Laplacian.

4. Critical points of the Lagrangians with small density

In this section we prove our main results.

We consider, as above, the orthonormal frame $\{e_1, \ldots, e_n\}$ of tangent vectors to (M, g) and the orthonormal frame $\{\nu_1, \ldots, \nu_p\}$ of the normal bundle $\nu(M)$ such that $H = h\nu_1$. We write $\alpha(e_i, e_j) = \sum_{r=1}^{p} h_{ij}^r \nu_r$. Then we have that

$$h = \sum_{i=1}^{n} h_{ii}^1 \text{ and } \sum_{i=1}^{n} h_{ii}^r = 0 \text{ for } 2 \leq r \leq p. \tag{16}$$

We denote by $\langle \cdot, \cdot \rangle$ the standard metric on the space form S_c^{n+p}. For immersions into it, the Ricci tensor and scalar curvature of (M, g) are such that

$$r_g(e_j, e_k) = (n-1)c\langle e_j, e_k\rangle + \left(h h_{jk}^1 - \sum_{i,r} h_{ij}^r h_{ik}^r \right), \tag{17}$$

and

$$s_g = n(n-1)c + h^2 - \|\alpha\|^2, \tag{18}$$

respectively. Thus, under the hypothesis of Theorems 2 and 3 we immediately derive that $s_g \geq 0$ in the case when $c > 0$.

Proof of Theorem 2: Suppose that $M \hookrightarrow \mathbb{S}^{n+p}$ is a critical point of Ψ of constant mean curvature function. By Theorem 7, we must have that

$$0 = 2\int h\Delta h \, d\mu_g = \int h^2(2n - 2\|\nabla_{e_i}^{\nu} \nu_1\|^2 - h^2 + 2\,\mathrm{trace}\, A_{\nu_1}^2)\, d\mu_g. \tag{19}$$

Now the trace of $A_{\nu_1}^2$ is nonnegative and bounded above by $\|\alpha\|^2$. By (4) it follows that

$$\left(\frac{1}{2} - \lambda\right) h^2 \leq n - \|\nabla_{e_i}^{\nu} \nu_1\|^2 - \frac{1}{2}h^2 + \mathrm{trace}\, A_{\nu_1}^2 \leq n - \|\nabla_{e_i}^{\nu} \nu_1\|^2 - \frac{1}{2}h^2 + \|\alpha\|^2$$

is nonnegative and cannot be zero if $h \neq 0$. If $h = 0$ then $\nabla^\nu \nu_1 = 0$, and we have then that $0 \leq \|\alpha\|^2 \leq n$. The desired result now follows by Theorem 1. Indeed, the Clifford torus $\mathbb{S}^m(\sqrt{m/n}) \times \mathbb{S}^{n-m}(\sqrt{(n-m)/n})$, $1 \leq m < n$, has principal curvatures $\pm\sqrt{(n-m)/m}$ and $\mp\sqrt{m/(n-m)}$ with multiplicities m and $n-m$, respectively. In addition, by (17), its Ricci curvature is bounded between $n(m-1)/m$ and $n(n-m-1)/(n-m)$, while the real projective plane is embedded into $\mathbb{S}^4$ with an Einstein metric of scalar curvature $2/3$. □

We pause briefly to derive an elementary result to be used in the proof of Theorem 3. This will unravel the geometric content of the constants that appear in the estimates (5), and the inequality of Theorem 6.

Lemma 11. *Let (M, g) be a Riemannian manifold isometrically immersed into a background manifold $(\tilde{M}, \tilde{g})$, and consider the degree* 1 *homogeneous function*

$$\frac{\operatorname{trace}\left(A_{\nu_1}\sum_j A_{\nu_j}^2\right)}{\|\alpha\|^2}.$$

At a critical point we have that

$$\operatorname{trace}\left(A_{\nu_1}\sum_j A_{\nu_j}^2\right) = \frac{\|H\|(\|\alpha\|^2 + 2\|A_{\nu_1}\|^2)}{n + 2\|H\|^2/\|\alpha\|^2},$$

the maximum occurs when $\|\alpha\|^2 = \|A_{\nu_1}\|^2$, and so

$$\operatorname{trace}\left(A_{\nu_1}\sum_j A_{\nu_j}^2\right) \leq \frac{3\|H\|\|\alpha\|^2}{2}.$$

Proof: We have that

$$\operatorname{trace}\left(A_{\nu_1}\sum_j A_{\nu_j}^2\right) = \sum_{k=1}^{p}\sum_{i,l,s=1}^{n} h_{is}^1 h_{sl}^k h_{li}^k,$$

and so the function of the h_{ij}^ks under consideration is defined by

$$\frac{\operatorname{trace}\left(A_{\nu_1}\sum_j A_{\nu_j}^2\right)}{\|\alpha\|^2} = \frac{\sum_{k=1}^{p}\sum_{i,l,s=1}^{n} h_{is}^1 h_{sl}^k h_{li}^k}{\sum_k \sum_{ij}(h_{ij}^k)^2}$$

outside the origin, and extended by continuity everywhere. Its critical points subject to the constraints (16) are the solutions of the system of equations

$$\sum_{k=1}^{p}\sum_{l=1}^{n} h^k_{ul}h^k_{lv}\delta_{1r} + \sum_{i=1}^{n} h^1_{ui}h^r_{iv} + \sum_{s=1}^{n} h^1_{vs}h^r_{su} = \left(\sum_k\sum_{ij}(h^k_{ij})^2\right)^2 \delta_{uv}\sum_{j=1}^{p}\lambda_j\delta_{jr},$$

where $\lambda_1, \dots, \lambda_p$ are the Lagrange multipliers. Here δ is the Kronecker symbol. If we multiply by h^r_{uv} and add in u, v, and r, we obtain the relation

$$\|\alpha\|^2 \operatorname{trace}\left(A_{\nu_1}\sum_j A^2_{\nu_j}\right) = \lambda_1 h\|\alpha\|^4,$$

while if we set $u = v$, $r = 1$ and add in u, we obtain that

$$\|\alpha\|^2(\|\alpha\|^2 + 2\|A^2_{\nu_1}\|^2) - 2h \operatorname{trace}\left(A_{\nu_1}\sum_j A^2_{\nu_j}\right) = \lambda_1 n\|\alpha\|^4.$$

A simple algebraic manipulation yields the stated equality at critical points, and the statement about the maximum is then clear. As we have assumed that $n \geq 2$, the inequality follows. □

Proof of Theorem 3: Proceeding as above, by Theorem 7 we must have that

$$\begin{aligned} 0 &= 2\int h\Delta h \, d\mu_g \\ &= \int h^2\left(2 - 2\|\nabla^\nu_{e_i}\nu_1\|^2 - \|\alpha\|^2 + 2\operatorname{trace}\frac{1}{h}A_{\nu_1}\sum_k A^2_{\nu_k}\right) d\mu_g. \end{aligned} \tag{20}$$

If estimates (5) hold for $\lambda \in [0,1)$, then $h = 0$ and so M is a minimal submanifold, and $\nabla^\nu\nu_1 = 0$. The desired result now follows by Theorem 1. The argument is parallel to the one already used in the proof of Theorem 2. It suffices to add that the symmetric Clifford torus $\mathbb{S}^m(\sqrt{1/2}) \times \mathbb{S}^m(\sqrt{1/2}) \subset \mathbb{S}^{2m+1}$ is Einstein. □

Proof of Corollary 4: There only needs to be observed why these Riemannian manifolds are critical points of the total scalar curvature under deformations of the metric g in M that can be realized by an isometric immersions of M into $\mathbb{S}^{n+p}$. By (10), we have that

$$\int s_g \, d\mu_g = n(n-1)\mu_g(M) + \Psi(M) - \Pi(M) = \Theta(M) - \Pi(M),$$

where $\mu_g(M)$ is the volume of the isometrically immersed manifold. The critical manifold M is separately a critical point of the functionals $\mu_g(M)$, $\Psi(M)$, and $\Pi(M)$ under deformations of the immersion, hence a critical point of their linear combination above. $\square$

Remark 12. It is worth observing that the Klein bottle, a nonoriented manifold of zero Euler characteristic, can be embedded into $\mathbb{R}^4$ and therefore into $\mathbb{S}^4$. But no embedding of such can be a critical point of Π of constant mean curvature function satisfying estimates (5).

Remark 13. The attentive reader may have observed that we present Theorems 2 and 3 for immersions into spheres in order to reinterpret the classical gap theorem. If we were to use the characterization of critical points given by Theorem 7 for immersions into parabolic spaces ($c = 0$), we would obtain corresponding gap theorem results for immersions into this type of spaces that have constant mean curvature function and satisfy the pointwise inequality $-\lambda\|H\|^2 \leq \|\alpha\|^2 - \|H\|^2 - \|\nabla^\nu \nu_H\|^2$ instead. In this case, dealing with critical points of (1) or (2) allows for the different range of values of λ for which we can still draw the conclusion. The detailed formulation of the statements so obtained are left to that astute reader.

Proof of Theorem 5: By Theorem 7 we have that

$$0 \leq 2\int h\Delta h \, d\mu_g = \int h^2(-2n - 2\|\nabla^\nu_{e_i}\nu_1\|^2 - h^2 + 2\,\mathrm{trace}\, A^2_{\nu_1})\, d\mu_g. \tag{21}$$

But $0 \leq \mathrm{trace}\, A^2_{\nu_1} \leq \|\alpha\|^2$, and so

$$-2n - h^2 + 2\,\mathrm{trace}\, A^2_{\nu_1} \leq -2n - h^2 + 2\|\alpha\|^2 \leq 0.$$

It follows that either $h = 0$, or that

$$-2n - h^2 + 2\,\mathrm{trace}\, A^2_{\nu_1} = -2n - h^2 + 2\|\alpha\|^2 = 0, \quad \nabla^\nu_{e_i}\nu_1 = 0.$$

In the latter case, we have that $h^r_{ij} = 0$ for all $r \geq 2$, and the vector $H = h\nu_1$ is a covariantly constant section of the normal bundle $\nu(M)$. The desired result follows. $\square$

Proof of Theorem 6: We use once again the critical point equation given by Theorem 7, and obtain that

$$\begin{aligned} 0 &\leq 2\int h\Delta h \, d\mu_g \\ &= \int h^2\left(-2 - 2\|\nabla^\nu_{e_i}\nu_1\|^2 - \|\alpha\|^2 + 2\frac{1}{h}\,\mathrm{trace}\, A_{\nu_1}\sum_j A^2_{\nu_j}\right) d\mu_g. \end{aligned}$$

By Lemma 11, we have that

$$-2 - \|\alpha\|^2 + 2\frac{1}{h}\,\mathrm{trace}\, A_{\nu_1} \sum_j A_{\nu_j}^2 \leq -2 - \|\alpha\|^2 + 2\frac{3\|\alpha\|^2}{n + 2\|H\|^2/\|\alpha\|^2},$$

and the stated inequality is equivalent to the right side of this expression being nonpositive. Thus, either $h = 0$ or the right hand side of the expression above vanishes and the equality holds, and $\nabla^\nu \nu_1 = 0$. In the latter case, $h_{ij}^r = 0$ for all $r \geq 2$ and $H = h\nu_1$ is a covariantly constant section of $\nu(M)$. □

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