RIGHT ENGEL ELEMENTS OF STABILITY GROUPS
OF GENERAL SERIES IN VECTOR SPACES

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Abstract: Let $V$ be an arbitrary vector space over some division ring $D$, $L$ a general series of subspaces of $V$ covering all of $V \setminus \{0\}$ and $S$ the full stability subgroup of $L$ in $\text{GL}(V)$. We prove that always the set of bounded right Engel elements of $S$ is equal to the $\omega$-th term of the upper central series of $S$ and that the set of right Engel elements of $S$ is frequently equal to the hypercentre of $S$.

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Throughout this paper we keep to the following notation. Let $V$ be a vector space over a division ring $D$ and $L = \{ (\Lambda_\alpha, V_\alpha) : \alpha \in A \}$ a series of subspaces of $V$ running from $\{0\}$ to $V$ (see [1] for the definition and basic properties of general series). Thus in particular $A$ is a linearly ordered set, the $\Lambda_\alpha/V_\alpha$ are the jumps of the series, $V \setminus \{0\} = \bigcup_{\alpha \in A} \Lambda_\alpha \setminus V_\alpha$ and $\Lambda_\alpha \leq V_\beta$ whenever $\alpha < \beta$ ($\alpha > \beta$ just for descending series, which we usually order from the top rather than from the bottom).

The completion of $L$ we denote by $L^*$. Set $S = \text{Stab}(L)$, the full stability group of $L$ in $\text{GL}(V)$; that is, $S = \cap_{\alpha} C_{\text{GL}(V)}(\Lambda_\alpha/V_\alpha)$. In [2] we study the set of left Engel elements of $S$. In [4] we consider the right Engel elements $S$, but only for ascending or descending series. Here we consider what we can say for general series.

Theorem 1. The set $R^-(S)$ of bounded right Engel elements of $S$ is equal to the $\omega$-th term $\zeta_\omega(S)$ of the upper central series of $S$.

In [4] we prove Theorem 1, but only in the special cases of ascending or descending series. Our proof of Theorem 1 proceeds by applying these special cases to certain ascending and descending subseries of $L$. We also use Theorem 3 below in the proof of Theorem 1. Note that the left analogue of Theorem 1, namely that the set $L^-(S)$ of bounded left Engel elements of $S$ is equal to the Fitting subgroup $\text{Fitt}(S)$ of $S$, is also valid, see [2, Theorem A].
The sets $R(S)$ and $L(S)$ of right and left Engel elements of $S$ are much harder to compute and we only have partial results. Suppose that either $\dim_D V$ is countable or $V$ has an $L$-basis $B$, meaning that $B \cap L$ is a basis of $L$ for each subspace $L$ belonging to $L$. Now $L(S) = \text{Fitt}(S) = L^-(S)$, see [2, Theorem B], and if $L$ is a descending series then $R(S) = \zeta(S)$, see [4, 1.2]. However if $L$ is an ascending series, then although $R(S)$ is equal to the hypercentre $\zeta(S)$ of $S$, it is frequently not equal to $\zeta(S)$, see [3, 1.2]. Thus the obvious right analogue of the left Engel case is not valid, even for ascending series. However we do have the following two special cases.

**Theorem 2.** Suppose that either $\dim_D V$ is countable or $V$ has an $L$-basis. Then set $R(S)$ of right Engel elements of $S$ is equal to the hypercentre $\zeta(S)$ of $S$.

**Theorem 3.** Suppose either $L$ has no top jump or $L$ has no bottom jump (meaning that either $V = \Lambda_\alpha$ implies $V = V_\alpha$ or $\{0\} = V_\alpha$ implies $\{0\} = \Lambda_\alpha$). Then:

a) $R^-(S) = \{1\}$.

b) If either $\dim_D V$ is countable or $V$ has an $L$-basis $B$, then $R(S) = \{1\}$.

1. **The proofs**

We keep our notation above, in particular we have our series $L$ and $L^*$ and stability group $S$. We always assume that $V$ is a left vector space over $D$. Clearly there are analogous results for right vector spaces. Also clearly we may remove all trivial jumps $\Lambda_\alpha = V_\alpha$ from $L$ without affecting the conclusions of the theorems. Thus below assume that $\Lambda_\alpha > V_\alpha$ for all $\alpha$ in $A$. This makes various statements simpler. For example, the hypothesis of Theorem 3 is now that $A$ has either no maximal member or no minimal member. Let $\text{Fitt}(L)$ denote the set of elements of $S$ that stabilize some finite subseries of $L^*$ running from $\{0\}$ to $V$; $\text{Fitt}(L)$ is a normal subgroup of $S$ contained in $\text{Fitt}(S)$.

**Lemma 1.** Let $x \in R(S) \cap \text{Fitt}(L)$ and set $X = \cup \{\Lambda_\alpha : \Lambda_\alpha(x - 1) = \{0\}\}$; $X$ is an element of $L^*$. Then $C = \{\alpha \in A : X \leq V_\alpha\}$ is inversely well-ordered (so $\{(\Lambda_\alpha, V_\alpha) : X \leq V_\alpha\}$ is a descending series).

**Proof:** Clearly we may assume that $x \neq 1$. Now $x$ stabilizes a finite subseries $\{0\} = X_0 < X_1 < \cdots < X_r = V$ of $L^*$, where clearly $r \geq 2$ and we may assume that $X_1 = X$. If $C$ is not inversely well-ordered there exist a $j \geq 1$ and an infinite subsequence $\alpha(1) < \alpha(2) < \cdots < \alpha(i) < \cdots$
of $\mathbf{C}$ with $X_j \leq V_{\alpha(1)}$ and $Y = \bigcup_i \Lambda_{\alpha(i)} \leq X_{j+1}$. Let $\mathbf{M}$ denote the ascending subseries

$$\{0\} < X_1 < \cdots < X_j \leq V_{\alpha(1)} \leq \Lambda_{\alpha(1)} \leq \cdots \leq V_{\alpha(j+1)} < \cdots \leq Y$$

of $\mathbf{L}^*$. Clearly $x|_Y \in \text{Stab}(\mathbf{M}) = T$ say and $T \leq \text{GL}(Y)$. Choose a subspace $W$ of $V$ with $V = W \oplus Y$ and extend the action of $T$ on $Y$ to one on $V$ by making $T$ centralize $W$. Then $T \leq S$. Also if $t \in T$, then $[x|_Y, kt]|_Y = [x, kt]|_Y$ for all $k \geq 1$ and so $x|_Y \in R(T)$. But then $x|_Y = 1$ by 1.1c) of [4]. This contradicts the choice of $X_1$ and completes the proof of Lemma 1. \qed

**Lemma 2.** Let $x \in R(S)$ and define $X$ and $\mathbf{C}$ as in Lemma 1.

a) If $x \in R^-(S)$ then $\mathbf{C}$ is inversely well-ordered.

b) If either $\dim_D V$ is countable or $V$ has an $\mathbf{L}$-basis, then $\mathbf{C}$ is inversely well-ordered.

**Proof:** a) $R^-(S) \subseteq L^-(S)^{-1} \subseteq \text{Fitt}(\mathbf{L})$ by [1, 7.11] and [2, Theorem A]. Thus Lemma 1 applies.

b) Here $R(S) \subseteq L(S)^{-1} \subseteq \text{Fitt}(\mathbf{L})$ by [1, 7.11] and [2, Theorem B] and again Lemma 1 applies. \qed

**Lemma 3.** Let $x \in R^-(S) \cap \text{Fitt}(\mathbf{L})$ and set $X = \cap \{ V_\alpha : V(x-1) \leq V_\alpha \}$; $X$ is an element of $\mathbf{L}^*$. Then $\mathbf{C} = \{ \alpha \in \mathbf{A} : \Lambda_\alpha \leq X \}$ is well-ordered (so $\{(\Lambda_\alpha, V_\alpha) : \Lambda_\alpha \leq X \}$ is an ascending series).

**Proof:** Again assume that $x \neq 1$, so $x$ stabilizes a finite subseries $\{0\} = X_0 < X_1 < \cdots < X_r = V$ of $\mathbf{L}^*$, where $r \geq 2$ and $X_{r-1} = X$. If $\mathbf{C}$ is not well-ordered there exist a $j < r$ and an infinite subsequence $\alpha(1) > \alpha(2) > \cdots > \alpha(i) > \cdots$ of $\mathbf{C}$ with $X_j \geq \Lambda_{\alpha(1)}$ and $Y = \cap_i V_{\alpha(i)} \geq X_{j+1}$. Let $\mathbf{M}$ denote the descending subseries of $V/Y$ consisting of the $X_k/Y$ for $j \leq k \leq r$ and the $\Lambda_{\alpha(i)}/Y$ and the $V_{\alpha(i)}/Y$ for $i \geq 1$. (This is a descending subseries of length $\omega$ of $\mathbf{L}^*$ taken modulo the element $Y$ of $\mathbf{L}^*$.)

Clearly $x|_{V/Y} \in T = \text{Stab}(\mathbf{M})$. Pick a subspace $W$ of $V$ with $V = W \oplus Y$ and let $T$ act on $W$ via its given action on $V/Y$ and the natural isomorphism of $V/Y$ onto $W$. Then extend this action of $T$ on $W$ to one on $V$ by making $T$ centralize $Y$. Clearly $T \leq S$ and $[x|_{V/Y}, kt]|_{V/Y} = [x, kt]|_{V/Y}$ for all $k \geq 1$ and all $t \in T$. Hence $x|_{V/Y} \in R^-(T)$. But then $x|_{V/Y} \in \zeta(T)$ by [4, 1.2a)] and $\zeta(T) = \langle 1 \rangle$ by [3, 1.1]. Lemma 3 follows. \qed
Lemma 4. If \( x \in R^-(S) \) and if \( C \) is as in Lemma 3, then \( C \) is well-ordered.

Proof: Now \( R^-(S) \subseteq \text{Fitt}(L) \) by [1, 7.11] and [2, Theorem A]. Thus Lemma 3 applies.

Lemma 5. Suppose either \( \dim_D V \) is countable or \( V \) has an \( L \)-basis. If \( x \in R(S) \) and if \( X = \cap \{ V_\alpha : V(x-1) \leq V_\alpha \} \), then \( C = \{ \alpha \in A : \Lambda_\alpha \leq X \} \) is well-ordered.

Proof: As before \( R(S) \subseteq L(S)^{-1} \subseteq \text{Fitt}(L) \) by [1, 7.11] and [2, Theorem B]. Now repeat the proof of Lemma 3. Here we only obtain that \( x|_{V/Y} \in R(T) \). But here \( R(T) = \{ 1 \} \) by [4, 1.2c)]. Lemma 5 now follows.

Lemma 6. Suppose that \( A \) either has no maximal member or has no minimal member. Then \( R^-(S) = \{ 1 \} \). If also either \( \dim_D V \) is countable or \( V \) has an \( L \)-basis, then \( R(S) = \{ 1 \} \).

This lemma completes the proof of Theorem 3.

Proof: If \( A \) has no maximal member, then in Lemma 2 always \( C \) is empty. Thus in Lemma 2, case a) we have \( R^-(S) = \{ 1 \} \) and in Lemma 2, case b) we have \( R(S) = \{ 1 \} \).

Now assume \( A \) has no minimal member. Then \( C \) is always empty in Lemmas 4 and 5. Thus in Lemma 4 we have \( R^-(S) = \{ 1 \} \) and in Lemma 5 we have \( R(S) = \{ 1 \} \). Lemma 6 follows.

Further notation. The series \( L^* \) has a maximal ascending segment starting at \( \{ 0 \} \); specifically there is an ordinal number \( \lambda = \mu + n \), where \( n \geq 0 \) is an integer, \( \mu \) is zero or a limit ordinal, and

\[
\{ 0 \} = \Lambda_0 < \Lambda_1 < \cdots < \Lambda_\gamma < \cdots < \Lambda_\lambda
\]

is a subseries of \( L^* \) with each \( \Lambda_{\gamma+1}/\Lambda_\gamma \) a jump of \( L \) and \( \Lambda_\lambda \neq V_\alpha \) for all \( \alpha \in A \).

In the same way \( L^* \) has a maximal descending segment

\[
V = V_0 > V_1 > \cdots > V_\gamma > \cdots > V_{\lambda'},
\]

where \( \lambda' = \mu'/n' \) is an ordinal number, \( n' \geq 0 \) is an integer, \( \mu' \) is zero or a limit ordinal, each \( V_\gamma/V_{\gamma+1} \) is a jump of \( L \) and \( V_{\lambda'} \neq \Lambda_\alpha \) for all \( \alpha \in A \).

Clearly \( V_{\mu'} \geq \Lambda_\lambda \) and \( V_{\lambda'} \geq \Lambda_\mu \). Thus we have two possibilities. Firstly we could have that \( V_{\lambda'} > \Lambda_\lambda \). Secondly if this is not the case then \( V_{\mu'} = \Lambda_\lambda \) and \( V_{\lambda'} = \Lambda_\mu \).
Proof of Theorem 1: We have to prove that $R^-(S) \subseteq \zeta_\omega(S)$. By [4, 1.1 and 1.2] we may assume that $L$ is neither an ascending series nor a descending series. Set $R = R^-(S)$. Then $R$ is a subgroup of $S$ by [4, Theorem 1.3] and $R$ centralizes $V/\Lambda_\lambda$ and $V_{\lambda'}$ by Theorem 3 (and [2, 1.5]). Thus if $x \in R$, then $x\phi: v \mapsto v(x-1)$ is a linear homomorphism of $V$ into $\Lambda_\lambda$ with $V_{\lambda'} \leq \ker(x\phi)$. Thus $\phi$ is effectively an embedding of $R$ into $\text{Hom}_D(V/V_{\lambda'}, \Lambda_\lambda)$. We are not claiming at this stage that $\phi$ is a homomorphism of $R$; that is, possibly $(xy)\phi \neq x\phi + y\phi$, for some $x$ and $y$ in $R$.

Let $M$ denote the descending subseries $V = V_0 > V_1 > \cdots > V_\gamma > \cdots > V_{\lambda'} > \{0\}$ of $L^*$. Then $R \leq T = \text{Stab}(M) \leq S$, so $R \leq R^-(T)$. Then $R \leq \zeta_\omega(T)$ by [4, 1.2] and hence by [3, 4.1] (and its proof) $R\phi$ is contained in the subset of $\text{End}_D V$ of all $\theta$ such that $V\theta \leq V_{\mu'}$ and there exists $i < \omega$ with $V_i\theta = \{0\}$. (Possibly $V_{\mu'} = V$, in which case $\mu' = 0$ and we can set $i = \lambda'$.)

Let $N$ denote the ascending subseries $\{0\} = \Lambda_0 < \Lambda_1 < \cdots < \Lambda_\gamma < \cdots < \Lambda_\lambda < V$ of $L^*$. Then $R \leq R^-(U)$ for $U = \text{Stab}(N)$ and so $R \leq \zeta_\omega(U)$ by [4, 1.1]. Consequently if $\theta \in R\phi$, then for some $j < \omega$ we have $V\theta \leq \Lambda_j$ (and $\Lambda_\mu \theta = \{0\}$, which here we already know), see the last line of the proof of 3.5 of [4], where $\zeta_\omega(U)$ is identified, in the notation of [4], with $T_{\omega^-}$. Again, if $\Lambda_\mu = \{0\}$, then $\mu = 0$ and we can choose $j = \lambda$.

Set $K_{ij} = \{\theta \in \text{End}_D V : V_i\theta = \{0\}$ and $V\theta \leq \Lambda_j\}$, where $0 \leq i, j < \omega$, $i \leq \lambda'$, and $j \leq \lambda$. If $\theta \in K_{ij}$, $v \in V$, and $g \in S$, then

$$v[\theta, g] = vg^{-1}\theta g - v\theta = v(g^{-1} - 1)\theta g + v\theta(g-1).$$

If also $i \geq 1$, then $V_{i-1}(g^{-1} - 1)\theta g \leq V_i\theta g = \{0\}$ and $V(g^{-1} - 1)\theta g \leq V\theta g \leq \Lambda_j$. Further if $j \geq 1$, then $V\theta(g-1) \leq \Lambda_j(g-1) \leq \Lambda_{j-1}$ and $V_i\theta(g-1) = \{0\}$. Consequently $[K_{ij}, S] \leq K_{i-1,j} + K_{i,j-1}$ whenever $i, j \geq 1$. Also $K_{ij} = \{0\}$ if either $i = 0$ or $j = 0$. Set $L_r = \sum_{i+j \leq r} K_{ij}$. Then

$$\{0\} = L_0 = L_1 \leq L_2 \leq \cdots \leq L_r \leq \cdots$$

with $[L_{r+1}, S] \leq L_r$ for all $r \geq 1$. Also $R\phi \subseteq L = \cup_r L_r$.

Suppose $\mu'$ is infinite. Then all $V_i \geq V_{\mu'}$ and $[V_{\mu'}, R] = \{0\}$; also $[V, R] \leq \Lambda_\lambda \leq V_{\mu'}$. Thus in this case $\phi$ is an $S$-monomorphism of $R$ and hence $R \leq \zeta_\omega(S)$. Suppose $\mu$ is infinite. Then all $\Lambda_j \leq \Lambda_\mu$ and $[V, R] \leq \Lambda_{\mu}$; also $[\Lambda_{\mu}, R] \leq [V_{\lambda'}, R] = \{0\}$. Thus here too $\phi$ is an $S$-monomorphism of $R$ and again $R \leq \zeta_\omega(S)$. Finally suppose $\mu = 0 = \mu'$. Then $\Lambda_\lambda < V_{\lambda'}$ (recall $L$ here is not ascending, or for that matter descending). But then $[V, R] \leq \Lambda_\lambda$, $[V_{\lambda'}, R] = \{0\}$, $\phi$ is an $S$-monomorphism and $R \leq \zeta_\omega(S)$ (actually in this case $R \leq \zeta_{\lambda+\lambda'}(S)$).
Proof of Theorem 2: Here \( \dim_D V \) is countable or \( V \) has an \( \mathbf{L} \)-basis. As far as we can, we follow the strategy of the proof of Theorem 1. Let \( R = R(S) \). We need only prove that \( R \subseteq \zeta(S) \). Again we may assume that \( \mathbf{L} \) is neither ascending nor descending by [4, 1.1 and 1.2]. Also \( R \) is a normal subgroup of \( S \) by [4, 1.3] and \( R \) centralizes \( V/\Lambda_\lambda \) and \( V_\lambda \) by Theorem 3. If \( x \in R \) and \( x\phi: v \mapsto v(x - 1) \), then \( \phi \) embeds \( R \) into \( \text{Hom}_D(V/V_\lambda, \Lambda_\lambda) \), which we regard as a subset of \( \text{End}_D(V) \) in the usual way.

With \( \mathbf{M} \) and \( T = \text{Stab}(\mathbf{M}) \) as in the proof of Theorem 1, we have \( R \leq T \leq S \) and \( R \leq R(T) = \zeta_\omega(T) \), the latter by [4, 1.2]. Hence [3, 4.2] yields that for each \( x \) in \( R \) there exists \( i < \omega \) with \( V_i(x - 1) = \{0\} \). Again if \( \mu' = 0 \) then \( \lambda' \) is finite and we can choose \( i = \lambda' \) for all such \( x \). To avoid two formally different cases, if \( \mu' = 0 \) set \( V_i = V_{\lambda'} \) for \( \lambda' < i \leq \omega \).

Thus in both cases we have \( R\phi \subseteq \text{Hom}_D(V/V_\omega, \Lambda_\lambda) \); in fact we have

\[
R\phi \subseteq \bigcup_{i < \omega} \text{Hom}_D(V/V_i, \Lambda_\lambda).
\]

Let \( \mathbf{N} \) be as in the proof of Theorem 1 and again set \( U = \text{Stab}(\mathbf{N}) \). Then \( R \leq R(U) = \zeta(U) \) by [4, 1.1]. From now on we can no longer proceed as in the proof of Theorem 1 since \( \zeta(U) \) has a more complicated structure than \( \zeta_\omega(U) \). Set \( P = R \cap C_U(V/\Lambda_\mu) \). Then \( P \leq C_U(V_\lambda) \leq C_U(\Lambda_\mu) \) and \( V_\omega \geq \Lambda_\mu \). Hence \( \phi: P \to \text{Hom}_D(V/V_\omega, \Lambda_\mu) \) is a group embedding.

Let \( H_\gamma = \text{Hom}_D(V/V_\omega, \Lambda_\gamma) \). Then \( \{P\phi \cap H_\gamma\}_{\gamma \leq \mu} \) is an ascending \( U \)-hypercentral series of \( P\phi \) by [3, 3.2 and 3.3]. Also \( (P\phi \cap H_{\gamma + 1})/(P\phi \cap H_\gamma) \) embeds into \( \text{Hom}_D(V/V_\omega, \Lambda_\gamma/\Lambda_\gamma) \). Let \( K_i = \text{Hom}_D(V/V_i, \Lambda_1) \).

Now \( P\phi \cap H_1 \leq \bigcup_{i < \omega} K_i \) since for each \( x \in R \) there exists \( i < \omega \) with \( V(x - 1) \leq V_i \). Also \( K_{i+1}/K_i \cong \text{Hom}_D(V_i/V_{i+1}, \Lambda_1) \) and the latter is centralized by \( S \). Consequently \( P\phi \cap H_1 \) is \( S \)-hypercentral. In view of [2, 1.5] we may apply this to \( V/\Lambda_\gamma \) for each \( \gamma < \mu \) and thus each \( (P\phi \cap H_{\gamma + 1})/(P\phi \cap H_\gamma) \) is \( S \)-hypercentral. Therefore \( P \leq \zeta(S) \).

We now need to consider \( R/P \). This acts faithfully on \( V/\Lambda_\mu \). In view of [2, 1.5] we may simplify our notation by assuming that \( \Lambda_\mu = \{0\} \); that is, that \( \mu = 0 \). If \( V_{\lambda'} = \Lambda_\mu \) then \( \mathbf{L} \) is a descending series; in which case we already know that \( R \leq \zeta(S) \). If \( V_{\lambda'} \neq \Lambda_\mu \), then we have \( V \geq V_\omega \geq V_{\mu'} \geq V_{\lambda'} > \Lambda_\lambda \geq \Lambda_\mu = \{0\} \). Also \( [V, R] \leq \Lambda_\lambda \) and \( [V_\omega, R] = \{0\} \). Consequently \( \phi: R \to \text{Hom}_D(V/V_\omega, \Lambda_\lambda) \) is a group embedding. But in fact \( \phi \) embeds \( R \) into \( \bigcup_{i < \omega} \text{Hom}_D(V/V_i, \Lambda_\lambda) \) and the latter is \( S \)-hypercentral; it has an ascending series with factors isomorphic to the \( S \)-trivial modules \( \text{Hom}_D(V_i/V_{i+1}, \Lambda_{j+1}/\Lambda_j) \). Consequently \( R \leq \zeta(S) \) and the proof of Theorem 2 is complete. \( \square \)
References


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