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RIGHT ENGEL ELEMENTS OF STABILITY GROUPS OF GENERAL SERIES IN VECTOR SPACES

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Abstract: Let V be an arbitrary vector space over some division ring D, $\mathbf{L}$ a general series of subspaces of V covering all of $V\backslash\{0\}$ and S the full stability subgroup of $\mathbf{L}$ in $\mathrm{GL}(V)$. We prove that always the set of bounded right Engel elements of S is equal to the ω-th term of the upper central series of S and that the set of right Engel elements of S is frequently equal to the hypercentre of S.

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Throughout this paper we keep to the following notation. Let V be a vector space over a division ring D and $\mathbf{L} = \{(\Lambda_\alpha, V_\alpha) : \alpha \in \mathbf{A}\}$ a series of subspaces of V running from $\{0\}$ to V (see [**1**] for the definition and basic properties of general series). Thus in particular $\mathbf{A}$ is a linearly ordered set, the Λ_α/V_α are the jumps of the series, $V\backslash\{0\} = \cup_{\alpha\in\mathbf{A}}\Lambda_\alpha\backslash V_\alpha$ and $\Lambda_\alpha \leq V_\beta$ whenever $\alpha < \beta$ ($\alpha > \beta$ just for descending series, which we usually order from the top rather than from the bottom). The completion of $\mathbf{L}$ we denote by $\mathbf{L}^*$. Set $S = \mathrm{Stab}(\mathbf{L})$, the full stability group of $\mathbf{L}$ in $\mathrm{GL}(V)$; that is, $S = \cap_\alpha C_{\mathrm{GL}(V)}(\Lambda_\alpha/V_\alpha)$. In [**2**] we study the set of left Engel elements of S. In [**4**] we consider the right Engel elements S, but only for ascending or descending series. Here we consider what we can say for general series.

Theorem 1. *The set $R^-(S)$ of bounded right Engel elements of S is equal to the ω-th term $\zeta_\omega(S)$ of the upper central series of S.*

In [**4**] we prove Theorem 1, but only in the special cases of ascending or descending series. Our proof of Theorem 1 proceeds by applying these special cases to certain ascending and descending subseries of $\mathbf{L}$. We also use Theorem 3 below in the proof of Theorem 1. Note that the left analogue of Theorem 1, namely that the set $L^-(S)$ of bounded left Engel elements of S is equal to the Fitting subgroup $\mathrm{Fitt}(S)$ of S, is also valid, see [**2**, Theorem A].

The sets $R(S)$ and $L(S)$ of right and left Engel elements of S are much harder to compute and we only have partial results. Suppose that either $\dim_D V$ is countable or V has an $\mathbf{L}$-basis $\mathbf{B}$, meaning that $\mathbf{B} \cap L$ is a basis of L for each subspace L belonging to $\mathbf{L}$. Now $L(S) = \mathrm{Fitt}(S) = L^-(S)$, see [**2**, Theorem B], and if $\mathbf{L}$ is a descending series then $R(S) = \zeta_\omega(S)$, see [**4**, 1.2]. However if $\mathbf{L}$ is an ascending series, then although $R(S)$ is equal to the hypercentre $\zeta(S)$ of S, it is frequently not equal to $\zeta_\omega(S)$, see [**3**, 1.2]. Thus the obvious right analogue of the left Engel case is not valid, even for ascending series. However we do have the following two special cases.

Theorem 2. *Suppose that either* $\dim_D V$ *is countable or* V *has an* $\mathbf{L}$*-basis. Then set* $R(S)$ *of right Engel elements of* S *is equal to the hypercentre* $\zeta(S)$ *of* S.

Theorem 3. *Suppose either* $\mathbf{L}$ *has no top jump or* $\mathbf{L}$ *has no bottom jump (meaning that either* $V = \Lambda_\alpha$ *implies* $V = V_\alpha$ *or* $\{0\} = V_\alpha$ *implies* $\{0\} = \Lambda_\alpha$*). Then:*

a) $R^-(S) = \{1\}$.

b) *If either* $\dim_D V$ *is countable or* V *has an* $\mathbf{L}$*-basis* $\mathbf{B}$*, then* $R(S) = \{1\}$.

1. The proofs

We keep our notation above, in particular we have our series $\mathbf{L}$ and $\mathbf{L}^*$ and stability group S. We always assume that V is a left vector space over D. Clearly there are analogous results for right vector spaces. Also clearly we may remove all trivial jumps $\Lambda_\alpha = V_\alpha$ from $\mathbf{L}$ without affecting the conclusions of the theorems. Thus below assume that $\Lambda_\alpha > V_\alpha$ for all α in $\mathbf{A}$. This makes various statements simpler. For example, the hypothesis of Theorem 3 is now that $\mathbf{A}$ has either no maximal member or no minimal member. Let $\mathrm{Fitt}(L)$ denote the set of elements of S that stabilize some finite subseries of $\mathbf{L}^*$ running from $\{0\}$ to V; $\mathrm{Fitt}(L)$ is a normal subgroup of S contained in $\mathrm{Fitt}(S)$.

Lemma 1. *Let* $x \in R(S) \cap \mathrm{Fitt}(\mathbf{L})$ *and set* $X = \cup\{\Lambda_\alpha : \Lambda_\alpha(x-1) = \{0\}\}$*;* X *is an element of* $\mathbf{L}^*$*. Then* $\mathbf{C} = \{\alpha \in \mathbf{A} : X \leq V_\alpha\}$ *is inversely well-ordered (so* $\{(\Lambda_\alpha, V_\alpha) : X \leq V_\alpha\}$ *is a descending series).*

Proof: Clearly we may assume that $x \neq 1$. Now x stabilizes a finite subseries $\{0\} = X_0 < X_1 < \cdots < X_r = V$ of $\mathbf{L}^*$, where clearly $r \geq 2$ and we may assume that $X_1 = X$. If $\mathbf{C}$ is not inversely well-ordered there exist a $j \geq 1$ and an infinite subsequence $\alpha(1) < \alpha(2) < \cdots < \alpha(i) < \cdots$

of $\mathbf{C}$ with $X_j \leq V_{\alpha(1)}$ and $Y = \cup_i \Lambda_{\alpha(i)} \leq X_{j+1}$. Let $\mathbf{M}$ denote the ascending subseries

$$\{0\} < X_1 < \cdots < X_j \leq V_{\alpha(1)} < \Lambda_{\alpha(1)} \leq \cdots \leq V_{\alpha(i)} < \Lambda_{\alpha(i)} \leq V_{\alpha(j+1)} < \cdots \leq Y$$

of $\mathbf{L}^*$. Clearly $x|_Y \in \mathrm{Stab}(\mathbf{M}) = T$ say and $T \leq \mathrm{GL}(Y)$. Choose a subspace W of V with $V = W \oplus Y$ and extend the action of T on Y to one on V by making T centralize W. Then $T \leq S$. Also if $t \in T$, then $[x|_Y, {}_k t]|_Y = [x, {}_k t]|_Y$ for all $k \geq 1$ and so $x|_Y \in R(T)$. But then $x|_Y = 1$ by 1.1c) of [**4**]. This contradicts the choice of X_1 and completes the proof of Lemma 1. □

Lemma 2. *Let $x \in R(S)$ and define X and $\mathbf{C}$ as in Lemma 1.*

a) *If $x \in R^-(S)$ then $\mathbf{C}$ is inversely well-ordered.*

b) *If either $\dim_D V$ is countable or V has an $\mathbf{L}$-basis, then $\mathbf{C}$ is inversely well-ordered.*

Proof: a) $R^-(S) \subseteq L^-(S)^{-1} \subseteq \mathrm{Fitt}(\mathbf{L})$ by [**1**, 7.11] and [**2**, Theorem A]. Thus Lemma 1 applies.

b) Here $R(S) \subseteq L(S)^{-1} \subseteq \mathrm{Fitt}(\mathbf{L})$ by [**1**, 7.11] and [**2**, Theorem B] and again Lemma 1 applies. □

Lemma 3. *Let $x \in R^-(S) \cap \mathrm{Fitt}(\mathbf{L})$ and set $X = \cap\{V_\alpha : V(x-1) \leq V_\alpha\}$; X is an element of $\mathbf{L}^*$. Then $\mathbf{C} = \{\alpha \in \mathbf{A} : \Lambda_\alpha \leq X\}$ is well-ordered (so $\{(\Lambda_\alpha, V_\alpha) : \Lambda_\alpha \leq X\}$ is an ascending series).*

Proof: Again assume that $x \neq 1$, so x stabilizes a finite subseries $\{0\} = X_0 < X_1 < \cdots < X_r = V$ of $\mathbf{L}^*$, where $r \geq 2$ and $X_{r-1} = X$. If $\mathbf{C}$ is not well-ordered there exist a $j < r$ and an infinite subsequence $\alpha(1) > \alpha(2) > \cdots . > \alpha(i) > \cdots$ of $\mathbf{C}$ with $X_j \geq \Lambda_{\alpha(1)}$ and $Y = \cap_i V_{\alpha(i)} \geq X_{j+1}$. Let $\mathbf{M}$ denote the descending subseries of V/Y consisting of the X_k/Y for $j \leq k \leq r$ and the $\Lambda_{\alpha(i)}/Y$ and the $V_{\alpha(i)}/Y$ for $i \geq 1$. (This is a descending subseries of length ω of $\mathbf{L}^*$ taken modulo the element Y of $\mathbf{L}^*$.)

Clearly $x|_{V/Y} \in T = \mathrm{Stab}(\mathbf{M})$. Pick a subspace W of V with $V = W \oplus Y$ and let T act on W via its given action on V/Y and the natural isomorphism of V/Y onto W. Then extend this action of T on W to one on V by making T centralize Y. Clearly $T \leq S$ and $[x|_{V/Y}, {}_k t]|_{V/Y} = [x, {}_k t]|_{V/Y}$ for all $k \geq 1$ and all $t \in T$. Hence $x|_{V/Y} \in R^-(T)$. But then $x|_{V/Y} \in \zeta(T)$ by [**4**, 1.2a)] and $\zeta(T) = \langle 1 \rangle$ by [**3**, 1.1]. Lemma 3 follows. □

Lemma 4. *If $x \in R^{-}(S)$ and if* $\mathbf{C}$ *is as in Lemma 3, then* $\mathbf{C}$ *is well-ordered.*

Proof: Now $R^{-}(S) \subseteq \mathrm{Fitt}(\mathbf{L})$ by [**1**, 7.11] and [**2**, Theorem A]. Thus Lemma 3 applies. □

Lemma 5. *Suppose either* $\dim_D V$ *is countable or* V *has an* $\mathbf{L}$*-basis. If* $x \in R(S)$ *and if* $X = \cap\{V_\alpha : V(x-1) \leq V_\alpha\}$*, then* $\mathbf{C} = \{\alpha \in \mathbf{A} : \Lambda_\alpha \leq X\}$ *is well-ordered.*

Proof: As before $R(S) \subseteq L(S)^{-1} \subseteq \mathrm{Fitt}(L)$ by [**1**, 7.11] and [**2**, Theorem B]. Now repeat the proof of Lemma 3. Here we only obtain that $x|_{V/Y} \in R(T)$. But here $R(T) = \{1\}$ by [**4**, 1.2c)]. Lemma 5 now follows. □

Lemma 6. *Suppose that* $\mathbf{A}$ *either has no maximal member or has no minimal member. Then* $R^{-}(S) = \{1\}$*. If also either* $\dim_D V$ *is countable or* V *has an* $\mathbf{L}$*-basis, then* $R(S) = \{1\}$*.*

This lemma completes the proof of Theorem 3.

Proof: If $\mathbf{A}$ has no maximal member, then in Lemma 2 always $\mathbf{C}$ is empty. Thus in Lemma 2, case a) we have $R^{-}(S) = \{1\}$ and in Lemma 2, case b) we have $R(S) = \{1\}$.

Now assume $\mathbf{A}$ has no minimal member. Then $\mathbf{C}$ is always empty in Lemmas 4 and 5. Thus in Lemma 4 we have $R^{-}(S) = \{1\}$ and in Lemma 5 we have $R(S) = \{1\}$. Lemma 6 follows. □

Further notation. The series $\mathbf{L}^*$ has a maximal ascending segment starting at $\{0\}$; specifically there is an ordinal number $\lambda = \mu + n$, where $n \geq 0$ is an integer, μ is zero or a limit ordinal, and

$$\{0\} = \Lambda_0 < \Lambda_1 < \cdots < \Lambda_\gamma < \cdots < \Lambda_\lambda$$

is a subseries of $\mathbf{L}^*$ with each $\Lambda_{\gamma+1}/\Lambda_\gamma$ a jump of $\mathbf{L}$ and $\Lambda_\lambda \neq V_\alpha$ for all $\alpha \in \mathbf{A}$.

In the same way $\mathbf{L}^*$ has a maximal descending segment

$$V = V_0 > V_1 > \cdots > V_\gamma > \cdots > V_{\lambda'},$$

where $\lambda' = \mu' + n'$ is an ordinal number, $n' \geq 0$ is an integer, μ' is zero or a limit ordinal, each $V_\gamma/V_{\gamma+1}$ is a jump of $\mathbf{L}$ and $V_{\lambda'} \neq \Lambda_\alpha$ for all $\alpha \in \mathbf{A}$.

Clearly $V_{\mu'} \geq \Lambda_\lambda$ and $V_{\lambda'} \geq \Lambda_\mu$. Thus we have two possibilities. Firstly we could have that $V_{\lambda'} > \Lambda_\lambda$. Secondly if this is not the case then $V_{\mu'} = \Lambda_\lambda$ and $V_{\lambda'} = \Lambda_\mu$.

Proof of Theorem 1: We have to prove that $R^-(S) \subseteq \zeta_\omega(S)$. By [**4**, 1.1 and 1.2] we may assume that $\mathbf{L}$ is neither an ascending series nor a descending series. Set $R = R^-(S)$. Then R is a subgroup of S by [**4**, Theorem 1.3] and R centralizes V/Λ_λ and $V_{\lambda'}$ by Theorem 3 (and [**2**, 1.5]). Thus if $x \in R$, then $x\phi\colon v \mapsto v(x-1)$ is a linear homomorphism of V into Λ_λ with $V_{\lambda'} \leq \ker(x\phi)$. Thus ϕ is effectively an embedding of R into $\mathrm{Hom}_D(V/V_{\lambda'}, \Lambda_\lambda)$. We are not claiming at this stage that ϕ is a homomorphism of R; that is, possibly $(xy)\phi \neq x\phi + y\phi$, for some x and y in R.

Let $\mathbf{M}$ denote the descending subseries $V = V_0 > V_1 > \cdots > V_\gamma > \cdots > V_{\lambda'} > \{0\}$ of $\mathbf{L}^*$. Then $R \leq T = \mathrm{Stab}(\mathbf{M}) \leq S$, so $R \leq R^-(T)$. Then $R \leq \zeta_\omega(T)$ by [**4**, 1.2] and hence by [**3**, 4.1] (and its proof) $R\phi$ is contained in the subset of $\mathrm{End}_D V$ of all θ such that $V\theta \leq V_{\mu'}$ and there exists $i < \omega$ with $V_i\theta = \{0\}$. (Possibly $V_{\mu'} = V$, in which case $\mu' = 0$ and we can set $i = \lambda'$.)

Let $\mathbf{N}$ denote the ascending subseries $\{0\} = \Lambda_0 < \Lambda_1 < \cdots < \Lambda_\gamma < \cdots < \Lambda_\lambda < V$ of $\mathbf{L}^*$. Then $R \leq R^-(U)$ for $U = \mathrm{Stab}(\mathbf{N})$ and so $R \leq \zeta_\omega(U)$ by [**4**, 1.1]. Consequently if $\theta \in R\phi$, then for some $j < \omega$ we have $V\theta \leq \Lambda_j$ (and $\Lambda_\mu\theta = \{0\}$, which here we already know), see the last line of the proof of 3.5 of [**4**], where $\zeta_\omega(U)$ is identified, in the notation of [**4**], with T_{ω_-}. Again, if $\Lambda_\mu = \{0\}$, then $\mu = 0$ and we can choose $j = \lambda$.

Set $K_{ij} = \{\theta \in \mathrm{End}_D V : V_i\theta = \{0\} \text{ and } V\theta \leq \Lambda_j\}$, where $0 \leq i, j < \omega$, $i \leq \lambda'$, and $j \leq \lambda$. If $\theta \in K_{ij}$, $v \in V$, and $g \in S$, then

$$v[\theta, g] = vg^{-1}\theta g - v\theta = v(g^{-1} - 1)\theta g + v\theta(g - 1).$$

If also $i \geq 1$, then $V_{i-1}(g^{-1} - 1)\theta g \leq V_i\theta g = \{0\}$ and $V(g^{-1} - 1)\theta g \leq V\theta g \leq \Lambda_j$. Further if $j \geq 1$, then $V\theta(g-1) \leq \Lambda_j(g-1) \leq \Lambda_{j-1}$ and $V_i\theta(g-1) = \{0\}$. Consequently $[K_{ij}, S] \leq K_{i-1,j} + K_{i,j-1}$ whenever $i, j \geq 1$. Also $K_{ij} = \{0\}$ if either $i = 0$ or $j = 0$. Set $L_r = \sum_{i+j \leq r} K_{ij}$. Then

$$\{0\} = L_0 = L_1 \leq L_2 \leq \cdots \leq L_r \leq \cdots$$

with $[L_{r+1}, S] \leq L_r$ for all $r \geq 1$. Also $R\phi \subseteq L = \cup_r L_r$.

Suppose μ' is infinite. Then all $V_i \geq V_{\mu'}$ and $[V_{\mu'}, R] = \{0\}$; also $[V, R] \leq \Lambda_\lambda \leq V_{\mu'}$. Thus in this case ϕ is an S-monomorphism of R and hence $R \leq \zeta_\omega(S)$. Suppose μ is infinite. Then all $\Lambda_j \leq \Lambda_\mu$ and $[V, R] \leq \Lambda_\mu$; also $[\Lambda_\mu, R] \leq [V_{\lambda'}, R] = \{0\}$. Thus here too ϕ is an S-monomorphism of R and again $R \leq \zeta_\omega(S)$. Finally suppose $\mu = 0 = \mu'$. Then $\Lambda_\lambda < V_{\lambda'}$ (recall $\mathbf{L}$ here is not ascending, or for that matter descending). But then $[V, R] \leq \Lambda_\lambda$, $[V_{\lambda'}, R] = \{0\}$, ϕ is an S-monomorphism and $R \leq \zeta_\omega(S)$ (actually in this case $R \leq \zeta_{\lambda+\lambda'}(S)$). $\square$

Proof of Theorem 2: Here $\dim_D V$ is countable or V has an $\mathbf{L}$-basis. As far as we can, we follow the strategy of the proof of Theorem 1. Let $R = R(S)$. We need only prove that $R \subseteq \zeta(S)$. Again we may assume that $\mathbf{L}$ is neither ascending nor descending by [**4**, 1.1 and 1.2]. Also R is a normal subgroup of S by [**4**, 1.3] and R centralizes V/Λ_λ and $V_{\lambda'}$ by Theorem 3. If $x \in R$ and $x\phi\colon v \mapsto v(x-1)$, then ϕ embeds R into $\mathrm{Hom}_D(V/V_{\lambda'}, \Lambda_\lambda)$, which we regard as a subset of $\mathrm{End}_D(V)$ in the usual way.

With $\mathbf{M}$ and $T = \mathrm{Stab}(\mathbf{M})$ as in the proof of Theorem 1, we have $R \leq T \leq S$ and $R \leq R(T) = \zeta_\omega(T)$, the latter by [**4**, 1.2]. Hence [**3**, 4.2] yields that for each x in R there exists $i < \omega$ with $V_i(x-1) = \{0\}$. Again if $\mu' = 0$ then λ' is finite and we can choose $i = \lambda'$ for all such x. To avoid two formally different cases, if $\mu' = 0$ set $V_i = V_{\lambda'}$ for $\lambda' < i \leq \omega$. Thus in both cases we have $R\phi \subseteq \mathrm{Hom}_D(V/V_\omega, \Lambda_\lambda)$; in fact we have

$$R\phi \subseteq \bigcup_{i<\omega} \mathrm{Hom}_D(V/V_i, \Lambda_\lambda).$$

Let $\mathbf{N}$ be as in the proof of Theorem 1 and again set $U = \mathrm{Stab}(\mathbf{N})$. Then $R \leq R(U) = \zeta(U)$ by [**4**, 1.1]. From now on we can no longer proceed as in the proof of Theorem 1 since $\zeta(U)$ has a more complicated structure than $\zeta_\omega(U)$. Set $P = R \cap C_U(V/\Lambda_\mu)$. Then $P \leq C_U(V_{\lambda'}) \leq C_U(\Lambda_\mu)$ and $V_\omega \geq \Lambda_\mu$. Hence $\phi\colon P \to \mathrm{Hom}_D(V/V_\omega, \Lambda_\mu)$ is a group embedding.

Let $H_\gamma = \mathrm{Hom}_D(V/V_\omega, \Lambda_\gamma)$. Then $\{P\phi \cap H_\gamma\}_{\gamma \leq \mu}$ is an ascending U-hypercentral series of $P\phi$ by [**3**, 3.2 and 3.3]. Also $(P\phi \cap H_{\gamma+1})/(P\phi \cap H_\gamma)$ embeds into $\mathrm{Hom}_D(V/V_\omega, \Lambda_{\gamma+1}/\Lambda_\gamma)$. Let $K_i = \mathrm{Hom}_D(V/V_i, \Lambda_1)$. Now $P\phi \cap H_1 \leq \cup_{i<\omega} K_i$ since for each $x \in R$ there exists $i < \omega$ with $V(x-1) \leq V_i$. Also $K_{i+1}/K_i \cong \mathrm{Hom}_D(V_i/V_{i+1}, \Lambda_1)$ and the latter is centralized by S. Consequently $P\phi \cap H_1$ is S-hypercentral. In view of [**2**, 1.5] we may apply this to V/Λ_γ for each $\gamma < \mu$ and thus each $(P\phi \cap H_{\gamma+1})/(P\phi \cap H_\gamma)$ is S-hypercentral. Therefore $P \leq \zeta(S)$.

We now need to consider R/P. This acts faithfully on V/Λ_μ. In view of [**2**, 1.5] we may simplify our notation by assuming that $\Lambda_\mu = \{0\}$; that is, that $\mu = 0$. If $V_{\lambda'} = \Lambda_\mu$ then $\mathbf{L}$ is a descending series; in which case we already know that $R \leq \zeta(S)$. If $V_{\lambda'} \neq \Lambda_\mu$, then we have $V \geq V_\omega \geq V_{\mu'} \geq V_{\lambda'} > \Lambda_\lambda \geq \Lambda_\mu = \{0\}$. Also $[V, R] \leq \Lambda_\lambda$ and $[V_\omega, R] = \{0\}$. Consequently $\phi\colon R \to \mathrm{Hom}_D(V/V_\omega, \Lambda_\lambda)$ is a group embedding. But in fact ϕ embeds R into $\cup_{i<\omega} \mathrm{Hom}_D(V/V_i, \Lambda_\lambda)$ and the latter is S-hypercentral; it has an ascending series with factors isomorphic to the S-trivial modules $\mathrm{Hom}_D(V_i/V_{i+1}, \Lambda_{j+1}/\Lambda_j)$. Consequently $R \leq \zeta(S)$ and the proof of Theorem 2 is complete. □

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