Publ. Mat. **64** (2020), 143–155
DOI: 10.5565/PUBLMAT6412006

SUMS, PRODUCTS, AND RATIOS ALONG THE EDGES OF A GRAPH

Noga Alon, Imre Ruzsa, and József Solymosi

Abstract: In their seminal paper Erdős and Szemerédi formulated conjectures on the size of sumset and product set of integers. The strongest form of their conjecture is about sums and products along the edges of a graph. In this paper we show that this strong form of the Erdős–Szemerédi conjecture does not hold. We give upper and lower bounds on the cardinalities of sumsets, product sets, and ratio sets along the edges of graphs.

2010 Mathematics Subject Classification: 11P99.

Key words: sumset, sum-product problems, incidence geometry.

1. Introduction

1.1. Sum-product problems. Given a finite set $\mathcal{A}$ of a ring, the *sumset* and the *product set* are defined by

$$\mathcal{A}+\mathcal{A}=\{A+B : A,B\in\mathcal{A}\}$$

and

$$\mathcal{A}\mathcal{A}=\{AB : A,B\in\mathcal{A}\}.$$

Erdős and Szemerédi raised the following conjecture:

Conjecture 1 ([**6**])**.** *Every finite set of integers $\mathcal{A}$ having large enough cardinality satisfies*

$$\max(|\mathcal{A}+\mathcal{A}|,|\mathcal{A}\mathcal{A}|)\geq|\mathcal{A}|^{2-\varepsilon}, \tag{1}$$

where $\varepsilon\to 0$ as $|A|\to\infty$.

They proved that

$$\max(|\mathcal{A}+\mathcal{A}|,|\mathcal{A}\mathcal{A}|)=\Omega(|\mathcal{A}|^{1+\delta}) \tag{2}$$

The first named author is supported in part by NSF grant DMS-1855464, ISF grant 281/17, and the Simons Foundation. The second named author is supported in part by an OTKA NK 104183 grant. The third named author is supported in part by a NSERC and an OTKA NK 104183 grant.

for some $\delta > 0$. Here and in what follows we use the asymptotic notation $\Omega(\cdot), O(\cdot)$, and $\Theta(\cdot)$. For two functions over the reals, $f(x)$ and $g(x)$, we write $f(x) = \Omega(g(x))$ if there is a positive constant $B > 0$ and a threshold D such that $f(x) \geq B \cdot g(x)$ for all $x \geq D$. We write $f(x) = O(g(x))$ if there is a positive constant $B > 0$ and a threshold D such that $f(x) \leq B \cdot g(x)$ for all $x \geq D$. Finally, $f(x) = \Theta(g(x))$ if $f(x) = O(g(x))$ and $f(x) = \Omega(g(x))$.

Erdős and Szemerédi formulated an even stronger conjecture. In this variant one considers a subset of the possible pairs in the sumset and product set. Let G_n be a graph on n vertices $v_1, v_2, \ldots, v_n$ with n^{1+c} edges for some real $c > 0$. Let $\mathcal{A}$ be an n-element set of real numbers, say $\mathcal{A} = \{a_1, a_2, \ldots, a_n\}$. The *sumset of* $\mathcal{A}$ *along* G_n, denoted by $\mathcal{A} +_{G_n} \mathcal{A}$, is the set $\{a_i + a_j \mid (i,j) \in E(G_n)\}$. The product set along G_n is defined similarly:

$$\mathcal{A} \cdot_{G_n} \mathcal{A} = \{a_i \cdot a_j \mid (i,j) \in E(G_n)\}.$$

The Strong Erdős–Szemerédi Conjecture is the following:

Conjecture 2 ([**6**])**.** *For every $c > 0$ and $\varepsilon > 0$ there is a threshold n_0 such that if $n \geq n_0$, then, for any n-element subset of integers $\mathcal{A} \subset \mathbb{N}$ and any graph G_n with n vertices and at least n^{1+c} edges, one has*

$$|\mathcal{A} +_{G_n} \mathcal{A}| + |\mathcal{A} \cdot_{G_n} \mathcal{A}| \geq |\mathcal{A}|^{1+c-\varepsilon}.$$

The original conjecture, inequality (1), would follow from this stronger conjecture by taking the complete graph $G_n = K_n$.

The problem of finding sets and graphs with small sumsets and product sets along the edges is a way to analyze to what extent the additive and multiplicative structure can intervene in a set. In the other direction, Balog and Wooley ([**2**]) showed that any finite set of real numbers can be partitioned into a highly non-additive part and a highly non-multiplicative part. (See also [**11**] for some related work.)

For more details on the sum-product problem we refer to a recent survey [**7**].

Here we refute Conjecture 2 by giving constructions with small sumsets along a graph where the product set is also small. A similar problem – which is closely related to the original sum-product conjecture – is to bound the number of sums and *ratios* along the edges of a graph. We give upper and lower bounds on these quantities.

2. Products

In the next construction we define a set and a graph with many edges such that both the sumset and the product set are small.

Theorem 3. *For arbitrary large m_0 there is a set of integers $\mathcal{A}$ and a graph G_m on $|\mathcal{A}| = m \geq m_0$ vertices with $\Omega(m^{5/3}/\log^{1/3} m)$ edges such that*

$$|\mathcal{A} +_{G_m} \mathcal{A}| + |\mathcal{A} \cdot_{G_m} \mathcal{A}| = O\Big((|\mathcal{A}| \log |\mathcal{A}|)^{4/3}\Big).$$

Proof: It is easier to describe our construction using rational numbers instead of integers. Multiplying then with the least common multiple of the denominators will not affect the size of the sumset or the product set, giving a construction for integers.

We define the set $\mathcal{A}$ first and then the graph. Below, the function $\mathrm{lpf}(m)$ denotes the least prime factor of m. We write $(v, w) = 1$ if v and w are relatively prime. Let

$$\mathcal{A} := \Big\{ \frac{uw}{v} \mid u, v, w \in \mathbb{N}, \text{ where } v, w \leq n^{1/6},\ u \leq n^{2/3},\ (v, w) = 1, \text{ and } \mathrm{lpf}(u) > n^{1/6} \Big\}.$$

The number of u, w, v triples with $v, w \leq n^{1/6}$, $u \leq n^{2/3}$ is about n, but there are further restrictions. The $\mathrm{lpf}(u) > n^{1/6}$, $u \leq n^{2/3}$ conditions allow us to select about $n^{2/3}/\log(n^{1/6})$ numbers for u, and there are $\sim 6n^{1/3}/\pi^2$ coprime pairs v, w up to $n^{1/6}$. We are going to define a graph G_m with vertex set $\mathcal{A}$, where $|\mathcal{A}| = m = O(n/\log n)$. Two elements $a, b \in \mathcal{A}$ are connected by an edge if, in the definition of $\mathcal{A}$ above, $a = \frac{wu}{v}$ and $b = \frac{vz}{w}$. There are at least

$$\Omega(n^{1/6} n^{1/6} (n^{2/3}/\log(n^{1/6}))^2) = \Omega(n^{5/3}/\log^2 n)$$

edges (the number of quadruples u, v, w, z satisfying $v, w \leq n^{1/6}$, $u, z \leq n^{2/3}$, $(v, w) = 1$, and $\mathrm{lpf}(z), \mathrm{lpf}(u) > n^{1/6}$).

The products of pairs of elements of $\mathcal{A}$ along an edge of G_m are integers of size at most $n^{4/3}$. The sums along the edges are of the form

$$\frac{wu}{v} + \frac{vz}{w} = \frac{w^2 u + v^2 z}{vw}.$$

The denominator is a positive integer of size at most $n^{1/3}$ and the numerator is a positive integer of size at most $2n$. Hence the number of sums is at most $2n^{4/3}$. □

Modifying the construction above we give a counterexample to the Strong Erdős–Szemerédi Conjecture for every $1 > c > 0$. For the sake of simplicity we will ignore logarithmic multipliers, using the asymptotic notations $\Omega_l(\cdot)$, $O_l(\cdot)$, and $\Theta_l(\cdot)$. For two functions over the reals, $f(x)$ and $g(x)$, we write $f(x) = \Omega_l(g(x))$ if there is a constant $B \leq 0$ and a threshold D such that $f(x) \geq \log^B x \cdot g(x)$ for all $x \geq D$. We write $f(x) = O_l(g(x))$ if there is a positive constant $B \geq 0$ and a threshold D such that $f(x) \leq \log^B x \cdot g(x)$ for all $x \geq D$. We write $f(x) = \Theta_l(g(x))$ if $f(x) = O_l(g(x))$ and $f(x) = \Omega_l(g(x))$.

Theorem 4. *For every* $1 > c > 0$ *there is a* $\delta > 0$ *such that, for arbitrary large* n*, there is an* n*-element subset of integers* $A \subset \mathbb{N}$ *and a graph* H_n *with* $\Omega_l(n^{1+c})$ *edges such that*

$$|\mathcal{A} +_{H_n} \mathcal{A}| + |\mathcal{A} \cdot_{H_n} \mathcal{A}| = O_l(|\mathcal{A}|^{1+c-\delta}).$$

Proof: We consider two cases separately, when $0 < c \leq 2/3$ and when $2/3 < c < 1$.

Case 1: $(2/3 < c < 1)$. We define $\mathcal{A}$ similar to the previous construction, but now the ranges of u, v, and w are different:

$$\mathcal{A} := \left\{ \frac{uw}{v} \mid v, w \leq n^{\frac{1-c}{2}},\ (v,w) = 1,\ u \leq n^c,\ \mathrm{lpf}(u) > n^{\frac{1-c}{2}} \right\}.$$

The number of u, w, v triples satisfying the conditions is $\Omega_l(n)$. If $|\mathcal{A}| = m$, then let H_m be the graph with vertex set $\mathcal{A}$. Two elements $a, b \in \mathcal{A}$ are connected by an edge if they can be written as $a = \frac{uw}{v}$ and $b = \frac{vz}{w}$. There are at least $\Omega_l(n^{1+c})$ edges in H_m. The products of two such elements of $\mathcal{A}$ are integers of size at most n^{2c}. A typical sum is

$$\frac{uw}{v} + \frac{vz}{w} = \frac{w^2u + v^2z}{vw}.$$

The numerator is an integer of size at most $2n$ and the denominator is an integer of size at most n^{1-c}. Therefore, the sumset along the edges of H_m has size at most $2n^{2-c}$. Since $2c > 2 - c$ in this range of c, we set $\delta = 1 - c$. (Note that $n = O_l(m)$.)

Case 2: $(0 < c \leq 2/3)$. It is possible to describe a construction similar to that in the first case, but we prefer to take a subgraph of the graph G_m in Theorem 3. Let p be a parameter satisfying $0 < p \leq 1$, to be specified later. In G_m take first the edges with the $pm^{4/3}$ most popular products. (The *popularity* of an element $e \in A \cdot A$ is the number of pairs $a, b \in A$ such that $a \cdot b = e$.) This gives a graph G'_m with at least $\Omega_l(pm^{5/3})$ edges, with $O_l(pm^{4/3})$ products, and at most $O_l(m^{4/3})$ sums. Now, in G'_m take the most popular $pm^{4/3}$ sums to get the subgraph H_m with at least $\Omega_l(p^2m^{5/3})$ edges, with $O_l(pm^{4/3})$ products, and $O_l(pm^{4/3})$ sums. Choosing p to be $n^{c/2}/n^{1/3}$ we get $\Omega_l(n^{1+c})$ edges and $O_l(n^{1+c/2})$ sums and products. □

3. Ratios

In this section we consider a problem similar to the Strong Erdős–Szemerédi Conjecture, but we change products to ratios. Define

$$\mathcal{A}/_{G_n}\mathcal{A} = \{a_i/a_j \mid (i,j) \in E(G_n)\}.$$

(Note that each edge (i,j) here provides two ratios: a_i/a_j and a_j/a_i.) Changing product to ratio is a common technique in sum-product bounds. When one is using the multiplicative energy (like in [**13**] and [**8**], for example) then the role of product and ratio are interchangeable. The multiplicative energy of a set $\mathcal{A}$ is the number of quadruples $(a,b,c,d) \in \mathcal{A}^4$ such that $ab = cd$, which is clearly the same as the number of quadruples where $a/c = d/b$. But the symmetry fails in the Strong Erdős–Szemerédi Conjecture. We are going to show examples when the sumsets and ratio sets are even smaller than in the previous construction.

3.1. Connection to the original conjecture. What is the connection of the Strong Erdős–Szemerédi Conjecture to the original conjecture (when $G_n = K_n$)? Similar questions were investigated in [**3**]. Here we consider the connections to the sum-ratio problem along a graph. If there was a counterexample to Conjecture 1, that would imply the existence of a set with very small sumset and ratio set along a dense graph. In our first result let us suppose that both the product set and ratio set are small.

Theorem 5. *Let us suppose that there is a set $\mathcal{A}$ of n real numbers such that $|\mathcal{A}+\mathcal{A}| \le n^{2-\alpha}$, $|\mathcal{A}\mathcal{A}| = \Theta(n^{2-\beta})$, and $|\mathcal{A}/\mathcal{A}| \le n^{2-\beta}$ for some $\alpha, \beta > 0$ real numbers. Then, there is a set $\mathcal{B}$ with $N > n$ elements and a graph G_N with $\Omega(N^{\frac{3}{3-\beta}})$ edges such that*

$$|\mathcal{B}/_{G_N}\mathcal{B}| = O(N)$$

and

$$|\mathcal{B} +_{G_N} \mathcal{B}| = O(N^{\frac{2-\alpha}{3-\beta}}).$$

Proof: Let $\mathcal{B} = \{a \pm \zeta bc \mid a,b,c \in \mathcal{A}\}$, where $\zeta \in \mathbb{R}$ is selected such that all sums are distinct. Note that $\mathcal{B}$ has cardinality $2|\mathcal{A}||\mathcal{A}\mathcal{A}| = N = \Theta(n^{3-\beta})$. In the graph G_N every $a - \zeta ac$ is connected to $b + \zeta ac$ by an edge. The number of edges is $n^3 = \Omega(N^{\frac{3}{3-\beta}})$. The number of sums along the edges is $|\mathcal{A}+\mathcal{A}| = O(N^{\frac{2-\alpha}{3-\beta}})$, and ratios along the edges have the form

$$\frac{a+\zeta ac}{b-\zeta ac} = \frac{1+\zeta c}{b/a - \zeta c},$$

so the cardinality of the ratio set is at most $|\mathcal{A}||\mathcal{A}/\mathcal{A}| = O(N)$. □

A bound on the cardinality of the product set does not imply a similar bound on the ratio set. Or, equivalently, a bound on the cardinality of the sumset does not imply a similar bound on the difference set. A classical construction of the second author in [**10**] is an example for that. It uses the observation that $S = \{0, 1, 3\}$ satisfies $|S + S| = 6$ and $|S - S| = 7$. If we consider the set of numbers A, of the form $a = \sum_{i=0}^{k-1} \alpha_i(a) 10^i$, where $\alpha_i(a) \in S$, then $|A| = 3^k$, $|A + A| = 6^k$, and $|A - A| = 7^k$.

Note that, in this construction, the multiplicity of a member $a - a' = \sum_{i=0}^{k-1} (\alpha(a) - \alpha(a'))10^i$ of $A - A$ along the edges of the complete graph on A is 3^r, where r is the number of indices satisfying $\alpha(a) = \alpha(a')$. It is easy to see that, for every fixed small $\delta > 0$, the fraction of edges in which the parameter r exceeds $(1/3+\delta)k$ is at most $e^{-\Omega(\delta^2 k)}$. Therefore, any graph on A with at least $(9^k)^{1-c\delta^2}$ edges, for an appropriate absolute positive constant c, has on at least half of its edges a value of the difference with multiplicity at most $3^{(1/3+\delta)k}$, implying that the number of distinct differences along the edges is at least

$$0.5 \frac{(9^k)^{1-c\delta^2}}{3^{(1/3+\delta)k}}.$$

As $9/3^{1/3} > 6$ and the number of sums is only 6^k, this shows that for small δ the number of differences along the edges of any such graph is significantly larger than the number of sums.

The above discussion shows that we need a modified statement to transform a possible counterexample to Conjecture 1 to a statement about few sums and ratios along a graph. We are going to apply the following lemma.

Lemma 6. *Let A be an n-element subset of an abelian group and suppose that $|A + A| \le K|A|$. Then, there is an integer parameter M and a graph H_n with vertex set A and at least $\sqrt{\frac{M|A|^3}{4K \log |A|}}$ edges such that $|A -_{H_n} A| \le M$. Moreover, M satisfies the following inequalities:*

$$\frac{|A|}{K \log |A|} \le M \le 4K \log |A||A|.$$

Proof: The additive energy of A, denoted by $E(A)$, is the number of a, b, c, d quadruples from A such that $a + b = c + d$. This is the same as the number of quadruples satisfying $a - c = d - b$. By the Cauchy–Schwarz inequality we have $E(A) \ge |A|^3/K$. Denote the elements of the difference set as follows: $A - A = \{t_1, t_2, \ldots, t_\ell\}$. For every element t_i, we can define its multiplicity by $m(t_i) = |\{a, b \in A \mid a - b = t_i\}|$. With these notations we can write the additive energy as

$$E(A) = \sum_{i=1}^{\ell} m^2(t_i) = \sum_{k=1}^{\log n} \sum_{2^k \leq m(t_i) < 2^{k+1}} m^2(t_i).$$

There is a k such that

$$\sum_{2^k \leq m(t_i) < 2^{k+1}} m^2(t_i) \geq \frac{|A|^3}{K \log |A|}. \tag{3}$$

Let $T_k = \{t_i \in A - A \mid 2^k \leq m(t_i) < 2^{k+1}\}$, and set $M = |T_k|$. The edges of H_n are defined as follows: $(a, b) \in A^2$ is an edge iff $a - b = t_i \in T_k$. The number of the edges is $\sum_{t_i \in T_k} m(t_i)$. From inequality (3) we have a lower bound on the $m(t_i)$s,

$$m(t_i) \geq \frac{|A|^{3/2}}{2\sqrt{MK \log |A|}},$$

so the number of edges is at least

$$\sqrt{\frac{M|A|^3}{4K \log |A|}}.$$

In order to bound the magnitude of M, note that since $\sum_{t_i \in T_k} m(t_i) \leq |A|^2$, the largest $m(t_i)$ for an element t_i in T_k satisfies the trivial inequality $\max_{t_i \in T_k}(m(t_i)) \leq 2|A|^2/M$. Replacing the $m(t_i)$s by $2|A|^2/M$ on the left hand side of inequality (3) we get the desired upper bound on M. The lower bound follows from the same inequality and from the fact that $m(t_i) \leq |A|$ for every $t_i \in A - A$. □

In the proof of Theorem 5, in G_N, the edges were defined by pairs of vertices having the form $(a - \zeta ac, b + \zeta ac)$. If we have a bound on the product set only, $|\mathcal{A}\mathcal{A}| \leq n^{2-\beta}$, then in our new graph, G'_N, we connect $a - \zeta ac$ and $b + \zeta ac$ only if (a, b) is an edge in H_n, where H_n is defined in Lemma 6 applied to the set $\mathcal{A}$ in the multiplicative group. This guarantees that the ratio set along the edges is not (much) larger than the product set along edges of G_N. The new graph G'_N is a subgraph of the graph G_N in Theorem 5.

The new parameter M makes the description of our next result a bit complicated, but the important feature of this construction is that it shows that if there is a counterexample to Conjecture 1, then there is a set of numbers $\mathcal{B}$ and graph G'_N with many edges so that the sumset and the ratio set are both small. The number of edges might be less than in

Theorem 5, but then the size of the ratio set along this graph is much smaller. We will state a simpler, but weaker, statement in a corollary below.

Theorem 7. *Let us suppose that there is a set of n real numbers $\mathcal{A}$ such that $|\mathcal{A}+\mathcal{A}| \le n^{2-\alpha}$ and $|\mathcal{A}\mathcal{A}| = \Theta_l(n^{2-\beta})$ for some $\alpha > 0$, $\beta > 1/2$ real numbers. Then, there is a set $\mathcal{B}$ with $N > n$ elements, a parameter M in the range*

$$\Omega_l(N^{\frac{\beta}{3-\beta}}) \le M \le O_l(N^{\frac{2-\beta}{3-\beta}}),$$

and a graph G'_N with $\Omega_l(M^{\frac{1}{2}}N^{\frac{4+\beta}{6-2\beta}})$ edges such that

$$|\mathcal{B}/_{G'_N}\mathcal{B}| = O_l(MN^{\frac{1}{3-\beta}})$$

and

$$|\mathcal{B}+_{G'_N}\mathcal{B}| = O_l(N^{\frac{2-\alpha}{3-\beta}}).$$

Note that the number of edges in G'_N is at least $\Omega_l(N^{\frac{2+\beta}{3-\beta}})$, which is bigger than the cardinalities of the sumset and ratio set along its edges.

Corollary 8. *Let us suppose that there is a set of n real numbers $\mathcal{A}$ such that $|\mathcal{A}+\mathcal{A}| \le n^{2-\alpha}$ and $|\mathcal{A}\mathcal{A}| = \Theta_l(n^{2-\beta})$ for some $\alpha > 0$, $\beta > 1/2$ real numbers. Then, there is a set $\mathcal{B}$ with $N > n$ elements and a graph G'_N with $\Omega_l(N^{\frac{2+\beta}{3-\beta}})$ edges such that*

$$|\mathcal{B}/_{G'_N}\mathcal{B}| = O_l(N)$$

and

$$|\mathcal{B}+_{G'_N}\mathcal{B}| = O_l(N^{\frac{2-\alpha}{3-\beta}}).$$

Proof of Theorem 7: Applying Lemma 6 to the multiplicative subgroup of real numbers with $K = |\mathcal{A}|^{1-\beta}$ we get a graph H_n as in the lemma. We connect $a - \zeta ac$ and $b + \zeta ac$ in G'_N only if (a,b) is an edge in H_n. For given a and b we can choose any $c \in \mathcal{A}$, so the number of edges is $\Omega_l(|\mathcal{A}|\sqrt{M|\mathcal{A}|^{2+\beta}}) = \Omega_l(\sqrt{M|\mathcal{A}|^{4+\beta}})$. The size of the ratio set along the edges is $|\mathcal{A}|M$, and the sumset is not larger than $O(N^{\frac{2-\alpha}{3-\beta}})$ since G'_N is a subgraph of G_N. □

3.2. Constructions. In the next construction we define a set and a graph with many edges such that both the sumset and the ratio set are very small. It can be viewed as a special case of Theorem 5 with $\alpha = 0$ and $\beta = 1$. An alternative construction with similar parameters is described in [**2**].

Theorem 9. *For arbitrary large n there is a set of reals $\mathcal{A}$ of cardinality n and a graph G_n with $\Omega(n^{3/2})$ edges such that*

$$|\mathcal{A} +_{G_n} \mathcal{A}| + |\mathcal{A}/_{G_n}\mathcal{A}| \leq O(|\mathcal{A}|).$$

Proof: As before, in the construction we define the set $\mathcal{A}$ first and then G_n. Let $\mathcal{A} = \{\pm(2^i - 2^j) \mid 1 \leq j < i \leq \sqrt{n}\}$. Two elements, $2^i - 2^j$ and $-(2^k - 2^\ell)$, are connected by an edge iff $j = \ell$. Along this G_n both the sumsets and the ratio sets are small,

$$|\mathcal{A} +_{G_n} \mathcal{A}| = |\{2^i - 2^k \mid 1 \leq i, k \leq \sqrt{n}\}| \leq n,$$

and

$$|\mathcal{A}/_{G_n}\mathcal{A}| = |\{-(2^s - 1)/(2^t - 1) \mid 1 \leq s, t \leq \sqrt{n} - 1\}| < n.$$

In this construction,

$$|\mathcal{A}| = 2\binom{\lfloor\sqrt{n}\rfloor}{2} \sim n$$

and the number of edges is a little more than

$$2\binom{\lfloor\sqrt{n}\rfloor}{3} \geq \left(\frac{1}{3} - o(1)\right) n^{3/2}. \qquad \square$$

3.3. Matchings. Erdős and Szemerédi mentioned in their paper that maybe even for a linear number of edges (when $c = 0$) the Strong Erdős–Szemerédi Conjecture holds, but noted that it is not true for reals. For any even integer n there is an n-element set of reals $\mathcal{A}$, such that G_n is a perfect matching and

$$|\mathcal{A} +_{G_n} \mathcal{A}| + |\mathcal{A} \cdot_{G_n} \mathcal{A}| = O(|\mathcal{A}|^{1/2}).$$

It was shown by Alon, Angel, Benjamini, and Lubetzky in [**1**] that if we assume the Bombieri–Lang conjecture (see details in [**3**]), then for any set of integers $\mathcal{A}$, if G_n is a matching, then

$$|\mathcal{A} +_{G_n} \mathcal{A}| + |\mathcal{A} \cdot_{G_n} \mathcal{A}| = \Omega(|\mathcal{A}|^{4/7}).$$

It is possible that $4/7$ can be improved to a number close to 1, but if we change multiplication to ratio, then just the trivial bound $\Omega(\sqrt{n})$ holds.

A simple construction demonstrating this is the following. Take $n = k^2$ and distinct primes, $p_1, \ldots, p_k, q_1, \ldots, q_k$. The matching consists of all pairs $(p_i/q_j, (q_j-1)p_i/q_j)$ $(i, j = 1, \ldots, k)$, and $\mathcal{A}$ is the collection of these pairwise distinct rationals. The sums along the matching edges are the p_is, the quotients (of large divided by small) along the edges are $(q_j - 1)$. For this set $\mathcal{A}$ of the quotients above and for the matching G_n we have

$$|\mathcal{A} +_{G_n} \mathcal{A}| + |\mathcal{A}/_{G_n}\mathcal{A}| = O(|\mathcal{A}|^{1/2}),$$

which is as small as possible.

4. Lower bounds

Lower bounds on the number of sums and products along graphs were obtained in [**1**]. Under assuming the Bombieri–Lang conjecture they proved that if $\mathcal{A}$ is an n-element set of integers and G_n a graph with m edges then

$$|\mathcal{A} +_{G_n} \mathcal{A}| + |\mathcal{A} \cdot_{G_n} \mathcal{A}| = \Omega\left(\min\left(\frac{m^{8/14}}{n^{1/14}}, \frac{m}{n^{1/2}}\right)\right). \tag{4}$$

For the unconditional case (without the Bombieri–Lang conjecture) they proved that

$$|\mathcal{A} +_{G_n} \mathcal{A}| + |\mathcal{A} \cdot_{G_n} \mathcal{A}| = \Omega\left(\frac{m^{19/9-o(1)}}{n^{28/9+o(1)}}\right). \tag{5}$$

We will apply a variant of Elekes' classical proof, used in his sum-product estimate in [**5**] to get better estimates.

Theorem 10. *Let $\mathcal{A}$ be an n-element set of reals and G_n a graph with m edges. Then*

$$|\mathcal{A} +_{G_n} \mathcal{A}| + |\mathcal{A} \cdot_{G_n} \mathcal{A}| \geq \Omega\left(\frac{m^{3/2}}{n^{7/4}}\right).$$

Proof: Let us consider the Cartesian product $(\mathcal{A} +_{G_n} \mathcal{A}) \times (\mathcal{A} \cdot_{G_n} \mathcal{A})$, where the sums and the products are considered along a graph G_n. Define a set of n^2 lines L, where the lines are $y = (x-a)b$ for every $a, b \in \mathcal{A}$. For an element of $\mathcal{A}$, say u, if in the graph u has two neighbours w_1, w_2, then $((u+w_1)-w_1)w_2$ is in $\mathcal{A} \cdot_{G_n} \mathcal{A}$. Thus $(u+w_1, uw_2)$ lies on the line $y = (x - w_1)w_2$ and hence the lines give at least the sum of the squares of degrees incidences in the Cartesian product $(\mathcal{A} +_{G_n} \mathcal{A}) \times (\mathcal{A} \cdot_{G_n} \mathcal{A})$. If the graph has m edges, then, by the Cauchy–Schwarz inequality, the number of incidences is at least $n(2m/n)^2$. On the other hand, by the Szemerédi–Trotter Theorem [**14**], we have at most $O(n^{4/3}(|\mathcal{A} + \mathcal{A}||\mathcal{A}/\mathcal{A}|)^{2/3})$ incidences. We conclude that $m^2/n < cn^{4/3}(|\mathcal{A}+\mathcal{A}||\mathcal{A}/\mathcal{A}|)^{2/3}$. This implies the required $\Omega((m^6/n^7)^{1/4})$ lower bound on $|\mathcal{A} +_{G_n} \mathcal{A}| + |\mathcal{A} \cdot_{G_n} \mathcal{A}|$. $\square$

Since the values of the product and sum along an edge determine the values in the end-vertices, there is an obvious lower bound, $|\mathcal{A} +_{G_n} \mathcal{A}| + |\mathcal{A} \cdot_{G_n} \mathcal{A}| \geq \sqrt{m}$. Note that Theorem 10 gives stronger bound only if the number of edges is larger than $n^{7/4}$. Our result improves the (conditional) inequality in (4) if the number of edges is larger than $n^{47/26} \sim n^{1.8}$, and it is always stronger than the bound in (5).

The very same technique can be applied to give a similar lower bound on the number of sums and ratios. Consider the Cartesian product $(\mathcal{A} +_{G_n} \mathcal{A}) \times (\mathcal{A}/_{G_n}\mathcal{A})$ and the set of lines L, where the lines are $y = (x-a)/b$ for every $a, b \in \mathcal{A}$. Applying the Szemerédi–Trotter Theorem as above, we have

$$|\mathcal{A} +_{G_n} \mathcal{A}| + |\mathcal{A}/_{G_n}\mathcal{A}| \geq \Omega\left(\frac{m^{3/2}}{n^{7/4}}\right).$$

Elekes' bound was improved in [**12**] using the Szemerédi–Trotter Theorem in a different way. The argument there can be modified to bound the number of sums and ratios along a graph. The proof is rather technical, it follows [**12**] step by step but with more parameters in order to deal with the density version of the original proof. We do not think that this estimate is close to the truth and it is just slightly better, in a small range when $m \gg n^{11/6}$, than the simple bound above. We state the bound without the detailed proof.

Claim 11. *Let $\mathcal{A}$ be an n-element set of reals and G_n a graph with m edges. Then*

$$|\mathcal{A} +_{G_n} \mathcal{A}| + |\mathcal{A}/_{G_n}\mathcal{A}| \geq \Omega\left(\frac{m^{18/11}}{n^2}\right).$$

5. Arrangements of pencils

The following question was asked by Misha Rudnev [**9**]. An n-pencil in the plane is a set of n concurrent lines. The center of a pencil is the common intersection point of its lines.

Problem 12. *If the centers of four n-pencils are not collinear, then what is the maximum possible number of points with four incident lines (one from each pencil)?*

Chang and Solymosi proved in [**4**] that the number of such points is at most $O(n^{2-\delta})$ for some $\delta > 0$. (They did not calculate δ explicitly.) Using the construction in Theorem 9 we show that $\delta \leq 1/2$.

Claim 13. *For arbitrary large n there are arrangements of four non-collinear n-pencils which determine $\Omega(n^{3/2})$ points incident to four lines.*

Proof: In this construction we refer to Theorem 9. If a set of reals A has small sumset, then the geometric interpretation of this fact is that the points of the Cartesian product $A \times A$ can be covered by a small number of slope -1 lines. Similarly, if the ratio set is small, then $A \times A$ can be covered by a small number of lines through the origin. The set of points where four lines intersect is defined as

$$P := \{(2^i - 2^j, -(2^k - 2^j)) \in \mathbb{R}^2 \mid 1 \le j \le i, k \le \sqrt{n}\}.$$

The four pencils are:

- The vertical lines with a point in P:
$$L_1 := \{x = 2^i - 2^j \mid 1 \le j \le i \le \sqrt{n}\}.$$
- The horizontal lines with a point in P:
$$L_2 := \{y = -2^i + 2^j \mid 1 \le j \le i \le \sqrt{n}\}.$$
- The slope -1 lines with a point in P:
$$L_3 := \{x - (2^i - 2^j) = -(y + (2^k - 2^j)) \mid 1 \le j \le i, k \le \sqrt{n}\}.$$
- Lines through the origin with a point in P:
$$L_4 := \left\{y = -\frac{2^{i-j} - 1}{2^{k-j} - 1} x \mid 1 \le j \le i, k \le \sqrt{n}\right\}.$$

Note that in the definition of L_3 and L_4 the same lines are listed multiple times. Ignoring these repetitions it is easy to see that all four families have size approximately n, and $|P| = \Omega(n^{3/2})$. One can apply a projective transformation to shift the centers of the pencils from infinity to $\mathbb{R}^2$. □

Acknowledgement

We are thankful to Misha Rudnev, whose question on pencils initiated this work.

References

[1] N. ALON, O. ANGEL, I. BENJAMINI, AND E. LUBETZKY, Sums and products along sparse graphs, *Israel J. Math.* **188(1)** (2012), 353–384. DOI: 10.1007/s11856-011-0170-x.

[2] A. BALOG AND T. D. WOOLEY, A low-energy decomposition theorem, *Q. J. Math.* **68(1)** (2017), 207–226. DOI: 10.1093/qmath/haw023.

[3] L. CAPORASO, J. HARRIS, AND B. MAZUR, Uniformity of rational points, *J. Amer. Math. Soc.* **10(1)** (1997), 1–35. DOI: 10.1090/S0894-0347-97-00195-1.

[4] M.-C. CHANG AND J. SOLYMOSI, Sum-product theorems and incidence geometry, *J. Eur. Math. Soc. (JEMS)* **9(3)** (2007), 545–560. DOI: 10.4171/JEMS/87.

[5] G. ELEKES, On the number of sums and products, *Acta Arith.* **81(4)** (1997), 365–367. DOI: 10.4064/aa-81-4-365-367.

[6] P. ERDŐS AND E. SZEMERÉDI, On sums and products of integers, in: *"Studies in Pure Mathematics"*, Birkhäuser, Basel, 1983, pp. 213–218. DOI: 10.1007/978-3-0348-5438-2_19.

[7] A. Granville and J. Solymosi, Sum-product formulae, in: *"Recent Trends in Combinatorics"*, IMA Vol. Math. Appl. **159**, Springer, Cham, 2016, pp. 419–451. DOI: 10.1007/978-3-319-24298-9_18.

[8] S. V. Konyagin and I. D. Shkredov, On sum sets of sets having small product set, *Proc. Steklov Inst. Math.* **290(1)** (2015), 288–299. DOI: 10.1134/S0081543815060255.

[9] M. Rudnev, Few products, many sums, Workshop on Algebraic Methods in Combinatorics, November 13–17, 2017, Harvard University, Center of Mathematical Sciences and Applications, https://www.youtube.com/watch?v=CsUieMKTd0O.

[10] I. Z. Ruzsa, Sums of finite sets, in: *"Number Theory: New York Seminar 1991–1995"*, Springer, New York, 1996, pp. 281–293. DOI: 10.1007/978-1-4612-2418-1_21.

[11] I. D. Shkredov, Some remarks on the Balog–Wooley decomposition theorem and quantities D^+, $D^\times$, *Proc. Steklov Inst. Math.* **298**, suppl. 1 (2017), S74–S90. DOI: 10.1134/S0081543817070057.

[12] J. Solymosi, On the number of sums and products, *Bull. London Math. Soc.* **37(4)** (2005), 491–494. DOI: 10.1112/S0024609305004261.

[13] J. Solymosi, Bounding multiplicative energy by the sumset, *Adv. Math.* **222(2)** (2009), 402–408. DOI: 10.1016/j.aim.2009.04.006.

[14] E. Szemerédi and W. T. Trotter, Jr., Extremal problems in discrete geometry, *Combinatorica* **3(3–4)** (1983), 381–392. DOI: 10.1007/BF02579194.

Noga Alon
Schools of Mathematics and Computer Science, Tel Aviv University, Tel Aviv 69978, Israel and
Department of Mathematics, Princeton University, Princeton, NJ 08544, USA
E-mail address: nogaa@tau.ac.il

Imre Ruzsa
Alfréd Rényi Institute of Mathematics, Hungarian Academy of Sciences, POB 127, H-1364 Budapest, Hungary
E-mail address: ruzsa.z.imre@renyi.mta.hu

József Solymosi
Department of Mathematics, University of British Columbia, 1984 Mathematics Road, Vancouver, BC, V6T 1Z2, Canada
E-mail address: solymosi@math.ubc.ca

Primera versió rebuda el 26 de febrer de 2018,
darrera versió rebuda el 19 de juliol de 2018.