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THE GENERIC DIMENSION OF SPACES OF A-HARMONIC POLYNOMIALS

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Abstract: Let $A_1, \ldots, A_r$ be linear partial differential operators in N variables, with constant coefficients in a field $\mathbb{K}$ of characteristic 0. With $\mathbf{A} := (A_1, \ldots, A_r)$, a polynomial u is $\mathbf{A}$-harmonic if $\mathbf{A}u = 0$, that is, $A_1 u = \cdots = A_r u = 0$.

Denote by m_i the order of the first nonzero homogeneous part of A_i (initial part). The main result of this paper is that if $r \leq N$, the dimension over $\mathbb{K}$ of the space of $\mathbf{A}$-harmonic polynomials of degree at most d is given by an explicit formula depending only upon r, N, d, and $m_1, \ldots, m_r$ (but not $\mathbb{K}$) provided that the initial parts of $A_1, \ldots, A_r$ satisfy a simple generic condition. If $r > N$ and $A_1, \ldots, A_r$ are homogeneous, the existence of a generic formula is closely related to a conjecture of Fröberg on Hilbert functions.

The main result holds even if $A_1, \ldots, A_r$ have infinite order, which is unambiguous since they act only on polynomials. This is used to prove, as a corollary, the same formula when $A_1, \ldots, A_r$ are replaced with finite difference operators. Another application, when $\mathbb{K} = \mathbb{C}$ and $A_1, \ldots, A_r$ have finite order, yields dimension formulas for spaces of $\mathbf{A}$-harmonic polynomial-exponentials.

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1. Introduction

Throughout this paper, $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$ and $\mathbb{K}$ is a field of characteristic 0. We denote by $\mathcal{P} = \mathcal{P}(N)$ the space of polynomials in N variables with coefficients in $\mathbb{K}$. If $d \in \mathbb{Z}$, $\mathcal{P}_d$ is the subspace of $\mathcal{P}$ of polynomials of degree at most d and $\mathcal{P}_d^H$ is the subspace of homogeneous polynomials of degree d. Of course, $\mathcal{P}_d = \mathcal{P}_d^H = \{0\}$ if $d < 0$, but the notation will be convenient.

All the partial differential operators are linear with constant coefficients in $\mathbb{K}$ and they act on $\mathcal{P}$ (as opposed to larger spaces of functions). In that setting, it makes sense to define such an operator A by

$$A = \sum_{|\alpha| \geq 0} a_\alpha \partial^\alpha,$$

with no requirement that the sum be finite since Au is always a finite sum when $u \in \mathcal{P}$. Consistent with standard notation, $\alpha := (\alpha_1, \ldots, \alpha_N) \in$

$\mathbb{N}_0^N$, $|\alpha| := \alpha_1 + \cdots + \alpha_N$, and $\partial^\alpha := \partial_1^{\alpha_1} \cdots \partial_N^{\alpha_N}$ with $\partial_i := \partial/\partial x_i$, $1 \leq i \leq N$.

We shall give a fairly complete answer to the following question: If A_i are partial differential operators for $1 \leq i \leq r \leq N$, and $\mathbf{A} := (A_1, \ldots, A_r)$, what is the dimension (over $\mathbb{K}$) of

$$\mathcal{Z}_d(\mathbf{A}) := \ker \mathbf{A}_{|\mathcal{P}_d} = \bigcap_{i=1}^r \mathcal{Z}_d(A_i),$$

the space of solutions $u \in \mathcal{P}_d$ of the simultaneous equations $A_1 u = \cdots = A_r u = 0$ ($\mathbf{A}$-harmonic polynomials of degree at most d)? To date, this has been fully resolved when $r = 1$, but only a few special cases are known when $r > 1$; see further below.

The question is also of interest for finite difference operators ([**1**], [**8**], [**14**], among others) and one can kill two birds with one stone by noticing that finite difference operators on $\mathcal{P}$ can be rewritten as partial differential operators of infinite order. (See Section 6 for details.) Also, even when $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$, some insight can be gained from the general case $\operatorname{char} \mathbb{K} = 0$ (Remark 4.2).

The most notorious examples arise when $r = N$ and $\mathbf{A} = \nabla$, the gradient operator and when $r = 1$ and $\mathbf{A} = \Delta$, the Laplace operator. The former is of course trivial (especially when $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$) and, when $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$, the latter is expounded in virtually every text discussing spherical harmonics. The restrictions on $\mathbb{K}$ are important when analytical arguments are involved.

Horváth ([**13**]) seems to have provided the first explicit answer to the dimension question in 1958, when $\mathbf{A} = A$ is scalar ($r = 1$) and homogeneous, although this answer had already been found decades earlier (see the comments after Lemma 3.1). Horváth assumes $\mathbb{K} = \mathbb{R}$ but the algebraic argument is the same when $\operatorname{char} \mathbb{K} = 0$. A few years later, Matsuura settled the issue when, more generally, A has finite order ([**18**, Theorem 5.1]). This restriction is actually not needed in his proof.

Horváth's result has been rediscovered by other methods. See for instance [**8**], [**23**], or [**25**]. Both Stiller [**28**] and Dahmen and Micchelli [**8**] consider the case when $\mathbb{K} = \mathbb{R}$ and $A_1, \ldots, A_r$ are homogeneous, with primary focus on $r = N$ or $N = 2$ (for any r). Among other things, they give some formulas for $\dim \mathcal{Z}_d(\mathbf{A})$ in these two special cases (if d is large enough in the former) and when $r = 1$ (Horváth's formula).

In Pedersen [**24**] and Smith [**26**], r is arbitrary and $A_1, \ldots, A_r$ have finite order. Pedersen describes a procedure to find a basis of $\mathcal{Z}_d(\mathbf{A})$, but $\dim \mathcal{Z}_d(\mathbf{A})$ must be known. Smith finds $\dim \mathcal{Z}_d(\mathbf{A})$ when $\mathbb{K} = \mathbb{C}$ and d is large enough under restrictive assumptions; see Subsection 5.1. Yet, this seems to be the only known result when $r > 1$ and the operators are not necessarily homogeneous. It is strictly limited to the finite order case.

When $\mathbf{A}$ is an $s \times t$ system of finite order, Karachik ([**15**, Theorem 2]) relates $\dim \mathcal{Z}_d(\mathbf{A})$ (with $\mathcal{Z}_d(\mathbf{A})$ now a subspace of $\mathcal{P}_d^t$) to the dimensions of other spaces which are often equally difficult to calculate. In particular, this approach does not yield $\dim \mathcal{Z}_d(\mathbf{A})$ when $t=1$ and $s>1$ (i.e., $r > 1$ in this paper), except in simple enough examples that ad hoc calculations can be carried out. However, when $t = s$, it produces a generic formula for $\dim \mathcal{Z}_d(\mathbf{A})$ ([**15**, Corollary 1]; the given formula is for $\dim \mathcal{Z}_d(\mathbf{A}) - \dim \mathcal{Z}_{d-1}(\mathbf{A})$ and the author forgets that $t = s$). When $t = s = 1$, Matsuura's formula is recovered. We are not aware of more recent advances.

Given $m \in \mathbb{N}_0$, we set

$$\overline{\mathcal{D}}(m) := \left\{ A = \sum_{|\alpha| \geq m} a_\alpha \partial^\alpha : a_\alpha \in \mathbb{K} \right\}.$$

In other words, $\overline{\mathcal{D}}(m)$ consists of those operators with *initial part* (nonzero homogeneous part of lowest order) of order at least m. This ensures that $\overline{\mathcal{D}}(m)$ is a $\mathbb{K}$-vector space. An important role will be played by the subspace $\mathcal{D}^H(m)$ of $\overline{\mathcal{D}}(m)$ of homogeneous operators of order m, that is

$$\mathcal{D}^H(m) := \left\{ A = \sum_{|\alpha| = m} a_\alpha \partial^\alpha : a_\alpha \in \mathbb{K} \right\}.$$

If the constant term of A is not 0, it is plain – since every differentiation lowers the degree of polynomials – that $Au = 0$ has no nonzero polynomial solution. For that reason, we shall assume (except in Section 7) that the order of the initial part of A_i is strictly positive for $1 \leq i \leq r$. This means that we are only interested in operators $A_i \in \overline{\mathcal{D}}(m_i)$ with $m_i \geq 1$. If so, we denote by $A_i^{(m_i)}$ the homogeneous part of A_i of order m_i. This is the initial part of A_i if and only if $A_i^{(m_i)} \neq 0$.

With $\mathbf{m} = (m_1, \ldots, m_r) \in \mathbb{N}^r$, suppose then that

$$\mathbf{A} := (A_1, \ldots, A_r) \in \overline{\mathcal{D}}(\mathbf{m}) := \prod_{i=1}^r \overline{\mathcal{D}}(m_i),$$

whence

$$\mathbf{A}^{(\mathbf{m})} := (A_1^{(m_1)}, \ldots, A_r^{(m_r)}) \in \mathcal{D}^H(\mathbf{m}) := \prod_{i=1}^r \mathcal{D}^H(m_i).$$

Since $\mathcal{D}^H(\mathbf{m})$ is finite dimensional, it has a natural Zariski topology. The main result of this paper (Theorem 4.2) states that if $1 \leq r \leq N$, there are a (fully identified) nonempty Zariski open subset $\mathcal{R}(\mathbf{m})$ of $\mathcal{D}^H(\mathbf{m})$ and a positive integer $\nu_r(N, d; \mathbf{m})$, given by (1.2)/(1.3) below, such that

$$\dim \mathcal{Z}_d(\mathbf{A}) = \nu_r(N, d; \mathbf{m}), \quad \text{for all } d \in \mathbb{Z}, \tag{1.1}$$

provided that $\mathbf{A}^{(\mathbf{m})} \in \mathcal{R}(\mathbf{m})$. In particular, $\dim \mathcal{Z}_d(\mathbf{A})$ depends only on $\mathbf{A}^{(\mathbf{m})}$ and is independent of $\mathbb{K}$. The generic equality (1.1) breaks down when $r > N$. In fact, the open subset $\mathcal{R}(\mathbf{m})$, as it will be defined later, is empty in this case.

The integer $\nu_r(N, d; \mathbf{m})$ in (1.1) is given by

$$\nu_r(N, d; \mathbf{m}) := \dim \mathcal{P}_d + \sum_{j=1}^{r} (-1)^j \sum_{1 \leq i_1 < \cdots < i_j \leq r} \dim \mathcal{P}_{d-(m_{i_1}+\cdots+m_{i_j})}, \tag{1.2}$$

that is,

$$\nu_r(N, d; \mathbf{m}) = \binom{N+d}{N} + \sum_{j=1}^{r} (-1)^j \sum_{1 \leq i_1 < \cdots < i_j \leq r} \binom{N+d-(m_{i_1}+\cdots+m_{i_j})}{N}, \tag{1.3}$$

where

$$\binom{N+\ell}{N} := 0 \text{ if } \ell < 0. \tag{1.4}$$

The formula (1.2)/(1.3) makes sense for $r, N \in \mathbb{N}_0$, $d \in \mathbb{Z}$, and $\mathbf{m} \in \mathbb{N}_0^r$. If $d < 0$, then $\nu_r(N, d; \mathbf{m}) = 0$ and $\nu_0(N, d) = \binom{N+d}{N}$.

The proof of (1.1) proceeds in two main steps. The first one deals with the special case when $\mathbf{A} \in \mathcal{D}^H(\mathbf{m})$ (i.e., $\mathbf{A}^{(\mathbf{m})} = \mathbf{A}$). If so, (1.1) follows from the properties of the affine Hilbert function of the ideal $\langle A_1, \ldots, A_r \rangle$ (Theorem 3.2). In order to make the paper more easily accessible to non-experts in commutative algebra and algebraic geometry, the relevant concepts and (known) results are summarized in the next section.

The second step of the proof of (1.1) uses a double induction on r and d (Theorem 4.2). The induction procedure makes crucial use of Theorem 3.2 in the homogeneous case and of a relation between the functions ν_r and ν_{r-1} (Lemma 4.1).

In Section 5 we elaborate on the cases when $m_1 = \cdots = m_r = 1$ and when $r = N$. The former is related to the aforementioned work of Smith [**26**] and the latter to some results in the papers by Dahmen and

Micchelli [**8**] and Stiller [**28**]. When $r > N$ and $A_1, \dots, A_r$ are homogeneous, the problem has an intimate connection with a famous conjecture of Fröberg [**11**] and, if also $N = 2$, with other aspects of [**8**] and [**28**].

In Section 6 we show that Theorem 4.2 is also applicable when $A_1, \dots, A_r$ are finite difference operators. In Section 7 we use Theorem 4.2 with $\mathbb{K} = \mathbb{C}$ and $A_1, \dots, A_r$ of finite order, $r \leq N$, to find the dimension of the space $\mathcal{Z}_d(\mathbf{A}, \eta)$ of solutions of $\mathbf{A}u = 0$ of the form $u(x) := e^{(x|\bar{\eta})}v(x)$, where $\eta \in \mathbb{C}^N$, $v \in \mathcal{P}_d$, and $(\cdot|\cdot)$ denotes the standard inner product on $\mathbb{C}^N$.

2. Regular sequences and Hilbert functions

This section is devoted to a review of a few special topics in commutative algebra and algebraic geometry. It is intended for a non-expert audience and strictly limited to the material needed later. Everything is well-known, often in much greater generality than discussed below, but hard to find all in a single reference.

2.1. Regular sequences. A sequence $(\mathfrak{A}_1, \dots, \mathfrak{A}_r)$ of nonconstant (in particular, nonzero) *homogeneous* polynomials in $\mathcal{P}$ is said to be a *regular sequence* if the multiplication by $\mathfrak{A}_j$ is one-to-one on $\mathcal{P}/\langle \mathfrak{A}_1, \dots, \mathfrak{A}_{j-1} \rangle$ for every $2 \leq j \leq r$, where $\langle \mathfrak{A}_1, \dots, \mathfrak{A}_{j-1} \rangle$ denotes the ideal of $\mathcal{P}$ generated by $\mathfrak{A}_1, \dots, \mathfrak{A}_{j-1}$. If $r = 1$, this boils down to the already assumed $\mathfrak{A}_1 \neq 0$. This condition is unchanged if $\mathbb{K}$ is replaced with any field extension $\mathbb{L}$. Indeed, if E is a $\mathbb{K}$-vector space and $T \colon E \to E$ is linear, then T is one-to-one if and only if its extension to the $\mathbb{L}$-vector space $\mathbb{L} \bigotimes_{\mathbb{K}} E$ is one-to-one.

From the above, the sequence $\mathfrak{A}_1, \dots, \mathfrak{A}_r$ is regular if and only if it is regular when the coefficients of $\mathfrak{A}_1, \dots, \mathfrak{A}_r$ are viewed in any algebraically closed field extension $\mathbb{L}$ of $\mathbb{K}$ (for instance, the algebraic closure $\overline{\mathbb{K}}$ of $\mathbb{K}$, or $\mathbb{L} = \mathbb{C}$ if $\mathbb{K} = \mathbb{Q}$). If so, a simpler equivalent geometric definition is that $(\mathfrak{A}_1, \dots, \mathfrak{A}_r)$ is a regular sequence if and only if the projective subvariety $V := \bigcap_{i=0}^{r} \mathfrak{A}_i^{-1}(0)$ of $\mathbb{P}_{\mathbb{L}}^{N-1}$ has dimension $N - 1 - r$, where $\dim \emptyset := -1$ (thus, $V = \emptyset$ if $r = N$). A quick explanation follows. One of the many definitions of $\dim V$ is that it equals the Krull dimension of $\mathcal{P}/\langle \mathfrak{A}_1, \dots, \mathfrak{A}_r \rangle$ minus 1 and it is known (Macaulay's unmixedness Theorem [**16**, p. 187]) that the Krull dimension of $\mathcal{P}/\langle \mathfrak{A}_1, \dots, \mathfrak{A}_r \rangle$ is $N-r$ if and only if $(\mathfrak{A}_1, \dots, \mathfrak{A}_r)$ is a regular sequence.

In particular: (i) There is no regular sequence with length $r > N$. (ii) If $r \leq N$, $(\xi_1^{m_1}, \dots, \xi_r^{m_r})$ is a regular sequence since, in this case, $V = \mathbb{P}_{\mathbb{L}}^{N-1-r}$ (which is empty if $r = N$). (iii) Every permutation of a regular sequence is regular (anecdotally, this is notoriously false if the homogeneity assumption is dropped).

From the first definition, it is obvious that if $(\mathfrak{A}_1, \dots, \mathfrak{A}_r)$ is a regular sequence, then $(\mathfrak{A}_1, \dots, \mathfrak{A}_{r'})$ is also regular for every $1 \leq r' < r$. More generally, by the invariance under permutations, every subsequence $(\mathfrak{A}_{i_1}, \dots, \mathfrak{A}_{i_{r'}})$ is regular.

Most sequences with length $r \leq N$ are regular. More precisely, given $\mathbf{m} = (m_1, \dots, m_r) \in \mathbb{N}^r$, the set $\mathcal{R}(\mathbf{m}) \subset \mathcal{P}^H(\mathbf{m}) := \prod_{i=1}^r \mathcal{P}^H_{m_i}$ of regular sequences is open and nonempty in the Zariski topology of $\mathcal{P}^H(\mathbf{m})$. This is proved in Fröberg and Löfwall [**12**].

In spite of being essentially trivial, the following example is representative of the general case. It will be used later.

Example 2.1. Suppose $r \leq N$ and $m_1 = \cdots = m_r = 1$, so that $\mathfrak{A}_i(\xi) = \sum_{j=1}^N a_{ij}\xi_j$ with $a_{ij} \in \mathbb{K}$, $1 \leq i \leq r$, $1 \leq j \leq N$. Then, $(\mathfrak{A}_1, \dots, \mathfrak{A}_r) \in \mathcal{R}(\mathbf{m})$ if and only if one of the $r \times r$ minors of the matrix (a_{ij}) is nonzero. This property holds in a Zariski open subset of the space of $r \times N$ matrices, it is unaffected by permutation of $\mathfrak{A}_1, \dots, \mathfrak{A}_r$, and it remains true when r is replaced with $1 \leq r' < r$.

2.2. Hilbert functions. For any sequence (regular or not) $(\mathfrak{A}_1, \dots, \mathfrak{A}_r) \in \mathcal{P}^H(\mathbf{m})$ and $d \in \mathbb{Z}$, the *Hilbert function* $\chi_{\langle \mathfrak{A}_1, \dots, \mathfrak{A}_r \rangle}(d)$ of $\langle \mathfrak{A}_1, \dots, \mathfrak{A}_r \rangle$ is the number of linearly independent homogeneous polynomials of degree d in $\mathcal{P}/\langle \mathfrak{A}_1, \dots, \mathfrak{A}_r \rangle$. Of course, $\chi_{\langle \mathfrak{A}_1, \dots, \mathfrak{A}_r \rangle}(d) = 0$ if $d < 0$.

The formal power series $\sum_{d=0}^\infty \chi_{\langle \mathfrak{A}_1, \dots, \mathfrak{A}_r \rangle}(d)\lambda^d$ is called the Hilbert series of $\langle \mathfrak{A}_1, \dots, \mathfrak{A}_r \rangle$. When $r \leq N$ and $(\mathfrak{A}_1, \dots, \mathfrak{A}_r) \in \mathcal{R}(\mathbf{m})$, it is proved in Stanley [**27**, Corollary 3.3] that

$$\sum_{d=0}^\infty \chi_{\langle \mathfrak{A}_1, \dots, \mathfrak{A}_r \rangle}(d)\lambda^d = (1-\lambda)^{-N} \prod_{i=1}^r (1 - \lambda^{m_i})$$

for $|\lambda| < 1$. Evidently, the coefficient $\chi_r(N, d; \mathbf{m}) := \chi_{\langle \mathfrak{A}_1, \dots, \mathfrak{A}_r \rangle}(d)$ of λ^d in the Taylor series of the right-hand side is independent of $\mathbb{K}$ and of the specific $(\mathfrak{A}_1, \dots, \mathfrak{A}_r) \in \mathcal{R}(\mathbf{m})$. Since regular sequences are generic (when $r \leq N$), $\chi_r(N, d; \mathbf{m})$ is called a *generic Hilbert function* and

$$\sum_{d=0}^\infty \chi_r(N, d; \mathbf{m})\lambda^d = (1-\lambda)^{-N} \prod_{i=1}^r (1 - \lambda^{m_i}). \tag{2.1}$$

The Hilbert functions defined above are projective Hilbert functions. The *affine* Hilbert function ${}^a\chi_{\langle \mathfrak{A}_1, \dots, \mathfrak{A}_r \rangle}(d)$ of $\langle \mathfrak{A}_1, \dots, \mathfrak{A}_r \rangle$ is the number of linearly independent polynomials of degree at most d in $\mathcal{P}/\langle \mathfrak{A}_1, \dots, \mathfrak{A}_r \rangle$. This definition makes sense when $\mathfrak{A}_1, \dots, \mathfrak{A}_r$ are not necessarily homogeneous but, in the homogeneous case, ${}^a\chi_{\langle \mathfrak{A}_1, \dots, \mathfrak{A}_r \rangle}(d) = \sum_{\delta=0}^d \chi_{\langle \mathfrak{A}_1, \dots, \mathfrak{A}_r \rangle}(\delta)$ by [**7**, Theorem 12(i), p. 464]. From the above, if $r \leq N$, then ${}^a\chi_{\langle \mathfrak{A}_1, \dots, \mathfrak{A}_r \rangle} = {}^a\chi_r(N, d; \mathbf{m})$ is independent of $(\mathfrak{A}_1, \dots, \mathfrak{A}_r) \in \mathcal{R}(\mathbf{m})$ and ${}^a\chi_r(N, d; \mathbf{m}) = \sum_{\delta=0}^d \chi_r(N, \delta; \mathbf{m})$ (*generic affine Hilbert function*).

The affine Hilbert function of any ideal of $\mathcal{P}$ equals the Hilbert function of its homogenization ([**7**, Theorem 12(ii), p. 464]). For a homogeneous ideal $\langle \mathfrak{A}_1, \ldots, \mathfrak{A}_r \rangle$, homogenization changes nothing beyond adding one variable. This means that the homogenization of $\langle \mathfrak{A}_1, \ldots, \mathfrak{A}_r \rangle$ is the ideal $\langle \mathfrak{A}_1, \ldots, \mathfrak{A}_r \rangle\widetilde{}$ of $\mathcal{P}(N+1)$ generated by $\mathfrak{A}_1, \ldots, \mathfrak{A}_r$ (which must be checked since, in general, the homogenization need not be the ideal generated by the homogenized generators). Thus, ${}^a\chi_{\langle \mathfrak{A}_1, \ldots, \mathfrak{A}_r \rangle} = \chi_{\langle \mathfrak{A}_1, \ldots, \mathfrak{A}_r \rangle\widetilde{}}$. In particular, it follows at once from the geometric definition that if $r \leq N$, the sequence $(\mathfrak{A}_1, \ldots, \mathfrak{A}_r) \in \mathcal{P}^H(\mathbf{m})$ is regular if and only if it is regular when $\mathfrak{A}_1, \ldots, \mathfrak{A}_r$ are viewed as polynomials in $N+1$ variables. As a result, ${}^a\chi_r(N, d; \mathbf{m}) = \chi_r(N+1, d; \mathbf{m})$ if $r \leq N$.

On the other hand, if $r \leq N+1$, an explicit formula for $\chi_r(N+1, d; \mathbf{m})$ (hence also for ${}^a\chi_r(N, d; \mathbf{m})$ if $r \leq N$) is

$$\chi_r(N+1, d; \mathbf{m}) = \nu_r(N, d; \mathbf{m}), \quad \text{for all } d \in \mathbb{Z}, \tag{2.2}$$

with $\nu_r(N, d; \mathbf{m})$ given by (1.2)/(1.3). This formula is well-known to experts, but not widely reported in the literature. It can for instance be found in Chandler [**6**, p. 431], where it is inferred from the fact that the Koszul complex of a complete intersection is a free resolution ([**5**, Corollary 1.6.14]). An elementary proof of (2.2) making no use of homological algebra is given in the Appendix.

Remark 2.1. If $r \leq N$ and $(\mathfrak{A}_1, \ldots, \mathfrak{A}_r) \in \mathcal{P}^H(\mathbf{m})$ is *not* a regular sequence, $\chi_{\langle \mathfrak{A}_1, \ldots, \mathfrak{A}_r \rangle}(d) \geq \chi_r(N, d; \mathbf{m})$ for every d and the inequality is strict for at least one $d \in \mathbb{N}_0$. This follows from (2.1) and from Stanley [**27**, Corollary 3.2] (with $R = \mathcal{P}(N)$ in that corollary, so that $F(R, \lambda)$ there is just $(1-\lambda)^{-N}$). Therefore, ${}^a\chi_{\langle \mathfrak{A}_1, \ldots, \mathfrak{A}_r \rangle}(d) \geq {}^a\chi_r(N, d; \mathbf{m})$ for every d and the inequality is strict for d large enough. More is true; see the next subsection.

2.3. Hilbert polynomials. A final fact about $\chi_{\langle \mathfrak{A}_1, \ldots, \mathfrak{A}_r \rangle}(d)$ is that it is a polynomial $\pi_{\langle \mathfrak{A}_1, \ldots, \mathfrak{A}_r \rangle}(d)$ for d large enough (Hilbert polynomial of $\chi_{\langle \mathfrak{A}_1, \ldots, \mathfrak{A}_r \rangle}$), whose degree is the dimension of the projective variety V defined earlier ($\pi_{\langle \mathfrak{A}_1, \ldots, \mathfrak{A}_r \rangle} = 0$ if $V = \emptyset$). There are numerous sources for this result. An especially accessible and detailed account can be found in [**7**, Chapter 9]. In particular, if $r \leq N$, the Hilbert polynomial of $\chi_r(N, \cdot; \mathbf{m})$ has degree $N-1-r$, whereas $\deg \pi_{\langle \mathfrak{A}_1, \ldots, \mathfrak{A}_r \rangle} \geq N-r$ if $(\mathfrak{A}_1, \ldots, \mathfrak{A}_r) \notin \mathcal{R}(\mathbf{m})$ (for then $\dim V \neq N-1-r$, in which case $\dim V \geq N-r$ is the only option).

It follows that if $r \leq N$, the Hilbert polynomial of ${}^a\chi_r(N, \cdot; \mathbf{m}) = \chi_r(N+1, \cdot; \mathbf{m})$ has degree $N-r$, but the Hilbert polynomial of ${}^a\chi_{\langle \mathfrak{A}_1, \ldots, \mathfrak{A}_r \rangle} = \chi_{\langle \mathfrak{A}_1, \ldots, \mathfrak{A}_r \rangle\widetilde{}}$ has degree at least $N-r+1$ if $(\mathfrak{A}_1, \ldots, \mathfrak{A}_r) \notin$

$\mathcal{R}(\mathbf{m})$. Thus, in the latter case, there is a constant $c = c(\mathfrak{A}_1, \ldots, \mathfrak{A}_r) > 0$ such that ${}^a\chi_{\langle \mathfrak{A}_1,\ldots,\mathfrak{A}_r\rangle}(d) \geq cd^{N-r+1}$ for d large enough. Since ${}^a\chi_{\langle \mathfrak{A}_1,\ldots,\mathfrak{A}_r\rangle}$ is nondecreasing and ${}^a\chi_{\langle \mathfrak{A}_1,\ldots,\mathfrak{A}_r\rangle}(0) = 1$, this is even true for every $d \in \mathbb{N}_0$. In contrast, ${}^a\chi_r(N, d; \mathbf{m}) = O(d^{N-r})$ for large d.

2.4. Formulation in terms of partial differential operators. A sequence $\mathbf{A} = (A_1, \ldots, A_r) \in \mathcal{D}^H(\mathbf{m})$ determines a sequence $(\mathfrak{A}_1, \ldots, \mathfrak{A}_r) \in \mathcal{P}^H(\mathbf{m})$ and vice versa, by exchanging the roles of ∂^α and ξ^α. Since our focus is on partial differential operators, we shall henceforth say that $\mathbf{A}$ is a regular sequence if $(\mathfrak{A}_1, \ldots, \mathfrak{A}_r)$ is a regular sequence. More generally, except for the definition of the variety V in Subsection 2.1, everything above may (and will) be rephrased in terms of $A_1, \ldots, A_r$. For instance, the ideal $\langle \mathfrak{A}_1, \ldots, \mathfrak{A}_r\rangle$ of $\mathcal{P}$ becomes the ideal $\langle A_1, \ldots, A_r\rangle$ of the ring $\mathcal{D}$ of linear partial differential operators of finite order with coefficients in $\mathbb{K}$, the subset $\mathcal{R}(\mathbf{m})$ of regular sequences becomes open for the Zariski topology of $\mathcal{D}^H(\mathbf{m})$, the Hilbert function $\chi_{\langle \mathfrak{A}_1,\ldots,\mathfrak{A}_r\rangle}$ becomes the Hilbert function $\chi_{\langle A_1,\ldots,A_r\rangle}$, etc.

3. Main theorem for homogeneous operators

We shall make use of the material reviewed in the previous section and of the notation introduced there. Consistent with Subsection 2.4, everything will be expressed in terms of partial differential operators instead of polynomials.

An equivalent form of Lemma 3.1 below was proved by Fischer [**10**] over a century ago and rediscovered much later by Pedersen [**24**] (and perhaps others in the meantime). We give a proof for completeness, which is a more direct variant of Pedersen's.

Lemma 3.1 (Fischer)**.** *If $r \in \mathbb{N}$, $\mathbf{m} \in \mathbb{N}^r$, and $\mathbf{A} = (A_1, \ldots, A_r) \in \mathcal{D}^H(\mathbf{m})$, then*

$$\dim \mathcal{Z}_d(\mathbf{A}) = {}^a\chi_{\langle A_1,\ldots,A_r\rangle}(d), \quad \textit{for all } d \in \mathbb{Z},$$

where ${}^a\chi_{\langle A_1,\ldots,A_r\rangle}$ is the affine Hilbert function of the ideal $\langle A_1, \ldots, A_r\rangle$ of $\mathcal{D}$.

Proof: With no loss of generality, assume $d \in \mathbb{N}_0$. Recall the notation $\mathcal{D}$ for the space (ring) of linear partial differential operators of finite order with coefficients in $\mathbb{K}$. In this proof, $\mathcal{D}_d$ denotes the subspace of $\mathcal{D}$ of differential operators of order at most d. Define the bilinear form $b\colon \mathcal{P}_d \times \mathcal{D}_d \to \mathbb{K}$ by $b(u, B) := (Bu)(0)$ and let $J_d := \langle A_1, \ldots, A_r\rangle \cap \mathcal{D}_d$. The key remark is that $\mathcal{Z}_d(\mathbf{A}) = \{u \in \mathcal{P}_d : b(u, B) = 0, \text{ for all } B \in J_d\}$. Since "$\subset$" is trivial, it suffices to show that if $u \in \mathcal{P}_d$ and $u \notin \mathcal{Z}_d(\mathbf{A})$, there is $B \in J_d$ such that $b(u, B) \neq 0$.

Let then $u \in \mathcal{P}_d$ be such that $u \notin \mathcal{Z}_d(\mathbf{A})$, so that $A_i u \neq 0$ for some $1 \leq i \leq r$. This can only happen if $d \geq m_i$ and, if so, $A_i u \in \mathcal{P}_{d-m_i} \backslash \{0\}$. Consequently, $\partial^\alpha A_i u$ is a nonzero constant c for some α with $|\alpha| \leq d - m_i$ and then $b(u, \partial^\alpha A_i) = c \neq 0$. By the homogeneity of A_i, it follows that $B := \partial^\alpha A_i$ is of order $|\alpha| + m_i \leq d$. Thus, $B \in J_d$ and $b(u, B) \neq 0$, as desired.

The codimension of J_d in $\mathcal{D}_d$ is $\dim \mathcal{D}_d / J_d = {}^a\chi_{\langle A_1, \dots, A_r \rangle}(d)$. Call k this codimension for short and, with $n := \dim \mathcal{P}_d = \dim \mathcal{D}_d$, let $\{B_1, \dots, B_n\}$ be a basis of $\mathcal{D}_d$ such that $\{B_{k+1}, \dots, B_n\}$ is a basis of J_d.

It is plain that if $u \in \mathcal{P}_d$, then $b(u, B) = 0$ for every $B \in \mathcal{D}_d$ if and only if $u = 0$. As a result, there is a basis $\{v_1, \dots, v_n\}$ of $\mathcal{P}_d$ which is b-dual to $\{B_1, \dots, B_n\}$, insofar as $b(v_i, B_j) = \delta_{ij}$ for every $1 \leq i, j \leq n$. Thus, from the above,

$$\mathcal{Z}_d(\mathbf{A}) = \{u \in \mathcal{P}_d : b(u, B_j) = 0,\ k+1 \leq j \leq n\} = \operatorname{span}\{v_1, \dots, v_k\},$$

so that $\dim \mathcal{Z}_d(\mathbf{A}) = k$. $\square$

Fischer proved $\mathcal{P}_d^H = (\mathcal{Z}_d(\mathbf{A}) \cap \mathcal{P}_d^H) \oplus (\langle \mathfrak{A}_1, \dots, \mathfrak{A}_r \rangle \cap \mathcal{P}_d^H)$, whence $\chi_{\langle A_1, \dots, A_r \rangle}(d)\ (:= \chi_{\langle \mathfrak{A}_1, \dots, \mathfrak{A}_r \rangle}(d)) = \dim \mathcal{Z}_d(\mathbf{A}) \cap \mathcal{P}_d^H$. If $r = 1$, Lemma 3.1 yields at once $\dim \mathcal{Z}_d(A_1) = \dim \mathcal{P}_d - \dim \mathcal{P}_{d-m_1}$ (Horváth's formula). In positive characteristic, it is no longer true that if $v \in \mathcal{P} \backslash \{0\}$, some partial derivative of v is a nonzero constant and Lemma 3.1 breaks down.

The geometric characterization of regular sequences (Subsection 2.1) may be helpful to use the next theorem in practice. Recall the definition of $\nu_r(N, d; \mathbf{m})$ in (1.2)/(1.3).

Theorem 3.2. *Let $1 \leq r \leq N$ and $\mathbf{m} \in \mathbb{N}^r$ be given. If $\mathbf{A} \in \mathcal{D}^H(\mathbf{m})$, then*

$$\dim \mathcal{Z}_d(\mathbf{A}) = \nu_r(N, d; \mathbf{m}), \quad \textit{for all } d \in \mathbb{Z}, \tag{3.1}$$

if and only if $\mathbf{A} \in \mathcal{R}(\mathbf{m})$ (i.e., $\mathbf{A}$ is a regular sequence; in particular, $A_i \neq 0, 1 \leq i \leq r$). If $\mathbf{A} \notin \mathcal{R}(\mathbf{m})$, then $\dim \mathcal{Z}_d(\mathbf{A}) \geq \nu_r(N, d; \mathbf{m})$ for every $d \in \mathbb{Z}$, the inequality is strict for d large enough and there is a constant $c = c(\mathbf{A}) > 0$ such that $\dim \mathcal{Z}_d(\mathbf{A}) \geq c d^{N-r+1}$ for every $d \in \mathbb{N}_0$.

Proof: From Section 2, if $\mathbf{A} = (A_1, \dots, A_r) \in \mathcal{R}(\mathbf{m})$, then ${}^a\chi_{\langle A_1, \dots, A_r \rangle}(d) = \chi_r(N+1, d; \mathbf{m}) = \nu_r(N, d; \mathbf{m})$, so that (3.1) follows from Lemma 3.1.

Conversely, suppose that $\mathbf{A} \notin \mathcal{R}(\mathbf{m})$. From Lemma 3.1 and Remark 2.1, $\dim \mathcal{Z}_d(\mathbf{A}) = {}^a\chi_{\langle A_1, \dots, A_r \rangle}(d) \geq {}^a\chi_r(N, d; \mathbf{m}) = \chi_r(N+1, d; \mathbf{m}) = \nu_r(N, d; \mathbf{m})$ for every d, with strict inequality if d is large enough. The existence of a constant $c > 0$ such that $\dim \mathcal{Z}_d(\mathbf{A}) \geq c d^{N-r+1}$ for every $d \in \mathbb{N}_0$ follows from Subsection 2.3. $\square$

As the proof shows, (3.1) is just Lemma 3.1 *plus* ${}^{a}\chi_{\langle A_1,\dots,A_r\rangle}(d) = \nu_r(N,d;\mathbf{m})$ when $\mathbf{A} \in \mathcal{R}(\mathbf{m})$. The latter point has apparently been missed in the literature, in which no reference to regular sequences or, equivalently, complete intersections, has been made in prior discussions of $\dim \mathcal{Z}_d(\mathbf{A})$. Another property of regular sequences will be of major importance in the next section.

4. Main theorem

We now generalize Theorem 3.2 when $\mathbf{A} \in \overline{\mathcal{D}}(\mathbf{m})$.

Lemma 4.1. *If $N \in \mathbb{N}$ and $r \geq 2$, then*

$$\begin{aligned}(4.1)\quad \nu_r(N,d;m_1,\dots,m_r) &= \nu_{r-1}(N,d;m_1,\dots,\widehat{m_i},\dots,m_r)\\ &\quad - \nu_{r-1}(N,d-m_i;m_1,\dots,\widehat{m_i},\dots,m_r),\end{aligned}$$

for every $1 \leq i \leq r$, every $d \in \mathbb{Z}$, and every $(m_1,\dots,m_r) \in \mathbb{N}_0^r$, where the notation $\widehat{m_i}$ means that m_i has been removed.

Proof: Since $\nu_r(N,d;m_1,\dots,m_r)$ is clearly a symmetric function of $(m_1,\dots,m_r)$, it suffices to prove (4.1) when $i = r$. Rewrite the right-hand side of (1.2) by collecting all the terms not containing m_r (first bracketed term in (4.2) below) and those containing it (second bracketed term in (4.2) below):

$$\begin{aligned}(4.2)\quad &\nu_r(N,d;m_1,\dots,m_r)\\ &= \left[\dim \mathcal{P}_d + \sum_{j=1}^{r-1}(-1)^j \sum_{1\leq i_1<\cdots<i_j\leq r-1} \dim \mathcal{P}_{d-(m_{i_1}+\cdots+m_{i_j})}\right]\\ &- \left[\dim \mathcal{P}_{d-m_r} - \sum_{j=2}^{r}(-1)^j \sum_{1\leq i_1<\cdots<i_{j-1}\leq r-1} \dim \mathcal{P}_{d-m_r-(m_{i_1}+\cdots+m_{i_{j-1}})}\right].\end{aligned}$$

Above, the first bracketed term is $\nu_{r-1}(N,d;m_1,\dots,m_{r-1})$. After changing $j-1$ into j, the second one is

$$\begin{aligned}\dim \mathcal{P}_{d-m_r} + \sum_{j=1}^{r-1}(-1)^j \sum_{1\leq i_1<\cdots<i_j\leq r-1} \dim \mathcal{P}_{d-m_r-(m_{i_1}+\cdots+m_{i_j})}\\ = \nu_{r-1}(N,d-m_r;m_1,\dots,m_{r-1}).\end{aligned}$$

This proves (4.1) when $i = r$. □

Remark 4.1. In particular, (4.1) shows that $\nu_r(N,d;m_1,\dots,m_r) = 0$ if $m_i = 0$ for some $1 \leq i \leq r$ (and $r \geq 2$, but this is also true and trivial if $r = 1$).

The identity (4.1) shows that ν_r can be calculated from ν_{r-1}. When $r \leq N+1$, it is a relation between generic Hilbert functions, namely, $\chi_r(N{+}1, d; m_1, \dots, m_r) = \chi_{r-1}(N{+}1, d; m_1, \dots, \widehat{m_i}, \dots, m_r) - \chi_{r-1}(N{+}1, d{-}m_i; m_1, \dots, \widehat{m_i}, \dots, m_r)$. We did not come across it in the literature.

Recall that if $\mathbf{m} = (m_1, \dots, m_r)$ and $\mathbf{A} = (A_1, \dots, A_r) \in \overline{\mathcal{D}}(\mathbf{m})$, we defined $\mathbf{A}^{(\mathbf{m})} = (A_1^{(m_1)}, \dots, A_r^{(m_r)}) \in \mathcal{D}^H(\mathbf{m})$, where $A_i^{(m_i)}$ denotes the homogeneous part of A_i of order m_i.

Theorem 4.2. *Suppose* $1 \leq r \leq N$ *and let* $\mathbf{m} \in \mathbb{N}^r$ *and* $\mathbf{A} \in \overline{\mathcal{D}}(\mathbf{m})$ *be given.*

(i) *If* $\mathbf{A}^{(\mathbf{m})} \in \mathcal{R}(\mathbf{m})$ *(i.e.,* $\mathbf{A}^{(\mathbf{m})}$ *is a regular sequence; in particular,* $A_i^{(m_i)} \neq 0$ *is the initial part of* A_i*,* $1 \leq i \leq r$*), then*

$$\dim \mathcal{Z}_d(\mathbf{A}) = \nu_r(N, d; \mathbf{m}), \quad \textit{for all } d \in \mathbb{Z}. \tag{4.3}$$

(ii) $\dim \mathcal{Z}_d(\mathbf{A}) \geq \nu_r(N, d; \mathbf{m})$ *for every* $d \in \mathbb{Z}$.

Proof: (i) The proof is by induction on r. If $r = 1$, the result follows from Matsuura [**18**], but we give a proof in line with the argument we shall use below when $r > 1$. Assume then $r = 1$, so that $\mathbf{m} = m_1$ and $\mathbf{A} = A_1$. The assumption that $(A_1^{(m_1)})$ is a regular sequence just means that $A_1^{(m_1)} \neq 0$. Both A_1 and $A_1^{(m_1)}$ map $\mathcal{P}_d$ into $\mathcal{P}_{d-m_1}$ and, by Theorem 3.2, $\dim \mathcal{Z}_d(A_1^{(m_1)}) = \nu_1(N, d; m_1) = \dim \mathcal{P}_d - \dim \mathcal{P}_{d-m_1}$ by (1.2). Thus, $A_1^{(m_1)}$ maps $\mathcal{P}_d$ onto $\mathcal{P}_{d-m_1}$ and (4.3) when $r = 1$ amounts to saying that A_1 also maps $\mathcal{P}_d$ onto $\mathcal{P}_{d-m_1}$. This is obvious if $d \leq 0$ since $m_1 \geq 1$. When $d \in \mathbb{N}$, we prove it by induction.

Let $d \geq 1$ be fixed and suppose that A_1 maps $\mathcal{P}_{d-1}$ onto $\mathcal{P}_{d-1-m_1}$, which is true when $d = 1$. We show that if $p \in \mathcal{P}_{d-m_1}$, there is $u \in \mathcal{P}_d$ such that $A_1 u = p$.

Write $A_1 = A_1^{(m_1)} + B_1$. From the above, $A_1^{(m_1)}$ maps $\mathcal{P}_d$ onto $\mathcal{P}_{d-m_1}$ and so there is $w \in \mathcal{P}_d$ such that $A_1^{(m_1)} w = p$. Set $u = v + w$, so that $A_1 u = A_1 v + p + B_1 w$. Thus, $A_1 u = p$ if and only if $A_1 v = -B_1 w$. Note that $\deg B_1 w \leq d - 1 - m_1$ since $\deg w \leq d$ and the initial part of B_1 has order at least $m_1 + 1$. Therefore, by the hypothesis of induction, there is $v \in \mathcal{P}_{d-1}$ such that $A_1 v = -B_1 w$ and so $u = w + v \in \mathcal{P}_d$ and $A_1 u = p$. This proves (4.3) when $r = 1$.

Assume now $r \geq 2$ and that the theorem has been proved when r is replaced with $r - 1$. Start with

$$\dim \textstyle\bigcap_{j=1}^r \mathcal{Z}_d(A_j^{(m_j)}) = \nu_r(N, d; m_1, \dots, m_r), \tag{4.4}$$

by Theorem 3.2. We shall show that, in this formula, $A_r^{(m_r)}, \dots, A_1^{(m_1)}$ can be successively replaced with $A_r, \dots, A_1$. Suppose then $1 \le i \le r$ and that

$$\text{(4.5)} \quad \dim \bigcap_{j=1}^{i} \mathcal{Z}_d(A_j^{(m_j)}) \cap \bigcap_{j=i+1}^{r} \mathcal{Z}_d(A_j) = \nu_r(N, d; m_1, \dots, m_r),$$

which is (4.4) when $i = r$. By (4.1),

$$\nu_r(N, d; m_1, \dots, m_r) = \nu_{r-1}(N, d; m_1, \dots, \widehat{m_i}, \dots, m_r) - \nu_{r-1}(N, d - m_i; m_1, \dots, \widehat{m_i}, \dots, m_r).$$

Since $(A_1^{(m_1)}, \dots, \widehat{A_i^{(m_i)}}, \dots, A_r^{(m_r)})$ is a regular sequence (a crucial point; see Subsection 2.1), it follows from the theorem with r replaced with $r-1$ and for the sequence $(A_1^{(m_1)}, \dots, A_{i-1}^{(m_{i-1})}, A_{i+1}, \dots, A_r)$ that

$$\text{(4.6)} \quad \nu_{r-1}(N, d; m_1, \dots, \widehat{m_i}, \dots, m_r) = \dim \bigcap_{j=1}^{i-1} \mathcal{Z}_d(A_j^{(m_j)}) \cap \bigcap_{j=i+1}^{r} \mathcal{Z}_d(A_j)$$

and that

$$\text{(4.7)} \quad \nu_{r-1}(N, d - m_i; m_1, \dots, \widehat{m_i}, \dots, m_r) = \dim \bigcap_{j=1}^{i-1} \mathcal{Z}_{d-m_i}(A_j^{(m_j)}) \cap \bigcap_{j=i+1}^{r} \mathcal{Z}_{d-m_i}(A_j).$$

As a result, (4.5) reads

$$\text{(4.8)} \quad \dim \bigcap_{j=1}^{i} \mathcal{Z}_d(A_j^{(m_j)}) \cap \bigcap_{j=i+1}^{r} \mathcal{Z}_d(A_j) = \dim \bigcap_{j=1}^{i-1} \mathcal{Z}_d(A_j^{(m_j)}) \cap \bigcap_{j=i+1}^{r} \mathcal{Z}_d(A_j) - \dim \bigcap_{j=1}^{i-1} \mathcal{Z}_{d-m_i}(A_j^{(m_j)}) \cap \bigcap_{j=i+1}^{r} \mathcal{Z}_{d-m_i}(A_j).$$

Since differential operators with constant coefficients commute, it follows that

$$\text{(4.9)} \quad A_i, A_i^{(m_i)} \colon \bigcap_{j=1}^{i-1} \mathcal{Z}_d(A_j^{(m_j)}) \cap \bigcap_{j=i+1}^{r} \mathcal{Z}_d(A_j) \to \bigcap_{j=1}^{i-1} \mathcal{Z}_{d-m_i}(A_j^{(m_j)}) \cap \bigcap_{j=i+1}^{r} \mathcal{Z}_{d-m_i}(A_j)$$

are well defined and $A_i^{(m_i)}$ has kernel $\bigcap_{j=1}^{i} \mathcal{Z}_d(A_j^{(m_j)}) \cap \bigcap_{j=i+1}^{r} \mathcal{Z}_d(A_j)$. Thus, (4.8) means that $A_i^{(m_i)}$ is surjective. We claim that A_i is surjective as well. This is obvious if $d \le 0$. If $d > 0$, we argue by induction (as in the case $r = 1$, but with mild extra technicalities).

Suppose that $d > 0$ and that A_i maps

$$\bigcap_{j=1}^{i-1} \mathcal{Z}_{d-1}(A_j^{(m_j)}) \cap \bigcap_{j=i+1}^{r} \mathcal{Z}_{d-1}(A_j)$$

onto

$$\bigcap_{j=1}^{i-1} \mathcal{Z}_{d-1-m_i}(A_j^{(m_j)}) \cap \bigcap_{j=i+1}^{r} \mathcal{Z}_{d-1-m_i}(A_j),$$

which is true if $d = 1$. Split $A_i = A_i^{(m_i)} + B_i$ and let

$$p \in \bigcap_{j=1}^{i-1} \mathcal{Z}_{d-m_i}(A_j^{(m_j)}) \cap \bigcap_{j=i+1}^{r} \mathcal{Z}_{d-m_i}(A_j).$$

From the surjectivity of $A_i^{(m_i)}$ in (4.9), there is

$$w \in \bigcap_{j=1}^{i-1} \mathcal{Z}_{d}(A_j^{(m_j)}) \cap \bigcap_{j=i+1}^{r} \mathcal{Z}_{d}(A_j)$$

such that $A_i^{(m_i)} w = p$. Set $u = v + w$, so that $A_i u = p$ if and only if $A_i v = -B_i w$. Since B_i commutes with every $A_j^{(m_j)}$ and every A_j, it follows that $A_j^{(m_j)} B_i w = B_i A_j^{(m_j)} w = 0$ if $1 \le j \le i-1$ and that $A_j B_i w = B_i A_j w = 0$ if $i+1 \le j \le r$. Also, $\deg B_i w \le d - 1 - m_i$ since $\deg w \le d$ and the initial part of B_i has order at least $m_i + 1$. Thus,

$$B_i w \in \bigcap_{j=1}^{i-1} \mathcal{Z}_{d-1-m_i}(A_j^{(m_j)}) \cap \bigcap_{j=i+1}^{r} \mathcal{Z}_{d-1-m_i}(A_j)$$

and so, by the hypothesis of induction, there is

$$v \in \bigcap_{j=1}^{i-1} \mathcal{Z}_{d-1}(A_j^{(m_j)}) \cap \bigcap_{j=i+1}^{r} \mathcal{Z}_{d-1}(A_j) \subset \bigcap_{j=1}^{i-1} \mathcal{Z}_{d}(A_j^{(m_j)}) \cap \bigcap_{j=i+1}^{r} \mathcal{Z}_{d}(A_j)$$

such that $A_i v = -B_i w$. Therefore, $u = v + w \in \bigcap_{j=1}^{i-1} \mathcal{Z}_{d}(A_j^{(m_j)}) \cap \bigcap_{j=i+1}^{r} \mathcal{Z}_{d}(A_j)$ and $A_i u = p$.

This proves that A_i in (4.9) is onto and so its kernel $\bigcap_{j=1}^{i-1} \mathcal{Z}_{d}(A_j^{(m_j)}) \cap \bigcap_{j=i}^{r} \mathcal{Z}_{d}(A_j)$ has dimension

$$\begin{aligned}\dim \bigcap_{j=1}^{i-1} \mathcal{Z}_{d}(A_j^{(m_j)}) \cap \bigcap_{j=i}^{r} \mathcal{Z}_{d}(A_j) &= \dim \bigcap_{j=1}^{i-1} \mathcal{Z}_{d}(A_j^{(m_j)}) \cap \bigcap_{j=i+1}^{r} \mathcal{Z}_{d}(A_j) \\ &\quad - \dim \bigcap_{j=1}^{i-1} \mathcal{Z}_{d-m_i}(A_j^{(m_j)}) \cap \bigcap_{j=i+1}^{r} \mathcal{Z}_{d-m_i}(A_j).\end{aligned}$$

Therefore, by (4.6), (4.7), and (4.1),

$$\dim \bigcap_{j=1}^{i-1} \mathcal{Z}_{d}(A_j^{(m_j)}) \cap \bigcap_{j=i}^{r} \mathcal{Z}_{d}(A_j) = \nu_r(N, d; m_1, \ldots, m_r).$$

This is (4.5) with i replaced with $i-1$ and (4.3) follows when $i = 1$.

(ii) With no loss of generality, assume $d \in \mathbb{N}_0$. Given $M \geq \max_{1 \leq i \leq r} m_i$, denote by $\mathcal{D}_M(m_i)$ the (finite dimensional) subspace of $\overline{\mathcal{D}}(m_i)$ of operators of order at most M. Let $d \in \mathbb{N}_0$ be fixed and let $\mathbf{A} \in \mathcal{D}_M(\mathbf{m}) := \prod_{i=1}^r \mathcal{D}_M(m_i)$.

Since $\dim \mathcal{D}_M(\mathbf{m}) < \infty$, it is plain that $\dim \mathcal{Z}_d(\mathbf{A}) \geq \dim \mathcal{Z}_d(\mathbf{A}')$ for every $\mathbf{A}' = (A_1', \ldots, A_r')$ in some d-dependent Zariski open neighborhood $\mathcal{U}_d$ of $\mathbf{A}$ in $\mathcal{D}_M(\mathbf{m})$. By the genericity of regular sequences in $\mathcal{D}^H(\mathbf{m}) \subset \mathcal{D}_M(\mathbf{m})$ (see Section 2), there is a nonempty Zariski open subset $\mathcal{V}$ of $\mathcal{D}_M(\mathbf{m})$ such that $\mathbf{A}'^{(\mathbf{m})} := (A_1'^{(m_1)}, \ldots, A_r'^{(m_r)})$ is a regular sequence whenever $\mathbf{A}' \in \mathcal{V}$ (indeed, $\mathbf{A}' \in \mathcal{D}_M(\mathbf{m}) \mapsto \mathbf{A}'^{(\mathbf{m})} \in \mathcal{D}^H(\mathbf{m})$ is linear and all polynomial maps are continuous in the Zariski topology). If $\mathbf{A}' \in \mathcal{U}_d \cap \mathcal{V}$ (nonempty), then $\dim \mathcal{Z}_d(\mathbf{A}) \geq \dim \mathcal{Z}_d(\mathbf{A}')$ and $\dim \mathcal{Z}_d(\mathbf{A}') = \nu_r(N, d; \mathbf{m})$ by part (i), whence $\dim \mathcal{Z}_d(\mathbf{A}) \geq \nu_r(N, d; \mathbf{m})$.

This proves (ii) when $\mathbf{A} \in \mathcal{D}_M(\mathbf{m})$ for any $M \geq \max_{1 \leq i \leq r} m_i$. If now $\mathbf{A} \in \overline{\mathcal{D}}(\mathbf{m})$ and if $d \in \mathbb{N}_0$ is fixed, choose M such that, in addition, $M \geq d$. With $A_i := \sum_{|\alpha| \geq m_i} a_{i\alpha} \partial^\alpha$, set $A_{M,i} := \sum_{|\alpha|=m_i}^{M} a_{i\alpha} \partial^\alpha$ and $\mathbf{A}_M := (A_{M,1}, \ldots, A_{M,r}) \in \mathcal{D}_M(\mathbf{m})$. Since $\partial^\alpha = 0$ on $\mathcal{P}_d$ if $|\alpha| > M$, it is obvious that $\mathbf{A}_M$ and $\mathbf{A}$ coincide on $\mathcal{P}_d$. Thus, $\mathcal{Z}_d(\mathbf{A}) = \mathcal{Z}_d(\mathbf{A}_M)$ and so $\dim \mathcal{Z}_d(\mathbf{A}) = \dim \mathcal{Z}_d(\mathbf{A}_M) \geq \nu_r(N, d; \mathbf{m})$ from the above. $\square$

The assumption in part (i) that $\mathbf{A}^{(\mathbf{m})} \in \mathcal{R}(\mathbf{m})$ can only hold if $A_i \neq 0$ and m_i is the order of the initial part of A_i for every $1 \leq i \leq r$. This suffices when $r = 1$. In other words, when $r = 1$, (4.3) is *always* applicable with m_1 being the order of the initial part of $A_1 \neq 0$. Note also that $\nu_r(N, d; \mathbf{m})$ has nothing to do with ${}^a\chi_{\langle A_1, \ldots, A_r \rangle}(d)$ if $\mathbf{A} \notin \mathcal{D}^H(\mathbf{m})$ (compare with Lemma 3.1).

Unlike Theorem 3.2, Theorem 4.2 does not assert that (4.3) holds *only if* $\mathbf{A}^{(\mathbf{m})}$ is a regular sequence. Apparently, if $r > 1$, it cannot always be ruled out that $\dim \mathcal{Z}_d(\mathbf{A}) = \nu_r(N, d; \mathbf{m})$ for every $d \in \mathbb{Z}$ even if $\mathbf{A}^{(\mathbf{m})}$ is not regular, due to the possible impact of higher order terms, which do not exist when the operators are homogeneous.

Remark 4.2. With a self-explanatory notation, a more precise formulation of (4.3) is $\dim_{\mathbb{K}} \mathcal{Z}_d^{\mathbb{K}}(\mathbf{A}) = \nu_r(N, d; \mathbf{m})$ irrespective of $\mathbb{K}$. Retain the assumptions of part (i) of Theorem 4.2 and let $\mathbb{L}$ be any field extension of $\mathbb{K}$. Then, every basis of $\mathcal{Z}_d^{\mathbb{K}}(\mathbf{A})$ remains linearly independent over $\mathbb{L}$ and hence is also a basis of $\mathcal{Z}_d^{\mathbb{L}}(\mathbf{A})$ since $\dim_{\mathbb{L}} \mathcal{Z}_d^{\mathbb{L}}(\mathbf{A}) = \dim_{\mathbb{K}} \mathcal{Z}_d^{\mathbb{K}}(\mathbf{A}) < \infty$. It easily follows that $\mathcal{Z}^{\mathbb{L}}(\mathbf{A}) := \bigcup_{d \geq 0} \mathcal{Z}_d^{\mathbb{L}}(\mathbf{A})$ also has a basis consisting of polynomials with coefficients in $\mathbb{K}$. In particular, if the coefficients of $\mathbf{A}$ are rational ($\mathbb{Q} \subset \mathbb{K}$ since $\operatorname{char} \mathbb{K} = 0$) and $\mathbf{A}^{(\mathbf{m})} \in \mathcal{R}(\mathbf{m})$ (a property independent of the field being $\mathbb{Q}$ or $\mathbb{K}$; see Section 2), there are bases of $\mathcal{Z}_d^{\mathbb{K}}(\mathbf{A})$, $d \geq 0$, and of $\mathcal{Z}^{\mathbb{K}}(\mathbf{A})$ consisting of polynomials with integer coefficients.

5. Relationship with the literature

When $r = 1$, (4.3) is simply $\dim \mathcal{Z}_d(A_1) = \binom{N+d}{N} - \binom{N+d-m_1}{N}$ (Matsuura's formula). Below, we discuss two other special cases of Theorem 4.2 and their connection with results obtained by different methods in the literature. We also summarize what can be said when $r > N$.

5.1. Case $m_1 = \cdots = m_r = 1$. When the initial parts of $A_1, \dots, A_r$ have order 1, Theorem 4.2 takes the following simpler form:

Theorem 5.1. *Suppose $1 \le r \le N$ and write $A_i \in \overline{\mathcal{D}}(1)$ in the form $A_i := \sum_{j=1}^N a_{ij}\partial_j + \sum_{|\alpha|\ge 2} a_{i\alpha}\partial^\alpha$, $1 \le i \le r$. If the matrix $(a_{ij})_{1\le i\le r, 1\le j\le N}$ has rank r, then*

$$\dim \mathcal{Z}_d(\mathbf{A}) = \binom{N-r+d}{N-r}, \quad \text{for all } d \in \mathbb{Z}. \tag{5.1}$$

Furthermore, if the matrix $(a_{ij})_{1\le i\le r,1\le j\le N}$ has rank $r' < r$, then $\dim \mathcal{Z}_d(\mathbf{A}) \ge \binom{N-r+d}{N-r}$ for every $d \in \mathbb{Z}$.

Proof: From Example 2.1, $\mathbf{A}^{(1,\dots,1)} = (A_1^{(1)}, \dots, A_r^{(1)})$ is a regular sequence if and only if the matrix $(a_{ij})_{1\le i\le r,1\le j\le N}$ has rank r. Thus, everything follows from Theorem 4.2, except that it is not clear from (1.3) and (1.4) that $\nu_r(N, d; 1, \dots, 1) = \binom{N+d-r}{N-r}$. To see this, use Theorem 3.2 with $(\partial_1, \dots, \partial_r)$ and notice the obvious $\bigcap_{i=1}^r \mathcal{Z}_d(\partial_i) = \mathcal{P}_d(N-r)$, with dimension $\binom{N-r+d}{N-r}$. □

Theorem 5.1 is related to a result of Smith [**26**, Proposition 3.3(e)] when $\mathbb{K} = \mathbb{C}$ and $A_1, \dots, A_r$ have finite order, but without the *a priori* restriction $r \le N$. If so, $\mathfrak{A}_i(\xi) := \sum_{|\alpha|\ge 1} a_{i\alpha}\xi^\alpha$ is a polynomial, so that $X_{\mathbf{A}} := \bigcap_{i=1}^r \mathfrak{A}_i^{-1}(0) \subset \mathbb{C}^N$ is an algebraic subset. By algebro-geometric arguments, Smith shows that if (the ideal $\langle \mathfrak{A}_1, \dots, \mathfrak{A}_r\rangle$ is prime and) $X_{\mathbf{A}}$ is smooth at 0, then $\dim \mathcal{Z}_d(\mathbf{A}) = \binom{\dim X_{\mathbf{A}}+d}{\dim X_{\mathbf{A}}}$ for d large enough (unspecified). The assumption that $\langle \mathfrak{A}_1, \dots, \mathfrak{A}_r\rangle$ is prime is not explicitly made in [**26**], but the result is false otherwise, even if $X_{\mathbf{A}}$ is irreducible. For example, if $r = 1$ and $\mathbf{A} = \partial_N^2$, then $X_{\mathbf{A}} = \mathbb{C}^{N-1}$ is irreducible and smooth at 0 but, by Theorem 3.2, $\dim \mathcal{Z}_d(\mathbf{A}) = \dim \mathcal{P}_d - \dim \mathcal{P}_{d-2} > \binom{N-1+d}{N-1}$ for every $d > 0$.

In general, smoothness at 0 amounts to $\dim X_{\mathbf{A}} = \dim T_0 X_{\mathbf{A}}$ where the tangent space $T_0 X_{\mathbf{A}}$ at $0 \in X_{\mathbf{A}}$ is, by definition, the intersection of the null-spaces of the derivatives at 0 of all the polynomials in (a set of generators of) $\sqrt{\langle \mathfrak{A}_1, \dots, \mathfrak{A}_r\rangle}$. Since prime ideals are radical, it follows that $T_0X_{\mathbf{A}}$ is the intersection of the null-spaces of the homogeneous parts of degree 1 of $\mathfrak{A}_1, \dots, \mathfrak{A}_r$, that is, the null-space of the matrix $(a_{ij})_{1\leq i\leq r, 1\leq j\leq N}$ of Theorem 5.1, which is just the Jacobian matrix $J(\mathfrak{A}_1, \dots, \mathfrak{A}_r)(0)$. If this matrix has rank r, so that $r \leq N$ and $m_1 = \cdots = m_r = 1$, then $\dim X_{\mathbf{A}} = \dim T_0X = N - r$ and Smith's formula is (5.1), albeit only for d large enough and only in a special case (in Theorem 5.1, the ideal $\langle \mathfrak{A}_1, \dots, \mathfrak{A}_r \rangle$ need not be prime, as the example $r = 1$ and $\mathbf{A} = \partial_N + \partial_N^2$ shows, and it does not even exist if some A_i has infinite order).

5.2. Case $r = N$. We begin with a simple lemma.

Lemma 5.2. *If* $\mathbf{m} = (m_1, \dots, m_N) \in \mathbb{N}^N$, *then* $\nu_N(N, d; \mathbf{m}) = m_1 \cdots m_N$ *for every* $d \geq m_1 + \cdots + m_N - N$.

Proof: This follows from (2.2) and Bézout's Theorem (see [**7**, p. 475]), but we give an elementary self-contained proof. By Theorem 3.2, $\dim \bigcap_{i=1}^N \mathcal{Z}_d(\partial_i^{m_i}) = \nu_N(N, d; \mathbf{m})$ since $(\partial_1^{m_1}, \dots, \partial_N^{m_N})$ is a regular sequence. On the other hand, it is readily checked that if $u \in \mathcal{P}$, then $\partial_1^{m_1} u = \cdots = \partial_N^{m_N} u = 0$ if and only if u lies in the subspace generated by the $m_1 \cdots m_N$ (linearly independent) monomials x^α with $0 \leq \alpha_i \leq m_i - 1$. Since no such monomial has degree greater than $m_1 + \cdots + m_N - N$, they are all contained in $\mathcal{P}_d$ if $d \geq m_1 + \cdots + m_N - N$. Hence, $\bigcap_{i=1}^N \mathcal{Z}_d(\partial_i^{m_i})$ is the subspace generated by these monomials if $d \geq m_1 + \cdots + m_N - N$. □

By Remark 4.1, Lemma 5.2 remains true for every d if some m_i is 0. When $r = N$, we obtain the following form of Theorem 4.2.

Theorem 5.3. *Let* $\mathbf{m} = (m_1, \dots, m_N) \in \mathbb{N}^N$ *be given. If* $\mathbf{A} = (A_1, \dots, A_N) \in \overline{\mathcal{D}}(\mathbf{m})$, *denote by* $\mathfrak{A}_i^{(m_i)} \in \mathcal{P}_{m_i}^H$ *the homogeneous polynomial obtained by replacing* ∂^α *with* ξ^α *in* $A_i^{(m_i)}$, $1 \leq i \leq N$.

(i) *Let* $\overline{\mathbb{K}}$ *denote the algebraic closure of* $\mathbb{K}$. *If* $\mathfrak{A}_1^{(m_1)}(\xi) = \cdots = \mathfrak{A}_N^{(m_N)}(\xi) = 0$ *has no solution* $\xi \in \overline{\mathbb{K}}^N \backslash \{0\}$, *then*

$$\dim \mathcal{Z}_d(\mathbf{A}) = \nu_N(N, d; \mathbf{m}), \quad \textit{for all } d \in \mathbb{Z}. \tag{5.2}$$

In particular,

$$\dim \mathcal{Z}_d(\mathbf{A}) = m_1 \cdots m_N, \quad \textit{for all } d \geq m_1 + \cdots + m_N - N. \tag{5.3}$$

(ii) $\dim \mathcal{Z}_d(\mathbf{A}) \geq \nu_N(N, d; m_1, \dots, m_N)$, *for every* $d \in \mathbb{Z}$.

Proof: (i) The projective algebraic variety $\bigcap_{i=1}^N (\mathfrak{A}_i^{(m_i)})^{-1}(0) \subset \mathbb{P}_{\overline{\mathbb{K}}}^{N-1}$ is empty, whence $(\mathfrak{A}_1^{(m_1)}, \dots, \mathfrak{A}_N^{(m_N)})$ is a regular sequence (Subsection 2.1).

Thus, (5.2) holds by Theorem 4.2(i) and (5.3) follows from (5.2) and Lemma 5.2.

(ii) This follows at once from Theorem 4.2 (ii). □

Let $\mathcal{Z}(\mathbf{A})$ denote the space of *all* $\mathbf{A}$-harmonic polynomials. Under the condition of part (i) of Theorem 5.3, it follows from (5.3) that $\dim \mathcal{Z}(\mathbf{A}) = m_1 \cdots m_N$. When $\mathbb{K} = \mathbb{R}$ and $A_1, \ldots, A_N$ are homogeneous, so that $A_i = A_i^{(m_i)}$, this was proved by Dahmen and Micchelli [**8**, Corollary 2.4]. Earlier, Stiller ([**28**, Theorem 1.1]) had obtained the less explicit "curious" (sic) formula $\dim \mathcal{Z}(\mathbf{A}) = \sum_{d=0}^{\infty} \nu_N(N-1, d; \mathbf{m})$ by arguments from the cohomology of vector bundles.

When $\mathbb{K} = \mathbb{C}$ and $A_1, \ldots, A_N$ are homogeneous, (5.3) is in D'Almeida [**3**, Theorem 2.5]. Under these assumptions, he also shows that $\mathcal{Z}(\mathbf{A}) = \{Bu : B \in \mathcal{D}\}$ for a single polynomial $u \in \mathcal{Z}(\mathbf{A})$ of degree $m_1 + \cdots + m_N - N$.

In [**15**, Example 1], Karachik proves (5.2) when $N = 2$, $\mathbb{K} = \mathbb{C}$, and $\mathfrak{A}_1$ and $\mathfrak{A}_2$ are homogeneous of the same degree m with no common factor. If so, $\mathfrak{A}_1(\xi) = \mathfrak{A}_2(\xi) = 0$ has no solution $\xi \in \mathbb{C}^2 \backslash \{0\}$, as required in part (i) of Theorem 5.3.

Still when $A_1, \ldots, A_N$ are homogeneous, but if now the system $\mathfrak{A}_1(\xi) = \cdots = \mathfrak{A}_N(\xi) = 0$ has a solution $\xi \in \overline{\mathbb{K}}^N \backslash \{0\}$ (by homogeneity, $\mathfrak{A}_i^{(m_i)} = \mathfrak{A}_i$), then $\dim \mathcal{Z}(\mathbf{A}) = \infty$ by [**8**, Corollary 2.1] or [**28**, Proposition 2.1] (when $\mathbb{K} = \mathbb{R}$). Since the projective subvariety $\bigcap_{i=1}^{N} \mathfrak{A}_i^{-1}(0)$ of $\mathbb{P}_{\overline{\mathbb{K}}}^{N-1}$ is not empty, $\mathbf{A}$ is not a regular sequence and the last part of Theorem 3.2 with $r = N$ gives the more precise $\dim \mathcal{Z}_d(\mathbf{A}) \geq cd$ for every $d \in \mathbb{N}_0$ and some constant $c > 0$ irrespective of $\mathbb{K}$ (with $\operatorname{char} \mathbb{K} = 0$).

We did not find special cases of Theorem 5.3 in the literature when $N > 1$ and $A_1, \ldots, A_N$ are not homogeneous, except those with $m_1 = \cdots = m_N = 1$ and d large enough already mentioned in Subsection 5.1 (yielding $\dim \mathcal{Z}_d(\mathbf{A}) = 1$ for d large enough, whence $\mathcal{Z}_d(\mathbf{A}) = \mathbb{K}$ for every $d \geq 0$, consistent with Theorem 5.1).

5.3. Case $r > N$. If $r > N$ and $\mathbf{m} = (m_1, \ldots, m_r) \in \mathbb{N}^r$, no sequence $\mathbf{A} = (A_1, \ldots, A_r) \in \mathcal{D}^H(\mathbf{m})$ is regular and $\nu_r(N, d; \mathbf{m})$ is no longer a nondecreasing function of d (it even turns negative), so that $\dim \mathcal{Z}_d(\mathbf{A}) = \nu_r(N, d; \mathbf{m})$ cannot hold for every d. However, by Lemma 3.1, the calculation of $\dim \mathcal{Z}_d(\mathbf{A})$ still amounts to the calculation of the affine Hilbert function ${}^a\chi_{\langle A_1, \ldots, A_r \rangle}(d) = \sum_{\delta=0}^{d} \chi_{\langle A_1, \ldots, A_r \rangle}(\delta)$ (Subsections 2.2 and 2.4). Thus, everything boils down to calculating $\chi_{\langle A_1, \ldots, A_r \rangle}$.

Lemma 5.4. *If $r > N$ and $\mathbf{m} = (m_1, \dots, m_r) \in \mathbb{N}^r$, then $\nu_r(N, d; \mathbf{m}) = 0$ when $d \geq m_1 + \cdots + m_r - N$.*

Proof: By (4.1), $\nu_{N+1}(N, d; m_1, \dots, m_{N+1}) = \nu_N(N, d; m_1, \dots, m_N) - \nu_N(N, d - m_{N+1}; m_1, \dots, m_N) = 0$ if $d \geq m_1 + \cdots + m_{N+1} - N$ by Lemma 5.2. Next, by (4.1),

$$\nu_{N+2}(N, d; m_1, \dots, m_{N+2}) = \nu_{N+1}(N, d; m_1, \dots, m_{N+1}) - \nu_{N+1}(N, d - m_{N+2}; m_1, \dots, m_{N+1}) = 0$$

if $d \geq m_1 + \cdots + m_{N+2} - N$ for then, by the previous step,

$$\nu_{N+1}(N, d; m_1, \dots, m_{N+1}) = \nu_{N+1}(N, d - m_{N+2}; m_1, \dots, m_{N+1}) = 0.$$

More generally, by (4.1) and induction, $\nu_r(N, d; m_1, \dots, m_r) = 0$ if $r > N$ and $d \geq m_1 + \cdots + m_r - N$. □

When $r > N$, it is a conjecture of Fröberg [**11**] that, for a generic sequence $(A_1, \dots, A_r) \in \mathcal{D}^H(\mathbf{m})$,

$$\chi_{\langle A_1, \dots, A_r \rangle}(d) = \begin{cases} \nu_r(N-1, d; \mathbf{m}) & \text{if } d \leq \widetilde{d}, \\ 0 & \text{if } d > \widetilde{d}, \end{cases}$$

where $\widetilde{d} \in \mathbb{N}_0$ is the largest integer such that $\nu_r(N-1, d; \mathbf{m}) > 0$ for $0 \leq d \leq \widetilde{d}$. Note that $\widetilde{d} \geq 0$ since $\nu_r(N-1, 0; \mathbf{m}) = 1$ and that, by Lemma 5.4, $\widetilde{d} \leq m_1 + \cdots + m_r - N$ if $r \geq N$.

Accordingly, ${}^a\chi_{\langle A_1, \dots, A_r \rangle}(d) = \sum_{\delta=0}^{\min\{d, \widetilde{d}\}} \nu_r(N-1, \delta; \mathbf{m})$. Now, by (1.3) and Pascal's rule $\binom{N+\ell}{N} - \binom{N+\ell-1}{N} = \binom{N+\ell-1}{N-1}$ (still valid when (1.4) is used), $\nu_r(N-1, \delta; \mathbf{m}) = \nu_r(N, \delta; \mathbf{m}) - \nu_r(N, \delta-1; \mathbf{m})$. Since $\nu_r(N, -1; \mathbf{m}) = 0$, a telescopic sum argument yields $\sum_{\delta=0}^{d} \nu_r(N-1, \delta; \mathbf{m}) = \nu_r(N, d; \mathbf{m})$ for every $d \in \mathbb{N}_0$.

In summary, if $r > N$, we get (assuming the Fröberg Conjecture)

$$\dim \mathcal{Z}_d(\mathbf{A}) = \widetilde{\nu}_r(N, d; \mathbf{m}) := \begin{cases} \nu_r(N, d; \mathbf{m}) & \text{if } d \leq \widetilde{d}, \\ \nu_r(N, \widetilde{d}; \mathbf{m}) & \text{if } d > \widetilde{d}, \end{cases} \tag{5.4}$$

for generic $\mathbf{A} \in \mathcal{D}^H(\mathbf{m})$. This generalizes Theorem 3.2, but now genericity is no longer expressed in terms of regular sequences.

The Fröberg Conjecture is known to be true if $r = N + 1$, or if $N \leq 3$ and in some other cases; see [**6**] or [**19**] for details and references and a recent new result in [**20**]. A direct (not based on (5.4)) but not fully conclusive treatment of $r > N = 2$ can be found in [**28**] when $\mathbb{K} = \mathbb{R}$. However, the method requires finding some "admissible" sequence of integers for which no general algorithm is provided. To our best knowledge, the other cases when $r > N$ and (5.4) holds (especially $N = 3$ or $r = N+1$) have not been discussed elsewhere.

When $r > N > 1$ and $A_1, \dots, A_r$ are not necessarily homogeneous, the question remains completely open. Since the crucial Lemma 4.1 breaks down if ν_r is replaced with $\widetilde{\nu}_r$, the proof of Theorem 4.2 cannot be paralleled when (5.4) holds (generically) in the homogeneous case. In particular, it is unknown whether $\dim \mathcal{Z}_d(\mathbf{A})$ continues to depend solely upon the initial parts of $A_1, \dots, A_r$ when these initial parts satisfy some generic condition (it does when $N = 1$).

6. Finite difference operators

We give a simple example when the operators A_i of Theorem 4.2 are of infinite order. Let $\{e_1, \dots, e_N\}$ denote the standard basis of $\mathbb{K}^N$ and let $h_j \in \mathbb{K}\backslash\{0\}$, $1 \le j \le N$. Define the "elementary" finite difference operator Δ_{h_j} on $\mathcal{P}$ by

$$\Delta_{h_j} u(x) := h_j^{-1}(u(x + h_j e_j) - u(x)), \tag{6.1}$$

for $u \in \mathcal{P}$. This definition includes the classical "forward" ($h_j > 0$) and "backward" ($h_j < 0$) operators. If $\alpha \in \mathbb{N}_0^N$, set $h := (h_1, \dots, h_N)$ and

$$\Delta_h^\alpha := \Delta_{h_1}^{\alpha_1} \cdots \Delta_{h_N}^{\alpha_N}. \tag{6.2}$$

Given $r \in \mathbb{N}$ and $(m_1, \dots, m_r) \in \mathbb{N}^r$, define the finite difference operators on $\mathcal{P}$

$$\Lambda_{h,i} := \sum_{|\alpha| \ge m_i} a_{i\alpha} \Delta_h^\alpha, \quad 1 \le i \le r, \tag{6.3}$$

where $a_{i\alpha} \in \mathbb{K}$. In practice, $\Lambda_{h,i}$ is a discretization of the partial differential operator

$$A_i := \sum_{|\alpha| \ge m_i} a_{i\alpha} \partial^\alpha \in \overline{\mathcal{D}}(m_i), \tag{6.4}$$

obtained as $h \to 0$ (if that makes sense in $\mathbb{K}$). As usual, we set

$$A_i^{(m_i)} := \sum_{|\alpha| = m_i} a_{i\alpha} \partial^\alpha \in \mathcal{D}^H(m_i). \tag{6.5}$$

Lastly, for $d \in \mathbb{Z}$, we define $\mathbf{\Lambda}_h := (\Lambda_{h,1}, \dots, \Lambda_{h,r})$ and

$$\mathcal{Z}_d(\mathbf{\Lambda}_h) := \{u \in \mathcal{P}_d : \ \mathbf{\Lambda}_h u = 0\} = \textstyle\bigcap_{i=1}^r \mathcal{Z}_d(\Lambda_{h,i}).$$

Theorem 6.1. *Assume* $1 \le r \le N$ *and let* $\mathbf{m} = (m_1, \dots, m_r) \in \mathbb{N}^r$.

(i) *If* $(A_1^{(m_1)}, \dots, A_r^{(m_r)}) \in \mathcal{D}^H(\mathbf{m})$ *(see (6.5)) is a regular sequence, then*

$$\dim \mathcal{Z}_d(\mathbf{\Lambda}_h) = \nu_r(N, d; \mathbf{m}), \quad \text{for all } d \in \mathbb{Z}.$$

(ii) $\dim \mathcal{Z}_d(\mathbf{\Lambda}_h) \ge \nu_r(N, d; \mathbf{m})$ *for every* $d \in \mathbb{Z}$.

Proof: On the space $\mathcal{P}$, the finite difference operator Δ_{h_j} in (6.1) coincides with the partial differential operator of infinite order $\sum_{k\ge1}(k!)^{-1}h_j^{k-1}\partial_j^k = \partial_j + \text{h.o.t.}$, where h.o.t. stands for "higher order terms" (use Taylor's formula). By (6.2), $\Delta_h^\alpha = \partial^\alpha + \text{h.o.t.}$ and so, by (6.3), $\Lambda_{h,i} = \sum_{|\alpha|\ge m_i}(a_\alpha \partial^\alpha + \text{h.o.t.}) = A_i^{(m_i)} + \text{h.o.t.}$ This shows that $\Lambda_{h,i}$ is a partial differential operator in $\overline{\mathcal{D}}(m_i)$ (of infinite order, even if A_i in (6.4) is homogeneous) with $\Lambda_{h,i}^{(m_i)} = A_i^{(m_i)}$. Thus, everything follows from Theorem 4.2. □

If the factor h_j^{-1} is omitted in (6.1), the only difference is that $A_i^{(m_i)}$ must be replaced with $A_{h,i}^{(m_i)} := \sum_{|\alpha|=m_i} a_{i\alpha} h^\alpha \partial_i^{m_i}$. If $h_1 = \dots = h_N$, this substitution is immaterial.

7. A-harmonic polynomial-exponentials

In this section, $\mathbb{K} = \mathbb{C}$ and $(\cdot|\cdot)$ denotes the standard inner product on $\mathbb{C}^N$. A polynomial-exponential is a function of the form $u(x) := e^{(x|\overline{\eta})}v(x)$ where $\eta \in \mathbb{C}^N$ and $v \in \mathcal{P}$. If $\mathbf{A} := (A_1, \dots, A_r) \in \mathcal{D}^r$ (i.e., each A_i has *finite order*), a polynomial-exponential u is $\mathbf{A}$-harmonic if $\mathbf{A}u = 0$. Note that the definition need not make sense if some A_i has infinite order. When x is restricted to $\mathbb{R}^N$, linear combinations of $\mathbf{A}$-harmonic polynomial-exponentials are dense in various spaces of $\mathbf{A}$-harmonic functions or distributions (Malgrange [**17**, Theorem 2.5]; see also Ehrenpreis [**9**], Palamodov [**22**]).

It is not assumed that $\mathbf{A}$ has zero constant term. We shall show that, given $\eta \in \mathbb{C}^N$, Theorem 4.2 can be used to find the dimension of the space

$$\mathcal{Z}_d(\mathbf{A}, \eta) := \{u = e^{(\cdot|\overline{\eta})}v : \mathbf{A}u = 0,\ v \in \mathcal{P}_d\}. \tag{7.1}$$

We denote by $\mathfrak{A}_i(\eta)$ the polynomial obtained by replacing ∂^α with η^α in A_i and, for $\eta \in \mathbb{C}^N$, we set

$$\mathbf{A}_\eta := (A_{1\eta}, \dots, A_{r\eta}),$$

where $A_{i\eta}$ is the differential operator with η-dependent constant coefficients

$$A_{i\eta} := \sum_\beta (\beta!)^{-1} \partial^\beta \mathfrak{A}_i(\eta) \partial^\beta. \tag{7.2}$$

Note that $A_{i\eta} = A_i$ if $\eta = 0$. We also set

$$X_{\mathbf{A}} := \bigcap_{i=1}^{r} \mathfrak{A}_i^{-1}(0) \subset \mathbb{C}^N.$$

Lemma 7.1. *If $\mathbf{A} \in \mathcal{D}^r$, the polynomial-exponential $u = e^{(\cdot|\bar{\eta})}v \neq 0$ is $\mathbf{A}$-harmonic if and only if $\eta \in X_{\mathbf{A}}$ and v is $\mathbf{A}_\eta$-harmonic.*

Proof: If $1 \le j \le N$, then $\partial_j u = e^{(\cdot|\bar{\eta})}(\eta_j + \partial_j)v$, so that $\partial^\alpha u = e^{(\cdot|\bar{\eta})}(\eta_1 + \partial_1)^{\alpha_1} \cdots (\eta_N + \partial_N)^{\alpha_N} v$ for every $\alpha = (\alpha_1, \ldots, \alpha_N) \in \mathbb{N}_0^N$. Thus, if $A = \sum_\alpha a_\alpha \partial^\alpha \in \mathcal{D}$ (hence has finite order), then $Au = e^{(\cdot|\bar{\eta})} A_\eta v$ where $A_\eta := \sum_\alpha \sum_{\beta \le \alpha} a_\alpha \binom{\alpha}{\beta} \eta^{\alpha-\beta} \partial^\beta$.

Since $\partial^\beta \eta^\alpha = \frac{\alpha!}{(\alpha-\beta)!}\eta^{\alpha-\beta}$ when $\beta \le \alpha$, the operator A_η may be written $A_\eta = \sum_\alpha \sum_{\beta \le \alpha} a_\alpha (\beta!)^{-1} \partial^\beta(\eta^\alpha)\partial^\beta$. Next, the restriction $\beta \le \alpha$ may be removed since $\partial^\beta(\eta^\alpha) = 0$ whenever $\beta_j > \alpha_j$ for some j, so that $A_\eta = \sum_\alpha \sum_\beta a_\alpha (\beta!)^{-1} \partial^\beta(\eta^\alpha)\partial^\beta$. Both sums are finite. Therefore, upon interchanging the order of summation, $A_\eta = \sum_\beta (\beta!)^{-1} \partial^\beta \mathfrak{A}(\eta) \partial^\beta$ where $\mathfrak{A}(\eta) := \sum_\alpha a_\alpha \eta^\alpha$.

By using this with $A = A_i, 1 \le i \le r$, it follows that $e^{(\cdot|\bar{\eta})}v$ is $\mathbf{A}$-harmonic if and only if v is $\mathbf{A}_\eta$-harmonic. Note that the constant term of $\mathbf{A}_\eta$ is $(\mathfrak{A}_1(\eta), \ldots, \mathfrak{A}_r(\eta))$. Thus, v must be 0 if $\mathfrak{A}_i(\eta) \neq 0$ for some $1 \le i \le r$. Since $v \neq 0$ is assumed, it follows that $\eta \in X_{\mathbf{A}}$. This completes the proof. $\square$

Lemma 7.1 claims to no novelty. That $\eta \in X_{\mathbf{A}}$ if $u = e^{(\cdot|\bar{\eta})}v \neq 0$ is $\mathbf{A}$-harmonic has been shown in various places by reduction to the trivial case $v = 1$, but the statement that v must be $\mathbf{A}_\eta$-harmonic with $A_{i\eta}$ given by (7.2) (rather than by the obvious $e^{-(\cdot|\bar{\eta})} A_i e^{(\cdot|\bar{\eta})}$) is harder to find.

Theorem 7.2. *Suppose $\mathbf{A} \in \mathcal{D}^r$ with $r \le N$. If $\eta \in X_{\mathbf{A}}$, let $A_{i\eta}^{(m_{i\eta})}$ denote the initial part of $A_{i\eta}$ in* (7.2) *(of order $m_{i\eta} \ge 1$) and let $\mathfrak{A}_{i\eta}^{(m_{i\eta})}$ be the polynomial obtained by replacing ∂^β with ξ^β in $A_{i\eta}^{(m_{i\eta})}$.*

(i) *If $(A_{1\eta}^{(m_{1\eta})}, \ldots, A_{r\eta}^{(m_{r\eta})})$ is a regular sequence, then* $\dim \mathcal{Z}_d(\mathbf{A}, \eta) = \nu_r(N, d; m_{1\eta}, \ldots, m_{r\eta})$ *for every $d \in \mathbb{Z}$.*
(ii) *If the Jacobian matrix $J(\mathfrak{A}_1, \ldots, \mathfrak{A}_r)(\eta)$ has rank r, then $m_{i\eta} = 1$ for $1 \le i \le r$ and so* $\dim \mathcal{Z}_d(\mathbf{A}, \eta) = \binom{N-r+d}{N-r}$ *for every $d \in \mathbb{Z}$.*
(iii) *If $r = N$ and $\mathfrak{A}_{1\eta}^{(m_{1\eta})}(\xi) = \cdots = \mathfrak{A}_{N\eta}^{(m_{N\eta})}(\xi) = 0$ has no solution $\xi \in \mathbb{C}\backslash\{0\}$, then* $\dim \mathcal{Z}_d(\mathbf{A}, \eta) = \nu_N(N, d; m_{1\eta}, \ldots, m_{N\eta})$ *for every $d \in \mathbb{Z}$. In particular, if $d \ge m_{1\eta} + \cdots + m_{N\eta} - N$, then* $\dim \mathcal{Z}_d(\mathbf{A}, \eta) = m_{1\eta} \cdots m_{N\eta}$.

Proof: Both (i) and (ii) follow at once from Lemma 7.1 and from Theorem 4.2 and Theorem 5.1, respectively (when $\mathbf{A}$ is replaced with $\mathbf{A}_\eta$, the matrix of Theorem 5.1 is just $J(\mathfrak{A}_1, \ldots, \mathfrak{A}_r)(\eta))$. Part (iii) follows from Lemma 7.1 and Theorem 5.3. □

Part (ii) is applicable with every $\eta \in X_\mathbf{A}$ if and only if $r \leq N$ and 0 is a regular value of $(\mathfrak{A}_1, \ldots, \mathfrak{A}_r)$ (example: $r = 1$ and $\mathbf{A}$ the heat operator $\partial_N - \sum_{i=1}^{N-1} \partial_i^2$).

Example 7.1. Let $\mathbf{A} = A = \Delta$ (Laplace operator) and let $\eta \in \mathbb{C}^N \backslash \{0\}$ be such that $\sum_{i=1}^N \eta_i^2 = 0$ (hence, $N \geq 2$). The operator $A_\eta := A_{1\eta}$ in (7.2) is $\sum_{j=1}^N 2\eta_j \partial_j + \Delta$. Since $\eta \neq 0$, it follows from Theorem 7.2(ii) with $r = 1$ that the space of harmonic polynomial-exponentials $e^{(\cdot|\bar{\eta})}v$ with $v \in \mathcal{P}_d$ has dimension $\binom{N-1+d}{N-1}$ independent of η. In contrast, when $\eta = 0$, the dimension of the subspace of harmonic polynomials in $\mathcal{P}_d$ is $\frac{N-1+2d}{N-1+d}\binom{N-1+d}{N-1}$, which is larger if $d \geq 1$.

We conclude with a few remarks on part (iii) of Theorem 7.2. Let $\mathcal{Z}(\mathbf{A}, \eta) = \bigcup_{d \geq 0} \mathcal{Z}_d(\mathbf{A}, \eta)$ denote the space of $\mathbf{A}$-harmonic functions of the form $e^{(\cdot|\bar{\eta})}v$ with $v \in \mathcal{P}$. If η is an isolated point of $X_\mathbf{A}$ (hence $r \geq N$), D'Almeida ([**2**, Theorem 1]) has shown that $\dim \mathcal{Z}(\mathbf{A}, \eta) = \mu_\eta$, the multiplicity of η in $X_\mathbf{A}$ (he assumes that $X_\mathbf{A}$ is finite, but the argument is the same). Under the assumption in (iii) that $\mathfrak{A}_{1\eta}^{(m_{1\eta})}(\xi) = \cdots = \mathfrak{A}_{N\eta}^{(m_{N\eta})}(\xi) = 0$ has no solution $\xi \in \mathbb{C} \backslash \{0\}$, $\dim \mathcal{Z}(\mathbf{A}, \eta) = m_{1\eta} \cdots m_{N\eta}$ since $\mathcal{Z}(\mathbf{A}, \eta) = \mathcal{Z}_d(\mathbf{A}, \eta)$ when $d \geq m_{1\eta} + \cdots + m_{N\eta} - N$ and it is not hard to check that η is an isolated point of $X_\mathbf{A}$. Altogether, $\mu_\eta = m_{1\eta} \cdots m_{N\eta}$. A proof of this formula by Brouwer's degree can be found, for holomorphic functions, in Yuzhakov and Tsikh [**29**, Theorem 2] who also show that $\mu_\eta > m_{1\eta} \cdots m_{N\eta}$ if η is an isolated point of $X_\mathbf{A}$ and $\mathfrak{A}_{1\eta}^{(m_{1\eta})}(\xi) = \cdots = \mathfrak{A}_{N\eta}^{(m_{N\eta})}(\xi) = 0$ has a solution $\xi \in \mathbb{C} \backslash \{0\}$.

When X_A is finite, the finite dimensionality of the space $\bigoplus_{\eta \in X_\mathbf{A}} \mathcal{Z}(\mathbf{A}, \eta)$ of all the $\mathbf{A}$-harmonic polynomial-exponentials has been proved by various authors ([**2**], [**4**], [**8**], [**21**]). If also $r = N$ and $X_\mathbf{A} \neq \emptyset$, the above yields $\sum_{\eta \in X_\mathbf{A}} m_{1\eta}, \ldots, m_{N\eta} \leq \dim \bigoplus_{\eta \in X_\mathbf{A}} \mathcal{Z}(\mathbf{A}, \eta) \leq \deg \mathfrak{A}_1 \cdots \deg \mathfrak{A}_N$ and the first (second) inequality is an equality if and only if $\mathfrak{A}_{1\eta}^{(m_{1\eta})}(\xi) = \cdots = \mathfrak{A}_{N\eta}^{(m_{N\eta})}(\xi) = 0$ has no solution $\xi \in \mathbb{C} \backslash \{0\}$ for every $\eta \in X_\mathbf{A}$ ($\mathfrak{A}_1 = \cdots = \mathfrak{A}_N = 0$ has no solution at infinity).

Appendix

We give an elementary proof of the formula $\chi_r(N{+}1, d; \mathbf{m}) = \nu_r(N, d; \mathbf{m})$ (see (2.2)) when $1 \leq r \leq N$ and $\mathbf{m} = (m_1, \ldots, m_r) \in \mathbb{N}^r$. From the equality $\chi_r(N+1,d;\mathbf{m}) = {}^a\chi_r(N,d;\mathbf{m}) = {}^a\chi_{\langle A_1,\dots,A_r\rangle}(d)$ when $\mathbf{A} := (A_1,\dots,A_r) \in \mathcal{D}^H(\mathbf{m})$ is a regular sequence and from Lemma 3.1, it suffices to show that $\dim \mathcal{Z}_d(\mathbf{A}) = \nu_r(N,d;\mathbf{m})$ for *one* choice of $\mathbb{K}$, say $\mathbb{K} = \mathbb{C}$, and *one* regular sequence $\mathbf{A}$, say $(\partial_1^{m_1},\dots,\partial_r^{m_r})$ (see Section 2), but of course without using Theorem 3.2, whose proof relies on the formula (2.2). Accordingly, we prove below that

$$\text{(A1)} \qquad \dim \textstyle\bigcap_{i=1}^r \mathcal{Z}_d(\partial_i^{m_i}) = \nu_r(N,d;\mathbf{m}) \text{ if } 1 \le r \le N.$$

Lemma A1. *If $N \in \mathbb{N}$ and $r \ge 2$, then for every $d \in \mathbb{Z}$ and every $(m_1,\dots,m_r) \in \mathbb{N}_0^r$,*

$$\text{(A2)} \quad \nu_r(N,d;m_1,\dots,m_r+1) = \nu_r(N,d;m_1,\dots,m_r) + \nu_{r-1}(N-1,d-m_r;m_1,\dots,m_{r-1})$$

and

$$\text{(A3)} \quad \nu_r(N,d;m_1,\dots,m_r) = \sum_{j=0}^{m_r-1} \nu_{r-1}(N-1,d-j;m_1,\dots,m_{r-1}).$$

Proof: Since $\nu_r(N,d;m_1,\dots,m_{r-1},0) = 0$ (Remark 4.1), (A3) follows at once from (A2) with m_r replaced with $m_r-1, m_r-2,\dots,0$. We now prove (A2).

By Lemma 4.1 with $i=r$ and m_r replaced with m_r+1,

$$\text{(A4)} \quad \nu_r(N,d;m_1,m_2,\dots,m_r+1) = \nu_{r-1}(N,d;m_1,\dots,m_{r-1}) - \nu_{r-1}(N,d-m_r-1;m_1,\dots,m_{r-1})$$

and, by (1.2),

$$\text{(A5)} \quad \nu_{r-1}(N,d-m_r-1;m_1,\dots,m_{r-1}) = \dim \mathcal{P}_{d-m_r-1}(N) + \sum_{j=1}^{r-1}(-1)^j \sum_{1\le i_1<\cdots<i_j\le r-1} \dim \mathcal{P}_{d-m_r-1-(m_{i_1}+\cdots+m_{i_j})}(N).$$

Now,

$$\dim \mathcal{P}_{k-1}(N) = \dim \mathcal{P}_k(N) - \dim \mathcal{P}_k(N-1)$$

(use $\mathcal{P}_k(N) = \mathcal{P}_{k-1}(N) \oplus \mathcal{P}_k^H(N)$ and $\dim \mathcal{P}_k^H(N) = \dim \mathcal{P}_k(N-1)$). Thus,

$$\dim \mathcal{P}_{d-m_r-1}(N) = \dim \mathcal{P}_{d-m_r}(N) - \dim \mathcal{P}_{d-m_r}(N-1)$$

and

$$\dim \mathcal{P}_{d-m_r-1-(m_{i_1}+\cdots+m_{i_j})}(N) = \dim \mathcal{P}_{d-m_r-(m_{i_1}+\cdots+m_{i_j})}(N) - \dim \mathcal{P}_{d-m_r-(m_{i_1}+\cdots+m_{i_j})}(N-1).$$

Hence, the right-hand side of (A5) is

$$\left[\dim \mathcal{P}_{d-m_r}(N)+\sum_{j=1}^{r-1}(-1)^j \sum_{1\le i_1<\cdots<i_j\le r-1} \dim \mathcal{P}_{d-m_r-(m_{i_1}+\cdots+m_{i_j})}(N)\right]$$

$$-\left[\dim \mathcal{P}_{d-m_r}(N-1)\right.$$

$$\left.+\sum_{j=1}^{r-1}(-1)^j \sum_{1\le i_1<\cdots<i_j\le r-1} \dim \mathcal{P}_{d-m_r-(m_{i_1}+\cdots+m_{i_j})}(N-1)\right]$$

$$=\nu_{r-1}(N,d-m_r;m_1,\dots,m_{r-1})-\nu_{r-1}(N-1,d-m_r;m_1,\dots,m_{r-1})$$

and so (A5) reads $\nu_{r-1}(N,d-m_r-1;m_1,\dots,m_{r-1}) = \nu_{r-1}(N,d-m_r;m_1,\dots,m_{r-1})-\nu_{r-1}(N-1,d-m_r;m_1,\dots,m_{r-1})$. After substitution into (A4), (A2) follows from Lemma 4.1. □

The proof of (A1) goes by induction on N. We change the notation $\mathcal{Z}_d(\partial_i^m)$ into $\mathcal{Z}_d(N;\partial_i^m)$ since it is important to keep track of the number of variables.

If $N=1$, then $r=1$, $\partial_1=\partial := d/dx$, and $\mathbf{m}=m\in\mathbb{N}$. Then, $\dim \mathcal{Z}_d(1,\partial^m)=\min\{(d+1)_+,m\}=\nu_1(1,d;m)$ by (1.2). Suppose now $N\ge 2$ and that (A1) holds when N is replaced with $N-1$, that is,

(A6) $\quad \dim \bigcap_{i=1}^s \mathcal{Z}_d(N-1;\partial_i^{m_i})=\nu_s(N-1,d;\mathbf{m})$ if $1\le s\le N-1$.

Let $u\in\bigcap_{i=1}^r \mathcal{Z}_d(N;\partial_i^{m_i})$ and set $x' := (x_1,\dots,\widehat{x_r},\dots,x_N)\in\mathbb{C}^{N-1}$. Since $u\in\mathcal{Z}_d(N;\partial_r^{m_r})$, it follows that $u(x)=\sum_{j=0}^{m_r-1}u_j(x')x_r^j$ with $u_j\in\mathcal{P}_{d-j}(N-1)$. Next, $u\in\mathcal{Z}_d(N;\partial_i^{m_i})$ with $1\le i\le r-1$ entails $\partial_i^{m_i}u_j=0$ for $0\le j\le m_r-1$ and $1\le i\le r-1$. Since u_j depends only upon $x'\in\mathbb{C}^{N-1}$, this means that $u_j\in\bigcap_{i=1}^{r-1}\mathcal{Z}_{d-j}(N-1;\partial_i^{m_i})$, $0\le j\le m_r-1$.

Conversely, if $u(x)=\sum_{j=0}^{m_r-1}u_j(x')x_r^j$ with $u_j\in\bigcap_{i=1}^{r-1}\mathcal{Z}_{d-j}(N-1;\partial_i^{m_i})$, it is obvious that $u\in\bigcap_{i=1}^r\mathcal{Z}_d(N;\partial_i^{m_i})$. This shows that

$$\bigcap_{i=1}^r \mathcal{Z}_d(N;\partial_i^{m_i})=\bigoplus_{j=0}^{m_r-1}\bigcap_{i=1}^{r-1}\mathcal{Z}_{d-j}(N-1;\partial_i^{m_i})\otimes x_r^j,$$

whence $\dim\bigcap_{i=1}^r\mathcal{Z}_d(N;\partial_i^{m_i})=\sum_{j=0}^{m_r-1}\dim\bigcap_{i=1}^{r-1}\mathcal{Z}_{d-j}(N-1;\partial_i^{m_i})$. By (A6) with $s=r-1\le N-1$,

$$\dim\bigcap_{i=1}^{r-1}\mathcal{Z}_{d-j}(N-1;\partial_i^{m_i})=\nu_{r-1}(N-1,d-j;m_1,\dots,m_{r-1}),$$

so that (A1) follows from (A3).

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