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KEY POLYNOMIALS OVER VALUED FIELDS

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Abstract: Let K be a field. For any valuation μ on $K[x]$ admitting key polynomials we determine the structure of the whole set of key polynomials in terms of a fixed key polynomial of minimal degree. We deduce a canonical bijection between the set of μ-equivalence classes of key polynomials and the maximal spectrum of the subring of elements of degree zero in the graded algebra of μ.

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Introduction

Key polynomials over a valued field (K, v) were introduced by S. MacLane as a tool to construct augmentations of discrete rank-one valuations on the polynomial ring $K[x]$ [**5**]. As an application, MacLane designed an algorithm to compute all extensions of the given valuation v on K to a finite field extension L/K [**6**].

This work was generalized to arbitrary valuations by M. Vaquié [**11**] and, independently, by F. J. Herrera, M. A. Olalla, and M. Spivakovsky [**4**].

In the non-discrete case *limit augmented valuations* arise. The structure of their graded algebra and the description of their sets of key polynomials are crucial questions linked with the study of the defect of a valuation in a finite extension and the local uniformization problem [**3, 7, 10, 12**].

In this paper we consider an arbitrary valuation μ on $K[x]$ admitting key polynomials and we describe its set of key polynomials $\mathrm{KP}(\mu)$ in terms of a fixed key polynomial of minimal degree. We also give some hints about the structure of the graded algebra of μ.

Some of the results of the paper can be found in [**11**], but only for augmented valuations. Also, in [**9**] some partial results are obtained for

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residually transcendental valuations, by using the fact that these valuations are determined by a minimal pair.

In our approach we do not make any assumption on μ, and we derive our results in a pure abstract form from the mere existence of key polynomials.

In Section 2 we study general properties of key polynomials, while in Section 3 we study specific properties of key polynomials of minimal degree. In Section 4 we determine the structure of the subring $\Delta \subset \mathcal{G}_\mu$ of elements of degree zero in the graded algebra of μ.

Section 5 is devoted to the introduction of residual polynomial operators, based on old ideas of Ore and MacLane [**8, 6**]. These operators yield a malleable and elegant tool, able to replace the onerous "lifting" techniques in the context of valuations constructed from minimal pairs.

In Section 6 we describe the set of key polynomials and we prove that a certain *residual ideal operator* sets a bijection

$$\mathrm{KP}(\mu)/\!\sim_\mu \longrightarrow \mathrm{Max}(\Delta)$$

between the set of μ-equivalence classes of key polynomials and the maximal spectrum of Δ. This result is inspired by [**2**], where it was proved for discrete rank-one valuations.

In Section 7 we single out a key polynomial of minimal degree for augmented and limit augmented valuations. In this way, all previous results can be applied to these valuations.

In Section 8 we obtain some partial results on the structure of the graded algebra.

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1. Graded algebra of a valuation on a polynomial ring

1.1. Graded algebra of a valuation. Let Γ be an ordered abelian group. Consider

$$w\colon L \longrightarrow \Gamma \cup \{\infty\}$$

a valuation on a field L, and denote

- $\mathfrak{m}_w \subset \mathcal{O}_w \subset L$, the maximal ideal and valuation ring of w.
- $k_w = \mathcal{O}_w/\mathfrak{m}_w$, the residue class field of w.
- $\Gamma_w = w(L^*)$, the group of values of w.

To any subring $A \subset L$ we may associate a graded algebra as follows.

For every $\alpha \in \Gamma_w$, consider the additive subgroups

$$\mathcal{P}_\alpha = \{a \in A \mid w(a) \geq \alpha\} \supset \mathcal{P}_\alpha^+ = \{a \in A \mid w(a) > \alpha\},$$

leading to the graded algebra

$$\mathrm{gr}_w(A) = \bigoplus\nolimits_{\alpha\in\Gamma_w} \mathcal{P}_\alpha/\mathcal{P}_\alpha^+.$$

The product of homogeneous elements is defined in an obvious way:

$$(a + \mathcal{P}_\alpha^+)(b + \mathcal{P}_\beta^+) = ab + \mathcal{P}_{\alpha+\beta}^+.$$

If the classes $a + \mathcal{P}_\alpha^+$, $b + \mathcal{P}_\beta^+$ are different from zero, then $w(a) = \alpha$, $w(b) = \beta$. Hence, $w(ab) = \alpha + \beta$, so that $ab + \mathcal{P}_{\alpha+\beta}^+$ is different from zero too.

Thus, $\mathrm{gr}_w(A)$ is an integral domain.

Consider the "initial term" mapping $H_w \colon A \to \mathrm{gr}_w(A)$, given by

$$H_w(0) = 0, \quad H_w(a) = a + \mathcal{P}_{w(a)}^+, \quad \text{for } a \in A,\ a \neq 0.$$

Note that $H_w(a) \neq 0$ if $a \neq 0$. For all $a, b \in A$ we have

$$(1) \qquad \begin{aligned} &H_w(ab) = H_w(a)H_w(b),\\ &H_w(a+b) = H_w(a) + H_w(b), \quad \text{if } w(a) = w(b) = w(a+b). \end{aligned}$$

Definition 1.1. Two elements $a, b \in A$ are said to be *w-equivalent* if $H_w(a) = H_w(b)$. In this case, we write $a \sim_w b$.

This is equivalent to $w(a - b) > w(b)$.

We say that a is *w-divisible* by b if $H_w(a)$ is divisible by $H_w(b)$ in $\mathrm{gr}_w(A)$. In this case, we write $b|_w a$.

This is equivalent to $a \sim_w bc$, for some $c \in A$.

1.2. Valuations on polynomial rings. General setting. Throughout the paper, we fix a field K and a valuation

$$\mu\colon K(x) \longrightarrow \Gamma_\mu \cup \{\infty\},$$

on the field $K(x)$ of rational functions in one indeterminate x.

We do not make any assumption on the rank of μ.

We denote by $v = \mu_{|K}$ the valuation on K obtained by the restriction of μ. The group of values of v is a subgroup of Γ_μ:

$$\Gamma_v = v(K^*) = \mu(K^*) \subset \Gamma_\mu.$$

For each one of the two valuations v, μ, we consider a different graded algebra:

$$\mathcal{G}_v := \mathrm{gr}_v(K), \quad \mathcal{G}_\mu := \mathrm{gr}_\mu(K[x]).$$

In the algebra $\mathcal{G}_v$, every non-zero homogeneous element is a unit, that is,

$$H_v(a)^{-1} = H_v(a^{-1}), \quad \text{for all } a \in K^*.$$

The subring of homogeneous elements of degree zero of $\mathcal{G}_v$ is k_v, so that $\mathcal{G}_v$ has a natural structure of k_v-algebra.

We have a natural embedding of graded algebras

$$\mathcal{G}_v \hookrightarrow \mathcal{G}_\mu, \quad a + \mathcal{P}^+_\alpha(v) \longmapsto a + \mathcal{P}^+_\alpha(\mu), \quad \text{for all } \alpha \in \Gamma_v \text{ and } a \in \mathcal{P}_\alpha(v).$$

The subring of $\mathcal{G}_\mu$ determined by the piece of degree zero is denoted

$$\Delta = \Delta_\mu = \mathcal{P}_0(\mu)/\mathcal{P}^+_0(\mu).$$

Since $\mathcal{O}_v \subset \mathcal{P}_0 = K[x] \cap \mathcal{O}_\mu$ and $\mathfrak{m}_v = \mathcal{P}^+_0 \cap \mathcal{O}_v \subset \mathcal{P}^+_0 = K[x] \cap \mathfrak{m}_\mu$, there are canonical injective ring homomorphisms

$$k_v \hookrightarrow \Delta \hookrightarrow k_\mu.$$

In particular, Δ and $\mathcal{G}_\mu$ are equipped with a canonical structure of k_v-algebra.

The aim of the paper is to analyze the structure of the graded algebra $\mathcal{G}_\mu$ and show that most of the properties of the extension μ/v are reflected in algebraic properties of the extension $\mathcal{G}_\mu/\mathcal{G}_v$.

For instance, an essential role is played by the *residual ideal operator*

$$\mathcal{R} = \mathcal{R}_\mu \colon K[x] \longrightarrow I(\Delta), \quad g \longmapsto (H_\mu(g)\mathcal{G}_\mu) \cap \Delta, \tag{2}$$

where $I(\Delta)$ is the set of ideals in Δ.

In Sections 5 and 6 we shall study in more detail this operator $\mathcal{R}$, which translates questions about the action of μ on $K[x]$ into ideal-theoretic problems in the ring Δ.

Commensurability. The *divisible hull* of an ordered abelian group Γ is

$$\mathbb{Q}\Gamma := \Gamma \otimes_\mathbb{Z} \mathbb{Q}.$$

This $\mathbb{Q}$-vector space inherits a natural structure of ordered abelian group, with the same rank as Γ.

The *rational rank* of Γ is defined as $\operatorname{rr}(\Gamma) = \dim_\mathbb{Q}(\mathbb{Q}\Gamma)$.

Since Γ has no torsion, it admits an order-preserving embedding $\Gamma \hookrightarrow \mathbb{Q}\Gamma$ into its divisible hull. For every $\gamma \in \mathbb{Q}\Gamma$ there exists a minimal positive integer e such that $e\gamma \in \Gamma$.

We say that our extension μ/v is *commensurable* if $\mathbb{Q}\Gamma_v = \mathbb{Q}\Gamma_\mu$ or, in other words, if $\operatorname{rr}(\Gamma_\mu/\Gamma_v) = 0$. This is equivalent to Γ_μ/Γ_v being a torsion group.

Actually, $\operatorname{rr}(\Gamma_\mu/\Gamma_v)$ takes only the values 0 or 1, as the following well-known inequality shows [**1**, Theorem 3.4.3]:

$$\text{(3)} \qquad \operatorname{tr.deg}(k_\mu/k_v) + \operatorname{rr}(\Gamma_\mu/\Gamma_v) \le \operatorname{tr.deg}(K(x)/K) = 1.$$

Finally, we fix some notation to be used throughout the paper.

Notation. For any positive integer m we denote

$$K[x]_m = \{a \in K[x] \mid \deg(a) < m\}.$$

For any polynomials $f, \chi \in K[x]$, with $\deg(\chi) > 0$, we denote the canonical χ-expansion of f by

$$f = \sum\nolimits_{0 \le s} f_s \chi^s,$$

being implicitly assumed that the coefficients $f_s \in K[x]$ have $\deg(f_s) < \deg(\chi)$.

2. Key polynomials. General properties

In this section we introduce the concept of key polynomial for μ and we study some general properties of key polynomials.

Definition 2.1. Let $\chi \in K[x]$.

We say that χ is μ-irreducible if $H_\mu(\chi)\mathcal{G}_\mu$ is a non-zero prime ideal.

We say that χ is μ-minimal if $\chi \nmid_\mu f$ for any non-zero $f \in K[x]$ with $\deg f < \deg \chi$.

The property of μ-minimality admits a relevant characterization, given in Proposition 2.3 below.

Lemma 2.2. *Let $f, \chi \in K[x]$. Consider a χ-expansion of $f \in K[x]$ as follows:*

$$f = \sum\nolimits_{0 \le s} a_s \chi^s, \quad a_s \in K[x], \quad \chi \nmid_\mu a_s, \text{ for all } s.$$

Then $\mu(f) = \operatorname{Min}\{\mu(a_s\chi^s) \mid 0 \le s\}$.

Proof: Write $f = a_0 + \chi q$ with $q \in K[x]$. Then $\mu(f) \ge \operatorname{Min}\{\mu(a_0), \mu(\chi q)\}$.

A strict inequality would imply $a_0 \sim_\mu -\chi q$, against our assumption. Hence equality holds, and the result follows from a recurrent argument. □

Proposition 2.3. *Let $\chi \in K[x]$ be a non-constant polynomial. The following conditions are equivalent:*

(1) *χ is μ-minimal.*
(2) *For any $f \in K[x]$ with χ-expansion $f = \sum_{0\le s} f_s\chi^s$, we have*

$$\mu(f) = \operatorname{Min}\{\mu(f_s\chi^s) \mid 0 \le s\}.$$

(3) *For any non-zero $f \in K[x]$ with χ-expansion $f = \sum_{0\le s} f_s\chi^s$, we have*

$$\chi \nmid_\mu f \iff \mu(f) = \mu(f_0).$$

Proof: The implication (1) $\Longrightarrow$ (2) follows from Lemma 2.2. In fact, if χ is μ-minimal, then $\chi \nmid_\mu f_s$ for all s, because $\deg(f_s) < \deg(\chi)$.

Let us deduce (3) from (2). Take a non-zero $f \in K[x]$ and write $f = f_0 + \chi q$ with $q \in K[x]$. By condition (2), we have $\mu(f) \le \mu(f_0)$.

If $\mu(f) < \mu(f_0)$, then $f \sim_\mu \chi q$, so $\chi \mid_\mu f$. Conversely, if $f \sim_\mu \chi g$ for some $g \in K[x]$, then $\mu(f - \chi g) > \mu(f)$. Since the χ-expansion of $f - \chi g$ has the same 0-th coefficient f_0, condition (2) shows that $\mu(f) < \mu(f - \chi g) \le \mu(f_0)$.

Finally, we show that (3) implies (1). If $\deg(f) < \deg(\chi)$, then the χ-expansion of f is $f = f_0$. By condition (3), $\chi \nmid_\mu f$. □

The property of μ-minimality is not stable under μ-equivalence. For instance, if χ is μ-minimal and $\mu(\chi) > 0$, then $\chi + \chi^2 \sim_\mu \chi$ and $\chi + \chi^2$ is not μ-minimal. However, for μ-equivalent polynomials of the same degree, μ-minimality is clearly preserved.

Definition 2.4. A *key polynomial* for μ is a monic polynomial in $K[x]$ which is μ-minimal and μ-irreducible.

The set of key polynomials for μ will be denoted by $\mathrm{KP}(\mu)$.

In the papers [**4**] and [**11**], different definitions for key polynomials are considered, which are somehow related, as shown by Mahboub in [**7**].

We are using the classical definition of MacLane–Vaquié given in [**11**].

Lemma 2.5. *Let $\chi \in \mathrm{KP}(\mu)$ and let $f \in K[x]$ a monic polynomial such that $\chi \mid_\mu f$ and $\deg(f) = \deg(\chi)$. Then $\chi \sim_\mu f$ and f is a key polynomial for μ too.*

Proof: The χ-expansion of f is $f = f_0 + \chi$, with $\deg(f_0) < \deg(\chi)$. Conditions (2) and (3) of Proposition 2.3 show that $\mu(f) < \mu(f_0)$. Hence, $H_\mu(f) = H_\mu(\chi)$ and f is μ-irreducible. Since $\deg(f) = \deg(\chi)$, f is μ-minimal too. □

Lemma 2.6. *Let $\chi \in \mathrm{KP}(\mu)$.*

(1) *For $a, b \in K[x]_{\deg(\chi)}$, let $ab = c + d\chi$ be the χ-expansion of ab. Then*
$$\mu(ab) = \mu(c) \le \mu(d\chi).$$
(2) *χ is irreducible in $K[x]$.*

Proof: For any $a, b \in K[x]_{\deg(\chi)}$, we have $\chi \nmid_\mu a$, $\chi \nmid_\mu b$ by the μ-minimality of χ. Hence, $\chi \nmid_\mu ab$ by the μ-irreducibility of χ. Thus, (1) follows from Proposition 2.3.

In particular, the equality $\chi = ab$ is impossible, so that χ is irreducible. □

Minimal expression of $H_\mu(f)$ in terms of χ-expansions.

Definition 2.7. For $\chi \in \mathrm{KP}(\mu)$ and a non-zero $f \in K[x]$, we let $s_\chi(f)$ be the largest integer s such that $\chi^s \mid_\mu f$.

Namely, $s_\chi(f)$ is the order with which the prime $H_\mu(\chi)$ divides $H_\mu(f)$ in $\mathcal{G}_\mu$.

Accordingly, by setting $s_\chi(0) := \infty$, we get

(4) $$s_\chi(fg) = s_\chi(f) + s_\chi(g), \quad \text{for all } f, g \in K[x].$$

Lemma 2.8. *Let $f \in K[x]$ with χ-expansion $f = \sum_{0 \le s} f_s \chi^s$. Denote*
$$I_\chi(f) = \{s \in \mathbb{Z}_{\ge 0} \mid \mu(f_s\chi^s) = \mu(f)\}.$$
Then $f \sim_\mu \sum_{s \in I_\chi(f)} f_s \chi^s$ and $s_\chi(f) = \mathrm{Min}(I_\chi(f))$.

Proof: Let $g = \sum_{s \in I_\chi(f)} f_s \chi^s$. By construction, $f - g = \sum_{s \notin I} f_s \chi^s$ has μ-value $\mu(f-g) > \mu(f)$. This proves $f \sim_\mu g$. In particular, $s_\chi(f) = s_\chi(g)$.

If $s_0 = \mathrm{Min}(I_\chi(f))$, we may write
$$g = \chi^{s_0}(f_{s_0} + \chi h),$$
for some $h \in K[x]$. By construction, $\mu(f_{s_0}) = \mu(f_{s_0} + \chi h) = \mu(g/\chi^{s_0})$.

By condition (3) of Proposition 2.3, $\chi \nmid_\mu (f_{s_0} + \chi h)$. Therefore, $s_\chi(g) = s_0$. □

Definition 2.9. For any $f \in K[x]$ we denote $s'_\chi(f) = \mathrm{Max}(I_\chi(f))$.

Denote for simpliciy $s = s_\chi(f)$ and $s' = s'_\chi(f)$. The homogeneous elements
$$\mathrm{irc}(f) := H_\mu(f_s) \quad \text{and} \quad \mathrm{lrc}(f) := H_\mu(f_{s'})$$
are the *initial residual coefficient* and the *leading residual coefficient* of f, respectively.

The next lemma shows that $s'_\chi(f)$ is an invariant of the μ-equivalence class of f.

Lemma 2.10. *If $f,g\in K[x]$ satisfy $f\sim_\mu g$, then $I_\chi(f)=I_\chi(g)$ and $f_s\sim_\mu g_s$ for all $s\in I_\chi(f)$. In particular,* $\operatorname{irc}(f)=\operatorname{irc}(g)$ *and* $\operatorname{lrc}(f)=\operatorname{lrc}(g)$.

Proof: Consider the χ-expansions $f=\sum_{0\le s}f_s\chi^s$, $g=\sum_{0\le s}g_s\chi^s$.

If $f\sim_\mu g$, then for any $s\ge 0$ we have

$$\mu(f)<\mu(f-g)\le\mu((f_s-g_s)\chi^s). \tag{5}$$

The condition $s\in I_\chi(f)$, $s\notin I_\chi(g)$ (or viceversa) contradicts (5). In fact,

$$\mu((f_s-g_s)\chi^s)=\mu(f),$$

because $\mu(f_s\chi^s)=\mu(f)$ and $\mu(g_s\chi^s)>\mu(g)=\mu(f)$.

Also, for all $s\in I_\chi(f)$, we have $\mu(f_s\chi^s)=\mu(f)$ and (5) shows that $f_s\chi^s\sim_\mu g_s\chi^s$. Thus, $f_s\sim_\mu g_s$. $\square$

We shall see in Section 3 that the equality

$$s'_\chi(fg)=s'_\chi(f)+s'_\chi(g),\quad \text{for all } f,g\in K[x],$$

holds if χ is a key polynomial of minimal degree.

Semivaluation attached to a key polynomial.

Lemma 2.11. *Let $\chi\in\mathrm{KP}(\mu)$. Consider the subset $\Gamma_v\subset\Gamma_{\deg(\chi)}\subset\Gamma_\mu$ defined as*

$$\Gamma_{\deg(\chi)}=\{\mu(a)\mid a\in K[x]_{\deg(\chi)},\ a\ne 0\}.$$

Then $\Gamma_{\deg(\chi)}$ is a subgroup of Γ_μ and $\langle\Gamma_{\deg(\chi)},\mu(\chi)\rangle=\Gamma_\mu$.

Proof: Since χ is μ-minimal, Proposition 2.3 shows that $\langle\Gamma_{\deg(\chi)},\mu(\chi)\rangle=\Gamma_\mu$.

By Lemma 2.6, $\Gamma_{\deg(\chi)}$ is closed under addition.

Take $\mu(a)\in\Gamma_{\deg(\chi)}$ for some non-zero $a\in K[x]_{\deg(\chi)}$. The polynomials a and χ are coprime, because χ is irreducible. Hence, they satisfy a Bézout identity

$$ab+\chi\, d=1,\quad \deg(b)<\deg(\chi),\quad \deg(d)<\deg(a)<\deg(\chi). \tag{6}$$

Since $ab=1-d\chi$ is the χ-expansion of ab, Lemma 2.6 shows that $\mu(ab)=\mu(1)=0$. Hence $-\mu(a)=\mu(b)\in\Gamma_{\deg(\chi)}$. This shows that $\Gamma_{\deg(\chi)}$ is a subgroup of Γ_μ. $\square$

Let $\chi\in\mathrm{KP}(\mu)$. Consider the prime ideal $\mathfrak{p}=\chi\, K[x]$ and the field $K_\chi=K[x]/\mathfrak{p}$. By the definition of $\Gamma_{\deg(\chi)}$, we get a well-defined onto mapping

$$v_\chi\colon K_\chi^*\twoheadrightarrow\Gamma_{\deg(\chi)},\quad v_\chi(f+\mathfrak{p})=\mu(f_0),\quad \text{for all } f\in K[x]\setminus\mathfrak{p},$$

where $f_0\in K[x]$ is the common 0-th coefficient of the χ-expansion of all polynomials in the class $f+\mathfrak{p}$.

This mapping v_χ depends on the pair μ, χ, and not only on χ.

Proposition 2.12. *The mapping v_χ is a valuation on K_χ extending v, with group of values $\Gamma_{\deg(\chi)}$.*

Proof: This mapping v_χ is a group homomorphism by Lemma 2.6. Finally,

$$v_\chi((f+g)+\mathfrak{p})=\mu(f_0+g_0)\geq \operatorname{Min}\{\mu(f_0),\mu(g_0)\}=\operatorname{Min}\{v_\chi(f+\mathfrak{p}),v_\chi(g+\mathfrak{p})\},$$

because $(f+g)_0=f_0+g_0$. Hence, v_χ is a valuation on K_χ. $\square$

Denote the maximal ideal, the valuation ring, and the residue class field of v_χ by

$$\mathfrak{m}_\chi\subset\mathcal{O}_\chi\subset K_\chi,\quad k_\chi=\mathcal{O}_\chi/\mathfrak{m}_\chi.$$

Let $\theta\in K_\chi=K[x]/\chi K[x]$ be the root of χ determined by the class of x.

With this notation we have $K_\chi=K(\theta)$, and

$$v_\chi(f(\theta))=\mu(f_0)=v_\chi(f_0(\theta)),\quad \text{for all } f\in K[x].$$

We abuse the language and denote still by v_χ the corresponding semivaluation

$$K[x]\twoheadrightarrow K_\chi\xrightarrow{v_\chi}\Gamma_{\deg(\chi)}\cup\{\infty\}$$

with support $\chi K[x]=v_\chi^{-1}(\infty)$.

According to the definition given in (2), the residual ideal $\mathcal{R}(\chi)$ of a key polynomial χ is a prime ideal in Δ. Let us show that it is actually a maximal ideal in Δ.

Proposition 2.13. *If χ is a key polynomial for μ, then $\mathcal{R}(\chi)$ is the kernel of the onto homomorphism*

$$\Delta\twoheadrightarrow k_\chi,\quad g+\mathcal{P}_0^+\longmapsto g(\theta)+\mathfrak{m}_\chi.$$

In particular, $\mathcal{R}(\chi)$ is a maximal ideal in Δ.

Proof: By Proposition 2.3, if $g\in\mathcal{P}_0$, we have $v_\chi(g(\theta))=\mu(g_0)\geq\mu(g)\geq 0$, so that $g(\theta)\in\mathcal{O}_\chi$. Thus, we get a well-defined ring homomorphism $\mathcal{P}_0\to k_\chi$.

This mapping is onto, because every element in k_χ can be represented as $h(\theta)+\mathfrak{m}_\chi$ for some $h\in K[x]_{\deg(\chi)}$ with $v_\chi(h(\theta))\geq 0$. Since $\mu(h)=v_\chi(h(\theta))\geq 0$, we see that h belongs to $\mathcal{P}_0$.

Finally, if $g\in\mathcal{P}_0^+$, then $v_\chi(g(\theta))\geq\mu(g)>0$; thus, the above homomorphism vanishes on $\mathcal{P}_0^+$ and it induces an onto mapping $\Delta\twoheadrightarrow k_\chi$.

The kernel of this mapping is the set of all elements $H_\mu(f)$ for $f\in K[x]$ satisfying $\mu(f_0)>\mu(f)=0$. By Proposition 2.3, this is equivalent to $\mu(f)=0$ and $\chi\mid_\mu f$. In other words, the kernel is $\mathcal{R}(\chi)$. $\square$

3. Key polynomials of minimal degree

In this section we study special properties of key polynomials of minimal degree. These objects are crucial for the resolution of the two main aims of the paper:

- Determine the structure of $\mathcal{G}_\mu$ as a k_v-algebra.
- Determine the structure of the quotient set $\mathrm{KP}(\mu)/\sim_\mu$.

Recall the embedding of graded k_v-algebras

$$\mathcal{G}_v \hookrightarrow \mathcal{G}_\mu.$$

Let $\xi \in \mathcal{G}_\mu$ be a non-zero homogeneous element which is algebraic over $\mathcal{G}_v$. Then ξ satisfies a homogeneous equation

$$\epsilon_0 + \epsilon_1\xi + \cdots + \epsilon_m\xi^m = 0,$$

with $\epsilon_0, \dots, \epsilon_m$ homogeneous elements in $\mathcal{G}_v$ such that $\deg(\epsilon_i\xi^i)$ is constant for all indices $0 \le i \le m$ for which $\epsilon_i \ne 0$.

Since all non-zero homogeneous elements in $\mathcal{G}_v$ are units, we have

(7) ξ algebraic over $\mathcal{G}_v \Longrightarrow \xi$ is a unit in $\mathcal{G}_\mu$, and ξ is integral over $\mathcal{G}_v$.

Lemma 3.1. *Let $\mathcal{G}_v \subset \mathcal{G}_v^{\mathrm{al}} \subset \mathcal{G}_\mu$ be the subalgebra generated by all homogeneous elements in $\mathcal{G}_\mu$ which are algebraic over $\mathcal{G}_v$.*

If a homogeneous element $\xi \in \mathcal{G}_\mu$ is algebraic over $\mathcal{G}_v^{\mathrm{al}}$, then it belongs to $\mathcal{G}_v^{\mathrm{al}}$.

Proof: Since all non-zero homogeneous elements in $\mathcal{G}_v^{\mathrm{al}}$ are units, the element ξ is integral over $\mathcal{G}_v^{\mathrm{al}}$. Hence, it is integral over $\mathcal{G}_v$, so that it belongs to $\mathcal{G}_v^{\mathrm{al}}$. □

Theorem 3.2. *Let $\phi \in K[x]$ be a monic polynomial of minimal degree n such that $H_\mu(\phi)$ is transcendental over $\mathcal{G}_v$. Then ϕ is a key polynomial for μ.*

Moreover, for $a, b \in K[x]_n$, let the ϕ-expansion of ab be

(8) $$ab = c + d\phi, \quad c, d \in K[x]_n.$$

Then $ab \sim_\mu c$.

Proof: Let us first show that ϕ is μ-minimal.

According to Proposition 2.3, the μ-minimality of ϕ is equivalent to

$$\mu(f) = \mathrm{Min}\{\mu(f_s\phi^s) \mid 0 \le s\},$$

for all $f \in K[x]$, being $f = \sum_{0\le s} f_s\phi^s$ its canonical ϕ-expansion.

For any given $f \in K[x]$, let $\delta = \operatorname{Min}\{\mu(f_s\phi^s) \mid 0 \le s\}$ and consider

$$I = \{0 \le s \mid \mu(f_s\phi^s) = \delta\}, \quad f_I = \sum_{s \in I} f_s\phi^s.$$

We have $\mu(f) \ge \delta$, and the desired equality $\mu(f) = \delta$ is equivalent to $\mu(f_I) = \delta$.

If $\#I = 1$, this is obvious. In the case $\#I > 1$, the equality $\mu(f_I) = \delta$ follows from the transcendence of $H_\mu(\phi)$ over $\mathcal{G}_v$. In fact,

$$\mu(f_I) > \delta \Longleftrightarrow \sum_{s \in I} H_\mu(f_s)H_\mu(\phi)^s = 0.$$

By the minimality of $n = \deg(\phi)$, all $H_\mu(f_s)$ are algebraic over $\mathcal{G}_v$. Hence, $\mu(f_I) > \delta$ would imply that $H_\mu(\phi)$ is algebraic over $\mathcal{G}_v^{\mathrm{al}}$, leading to $H_\mu(\phi)$ algebraic over $\mathcal{G}_v$ by Lemma 3.1. This ends the proof that ϕ is μ-minimal.

Let us now prove the last statement of the theorem.

For $a, b \in K[x]_n$ satisfying (8), Proposition 2.3 shows that

$$\mu(ab) = \operatorname{Min}\{\mu(c), \mu(d\phi)\},$$

because ϕ is μ-minimal. By equation (1), the inequality $\mu(c) \ge \mu(d\phi)$ implies

$$H_\mu(ab) = \begin{cases} H_\mu(d)H_\mu(\phi) & \text{if } \mu(c) > \mu(d\phi), \\ H_\mu(c) + H_\mu(d)H_\mu(\phi) & \text{if } \mu(c) = \mu(d\phi). \end{cases}$$

By the minimality of n, the elements $H_\mu(a)$, $H_\mu(b)$, $H_\mu(c)$, $H_\mu(d)$ are algebraic over $\mathcal{G}_v$. Hence, $H_\mu(\phi)$ would be algebraic over $\mathcal{G}_v^{\mathrm{al}}$, leading to $H_\mu(\phi)$ algebraic over $\mathcal{G}_v$ by Lemma 3.1. This contradicts our assumption on $H_\mu(\phi)$.

Therefore, we must have $\mu(c) < \mu(d\phi)$, leading to $ab \sim_\mu c$.

Finally, let us prove that ϕ is μ-irreducible.

Let $f, g \in K[x]$ be polynomials such that $\phi \nmid_\mu f$, $\phi \nmid_\mu g$. By Proposition 2.3,

$$\mu(f) = \mu(f_0), \quad \mu(g) = \mu(g_0),$$

where f_0, g_0 are the 0-th degree coefficients of the ϕ-expansions of f, g, respectively.

Let $f_0g_0 = c + d\phi$ be the ϕ-expansion of f_0g_0. As shown above, $f_0g_0 \sim_\mu c$, so that

$$\mu(fg) = \mu(f_0g_0) = \mu(c).$$

Since c is the 0-th coefficient of the ϕ-expansion of fg, the equality $\mu(fg) = \mu(c)$ shows that $\phi \nmid_\mu fg$, by Proposition 2.3. This ends the proof that ϕ is μ-irreducible. $\square$

Corollary 3.3. *Consider the following three natural numbers:*

$\ell =$ *minimal degree of* $f \in K[x]$ *such that* $H_\mu(f)$ *is not a unit in* $\mathcal{G}_\mu$,
$m =$ *minimal degree of a key polynomial for* μ,
$n =$ *minimal degree of* $f \in K[x]$ *such that*
$H_\mu(f)$ *is transcendental over* $\mathcal{G}_v$.

If one of these numbers exists, then all exist and $\ell = m = n$.

Proof: For any $f \in K[x]$ we have

$$f \in \mathrm{KP}(\mu) \Longrightarrow H_\mu(f) \notin \mathcal{G}_\mu^* \Longrightarrow H_\mu(f) \notin \mathcal{G}_v^{\mathrm{al}} \Longrightarrow \mathrm{KP}(\mu) \neq \emptyset,$$

where the last implication holds by Theorem 3.2. Hence, the conditions for the existence of these numbers are all equivalent:

$$\exists H_\mu(f) \notin \mathcal{G}_\mu^* \Longleftrightarrow \mathrm{KP}(\mu) \neq \emptyset \Longleftrightarrow \mathcal{G}_v^{\mathrm{al}} \subsetneq \mathcal{G}_\mu.$$

Suppose these conditions are satisfied. By Theorem 3.2, there exists a key polynomial ϕ of degree n. Also, for any $a \in K[x]_n$, the homogeneous element $H_\mu(a) \in \mathcal{G}_\mu$ is algebraic over $\mathcal{G}_v$, hence a unit in $\mathcal{G}_\mu$ by (7).

Since $H_\mu(\phi)$ is not a unit, this proves that $n = \ell$. Since there are no key polynomials in $K[x]_n$, this proves $n = m$ too. $\square$

Corollary 3.4. *If* ϕ *is a key polynomial of minimal degree, then for all* $f, g \in K[x]$ *we have*

$$\mathrm{lrc}(fg) = \mathrm{lrc}(f)\,\mathrm{lrc}(g), \quad s'_\phi(fg) = s'_\phi(f) + s'_\phi(g).$$

Proof: Consider the ϕ-expansions $f = \sum_{0 \le s} f_s \phi^s$, $g = \sum_{0 \le t} g_t \phi^t$.

We may write

$$fg = \sum_{0 \le j} b_j \phi^j, \quad b_j = \sum_{s+t=j} f_s g_t.$$

For each index j, there exist s_0, t_0 such that $s_0 + t_0 = j$ and

$$\mu(b_j \phi^j) \ge \mu(f_{s_0} \phi^{s_0} g_{t_0} \phi^{t_0}) \ge \mu(fg),$$

where the last inequality holds because $\mu(f_{s_0}\phi^{s_0}) \ge \mu(f)$ and $\mu(g_{t_0}\phi^{t_0}) \ge \mu(g)$. Hence,

$$fg \sim_\mu \sum_{j \in J} b_j \phi^j, \quad J = \{0 \le j \mid \mu(b_j \phi^j) = \mu(fg)\}.$$

For every $j \in J$, consider the set

$$I_j = \{(s,t) \mid s + t = j,\ s \in I_\phi(f),\ t \in I_\phi(g)\}.$$

Then, (1) and Theorem 3.2 show the existence of $c_{s,t} \in K[x]_n$ such that

$$H_\mu(b_j) = \sum_{(s,t)\in I_j} H_\mu(f_s g_t) = \sum_{(s,t)\in I_j} H_\mu(c_{s,t}) = H_\mu(c_j),$$

where $c_j = \sum_{(s,t)\in I_j} c_{s,t}$. Therefore, again by (1), we deduce that

$$fg \sim_\mu h, \quad h = \sum_{j\in J} c_j \phi^j.$$

Note that $J = I_\phi(h)$ by construction.

By Lemma 2.10, $I_\phi(fg) = I_\phi(h) = J$ and $\operatorname{lrc}(fg) = \operatorname{lrc}(h)$.

Thus, if $\ell = s'_\phi(f)$ and $m = s'_\phi(g)$, we need to show that $\operatorname{Max}(J) = \ell + m$ and $\operatorname{lrc}(h) = H_\mu(f_\ell)H_\mu(g_m)$.

If $j > \ell+m$, then for all pairs (s,t) with $s+t = j$ we have either $s > \ell$ or $t > m$. Thus, $\mu(f_s g_t \phi^j) = \mu(f_s\phi^s g_t \phi^t) > \mu(fg)$. Therefore, $j \notin J$.

For $j = \ell+m$, the same argument applies to all pairs (s,t) with $s+t = j$, except for the pair (s_ℓ, t_m), for which $\mu(f_{s_\ell} g_{t_m} \phi^j) = \mu(fg)$. Therefore, $j=\ell+m$ is the maximal index in J and $\operatorname{lrc}(h)=H_\mu(c_{\ell+m}) = H_\mu(b_{\ell+m}) = H_\mu(f_\ell g_m)$. □

Units and maximal subfield of Δ.

Proposition 3.5. *Let ϕ be a key polynomial of minimal degree n. For any non-zero $g \in K[x]$, with ϕ-expansion $g = \sum_{0\le s} g_s\phi^s$, the following conditions are equivalent:*

1. *$g \sim_\mu a$, for some $a \in K[x]_n$.*
2. *$H_\mu(g)$ is algebraic over $\mathcal{G}_v$.*
3. *$H_\mu(g)$ is a unit in $\mathcal{G}_\mu$.*
4. *$s_\phi(g) = s'_\phi(g) = 0$.*
5. *$g \sim_\mu g_0$.*

Proof: If $g \sim_\mu a$ for some $a \in K[x]_n$, then $H_\mu(g) = H_\mu(a)$ is algebraic over $\mathcal{G}_v$ by Corollary 3.3.

If $H_\mu(g)$ is algebraic over $\mathcal{G}_v$, then $H_\mu(g)$ is a unit by (7).

If $H_\mu(g)$ is a unit, there exists $f \in K[x]$ such that $fg \sim_\mu 1$. By Lemma 2.10, $I_\phi(fg) = I_\phi(1) = \{0\}$, so that $s_\phi(fg) = s'_\phi(fg) = 0$. By equation (4) and Corollary 3.4, we deduce that $s_\phi(g) = s'_\phi(g) = 0$.

This condition $s_\phi(g) = s'_\phi(g) = 0$ is equivalent to $I_\phi(g) = \{0\}$, which implies $g \sim_\mu g_0$ by Lemma 2.8.

This proves (1) $\Longrightarrow$ (2) $\Longrightarrow$ (3) $\Longrightarrow$ (4) $\Longrightarrow$ (5). Finally, (5) $\Longrightarrow$ (1) is obvious. □

As a consequence of this characterization of homogeneous algebraic elements, the subfield $\kappa \subset \Delta$ of all elements in Δ which are algebraic over $\mathcal{G}_v$ can be expressed as:

$$\kappa = \Delta^* \cup \{0\} = \{H_\mu(a) \mid a \in K[x]_n,\ \mu(a) = 0\} \cup \{0\}. \tag{9}$$

Since κ contains all units of Δ, it is the maximal subfield contained in Δ.

Since every $\xi \in \kappa$ is homogeneous of degree zero, any monic homogeneous algebraic equation of ξ over $\mathcal{G}_v$ has coefficients in the residue field k_v. Thus, κ coincides with the algebraic closure of k_v in Δ.

Proposition 3.6. *Let $\kappa \subset \Delta$ be the algebraic closure of k_v in Δ. For any key polynomial ϕ of minimal degree n, the composition of maps*

$$\kappa \hookrightarrow \Delta \twoheadrightarrow k_\phi$$

is an isomorphism.

Proof: The restriction to κ of the onto mapping $\Delta \to k_\phi$ described in Proposition 2.13 maps

$$H_\mu(a) \longmapsto a(\theta) + \mathfrak{m}_\phi, \quad \text{for all } a \in K[x]_n.$$

Since the images cover all k_ϕ, this mapping is an isomorphism between κ and k_ϕ. □

Upper bound for weighted values. Let us characterize μ-minimality of a polynomial $f \in K[x]$ in terms of its ϕ-expansion.

Proposition 3.7. *Let ϕ be a key polynomial of minimal degree n. For any $f \in K[x]$ with ϕ-expansion $f = \sum_{s=0}^{\ell} f_s\phi^s$, $f_\ell \neq 0$, the following conditions are equivalent:*

1. *f is μ-minimal.*
2. *$\deg(f) = s'_\phi(f)n$.*
3. *$\deg(f_\ell) = 0$ and $\mu(f) = \mu(f_\ell\phi^\ell)$.*

Proof: Since $\deg(f) = \deg(f_\ell) + \ell n$ and $s'_\phi(f) \le \ell$, condition (2) is equivalent to $\deg(f_\ell) = 0$ and $s'_\phi(f) = \ell$. Thus, (2) and (3) are equivalent.

Let us deduce (3) from (1). Since $\deg(f - f_\ell\phi^\ell) < \deg(f)$, the μ-minimality of f implies that $f - f_\ell\phi^\ell$ cannot be μ-equivalent to f. Hence $\mu(f_\ell\phi^\ell) = \mu(f)$.

In particular, $\ell = \operatorname{Max}(I_\phi(f))$. By Proposition 3.5, $H_\mu(f_s)$ is a unit for all $f_s \neq 0$. Take $b \in K[x]$ and $c_s \in K[x]_n$ such that $b\,f_\ell \sim_\mu 1$ and $bf_s \sim_\mu c_s$ for all $0 \le s < \ell$.

If we denote $c_\ell = 1$, Lemma 2.8 and equation (1) show that

$$bf \sim_\mu b \sum_{s\in I_\phi(f)} f_s\phi^s \Longrightarrow H_\mu(bf) = \sum_{s\in I_\phi(f)} H_\mu(bf_s\phi^s) = \sum_{s\in I_\phi(f)} H_\mu(c_s\phi^s).$$

Hence, $bf \sim_\mu g := \sum_{s\in I_\phi(f)} c_s\phi^s$. Since $f \mid_\mu g$ and f is μ-minimal,

$$\deg(f_\ell) + \ell n = \deg(f) \le \deg(g) = \ell n,$$

which implies $\deg(f_\ell) = 0$.

Finally, let us deduce (1) from (2). Take non-zero $g, h \in K[x]$ such that $g \sim_\mu fh$. By Lemma 2.10 and Corollary 3.4, $s'_\phi(g) = s'_\phi(fh) = s'_\phi(f) + s'_\phi(h)$, so that

$$\deg(f) = s'_\phi(f)\deg(\phi) \le s'_\phi(g)\deg(\phi) \le \deg(g).$$

Thus, f is μ-minimal. □

Corollary 3.8. *Suppose* KP$(\mu) \ne \emptyset$. *Take $f \in K[x]$ and m a positive integer. Then, f is μ-minimal if and only if f^m is μ-minimal.*

Proof: Let ϕ be a key polynomial of minimal degree. By Corollary 3.4, $s'_\phi(f^m) = ms'_\phi(f)$. Hence, condition (2) of Proposition 3.7 is equivalent for f and f^m. □

As another consequence of the criterion for μ-minimality, we may introduce an important numerical invariant of a valuation on $K[x]$ admitting key polynomials.

Theorem 3.9. *Let ϕ be a key polynomial of minimal degree for a valuation μ on $K[x]$. Then, for any monic non-constant $f \in K[x]$ we have*

$$\mu(f)/\deg(f) \le C(\mu) := \mu(\phi)/\deg(\phi),$$

and equality holds if and only if f is μ-minimal.

Proof: Since ϕ and f are monic, we may write

$$f^{\deg(\phi)} = \phi^{\deg(f)} + h, \quad \deg(h) < \deg(\phi)\deg(f).$$

By Proposition 2.3, $\mu(f^{\deg(\phi)}) \le \mu(\phi^{\deg(f)})$ or, equivalently, $\mu(f)/\deg(f) \le C(\mu)$.

By Proposition 3.7, equality holds if and only if $f^{\deg(\phi)}$ is μ-minimal, and this is equivalent to f being μ-minimal, by Corollary 3.8. □

4. Structure of Δ as a k_v-algebra

In this section we determine the structure of Δ as a k_v-algebra and we derive some specific information about the extension k_μ/k_v.

In Subsection 4.1 we deal with the case μ/v incommensurable. We show that $\Delta = k_\mu$. In this case all key polynomials have the same degree and they are all μ-equivalent.

In Subsection 4.2 we assume that μ admits no key polynomials. We have again $\Delta = k_\mu$. Also, we find several characterizations of the condition $\mathrm{KP}(\mu) = \emptyset$.

In Subsection 4.3 we deal with the case μ/v commensurable and $\mathrm{KP}(\mu) \neq \emptyset$, which corresponds to the classical situation in which μ is *residually transcendental.*

In this case Δ is isomorphic to a polynomial ring in one indeterminate with coefficients in κ, the algebraic closure of k_v in k_μ.

4.1. Case μ/v incommensurable. We recall that μ/v incommensurable means $\mathbb{Q}\Gamma_v \subsetneq \mathbb{Q}\Gamma_\mu$ or, equivalently, $\mathrm{rr}(\Gamma_\mu/\Gamma_v) > 0$.

Lemma 4.1. *Suppose μ/v is incommensurable. Let $\phi \in K[x]$ be a monic polynomial of minimal degree n satisfying $\mu(\phi) \notin \mathbb{Q}\Gamma_v$. Then, for all $f, g \in K[x]$ we have:*

1. *$f \sim_\mu a\phi^{s(f)}$, for some $a \in K[x]_n$ and a unique integer $s(f) \geq 0$.*
2. *ϕ is a key polynomial and $s(f) = s_\phi(f) = s'_\phi(f)$.*
3. *f is a μ-unit if and only if $s(f) = 0$.*
4. *$f \mid_\mu g$ if and only if $s(f) \leq s(g)$.*
5. *f is μ-irreducible if and only if $s(f) = 1$.*
6. *f is μ-minimal if and only if $\deg(f) = s(f)n$.*

Proof: Consider the ϕ-expansion $f = \sum_{0\leq s} f_s\phi^s$, with $f_s \in K[x]_n$ for all s. All monomials have different μ-value. In fact, an equality

$$\mu(f_s\phi^s) = \mu(f_t\phi^t) \Longrightarrow \mu(f_s) - \mu(f_t) = (t-s)\mu(\phi)$$

is possible only for $s = t$ because $\mu(\phi) \notin \mathbb{Q}\Gamma_v$ and $\mu(f_s)$, $\mu(f_t)$ belong to $\mathbb{Q}\Gamma_v$ by the minimality of n.

Hence, $f \sim_\mu a\phi^s$ for the monomial of least μ-value. The unicity of s follows from the same argument as above. This proves (1).

By Proposition 2.3, ϕ is μ-minimal. Let us show that ϕ is μ-irreducible.

Consider $f, g \in K[x]$ such that $\phi \nmid_\mu f$, $\phi \nmid_\mu g$. By property (1), $f \sim_\mu a$ and $g \sim_\mu b$ for some $a, b \in K[x]_n$. In particular,

$$\mu(fg) = \mu(ab) = \mu(a) + \mu(b) \in \mathbb{Q}\Gamma_v,$$

by the minimality of n. By property (1), $fg \sim_\mu c\phi^s$ for some $c \in K[x]_n$ and an integer $s \geq 0$. Since $\mu(\phi) \notin \mathbb{Q}\Gamma_v$, the condition $s\mu(\phi) = \mu(fg) - \mu(c) \in \mathbb{Q}\Gamma_v$ leads to $s = 0$, so that $fg \sim_\mu c$. Since ϕ is μ-minimal, we have $\phi \nmid_\mu fg$. Hence, ϕ is μ-irreducible.

Once we know that ϕ is a key polynomial, property (1) implies $I_\phi(f) = \{s(f)\}$ for all $f \in K[x]$. Thus, $s(f) = s_\phi(f) = s'_\phi(f)$. This proves (2).

If f is a μ-unit, then $\phi \nmid_\mu f$, so that $s(f) = 0$.

Conversely, if $s(f) = 0$, then $H_\mu(f) = H_\mu(a)$ for some $a \in K[x]_n$. Since the polynomials a and ϕ are coprime, they satisfy a Bézout identity

$$ab + \phi\, d = 1, \quad \deg(b) < n, \quad \deg(d) < \deg(a) < n.$$

Clearly, $ab = 1 - d\phi$ is the canonical ϕ-expansion of ab. Since we cannot have $ab \sim_\mu d\phi$, because $\mu(\phi) \notin \mathbb{Q}\Gamma_v$, we must have $ab \sim_\mu 1$. This proves (3).

The rest of statements follow easily from (1), (2), and (3). □

In particular, all results of the last section apply to our key polynomial ϕ, because it is a key polynomial of minimal degree.

Theorem 4.2. *Suppose μ/v incommensurable. Let $\phi \in K[x]$ be a monic polynomial of minimal degree n satisfying $\mu(\phi) \notin \mathbb{Q}\Gamma_v$. Then:*

(1) *ϕ is a key polynomial for μ.*
(2) *All key polynomials have degree n and are μ-equivalent to ϕ. More precisely,*

$$\mathrm{KP}(\mu) = \{\phi + a \mid a \in K[x]_n,\ \mu(a) > \mu(\phi)\}.$$

(3) *The natural inclusions determine equalities $\kappa = \Delta = k_\mu$.*
(4) *$\mathcal{R}(\phi) = 0$, and k_μ is a finite extension of k_v isomorphic to k_ϕ.*

Proof: By properties (2), (5), and (6) of Lemma 4.1, ϕ is a key polynomial for μ, all key polynomials have degree n, and are all μ-divisible by ϕ. By Lemma 2.5, they are μ-equivalent to ϕ. This proves (1) and (2).

Since $\mathrm{rr}(\Gamma_\mu/\Gamma_v) > 0$, the inequality in equation (3) shows that k_μ/k_v is an algebraic extension. Since $k_v \subset \Delta \subset k_\mu$, the ring Δ must be a field. In particular, $\kappa = \Delta$ by the remarks preceding Proposition 3.6.

Let us show that $\Delta = k_\mu$. An element in k_μ^* is of the form

$$(g/h) + \mathfrak{m}_\mu \in k_\mu^*,$$

with $g, h \in K[x]$ such that $\mu(g/h) = 0$.

By property (1) in Lemma 4.1, from $\mu(g) = \mu(h)$ we deduce that

$$g \sim_\mu a\phi^s, \quad h \sim_\mu b\phi^s,$$

for a certain integer $s \geq 0$ and polynomials $a, b \in K[x]_n$ such that $\mu(a) = \mu(b)$. Thus,

$$\frac{g}{h} + \mathfrak{m}_\mu = \frac{a\phi^s}{b\phi^s} + \mathfrak{m}_\mu = \frac{a}{b} + \mathfrak{m}_\mu.$$

By Lemma 4.1, $H_\mu(a)$ and $H_\mu(b)$ are units in $\mathcal{G}_\mu$, so that $H_\mu(a)/H_\mu(b) \in \Delta^*$ is mapped to $(g/h) + \mathfrak{m}_\mu$ under the embedding $\Delta \hookrightarrow k_\mu$. This proves (3).

By Proposition 2.13, $\mathcal{R}(\phi)$ is the kernel of the onto mapping $\Delta \twoheadrightarrow k_\phi$. Since Δ is a field, $\mathcal{R}(\phi) = 0$ and this mapping is an isomorphism.

This ends the proof of (4), because k_ϕ/k_v is a finite extension. $\square$

4.2. Valuations not admitting key polynomials.

Theorem 4.3. *If* $\mathrm{KP}(\mu) = \emptyset$, *the canonical embedding* $\Delta \hookrightarrow k_\mu$ *is an isomorphism.*

Proof: An element in k_μ^* is of the form

$$(f/g) + \mathfrak{m}_\mu \in k_\mu^*, \quad f, g \in K[x], \quad \mu(f/g) = 0.$$

If $\mathrm{KP}(\mu) = \emptyset$, then Corollary 3.3 shows that $H_\mu(f)$ and $H_\mu(g)$ are units in $\mathcal{G}_\mu$. Hence, $H_\mu(f)H_\mu(g)^{-1}$ is an element in Δ whose image in k_μ is $(f/g) + \mathfrak{m}_\mu$. $\square$

Theorem 4.4. *Let* μ *be a valuation on* $K[x]$ *extending* v. *The following conditions are equivalent:*

1. $\mathrm{KP}(\mu) = \emptyset$.
2. $\mathcal{G}_\mu$ *is algebraic over* $\mathcal{G}_v$.
3. *Every non-zero homogeneous element in* $\mathcal{G}_\mu$ *is a unit.*
4. μ/v *is commensurable and* k_μ/k_v *is algebraic.*
5. *The set of weighted values*

$$W = \{\mu(f)/\deg(f) \mid f \in K[x] \setminus K \text{ monic}\}$$

does not contain a maximal element.

Proof: By Corollary 3.3, conditions (1), (2), and (3) are equivalent.

Let us show that (1) implies (4). If $\mathrm{KP}(\mu) = \emptyset$, then μ/v is commensurable by Theorem 4.2, and k_μ/k_v is algebraic by Theorems 3.2 and 4.3.

Let us now deduce (5) from (4). Take ϕ an arbitrary monic polynomial in $K[x] \setminus K$. Let us show that $\mu(\phi)/\deg(\phi) \in W$ is not an upper bound for this set.

Since $\mathbb{Q}\Gamma_v = \mathbb{Q}\Gamma_\mu$, there exists a positive integer e such that $e\mu(\phi) \in \Gamma_v$. Thus, there exists $a \in K^*$ such that $\mu(a\phi^e) = 0$, so that $H_\mu(a\phi^e) \in k_\mu^*$.

By hypothesis, this element is algebraic over k_v. Hence $H_\mu(\phi)$ is algebraic over $\mathcal{G}_v$. As mentioned in (7), $H_\mu(\phi)$ is integral over $\mathcal{G}_v$. Consider a homogeneous equation

$$\epsilon_0 + \epsilon_1 H_\mu(\phi) + \cdots + \epsilon_{m-1} H_\mu(\phi)^{m-1} + H_\mu(\phi)^m = 0, \tag{10}$$

with $\epsilon_0, \dots, \epsilon_m$ homogeneous elements in $\mathcal{G}_v$ such that $\deg(\epsilon_i H_\mu(\phi)^i) = m\mu(\phi)$ for all indices $0 \le i < m$ for which $\epsilon_i \ne 0$.

By choosing $a_i \in K$ with $H_\mu(a_i) = \epsilon_i$ for all i, equation (10) is equivalent to

$$\mu(a_0 + a_1\phi + \cdots + a_{m-1}\phi^{m-1} + \phi^m) > \mu(\phi^m) = m\mu(\phi).$$

Hence, this monic polynomial $f = a_0 + a_1\phi + \cdots + a_{m-1}\phi^{m-1} + \phi^m$ has a larger weighted value

$$\mu(f)/\deg(f) > \mu(\phi^m)/\deg(f) = \mu(\phi)/\deg(\phi).$$

Hence the set W contains no maximal element.

Finally, the implication (5) $\Longrightarrow$ (1) follows from Theorem 3.9. $\square$

4.3. Case μ/v commensurable and $\mathrm{KP}(\mu) \ne \emptyset$.

Theorem 4.5. *Suppose μ/v commensurable and $\mathrm{KP}(\mu) \ne \emptyset$. The canonical embedding $\Delta \hookrightarrow k_\mu$ induces an isomorphism between the field of fractions of Δ and k_μ.*

Proof: Let $\chi \in K[x]$ be an arbitrary key polynomial for μ.

We must show that the induced morphism $\mathrm{Frac}(\Delta) \to k_\mu$ is onto.

An element in k_μ^* is of the form

$$(f/g) + \mathfrak{m}_\mu \in k_\mu^*, \quad f, g \in K[x], \quad \mu(f/g) = 0.$$

Set $\alpha = \mu(f) = \mu(g) \in \Gamma_\mu$. By Lemma 2.11, $\Gamma_\mu = \langle \Gamma_{\deg(\chi)}, \mu(\chi) \rangle$. Hence we may write

$$-\alpha = \beta + s\mu(\chi), \quad \beta \in \Gamma_{\deg(\chi)}, \quad s \in \mathbb{Z}.$$

Since $\mu(\chi) \in \mathbb{Q}\Gamma_{\deg(\chi)}$, we may assume that $0 \le s < e$ for some positive integer e. Take $a \in K[x]_{\deg(\chi)}$ such that $\mu(a) = \beta$. Then, the polynomial $h = a\chi^s$ satisfies $\mu(h) = -\alpha$.

Thus, $H_\mu(hf)$ and $H_\mu(hg)$ belong to Δ and the fraction $H_\mu(hf)/H_\mu(hg)$ is mapped to $(f/g) + \mathfrak{m}_\mu$ by the morphism $\mathrm{Frac}(\Delta) \to k_\mu$. $\square$

Theorem 4.6. *Suppose μ/v commensurable. Let ϕ be a key polynomial of minimal degree n and let e be a minimal positive integer such that $e\mu(\phi)\in\Gamma_{\deg(\phi)}$.*

Take $u\in K[x]_n$ such that $\mu(u\phi^e)=0$. Then $\xi=H_\mu(u\phi^e)\in\Delta$ is transcendental over k_v and $\Delta=\kappa[\xi]$.

Proof: The element ξ is not a unit, because it is divisible by the prime element $H_\mu(\phi)$. By (7), ξ is transcendental over k_v.

Consider $H_\mu(g)\in\Delta$, for some $g\in K[x]$ with $\mu(g)=0$. Let $g=\sum_{0\le s} g_s\phi^s$ be the ϕ-expansion of g.

Let $I=I_\phi(g)$ be the set of indices s such that $\mu(g_s\phi^s)=0$. For each $s\in I$, the equality $s\mu(\phi)=-\mu(g_s)\in\Gamma_{\deg(\phi)}$ implies that $s=ej_s$ for some integer $j_s\ge 0$.

By Lemma 2.8 and equation (1),

$$g\sim_\mu \sum\nolimits_{s\in I} g_s\phi^s,\quad H_\mu(g)=\sum\nolimits_{s\in I} H_\mu(g_s\phi^s). \tag{11}$$

Proposition 3.5 shows that $H_\mu(u)$ is a unit. For each $s\in I$, there exists $c_s\in K[x]_n$ such that $H_\mu(c_s)=H_\mu(g_s)H_\mu(u)^{-j_s}$. Hence,

$$H_\mu(g_s\phi^s)=H_\mu(g_s)H_\mu(u)^{-j_s}H_\mu(u)^{j_s}H_\mu(\phi^s)=H_\mu(c_s)\xi^{j_s}\in\kappa[\xi].$$

Hence $H_\mu(g)\in\kappa[\xi]$. This proves that $\Delta=\kappa[\xi]$. □

As a consequence of Theorems 4.2, 4.3, 4.4, 4.5, and 4.6, we obtain the following computation of the residue class field k_μ.

Corollary 4.7. *If* $\mathrm{KP}(\mu)=\emptyset$, *then $\kappa=\Delta=k_\mu$ is an algebraic extension of k_v.*

If μ/v is incommensurable, then $\kappa=\Delta=k_\mu$ is a finite extension of k_v.

If μ/v is commensurable and $\mathrm{KP}(\mu)\ne\emptyset$, *then $k_\mu\simeq\kappa(y)$, where y is an indeterminate.*

5. Residual polynomial operator

Suppose μ/v commensurable and $\mathrm{KP}(\mu)\ne\emptyset$.

Let us fix a key polynomial $\phi\in\mathrm{KP}(\mu)$ of minimal degree n. Let $q=H_\mu(\phi)$ be the corresponding prime element of $\mathcal{G}_\mu$.

Having in mind the description of the set $\mathrm{KP}(\mu)$ in Section 6, we introduce a *residual polynomial operator*

$$R\colon K[x]\longrightarrow\kappa[y],$$

which provides a decomposition of any homogeneous element $H_\mu(f) \in \mathcal{G}_\mu$ into a product of a unit, a power of q, and the degree-zero element $R(f)(\xi) \in \Delta = \kappa[\xi]$ (Theorem 5.3). As a consequence, the operator R provides a computation of the residual ideal operator (Theorem 5.7).

In Lemma 2.11 we proved that $\Gamma_\mu = \langle \Gamma_n, \mu(\phi)\rangle$, where Γ_n is the subgroup

$$\Gamma_n = \{\mu(a) \mid a \in K[x]_n,\ a \neq 0\} \subset \Gamma_\mu.$$

Let e be a minimal positive integer with $e\mu(\phi) \in \Gamma_n$. By Theorem 3.9, all key polynomials χ of degree n have the same μ-value $\mu(\chi) = \mu(\phi)$. Thus, this positive integer e does not depend on the choice of ϕ.

It will be called the *relative ramification index of* μ.

We fix a polynomial $u \in K[x]_n$ such that $\mu(u\phi^e) = 0$ and consider

$$\xi = H_\mu(u\phi^e) = H_\mu(u)q^e \in \Delta.$$

By Theorem 4.6, ξ is transcendental over k_v and $\Delta = \kappa[\xi]$.

Throughout this section, for any polynomial $f \in K[x]$ we denote

$$s(f) := s_\phi(f), \quad s'(f) := s'_\phi(f), \quad I(f) := I_\phi(f).$$

For $s \in I(f)$, the condition $\mu(f_s\phi^s) = \mu(f)$ implies that s belongs to a fixed class modulo e. In fact, for any pair $s, t \in I(f)$,

$$\mu(f_s\phi^s) = \mu(f_t\phi^t) \Longrightarrow (t-s)\mu(\phi) = \mu(f_s) - \mu(f_t) \in \Gamma_n \Longrightarrow t \equiv s \pmod{e}.$$

Hence $I(f) \subset \{s_0, s_1, \dots, s_d\}$, where

$$s_0 = s(f) = \operatorname{Min}(I(f)), \quad s_j = s_0 + je,\ 0 \le j \le d, \quad s_d = s'(f) = \operatorname{Max}(I(f)).$$

By Lemma 2.8, we may write

$$f \sim_\mu \sum_{s\in I(f)} f_s\phi^s \sim_\mu \phi^{s_0}(f_{s_0} + \cdots + f_{s_j}\phi^{je} + \cdots + f_{s_d}\phi^{de}), \tag{12}$$

taking into account only the monomials for which $s_j \in I(f)$.

Definition 5.1. Consider the residual polynomial operator

$$R := R_{\phi,u} \colon K[x] \longrightarrow \kappa[y], \quad R(f) = \zeta_0 + \zeta_1 y + \cdots + \zeta_{d-1}y^{d-1} + y^d,$$

for $f \neq 0$, where the coefficients $\zeta_j \in \kappa$ are defined by

$$\zeta_j = \begin{cases} H_\mu(f_{s_d})^{-1}H_\mu(u)^{d-j}H_\mu(f_{s_j}) & \text{if } s_j \in I(f), \\ 0 & \text{if } s_j \notin I(f). \end{cases} \tag{13}$$

Also, we define $R(0) = 0$.

For $s_j \in I(f)$, we have

$$\mu(f_{s_j}\phi^{je}) = \mu(f_{s_d}\phi^{de}) = \mu(f/\phi^{s_0}),$$

so that $\mu(f_{s_j}) = \mu(f_{s_d}) + (d-j)e\mu(\phi) = \mu(f_{s_d}) - (d-j)\mu(u)$.

Since the three homogeneous elements $H_\mu(f_{s_j})$, $H_\mu(f_{s_d})$, and $H_\mu(u)$ are units in $\mathcal{G}_\mu$, we deduce that $\zeta_j \in \Delta^* = \kappa^*$ for $s_j \in I(f)$.

Thus, the monic residual polynomial $R(f)$ is well defined, and it has degree

$$d(f) := \deg(R(f)) = d = (s'(f) - s(f))/e. \tag{14}$$

Note that $\zeta_0 \neq 0$, because $s_0 \in I(f)$. Thus, $R(f)(0) \neq 0$.

Example. For any monomial $f = a\phi^s$ with $a \in K[x]_n$, we have $R(f) = 1$.

Definition 5.2. With the above notation, the *normalized leading residual coefficient*

$$\mathrm{nlc}(f) = H_\mu(f_{s_d})H_\mu(u)^{-d} = \mathrm{lrc}(f)H_\mu(u)^{-d} \in \mathcal{G}_\mu^*$$

is a homogeneous unit in $\mathcal{G}_\mu$ of degree $\mu(f) - s(f)\mu(\phi)$.

For any $g \in K[x]$, from (4) and Corollary 3.4, we deduce that

$$d(fg) = d(f) + d(g), \quad \mathrm{nlc}(fg) = \mathrm{nlc}(f)\,\mathrm{nlc}(g), \quad \text{for all } f, g \in K[x].$$

By definition, for any $s_j \in I(f)$ we have

$$\mathrm{nlc}(f)\,\zeta_j\,\xi^j = H_\mu(f_{s_j})H_\mu(\phi^{je}).$$

Thus, (12) leads to the following identity, which is the "raison d'être" of $R(f)$.

Theorem 5.3. *For any $f \in K[x]$, we have $H_\mu(f) = \mathrm{nlc}(f)\, q^{s(f)} R(f)(\xi)$.* □

Note that $\mathrm{nlc}(f)$ is a unit, $q^{s(f)}$ the power of a prime element, and $R(f)(\xi) \in \Delta$.

Let us derive from Theorem 5.3 some basic properties of the residual polynomials.

Corollary 5.4. *For all $f, g \in K[x]$, we have $R(fg) = R(f)R(g)$.*

Proof: Since the functions H_μ and nlc are multiplicative, Theorem 5.3 shows that

$$R(fg)(\xi) = R(f)(\xi)R(g)(\xi).$$

By Theorem 4.6, we deduce that $R(fg) = R(f)R(g)$. □

Corollary 5.5. *For all $f, g \in K[x]$,*

$$f \sim_\mu g \iff I(f) = I(g),\ \mathrm{nlc}(f) = \mathrm{nlc}(g),\ \textit{and}\ R(f) = R(g);$$
$$f \mid_\mu g \iff s(f) \le s(g)\ \textit{and}\ R(f) \mid R(g)\ \textit{in}\ \kappa[y].$$

Proof: If $f \sim_\mu g$, then $I(f) = I(g)$ and $\mathrm{nlc}(f) = \mathrm{nlc}(g)$ by Lemma 2.10. Thus, $R(f)(\xi) = R(g)(\xi)$ by Theorem 5.3, leading to $R(f) = R(g)$ by Theorem 4.6.

Conversely, $I(f)=I(g)$ implies $s(f) = \mathrm{Min}(I(f)) = \mathrm{Min}(I(g)) = s(g)$. Thus, $H_\mu(f) = H_\mu(g)$ follows from Theorem 5.3.

If $f \mid_\mu g$, then $fh \sim_\mu g$ for some $h \in K[x]$. By the first item and Corollary 5.4, we get $R(g) = R(fh) = R(f)R(h)$, so that $R(f) \mid R(g)$.

Also, since $s(g) = s(f) + s(h)$, we deduce that $s(f) \le s(g)$.

Conversely, $s(f) \le s(g)$ and $R(f) \mid R(g)$ imply $H_\mu(f) \mid H_\mu(g)$ by Theorem 5.3, having in mind that $\mathrm{nlc}(f)$, $\mathrm{nlc}(g)$ are units in $\mathcal{G}_\mu$. ☐

Corollary 5.6. *Let $s \in \mathbb{Z}_{\ge 0}$, $\zeta \in \kappa^*$, and $\psi \in \kappa[y]$ a monic polynomial with $\psi(0) \ne 0$. Then there exists a polynomial $f \in K[x]$ such that*

$$s(f) = s, \quad \mathrm{nlc}(f) = \zeta, \quad R(f) = \psi.$$

Proof: Let $\psi = \zeta_0 + \zeta_1 y + \cdots + \zeta_{d-1} y^{d-1} + \zeta_d y^d$, with $\zeta_0, \dots, \zeta_{d-1} \in \kappa$ and $\zeta_d = 1$. Let I be the set of indices $0 \le j \le d$ with $\zeta_j \ne 0$.

By (9), for each $j \in I$ we may take $f_j \in K[x]_n$ such that $H_\mu(f_j) = \zeta H_\mu(u)^j \zeta_j$. Then $f = \phi^s(f_0 + \cdots + f_j \phi^{je} + \cdots + f_d \phi^{de})$ satisfies all our requirements. ☐

Theorem 5.7. *For any non-zero $f \in K[x]$,*

$$\mathcal{R}(f) = \xi^{\lceil s(f)/e \rceil} R(f)(\xi)\Delta.$$

Proof: By definition, an element in the ideal $\mathcal{R}(f)$ is of the form $H_\mu(h)$ for some $h \in K[x]$ such that $f \mid_\mu h$ and $\mu(h) = 0$.

The condition $\mu(h) = 0$ implies $e \mid s(h)$. By Theorem 5.3,

$$H_\mu(h) = \xi^{s(h)/e} H_\mu(u)^{-s(h)/e}\, \mathrm{nlc}(h)\, R(h)(\xi).$$

On the other hand, Corollary 5.5 shows that $s(f) \le s(h)$ and $R(f) \mid R(h)$. Therefore, $H_\mu(h)$ belongs to the ideal $\xi^{\lceil s(f)/e \rceil} R(f)(\xi)\Delta$.

Conversely, if $m = \lceil s(f)/e \rceil$, then Theorem 5.3 shows that

$$\begin{aligned}\xi^m R(f)(\xi) &= q^{me} H_\mu(u)^m R(f)(\xi) \\ &= H_\mu(f) q^{me - s(f)}\, \mathrm{nlc}(f)^{-1} H_\mu(u)^m \in \mathcal{R}(f),\end{aligned}$$

because $me \ge s(f)$ and $\mathrm{nlc}(f)$, $H_\mu(u)$ are units. ☐

5.1. Dependence of R on the choice of u. Let $u^* \in K[x]_n$ be another choice of a polynomial such that $\mu(u^*\phi^e) = 0$ and denote

$$\xi^* = H_\mu(u^*\phi^e) = H_\mu(u^*)q^e \in \Delta.$$

Since $\mu(u) = \mu(u^*)$ and $H_\mu(u)$, $H_\mu(u^*)$ are units in $\mathcal{G}_\mu$, we have

$$\xi^* = \sigma^{-1}\xi, \quad \text{where } \sigma = H_\mu(u)H_\mu(u^*)^{-1} \in \Delta^* = \kappa^*.$$

Let R^* be the residual polynomial operator associated with this choice of u^*.

For any $f \in K[x]$, suppose that $R^*(f) = \zeta_0^* + \zeta_1^* y + \cdots + \zeta_{d-1}^* y^{d-1} + y^d$. By the very definition (13) of the residual coefficients,

$$\zeta_j^* = \sigma^{d-j}\zeta_j, \quad 1 \le j \le d.$$

We deduce the following relationship between R and R^*:

$$R^*(f)(y) = \sigma^d R(f)(\sigma^{-1}y), \quad \text{for all } f \in K[x].$$

5.2. Dependence of R on the choice of ϕ. Let ϕ_* be another key polynomial with minimal degree n and denote $q_* = H_\mu(\phi_*)$.

By Theorem 3.9, $\mu(\phi_*) = \mu(\phi)$, so that

$$\phi_* = \phi + a, \quad a \in K[x]_n, \quad \mu(a) \ge \mu(\phi).$$

In particular, $\mu(u\phi_*^e) = 0$ and we may consider

$$\xi_* = H_\mu(u\phi_*^e) = H_\mu(u)q_*^e \in \Delta$$

as a transcendental generator of Δ as a κ-algebra.

Let R_* be the residual polynomial operator associated with this choice of ϕ_*.

Proposition 5.8. *Let ϕ_* be another key polynomial with minimal degree, and denote with a subindex $(\ )_*$ all objects depending on ϕ_*.*

1. *If $\phi_* \sim_\mu \phi$, then $q_* = q$, $\xi_* = \xi$, and $R_* = R$.*
2. *If $\phi_* \not\sim_\mu \phi$, then $e = 1$, $q_* = q + H_\mu(a)$, and $\xi_* = \xi + \tau$, where $\tau = H_\mu(ua) \in \kappa^*$. In this case, for any $f \in K[x]$ we have*

$$y^{s(f)}R(f)(y) = (y+\tau)^{s_*(f)}R_*(f)(y+\tau). \tag{15}$$

In particular, $s_*(f) = \operatorname{ord}_{y+\tau}(R(f))$ and $s(f) + d(f) = s_*(f) + d_*(f)$.

Proof: Suppose $\phi_* \sim_\mu \phi$. By definition,

$$q_* = H_\mu(\phi_*) = H_\mu(\phi) = q, \quad \xi_* = H_\mu(u\phi_*) = H_\mu(u\phi) = \xi.$$

Let $f \in K[x]$. By equation (1), we can replace ϕ with ϕ_* in equation (12) to obtain

$$f \sim_\mu \sum\nolimits_{s\in I(f)} f_s\phi_*^s \sim_\mu \phi_*^{s_0}(f_{s_0} + \cdots + f_{s_j}\phi_*^{je} + \cdots + f_{s_d}\phi_*^{de}).$$

Hence, (13) leads to the same residual coefficients, so that $R(f) = R_*(f)$.

Suppose $\phi_* \not\sim_\mu \phi$, that is, $\mu(a) = \mu(\phi)$. Then $e = 1$, $H_\mu(\phi_*) = H_\mu(\phi) + H_\mu(a)$, and

$$\xi_* = H_\mu(u\phi_*) = H_\mu(u)H_\mu(\phi) + H_\mu(u)H_\mu(a) = \xi + \tau.$$

Finally, let $f \in K[x]$, and denote $s = s(f)$, $s_* = s_*(f)$. By Theorem 5.3,

$$q^s \operatorname{nlc}(f)R(f)(\xi) = H_\mu(f) = q_*^{s_*} \operatorname{nlc}_*(f)R_*(f)(\xi_*).$$

Since $q = H_\mu(u)^{-1}\xi$ and $q_* = H_\mu(u)^{-1}\xi_* = H_\mu(u)^{-1}(\xi + \tau)$, we deduce

$$\xi^s R(f)(\xi) = \sigma(\xi + \tau)^{s_*} R^*(f)(\xi + \tau),$$

where $\sigma = H_\mu(u)^{s-s_*} \operatorname{nlc}_*(f) \operatorname{nlc}(f)^{-1} \in \kappa^*$, because

$$\deg(\operatorname{nlc}(f)H_\mu(u)^{-s}) = 0 = \deg(\operatorname{nlc}_*(f)H_\mu(u)^{-s_*}).$$

By Theorem 4.6, this implies $y^s R(f)(y) = \sigma(y + \tau)^{s_*} R^*(f)(y + \tau)$. Since $R(f)$ and $R_*(f)$ are monic polynomials, we have necessarily $\sigma = 1$. This proves (15). ☐

6. Key polynomials and unique factorization in $\mathcal{G}_\mu$

We keep the assumptions that μ/v is commensurable and $\mathrm{KP}(\mu) \neq \emptyset$. We keep dealing with a fixed key polynomial ϕ of minimal degree n, and we denote $q = H_\mu(\phi)$, e least positive integer with $e\mu(\phi) \in \Gamma_n$, $u \in K[x]_n$ such that $\mu(u\phi^e) = 0$, and $\xi = H_\mu(u\phi^e)$. Also, we denote

$$s(f) := s_\phi(f), \quad s'(f) := s'_\phi(f), \quad R(f) := R_{\phi,u}(f), \quad \text{for all } f \in K[x].$$

6.1. Homogeneous prime elements. By Theorem 4.6, the prime elements in Δ are those of the form $\psi(\xi)$ for $\psi \in \kappa[y]$ an irreducible polynomial.

An element in Δ, which is a prime in $\mathcal{G}_\mu$, is a prime in Δ, but the converse is not true. Let us now discuss what primes in Δ remain prime in $\mathcal{G}_\mu$.

Lemma 6.1. *Let $\psi \in \kappa[y]$ be a monic irreducible polynomial.*

1. *If $\psi \neq y$, then $\psi(\xi)$ is a prime element in $\mathcal{G}_\mu$.*
2. *If $\psi = y$, then ξ is a prime element in $\mathcal{G}_\mu$ if and only if $e = 1$.*

Proof: Suppose $\psi \neq y$. Since ψ is irreducible, we have $\psi(0) \neq 0$. By Corollary 5.6, there exists $f \in K[x]$ such that $s(f) = 0$, $\operatorname{nlc}(f) = 1$, and $R(f) = \psi$. By Theorem 5.3, $H_\mu(f) = \psi(\xi)$.

Suppose $\psi(\xi) = H_\mu(f)$ divides the product of two homogeneous elements in $\mathcal{G}_\mu$, say $f \mid_\mu gh$ for some $g, h \in K[x]$.

By Corollaries 5.5 and 5.4, $\psi = R(f)$ divides $R(gh) = R(g)R(h)$. Being ψ irreducible, it divides either $R(g)$ or $R(h)$, and this leads to $\psi(\xi)$ dividing either $H_\mu(g)$ or $H_\mu(h)$ in $\mathcal{G}_\mu$, by Theorem 5.3.

The element ξ is associate to q^e in $\mathcal{G}_\mu$. Since q is a prime element, its e-th power is a prime if and only if $e = 1$. □

Besides these prime elements belonging to Δ, we know that q is another prime element in $\mathcal{G}_\mu$, of degree $\mu(\phi)$.

The next result shows that there are no other homogeneous prime elements in $\mathcal{G}_\mu$ up to multiplication by units.

Proposition 6.2. *A polynomial $f \in K[x]$ is μ-irreducible if and only if one of the two following conditions is satisfied:*

(1) $s(f) = s'(f) = 1$.

(2) $s(f) = 0$ *and $R(f)$ is irreducible in $\kappa[y]$.*

In the first case, $H_\mu(f)$ is associate to q; in the second case, to $R(f)(\xi)$.

Proof: By Theorem 5.3, $H_\mu(f) = q^{s(f)} \operatorname{nlc}(f)\, R(f)(\xi)$. Since $\operatorname{nlc}(f)$ is a unit and q is a prime, $H_\mu(f)$ is a prime if and only if one of the two following conditions is satisfied:

(i) $s(f) = 1$ and $R(f)(\xi)$ is a unit.

(ii) $s(f) = 0$ and $R(f)(\xi)$ is a prime in $\mathcal{G}_\mu$.

The homogeneous element of degree zero $R(f)(\xi)$ is a unit in $\mathcal{G}_\mu$ if and only if it is a unit in Δ. By Theorem 4.6, this is equivalent to $\deg(R(f)) = 0$, which in turn is equivalent to $s(f) = s'(f)$ by (14). Thus, (i) is equivalent to (1), and $H_\mu(f)$ is associate to q in this case.

Since $R(f) \neq y$, (ii) is equivalent to (2) by Lemma 6.1. Clearly, $H_\mu(f)$ is associate to $R(f)(\xi)$ in this case. □

Putting together this characterization of μ-irreducibility with the characterization of μ-minimality from Proposition 3.7, we get the following characterization of key polynomials.

Proposition 6.3. *Let ϕ be a key polynomial for μ, of minimal degree n.*

A monic $\chi \in K[x]$ is a key polynomial for μ if and only if one of the two following conditions is satisfied:

(1) $\deg(\chi) = \deg(\phi)$ *and $\chi \sim_\mu \phi$.*

(2) $s(\chi) = 0$, $\deg(\chi) = en\deg(R(\chi))$, *and $R(\chi)$ is irreducible in $\kappa[y]$.*

In the first case, $\mathcal{R}(\chi) = \xi\Delta$; in the second case, $\mathcal{R}(\chi) = R(\chi)(\xi)\Delta$.

Proof: If χ satisfies (1), then χ is a key polynomial by Lemma 2.5.
Also, $\mathcal{R}(\chi)=\mathcal{R}(\phi)=\xi\Delta$ by Theorem 5.7, since $s(\phi)=1$ and $R(\phi)=1$.

If χ satisfies (2), then $\deg(R(\chi))=s'(\chi)/e$ by (14), so that $\deg(\chi)=s'(\chi)n$, and χ is μ-minimal by Proposition 3.7. Also, χ is μ-irreducible by Proposition 6.2.

Thus, χ is a key polynomial, and $\mathcal{R}(\chi)=R(\chi)(\xi)\Delta$ by Theorem 5.7.

Conversely, suppose χ is a key polynomial for μ. Since χ is μ-minimal, $\deg(\chi)=s'(\chi)n$, by Proposition 3.7.

Since χ is μ-irreducible, it satisfies one of the conditions of Proposition 6.2.

If $s(\chi)=s'(\chi)=1$, we get $\deg(\chi)=n$ and $\phi\mid_\mu\chi$. Thus, $\chi\sim_\mu\phi$ by Lemma 2.5, and hence ϕ satisfies (1).

If $s(\chi)=0$ and $R(\chi)$ is irreducible in $\kappa[y]$, then $\deg(R(\chi))=s'(\chi)/e$ by (14). Thus, $\deg(\chi)=s'(\chi)n=en\deg(R(\chi))$, and hence χ satisfies (2). □

Corollary 6.4. *Let ϕ be a key polynomial for μ of minimal degree n. Let $\chi\in K[x]$ be a key polynomial such that $\chi\not\sim_\mu\phi$. Then $\Gamma_{\deg(\chi)}=\Gamma_\mu$.*

Proof: By Lemma 2.11, $\Gamma_\mu=\langle\Gamma_{\deg(\chi)},\mu(\chi)\rangle$. Since $\deg(\chi)\ge n$, we clearly have $\Gamma_n\subset\Gamma_{\deg(\chi)}$. By Theorem 3.9 and Proposition 6.3,

$$\mu(\chi)=\frac{\deg(\chi)}{\deg(\phi)}\,\mu(\phi)=\deg(R(\chi))e\mu(\phi)\in\Gamma_n\subset\Gamma_{\deg(\chi)}.$$

Hence $\Gamma_\mu=\Gamma_{\deg(\chi)}$. □

Corollary 6.5. *The two following conditions are equivalent:*

(1) *$e>1$.*
(2) *All key polynomials of minimal degree are μ-equivalent.*

Proof: Let ϕ be a key polynomial of minimal degree n.

If $e>1$ and χ is a key polynomial not μ-equivalent to ϕ, then Proposition 6.3 shows that $\deg(\chi)=en\deg(R(\chi))>n$. Hence, all key polynomials of degree n are μ-equivalent to ϕ.

If $e=1$, then $\mu(\phi)\in\Gamma_n$, so that there exists $a\in K[x]_n$ with $\mu(a)=\mu(\phi)$. The monic polynomial $\chi=\phi+a$ has degree n and it is not μ-equivalent to ϕ. Also, $\deg(R(\chi))=1$, so that $R(\chi)$ is irreducible. Therefore, χ is a key polynomial for μ, because it satisfies condition (2) of Proposition 6.3. □

6.2. Unique factorization in $\mathcal{G}_\mu$. If χ is a key polynomial for μ, then $\mathcal{R}(\chi)$ is a maximal ideal of Δ, by Proposition 2.13. Let us study the fibers of the mapping $\mathcal{R}\colon \operatorname{KP}(\mu)\to \operatorname{Max}(\Delta)$.

Proposition 6.6. *Let ϕ be a key polynomial of minimal degree and let $R=R_{\phi,u}$.*

For any $\chi,\chi'\in\operatorname{KP}(\mu)$, the following conditions are equivalent:

(1) $\chi\sim_\mu\chi'$.
(2) $H_\mu(\chi)$ *and* $H_\mu(\chi')$ *are associate in* $\mathcal{G}_\mu$.
(3) $\chi\mid_\mu\chi'$.
(4) $\mathcal{R}(\chi)=\mathcal{R}(\chi')$.
(5) $R(\chi)=R(\chi')$.

Moreover, these conditions imply $\deg(\chi)=\deg(\chi')$.

Proof: The implications (1) $\Longrightarrow$ (2) $\Longrightarrow$ (3) are obvious.

Also, (3) $\Longrightarrow\mathcal{R}(\chi')\subset\mathcal{R}(\chi)\Longrightarrow$ (4), because $\mathcal{R}(\chi')$ is a maximal ideal.

Let us show that (4) implies (5). By Proposition 6.3, condition (4) implies that we have two possibilities for the pair χ, χ':

(i) $\chi\sim_\mu\phi\sim_\mu\chi'$, or
(ii) $s(\chi)=s(\chi')=0$.

In the first case, we deduce (5) from Corollary 5.5.

In the second case, condition (5) follows from Theorems 5.7, 4.6, and the fact that $R(\chi)$, $R(\chi')$ are monic polynomials.

Let us show that (5) implies (1). If $R(\chi)=R(\chi')=1$, then Proposition 6.3 shows that $\chi\sim_\mu\phi\sim_\mu\chi'$.

If $R(\chi)=R(\chi')\neq 1$, then Proposition 6.3 shows that $s(\chi)=s(\chi')=0$ and

$$\deg(\chi)=en\deg(R(\chi))=en\deg(R(\chi'))=\deg(\chi').$$

Also, $\chi\mid_\mu\chi'$ by the second equivalence of Corollary 5.5. Hence, $\chi\sim_\mu\chi'$ by Lemma 2.5.

This ends the proof of the equivalence of all conditions.

Finally, (1) implies $\deg(\chi)=\deg(\chi')$ by the μ-minimality of both polynomials. □

Theorem 6.7. *Suppose μ/v commensurable and* $\operatorname{KP}(\mu)\neq\emptyset$. *The residual ideal mapping*

$$\mathcal{R}\colon \operatorname{KP}(\mu)\longrightarrow\operatorname{Max}(\Delta)$$

induces a bijection between $\operatorname{KP}(\mu)/\sim_\mu$ *and* $\operatorname{Max}(\Delta)$.

Proof: Let ϕ be a key polynomial of minimal degree n.

By Proposition 6.6, $\mathcal{R}$ induces a 1-1 mapping between $\mathrm{KP}(\mu)/\sim_\mu$ and $\mathrm{Max}(\Delta)$.

Let us show that $\mathcal{R}$ is onto. By Theorem 4.6, a maximal ideal $\mathcal{L}$ in Δ is given by $\psi(\xi)\Delta$ for some monic irreducible polynomial $\psi \in \kappa[y]$.

If $\psi = y$, then $\mathcal{L} = \mathcal{R}(\phi)$ by Theorem 5.7. If $\psi \neq y$, then it suffices to show the existence of a key polynomial χ such that $R(\chi) = \psi$, by Proposition 6.3.

Let $d = \deg(\psi)$. By Lemma 5.6, there exists $\chi \in K[x]$ such that

$$s(\chi) = 0, \quad \mathrm{nlc}(\chi) = H_\mu(u)^{-d}, \quad R(\chi) = \psi.$$

Along the proof of that lemma, we saw that χ may be chosen to have ϕ-expansion:

$$\chi = a_0 + a_1\phi^e + \cdots + a_d\phi^{de}, \quad \deg(a_j) < n.$$

Also, the condition on a_d is $H_\mu(a_d) = \mathrm{nlc}(\chi)H_\mu(u)^d = 1_{\mathcal{G}_\mu}$. Thus, we may choose $a_d = 1$. Then $\deg(\chi) = den$, so that χ is a key polynomial because it satisfies condition (2) of Proposition 6.3. □

Theorem 6.8. *Let $\mathcal{P} \subset \mathrm{KP}(\mu)$ be a set of representatives of key polynomials under μ-equivalence. Then the set $H\mathcal{P} = \{H_\mu(\chi) \mid \chi \in \mathcal{P}\}$ is a system of representatives of homogeneous prime elements of $\mathcal{G}_\mu$ up to associates.*

Also, up to units in $\mathcal{G}_\mu$, for any non-zero $f \in K[x]$ there is a unique factorization:

$$f \sim_\mu \prod\nolimits_{\chi\in\mathcal{P}} \chi^{a_\chi}, \quad a_\chi = s_\chi(f). \tag{16}$$

Proof: All elements in $H\mathcal{P}$ are prime elements by the definition of μ-irreducibility. Also, they are pairwise non-associate by Proposition 6.6.

Let ϕ be a key polynomial of minimal degree n.

By Proposition 6.2, every homogeneous prime element is associate either to $H_\mu(\phi)$ (which belongs to $H\mathcal{P}$), or to $\psi(\xi)$ for some irreducible polynomial $\psi \in \kappa[y]$, $\psi \neq y$.

In the latter case, along the proof of Theorem 6.7 we saw the existence of $\chi \in \mathrm{KP}(\mu)$ such that $s(\chi) = 0$ and $R(\chi) = \psi$.

Therefore, $\psi(\xi) = R(\chi)(\xi)$ is associate to $H_\mu(\chi) \in H\mathcal{P}$, by Theorem 5.3.

Finally, every homogeneous element in $\mathcal{G}_\mu$ is associate to a product of homogeneous prime elements, by Theorem 5.3 and Lemma 6.1. □

7. Augmentation of valuations

We keep dealing with a valuation μ on $K[x]$ with $\mathrm{KP}(\mu) \neq \emptyset$.

There are different procedures to *augment* this valuation, in order to obtain valuations μ' on $K[x]$ such that

$$\mu(f) \le \mu'(f), \quad \text{for all } f \in K[x],$$

after embedding the value groups of μ and μ' in a common ordered group.

In this section we show how to single out a key polynomial for μ' of minimal degree.

As an application, all results of this paper apply to determine the structure of $\mathcal{G}_{\mu'}$ and the set $\mathrm{KP}(\mu')/\!\sim_{\mu'}$ in terms of this key polynomial.

7.1. Ordinary augmentations.

Definition 7.1. Take $\chi \in \mathrm{KP}(\mu)$. Let $\Gamma_\mu \hookrightarrow \Gamma'$ be an order-preserving embedding of Γ_μ into another ordered group and choose $\gamma \in \Gamma'$ such that $\mu(\chi) < \gamma$.

The *augmented valuation* of μ with respect to these data is the mapping

$$\mu' \colon K[x] \longrightarrow \Gamma' \cup \{\infty\},$$

assigning, to any $g \in K[x]$ with canonical χ-expansion $g = \sum_{0 \le s} g_s \chi^s$, the value

$$\mu'(g) = \mathrm{Min}\{\mu(g_s) + s\gamma \mid 0 \le s\}.$$

We use the notation $\mu' = [\mu; \chi, \gamma]$. Note that $\mu'(\chi) = \gamma$.

The following proposition collects several results of [**11**, Section 1.1].

Proposition 7.2.

(1) *The mapping $\mu' = [\mu; \chi, \gamma]$ is a valuation on $K[x]$ extending v, with value group $\Gamma_{\mu'} = \langle \Gamma_{\deg(\chi)}, \gamma \rangle$.*

(2) *For all $f \in K[x]$, we have $\mu(f) \le \mu'(f)$.*
Equality holds if and only if $\chi \nmid_\mu f$ or $f = 0$.

(3) *If $\chi \nmid_\mu f$, then $H_{\mu'}(f)$ is a unit in $\mathcal{G}_{\mu'}$.*

(4) *The polynomial χ is a key polynomial for μ'.*

(5) *The kernel of the canonical homomorphism*

$$\mathcal{G}_\mu \longrightarrow \mathcal{G}_{\mu'}, \quad a + \mathcal{P}^+_\alpha(\mu) \longmapsto a + \mathcal{P}^+_\alpha(\mu'), \ \textit{for all } \alpha \in \Gamma_\mu,$$

is the principal ideal of $\mathcal{G}_\mu$ generated by $H_\mu(\chi)$.

Corollary 7.3. *The polynomial χ is a key polynomial for μ' of minimal degree.*

Proof: For any polynomial $f \in K[x]$ with $\deg(f) < \deg(\chi)$, we have $\chi \nmid_\mu f$, because χ is μ-minimal. By (3) of Proposition 7.2, $H_{\mu'}(f)$ is a unit in $\mathcal{G}_{\mu'}$, so that f cannot be a key polynomial for μ'. □

7.2. Limit augmentations. Consider a totally ordered set A not containing a maximal element.

A *continuous MacLane chain* based on μ and parameterized by A is a family $(\mu_\alpha)_{\alpha\in A}$ of augmented valuations

$$\mu_\alpha = [\mu; \phi_\alpha, \gamma_\alpha], \quad \phi_\alpha \in \mathrm{KP}(\mu), \quad \mu(\phi_\alpha) < \gamma_\alpha \in \Gamma_\mu,$$

for all $\alpha \in A$, satisfying the following conditions:

(1) $\deg(\phi_\alpha) = d$ is independent of $\alpha \in A$.
(2) The mapping $A \to \Gamma_\mu$, $\alpha \mapsto \gamma_\alpha$ is an order-preserving embedding.
(3) For all $\alpha < \beta$ in A, we have

$$\phi_\beta \in \mathrm{KP}(\mu_\alpha), \quad \phi_\alpha \not\sim_{\mu_\alpha} \phi_\beta, \quad \mu_\beta = [\mu_\alpha; \phi_\beta, \gamma_\beta].$$

In Vaquié's terminology, $(\mu_\alpha)_{\alpha\in A}$ is a “famille continue de valuations augmentées itérées” [**11**].

Definition 7.4. A polynomial $f \in K[x]$ is *A-stable* if there exists $\alpha_0 \in A$ such that

$$\mu_{\alpha_0}(f) = \mu_\alpha(f), \quad \text{for all } \alpha \geq \alpha_0.$$

In this case, we denote by $\mu_A(f)$ this stable value.

Lemma 7.5. *If $f \in K[x]$ is not A-stable, then $\mu_\alpha(f) < \mu_\beta(f)$ for all $\alpha < \beta$ in A.*

Proof: Let us show that the equality $\mu_\alpha(f) = \mu_\beta(f)$ for some $\alpha < \beta$ in A implies that f is A-stable.

Since $\mu_\beta = [\mu_\alpha; \phi_\beta, \gamma_\beta]$, Proposition 7.2 shows that $\phi_\beta \nmid_{\mu_\alpha} f$ and $H_{\mu_\beta}(f)$ is a unit in $\mathcal{G}_{\mu_\beta}$. Hence, for all $\delta \geq \beta$, the image of $H_{\mu_\beta}(f)$ in $\mathcal{G}_{\mu_\delta}$ is a unit.

Since $\mu_\delta = [\mu_\beta; \phi_\delta, \gamma_\delta]$, property (5) of Proposition 7.2 shows that $\phi_\delta \nmid_{\mu_\beta} f$. Hence $\mu_\beta(f) = \mu_\delta(f)$, again by Proposition 7.2. Thus, f is A-stable. □

If all polynomials in $K[x]$ are A-stable, then $\mu_A = \lim_{\alpha\in A} \mu_\alpha$ has an obvious meaning, and μ_A is a valuation on $K[x]$.

If there are polynomials which are not A-stable, there are still some particular situations in which $(\mu_\alpha)_{\alpha\in A}$ converges to a valuation (or semivaluation) on $K[x]$.

However, regardless of the fact that $(\mu_\alpha)_{\alpha\in A}$ converges or not, nonstable polynomials may be used to define *limit augmented* valuations of this continuous MacLane chain.

Let us assume that not all polynomials in $K[x]$ are A-stable.

We take a monic $\phi \in K[x]$ which is not A-stable, and has minimal degree among all polynomials having this property.

Since the product of A-stable polynomials is A-stable, ϕ is irreducible in $K[x]$.

Lemma 7.6. *Let $f \in K[x]$ be a non-zero polynomial with canonical ϕ-expansion $f = \sum_{0\le s} f_s\phi^s$. Then, there exist an index s_0 and an element $\alpha_0 \in A$ such that*

$$\mu_\alpha(f) = \mu_\alpha(f_{s_0}\phi^{s_0}) < \mu_\alpha(f_s\phi^s), \quad \textit{for all } s \neq s_0 \textit{ and } \alpha \geq \alpha_0.$$

Proof: Since all coefficients f_s have degree less than $\deg(\phi)$, they are all A-stable. Let us take α_1 sufficiently large so that $\mu_\alpha(f_s) = \mu_A(f_s)$ for all $s \geq 0$ and all $\alpha \geq \alpha_1$.

For every $\alpha \in A$, $\alpha \geq \alpha_1$, let

$$\delta_\alpha = \operatorname{Min}\{\mu_\alpha(f_s\phi^s) \mid 0 \leq s\}, \quad I_\alpha = \{s \mid \mu_\alpha(f_s\phi^s) = \delta_\alpha\}, \quad s_\alpha = \operatorname{Min}(I_\alpha).$$

For any index s we have

$$(17)\quad \mu_\alpha(f_{s_\alpha}\phi^{s_\alpha}) = \mu_A(f_{s_\alpha}) + s_\alpha\mu_\alpha(\phi) \leq \mu_\alpha(f_s\phi^s) = \mu_A(f_s) + s\mu_\alpha(\phi).$$

Since ϕ is not A-stable, Lemma 7.5 shows that $\mu_\alpha(\phi) < \mu_\beta(\phi)$ for all $\beta > \alpha$, $\beta \in A$. Thus, if we replace μ_α with μ_β in (17), we get a strict inequality for all $s > s_\alpha$, because the left-hand side of (17) increases by $s_\alpha(\mu_\beta(\phi) - \mu_\alpha(\phi))$, while the right-hand side increases by $s(\mu_\beta(\phi) - \mu_\alpha(\phi))$.

Therefore, either $I_\beta = \{s_\alpha\}$, or $s_\beta = \operatorname{Min}(I_\beta) < s_\alpha$.

Since A contains no maximal element, we may consider a strictly increasing infinite sequence of values of $\beta \in A$. There must be an $\alpha_0 \in A$ such that

$$I_\alpha = \{s_0\}, \quad \text{for all } \alpha \geq \alpha_0,$$

because the set of indices s is finite. □

Definition 7.7. For any $f \in K[x]$, we say that f is A-divisible by ϕ, and we write $\phi \mid_A f$, if there exists $\alpha_0 \in A$ such that $\phi \mid_{\mu_\alpha} f$ for all $\alpha \geq \alpha_0$.

Lemma 7.8. *For any $f \in K[x]$ with canonical ϕ-expansion $f = \sum_{0\le s} f_s\phi^s$, the following conditions are equivalent:*

(1) *f is A-stable.*
(2) *There exists $\alpha_0 \in A$ such that*

$$\mu_\alpha(f) = \mu_\alpha(f_0) < \mu_\alpha(f_s\phi^s), \quad \textit{for all } s > 0 \textit{ and } \alpha \geq \alpha_0.$$

(3) $\phi \nmid_A f$.

Proof: By Lemma 7.6, there exist an index s_0 and an element $\alpha_0 \in A$ such that $\mu_\alpha(f_s) = \mu_A(f_s)$ for all s, and

$$\mu_\alpha(f) = \mu_\alpha(f_{s_0}\phi^{s_0}) = \mu_A(f_{s_0}) + s_0\mu_\alpha(\phi) < \mu_\alpha(f_s\phi^s), \tag{18}$$

for all $s \neq s_0$ and all $\alpha \geq \alpha_0$. By Lemma 7.5, $\mu_\alpha(\phi)$ grows strictly with α. Hence, condition (1) is equivalent to $s_0 = 0$, which is in turn equivalent to condition (2).

If $\mu_\alpha(f) = \mu_\beta(f)$ for some $\alpha < \beta$, we have necessarily $\phi \nmid_{\mu_\alpha} f$, because $\mu_\alpha(\phi) < \mu_\beta(\phi)$. Thus, (1) implies (3).

On the other hand, $s_0 > 0$ in (18) implies $\phi \nmid_{\mu_\alpha} f$. Thus, (3) implies (2). □

Definition 7.9. Take $\phi \in K[x]$ a monic polynomial with minimal degree among all polynomials which are not A-stable.

Let $\Gamma_\mu \hookrightarrow \Gamma'$ be an order-preserving embedding of Γ_μ into another ordered group, and suppose that there exists $\gamma \in \Gamma'$ such that $\mu_\alpha(\phi) < \gamma$ for all $\alpha \in A$.

The *limit augmented valuation* of the continuous MacLane chain $(\mu_\alpha)_{\alpha\in A}$ with respect to these data is the mapping

$$\mu' \colon K[x] \longrightarrow \Gamma' \cup \{\infty\},$$

assigning, to any $g \in K[x]$ with ϕ-expansion $g = \sum_{0\leq s} g_s\phi^s$, the value

$$\mu'(g) = \operatorname{Min}\{\mu_A(g_s) + s\gamma \mid 0 \leq s\}.$$

We use the notation $\mu' = [\mu_A; \phi, \gamma]$. Note that $\mu'(\phi) = \gamma$.

The following proposition collects Propositions 1.22 and 1.23 of [**11**].

Proposition 7.10.

(1) *The mapping $\mu' = [\mu_A; \phi, \gamma]$ is a valuation on $K[x]$ extending v.*
(2) *For all $f \in K[x]$, we have $\mu_\alpha(f) \leq \mu'(f)$ for all $\alpha \in A$.*
The condition $\mu_\alpha(f) < \mu'(f)$, for all $\alpha \in A$, is equivalent to $\phi \mid_A f$ and $f \neq 0$.

Lemma 7.11. *For $a, b \in K[x]_{\deg(\phi)}$, let the ϕ-expansion of ab be*

$$ab = c + d\phi, \quad c, d \in K[x]_{\deg(\phi)}.$$

Then $ab \sim_{\mu'} c$.

Proof: Since $\deg(a), \deg(b) < \deg(\phi)$, the polynomials a and b are A-stable. In particular, ab is A-stable. By Lemma 7.8, there exists $\alpha_0 \in A$ such that

$$\mu_A(c) = \mu_\alpha(c), \quad \text{and} \quad \mu_\alpha(ab) = \mu_\alpha(c) < \mu_\alpha(d\phi), \quad \text{for all } \alpha \geq \alpha_0 \text{ in } A.$$

Hence, $\mu_A(ab) = \mu_A(c) < \mu_A(d) + \mu_\alpha(\phi) < \mu_A(d) + \gamma$. By the definition of μ', we conclude that $\mu'(ab) = \mu'(c) < \mu'(d\phi)$. □

Corollary 7.12. *For all $a \in K[x]_{\deg(\phi)}$, $H_{\mu'}(a)$ is a unit in $\mathcal{G}_{\mu'}$.*

Proof: Since ϕ is irreducible in $K[x]$ and $\deg(a) < \deg(\phi)$, these two polynomials are coprime. Hence there is a Bézout identity

$$ac + d\phi = 1, \quad \deg(c) < \deg(\phi),\ \deg(d) < \deg(a) < \deg(\phi).$$

By Lemma 7.11, $ac \sim_{\mu'} 1$. □

Corollary 7.13. *The polynomial ϕ is a key polynomial for μ' of minimal degree.*

Proof: By the definition of μ' we have $\mu'(f) = \operatorname{Min}\{\mu'(f_s\phi^s) \mid 0 \le s\}$ for any $f = \sum_{0\le s} f_s\phi^s \in K[x]$. By Proposition 2.3, ϕ is μ'-minimal.

Let us show that ϕ is μ'-irreducible. Suppose $\phi \nmid_{\mu'} f$, $\phi \nmid_{\mu'} g$ for $f, g \in K[x]$. Let f_0, g_0 be the 0-th coefficients of the ϕ-expansion of f and g, respectively.

Since ϕ is μ'-minimal, Proposition 2.3 shows that $\mu'(f) = \mu'(f_0)$ and $\mu'(g) = \mu'(g_0)$.

Consider the ϕ-expansion

$$f_0g_0 = c + d\phi, \quad \deg(c), \deg(d) < \deg(\phi).$$

By Lemma 7.11, $\mu'(f_0g_0) = \mu'(c)$, so that

$$\mu'(fg) = \mu'(f_0g_0) = \mu'(c).$$

Since the polynomial c is the 0-th coefficient of the ϕ-expansion of fg, Proposition 2.3 shows that $\phi \nmid_{\mu'} fg$. Hence, ϕ is μ-irreducible.

This shows that ϕ is a key polynomial for μ'.

Finally, by Corollary 7.12, any polynomial in $K[x]$ of degree smaller than $\deg(\phi)$ cannot be a key polynomial for μ', because it is a μ'-unit. □

8. Some remarks on the structure of $\mathcal{G}_\mu$ as a graded algebra

Let μ be a valuation on $K[x]$ admitting key polynomials. Let n be the minimal degree of a key polynomial for μ.

The subgroup $\Gamma_n \subset \Gamma_\mu$ admits an intrinsic description as the subgroup of Γ_μ formed by all values α such that there is a unit in $\mathcal{G}$ of degree α.

Lemma 8.1. *Let μ be a valuation on $K[x]$ with $\mathrm{KP}(\mu) \neq \emptyset$. For any $\alpha \in \Gamma_\mu$, we have*

$$(\mathcal{P}(\alpha)/\mathcal{P}^+(\alpha)) \cap \mathcal{G}^* \neq \emptyset \iff \alpha \in \Gamma_n,$$

where n is the minimal degree of a key polynomial for μ.

Proof: Let $\alpha \in \Gamma_n$, and take $a \in K[x]_n$ such that $\mu(a) = \alpha$. By Proposition 3.5, $H_\mu(a)$ is a unit in $\mathcal{P}(\alpha)/\mathcal{P}^+(\alpha)$.

Let ϕ be a key polynomial of degree n and let $\gamma = \mu(\phi)$, so that $\Gamma_\mu = \langle \Gamma_n, \gamma \rangle$.

Any $\alpha \notin \Gamma_n$ can be written as

$$\alpha = \ell\gamma + \beta, \quad \ell \in \mathbb{Z}, \ell \neq 0, \quad \beta \in \Gamma_n.$$

If $\ell < 0$, then Proposition 2.3 shows that there is no polynomial in $K[x]$ with μ-value equal to α. Thus, $\mathcal{P}(\alpha)/\mathcal{P}^+(\alpha) = \{0\}$.

Suppose $\ell > 0$. By the previous argument, there exists a unit $z \in \mathcal{G}^*$ of degree β. Then $z\, H_\mu(\phi)^\ell$ has degree α and there is no unit in

$$\mathcal{P}(\alpha)/\mathcal{P}^+(\alpha) = (z\, H_\mu(\phi)^\ell)\Delta,$$

because $H_\mu(\phi)$ is a prime element. □

The structure of $\mathcal{G}_\mu$ is determined by that of the subalgebra

$$\mathcal{G}_{\mu,n} := \bigoplus\nolimits_{\alpha\in\Gamma_n} \mathcal{P}_\alpha/\mathcal{P}_\alpha^+ \subset \mathcal{G}_\mu,$$

as indicated in the next result.

Proposition 8.2. *Let μ be a valuation on $K[x]$ with $\mathrm{KP}(\mu) \neq \emptyset$. Let $\gamma = \mu(\phi)$ be the common μ-value of all key polynomials of minimal degree n.*

Consider the graded algebra $\mathcal{G}_{\mu,n}[x]$ obtained by assigning degree γ to the indeterminate x. Then:

(1) *If μ is incommensurable, then $\mathcal{G}_\mu \simeq \mathcal{G}_{\mu,n}[x]$ as $\mathcal{G}_v$-algebras.*
(2) *If μ is commensurable, then we have an isomorphism of $\mathcal{G}_v$-algebras*

$$\mathcal{G}_\mu \simeq \mathcal{G}_{\mu,n}[x]/(x^e - z\xi),$$

where e is the least positive integer such that $e\gamma \in \Gamma_n$, z is a unit in $\mathcal{G}_\mu$ of degree $e\gamma$, and ξ is a generator of Δ as a κ-algebra.

Proof: Let ϕ be any key polynomial of minimal degree. The inclusion $\mathcal{G}_{\mu,n} \subset \mathcal{G}_\mu$ determines a well-defined onto homomorphism of graded $\mathcal{G}_v$-algebras:

$$\mathcal{G}_{\mu,n}[x] \longrightarrow \mathcal{G}_\mu, \quad x \longmapsto H_\mu(\phi). \tag{19}$$

If μ is incommensurable, then any polynomial with coefficients in $\mathcal{G}_{\mu,n}$ has monomials with different degree. Hence, our mapping (19) has a trivial kernel and it is an isomorphism.

If μ is commensurable, there exists $u \in K[x]_n$ such that $\mu(u\phi^e) = 0$, and $\xi = H_\mu(u\phi^e)$ is a transcendental generator of Δ as a κ-algebra by Theorem 4.6.

Since $H_\mu(u)$ is a unit, we have $H_\mu(\phi)^e = z\xi$, where $z = H_\mu(u)^{-1}$. If we show that this is a minimal relation for $H_\mu(\phi)$, the proof of the proposition will be complete. Suppose that the polynomial

$$\sum\nolimits_{m\ge 0} a_m\, x^m \in \mathcal{G}_{\mu,n}[x]$$

lies in the kernel of (19). By applying $H_\mu(\phi)^e = z\xi$, we may assume that $0\le m < e$. Again, this sum cannot have two different monomials. In fact,

$$\deg(a\, x^m) = \deg(b\, x^\ell) \Longrightarrow (m-\ell)\gamma = \deg(b) - \deg(a) \in \Gamma_n.$$

From $(m-\ell)\gamma \in \Gamma_n$ we deduce $m \equiv \ell \pmod{e}$, and this implies $m = \ell$.

Thus, our relation takes the form $a\, H_\mu(\phi)^m = 0$. Since $\mathcal{G}$ is an integral domain, we necessarily have $a = 0$. □

The structure of the subalgebra $\mathcal{G}_{\mu,n}$ depends very much on how the valuation μ has been constructed. We determine its structure in two particular situations:

- Γ_n is a finitely-generated group, or
- μ has been obtained by a finite number of (ordinary) augmentations, starting with a valuation μ_0 such that $\Gamma_{\mu_0} = \Gamma_v$.

If Γ_n is finitely-generated, then it is a free abelian group. If N is the rank of this group, we can choose

$$a_i \in K[x]_n, \quad \gamma_i = \mu(a_i), \quad 1 \le i \le N,$$

such that $\gamma_1, \dots, \gamma_N$ is a $\mathbb{Z}$-basis of Γ_n. Then, every $\alpha \in \Gamma_n$ can be written in a unique way as $\alpha = m_1\gamma_1 + \cdots + m_N\gamma_N$, with $m_1, \dots, m_N \in \mathbb{Z}$.

Therefore, we may choose homogeneous units of each degree α

$$z^\alpha := H_\mu(a_1)^{m_1} \cdots H_\mu(a_N)^{m_N} \in \mathcal{G}^*_{\mu,n},$$

and these units satisfy

$$z^{\alpha+\beta} = z^\alpha z^\beta, \quad \text{for all } \alpha, \beta \in \Gamma_n.$$

As a consequence, $\mathcal{G}_{\mu,n}$ is isomorphic to the group algebra

$$\mathcal{G}_{\mu,n} \simeq \Delta[\Gamma_n],$$

since $\mathcal{P}(\alpha)/\mathcal{P}^+(\alpha) = z^\alpha \Delta$ for all $\alpha \in \Gamma_n$.

This is an explicit description of the structure of $\mathcal{G}_{\mu,n}$ as a graded algebra, because the structure of Δ has been explicitely determined in Section 4 in all cases.

Suppose now that μ_0 is a valuation on $K[x]$ extending v with $\Gamma_{\mu_0} = \Gamma_v$ and there is a finite sequence of augmentations (see Subsection 7.1)

$$(20) \qquad \mu_0 \xrightarrow{\phi_1,\gamma_1} \mu_1 \xrightarrow{\phi_2,\gamma_2} \cdots \xrightarrow{\phi_{r-1},\gamma_{r-1}} \mu_{r-1} \xrightarrow{\phi_r,\gamma_r} \mu_r = \mu,$$

with $\mu_i = [\mu_{i-1}; \phi_i, \gamma_i]$ for $1 \le i \le r$.

We may always assume that (20) is a *MacLane chain* of augmentations, that is,

$$\phi_{i+1} \nmid_{\mu_i} \phi_i, \quad 0 \le i < r.$$

By Proposition 7.2, ϕ_{r-1}, ϕ_r are key polynomials of minimal degree for μ_{r-1}, μ, respectively. In particular, $n = \deg(\phi_r)$.

Also, ϕ_r is another key polynomial for μ_{r-1} which is not μ_{r-1}-equivalent to ϕ_{r-1}. Hence, Corollary 6.4 shows that

$$\Gamma_{\mu_{r-1}} = \Gamma_{\mu_{r-1},n} = \Gamma_n,$$

and the last equality holds because $\mu_{r-1}(a) = \mu(a)$ for all $a \in K[x]_n$.

By Theorem 4.2, the valuations $\mu_1, \dots, \mu_{r-1}$ are necessarily commensurable. Let $e_1, \dots, e_{r-1}$ be their relative ramification indices. Then, we have an isomorphism of $\mathcal{G}_v$-algebras

$$\mathcal{G}_{\mu,n} \simeq (\mathcal{G}_v \otimes_{k_v} \Delta)[x_1, \dots, x_{r-1}],$$

where $x_1, \dots, x_{r-1}$ are homogeneous units of degree $\gamma_1, \dots, \gamma_{r-1}$ and satisfy relations

$$x_i^{e_i} = \eta_i, \quad 1 \le i < r.$$

The arguments are quite similar to those in Proposition 8.2. We omit the details.

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