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THE FIBERING METHOD APPROACH FOR A NON-LINEAR SCHRÖDINGER EQUATION COUPLED WITH THE ELECTROMAGNETIC FIELD

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Abstract: We study, with respect to the parameter $q \neq 0$, the following Schrödinger–Bopp–Podolsky system in $\mathbb{R}^3$

$$\begin{cases} -\Delta u + \omega u + q^2 \phi u = |u|^{p-2} u, \\ -\Delta \phi + a^2 \Delta^2 \phi = 4\pi u^2, \end{cases}$$

where $p \in (2, 3]$, $\omega > 0$, $a \geq 0$ are fixed. We prove, by means of the fibering approach, that the system has no solutions *at all* for large values of q and has two *radial* solutions for small q's. We give also qualitative properties about the energy level of the solutions and a variational characterization of these extremal values of q. Our results recover and improve some results in [**2, 5**].

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1. Introduction

The following system in $\mathbb{R}^3$

$$\begin{cases} -\Delta u + \omega u + q^2 \phi u = |u|^{p-2} u, \\ -\Delta \phi + a^2 \Delta^2 \phi = 4\pi u^2, \end{cases} \tag{1.1}$$

where $a, \omega > 0$, $q \neq 0$, and $p \in (2, 6)$, has been studied in [**2**] for the first time in the mathematical literature. The system appears when one looks for stationary solutions $u(x)e^{i\omega t}$ of the Schrödinger equation coupled with the Bopp–Podolski Lagrangian of the electromagnetic field in the purely electrostatic situation. Here, u represents the modulus of the wave function and ϕ the electrostatic potential. From a physical point of view, the parameter q has the meaning of the electric charge and a is the parameter of the Bopp–Podolski term.

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In the said paper, it has been shown that the problem can be addressed variationally. Indeed, introducing the Hilbert space

$$\mathcal{D} := \{\phi \in D^{1,2}(\mathbb{R}^3) : \Delta\phi \in L^2(\mathbb{R}^3)\}$$

normed by

$$\|\phi\|_{\mathcal{D}}^2 = a^2\|\Delta\phi\|_2^2 + \|\nabla\phi\|_2^2,$$

it can be proved that, for every $u \in H^1(\mathbb{R}^3)$ there is a unique solution $\phi_u \in \mathcal{D}$ of the second equation in the system, that is, an element $\phi_u \in \mathcal{D}$ satisfying

$$-\Delta\phi_u + a^2\Delta^2\phi_u = 4\pi u^2. \tag{1.2}$$

Moreover, it turns out that

$$\phi_u = \frac{1 - e^{-|\cdot|/a}}{|\cdot|} * u^2.$$

Observe from (1.2) that for every $u \in H^1(\mathbb{R}^3)$ one has

$$4\pi \int \phi_u u^2 = \|\phi_u\|_{\mathcal{D}}^2,$$

which will be used throughout the paper.

By using the by now classical *reduction argument* one is led to study, equivalently, the single equation

$$-\Delta u + \omega u + q^2\phi_u u = |u|^{p-2}u \quad \text{in } \mathbb{R}^3, \tag{1.3}$$

containing the nonlocal term $\phi_u u$. For these reasons, whenever we speak of the solution of system (1.1), from now on we mean just the solution u of the above equation since $\phi = \phi_u$ is univocally determined. The energy functional

$$\mathcal{J}_q(u) = \frac{1}{2}\|u\|^2 + \frac{q^2}{4}\int \phi_u u^2 - \frac{1}{p}\|u\|_p^p, \quad u \in H^1(\mathbb{R}^3),$$

which is well defined and C^1, is related to equation (1.3). In this way we are simply reduced to find critical points of $\mathcal{J}_q$. We denote (here and throughout the paper) by $\|u\|_p$ the L^p-norm and by

$$\|u\|^2 = \|\nabla u\|_2^2 + \omega\|u\|_2^2$$

the (squared) norm in $H^1(\mathbb{R}^3)$, where ω is a fixed positive constant.

In [**2**, Theorems 1.1 and 1.2] it is proved that if $p \in (3,6)$, then problem (1.1) admits a solution for every $q \neq 0$. On the other hand, if $p \in (2,3]$, the existence of a solution is proved just for $q \neq 0$ sufficiently small. As we can see, there is a difference in the result depending on the range where p varies. Indeed, in the case that p is small the value of q may prevent the existence of critical points for the functional $\mathcal{J}_q$.

Of course, if $a = 0$ system (1.1) reduces to the so called Schrödinger–Poisson system in $\mathbb{R}^3$

$$\begin{cases} -\Delta u + \omega u + q^2\phi u = |u|^{p-2}u, \\ -\Delta\phi = 4\pi u^2, \end{cases} \tag{1.4}$$

or, equivalently, to

$$-\Delta u + \omega u + q^2\phi_u^{\mathrm{SP}} u = |u|^{p-2}u,$$

where now

$$\phi_u^{\mathrm{SP}} = \frac{1}{|\cdot|} * u^2.$$

In the mathematical literature there is a huge number of papers concerning this last problem. However, here we cite [**1**], where the authors introduced for the first time the *reduction method* which allows to study a single equation instead of a system, and [**5**], where the author studies the problem depending on the parameter q^2. In particular, Ruiz ([**5**]) shows among other results that, in the case $p \in (2,3)$, the system (1.4) has two radial solutions for small q^2 and has no solutions at all (radial or not) for $q^2 \geq 1/4$. See also [**6**] for similar results related to the problem in bounded domains.

Motivated by the cited papers [**2, 5**], our aim here is to understand in a more satisfactory way the existence of solutions for (1.1) or (1.3), namely the behaviour of $\mathcal{J}_q$ concerning its critical points in the case $p \in (2,3]$ and how they are influenced by the value of q.

We prove two types of results. The first type is concerned with the smallness of q^2 as a necessary condition in order to have a nontrivial solution of the problem (the sufficiency being proved in [**2**]). Indeed, we show that for q^2 suitably large the problem has no solutions at all. See Theorem 1.1 below.

The second type concerns the existence of solutions for q^2 small. However, due to the technique used (we borrow some ideas from [**5**]), we are able to state such a result just in the radial case, where two solutions are obtained (in contrast to the result in [**2**] which states the existence of one solution in the nonradial case). See Theorem 1.2 below.

Before stating rigorously the theorems, we observe that the positive parameter q^2 appears in the problem. In view of this, the results are stated and proved for simplicity just for $q > 0$, with the understanding that they are valid by replacing q with $|q|$. More specifically, under the assumption $p \in (2,3]$ our main results in this work are the following.

Theorem 1.1. *There exists $q^* > 0$ such that, for every $q > q^*$ the problem admits only the trivial solution.*

Theorem 1.2. *There exist $\varepsilon > 0$, $q_0^* > 0$ satisfying $q_0^* + \varepsilon < q^*$ (with q^* given in Theorem 1.1) such that, for every $q \in (0, q_0^* + \varepsilon)$ the problem has two radial solutions.*

A few comments on these results are in order.

As we already said, we are reduced to find critical points of $\mathcal{J}_q$. We remark explicitly that no Pohozaev identity is involved in proving the nonexistence result in Theorem 1.1: this just follows from the properties of the fibering map.

To prove Theorem 1.2, we will use the Mountain Pass Theorem on the space of radial functions, that is, on $H_r^1(\mathbb{R}^3)$. We take advantage of the smallness of q to prove that the energy functional has the Mountain Pass Geometry. However, in contrast to [**2**], where the condition of q small was used in order to find a function where $\mathcal{J}_q$ is negative (and then apply the Mountain Pass Theorem in a standard way), the value q_0^* we find here is a threshold: for $q < q_0^*$ there is a function in $H_r^1(\mathbb{R}^3)$ where the functional is negative, while for $q \geq q_0^*$ the functional is non-negative. Hence the argument employed in both papers [**2, 5**] does not work for $q > q_0^*$. Nevertheless, also in this case we exhibit a Mountain Pass structure.

As shown in [**2**], the solutions we find in Theorem 1.2 are classical and positive by the Maximum Principle.

Our results hold for any fixed $a \geq 0$. We also notice that, in case $a = 0$ we do not need Lemma 2.1 and the inequality in Proposition 2.2 follows (up to some positive constant) by multiplying the second equation in (1.4) by $|u|$ and integrating, which is the relation used in [**5**]. In this sense our result recovers the one in [**5**] and even a better understanding of the fiber maps is given here, since the Mountain Pass structure for the functional related to (1.4) holds although the functional is non-negative for $q \geq q_0^*$.

Actually, we deduce Theorem 1.1 and Theorem 1.2 as consequences of the following result which gives additional information on the solutions.

Theorem 1.3. *Let $p \in (2, 3]$, $a \geq 0$ be fixed. There exist positive numbers ε, q_0, q_0^* satisfying $q_0^* + \varepsilon < q^*$ such that:*

1. *For each $q > q^*$ the functional $\mathcal{J}_q$ has no critical points in $H^1(\mathbb{R}^3)$ other than the zero function.*
2. *For each $q \in (0, q_0^* + \varepsilon)$ the functional $\mathcal{J}_q$ has two nontrivial critical points $u_q, w_q \in H_r^1(\mathbb{R}^3)$, where w_q is a Mountain Pass critical point with*

$$\mathcal{J}_q(w_q) > \max\{0, \mathcal{J}_q(u_q)\}.$$

More specifically,

(i) *if* $q \in (0, q_0^*]$, *then* u_q *is a global minimum with*

$$\mathcal{J}_q(u_q) < 0 \text{ if } q \in (0, q_0^*) \quad \text{and} \quad \mathcal{J}_{q_0^*}(u_{q_0^*}) = 0;$$

(ii) *if* $q \in (q_0^*, q_0^* + \varepsilon)$, *then* u_q *is a local minimum with*

$$\mathcal{J}_q(u_q) > 0.$$

As we can see, whenever $q = q_0^*$, then $\mathcal{J}_{q_0^*}$ is non-negative and we find a global minimizer at zero energy; then additional work is needed in order to show that this is not the zero function. This result is new also in the case $a = 0$.

Concerning the *extremal values* q_0^* and q^*, we say that they have a variational characterization (see Section 2). Furthermore, although we were not able to prove it, it seems plausible that for all $q \in (0, q^*)$, system (1.1) has two positive solutions satisfying the properties in Theorem 1.3 and q^* is in fact a bifurcation parameter where the two solutions (the local minimum and the Mountain Pass type solution) collapse.

Finally, we point out that similar results have been obtained in some nonlinear problems depending on a parameter in some recent papers; see Il'yasov and Silva [**4**], Silva and Macedo [**7**].

This paper is organized as follows.

In Section 2 some preliminaries and technical results (true in the general nonradial setting) are given. As a byproduct of these results, the proof of item 1 in Theorem 1.3 follows; see Corollary 2.8.

In Section 3 we prove Proposition 3.1. It concerns radial functions and is fundamental in the proof of Theorem 1.3, which is completed in Section 4.

Notation. We use the generic letters $C, C', \ldots$ to denote a positive constant, usually related to Sobolev embedding. Even though their value may also change from line to line, no confusion should arise.

2. Preliminaries and technical results

In [**2**], some properties of the solution ϕ_u are found. However, to deal with the case $p \in (2, 3]$ we need also the following ones. Of course this applies just for $a \neq 0$.

Lemma 2.1. *For each* $u \in H^1(\mathbb{R}^3)$ *we have*

(i) $\Delta \phi_u \in \mathcal{D}$.
(ii) $a^2 \Delta \phi_u \leq \phi_u$.

Proof: Let us fix $u \in H^1(\mathbb{R}^3)$ and let $\psi_u := -\Delta\phi_u$. Then

$$-a^2\Delta\psi_u + \psi_u = 4\pi u^2.$$

Since $u^2 \in L^2(\mathbb{R}^3)$, by standard results we have $\psi_u \in H^2(\mathbb{R}^3)$ and in particular

$$\psi_u \in L^6(\mathbb{R}^3), \quad \nabla\psi_u \in L^2(\mathbb{R}^3), \quad \text{and} \quad \Delta\psi_u \in L^2(\mathbb{R}^3).$$

This gives $\Delta\phi_u \in D^{1,2}(\mathbb{R}^3)$ and $\Delta^2\phi_u \in L^2(\mathbb{R}^3)$, namely $\Delta\phi_u \in \mathcal{D}$ proving (i).

On the other hand, if we set $v = -a^2\Delta\phi_u + \phi_u$, then

$$-\Delta v = 4\pi u^2 \geq 0,$$

$v \in D^{1,2}(\mathbb{R}^3)$, and it is continuous. Define $\Omega^- = \{x \in \mathbb{R}^3 : v(x) < 0\}$ and suppose that $\Omega^- \neq \emptyset$. Since v is continuous, the set Ω^- is open. Let $v^- = \max\{-v, 0\}$. It follows that

$$-\int_{\Omega^-} |\nabla v|^2 = \int \nabla v \nabla v^- \geq 0,$$

which is a contradiction. Therefore, $\Omega^- = \emptyset$ and $a^2\Delta\phi_u \leq \phi_u$ in $\mathbb{R}^3$, proving (ii). □

The next result will be useful to get a generalisation of [**5**, Formula (19)] to the case $a \neq 0$.

Proposition 2.2. *We have*

$$\int |u|^3 \leq \frac{1}{\pi}\|\phi_u\|_{\mathcal{D}}\|\nabla u\|_2 \quad \textit{for all } u \in H^1(\mathbb{R}^3).$$

Proof: For $u \in H^1(\mathbb{R}^3)$ fixed, let us consider equation (1.2). Since by Lemma 2.1 we have in particular that $\nabla\Delta\phi_u \in L^2(\mathbb{R}^3)$, by multiplying the equation (1.2) by $|u| \in H^1(\mathbb{R}^3)$ and integrating we get

$$\begin{aligned} 4\pi\int |u|^3 &= a^2\int \nabla(-\Delta\phi_u)\nabla|u| + \int \nabla\phi_u\nabla|u| \\ &\leq a^2\|\nabla(-\Delta\phi_u)\|_2\|\nabla u\|_2 + \|\nabla\phi_u\|_2\|\nabla u\|_2 \\ &= (a^2\|\nabla(-\Delta\phi_u)\|_2 + \|\nabla\phi_u\|_2)\|\nabla u\|_2 \\ &\leq (a^2\|\nabla(\Delta\phi_u)\|_2 + \|\phi_u\|_{\mathcal{D}})\|\nabla u\|_2. \end{aligned} \tag{2.1}$$

On the other hand, multiplying (1.2) by $\Delta\phi_u \in \mathcal{D}$ and making use of (ii) of Lemma 2.1 we get

$$\begin{aligned} a^2\|\nabla(\Delta\phi_u)\|_2^2 &= 4\pi \int \Delta\phi_u u^2 - \int \nabla\phi_u \nabla(\Delta\phi_u) \\ &\le \frac{1}{a^2}\|\phi_u\|_{\mathcal{D}}^2 + \frac{\varepsilon^2}{2}\|\nabla(\Delta\phi_u)\|_2^2 + \frac{1}{2\varepsilon^2}\|\nabla\phi_u\|_2^2 \\ &\le \frac{1}{a^2}\|\phi_u\|_{\mathcal{D}}^2 + \frac{\varepsilon^2}{2}\|\nabla(\Delta\phi_u)\|_2^2 + \frac{1}{2\varepsilon^2}\|\phi_u\|_{\mathcal{D}}^2. \end{aligned}$$

By choosing $\varepsilon = a$ above we conclude that, for all $u \in H^1(\mathbb{R}^3)$,

$$a^2\|\nabla(\Delta\phi_u)\|_2^2 \le \frac{2}{a^2}\|\phi_u\|_{\mathcal{D}}^2 + \frac{1}{a^2}\|\phi_u\|_{\mathcal{D}}^2 = \frac{3}{a^2}\|\phi_u\|_{\mathcal{D}}^2. \tag{2.2}$$

From (2.1) and (2.2) we conclude that

$$\int |u|^3 \le \frac{1}{\pi}\|\phi_u\|_{\mathcal{D}}\|\nabla u\|_2,$$

completing the proof. □

We conclude this section by showing a first simple property of the energy functional. The next result says that the functional $\mathcal{J}_q$ has a strict local minimum at 0, uniformly in q. However, to have the complete Mountain Pass structure q has to be small, as will be shown in Corollary 2.9.

Proposition 2.3. *There exist $\rho > 0$ and $M > 0$ such that*

$$\mathcal{J}_q(u) \ge M \quad \textit{for all } q \in \mathbb{R},\ u \in H^1(\mathbb{R}^3) \textit{ with } \|u\| = \rho.$$

Proof: Since

$$\mathcal{J}_q(u) \ge \frac{1}{2}\|u\|^2 - \frac{1}{p}\|u\|_p^p \ge \frac{1}{2}\|u\|^2 - C\|u\|^p,$$

the conclusion easily follows. □

We now establish some notations and study the geometry of the functional $\mathcal{J}_q$. We observe that $\phi_{tu} = t^2\phi_u$ and therefore, if $\psi_{q,u} \colon [0,\infty) \to \mathbb{R}$ is defined by $\psi_{q,u}(t) = \mathcal{J}_q(tu)$, we have that

$$\psi_{q,u}(t) = \frac{t^2}{2}\|u\|^2 + \frac{q^2t^4}{4}\int \phi_u u^2 - \frac{t^p}{p}\|u\|_p^p.$$

Whenever q and u are fixed, we will use for brevity also the notation $\psi := \psi_{q,u}$. A simple analysis shows that:

Proposition 2.4. *For each $q \in \mathbb{R}\setminus\{0\}$ and $u \in H^1(\mathbb{R}^3)\setminus\{0\}$, there are only three possibilities for the graph of ψ:*

(i) *The function ψ has only two critical points when $t > 0$, to wit, $0 < t_q^-(u) < t_q^+(u)$. Moreover, $t_q^-(u)$ is a local maximum with $\psi''(t_q^-(u)) < 0$ and $t_q^+(u)$ is a local minimum with $\psi''(t_q^+(u)) > 0$.*

(ii) *The function ψ has only one critical point when $t > 0$ at the value $t_q(u)$. Moreover, $\psi''(t_q(u)) = 0$ and ψ is increasing.*

(iii) *The function ψ is increasing and has no critical points.*

It is important to notice that (i) happens for q small, and (iii) for q large.

Let us consider the Nehari manifold associated with the functional $\mathcal{J}_q$, that is,

$$\mathcal{N}_q = \{u \in H^1(\mathbb{R}^3)\setminus\{0\} : \psi'_{q,u}(1) = 0\}.$$

Note that, for $u \in \mathcal{N}_q$

$$\|u\|^2 \leq \|u\|^2 + q^2\int \phi_u u^2 \leq C\|u\|^p,$$

and then all the Nehari manifolds are bounded away from zero uniformly in q in the sense that

$$\exists \widetilde{C} > 0 \text{ such that for all } q \in \mathbb{R},\ u \in \mathcal{N}_q \text{ one has } \|u\| \geq \widetilde{C}. \tag{2.3}$$

Moreover, since

$$\mathcal{N}_q = \mathcal{N}_q^+ \cup \mathcal{N}_q^0 \cup \mathcal{N}_q^-,$$

where

$$\begin{aligned}\mathcal{N}_q^+ &= \{u \in \mathcal{N}_q : \psi''(1) > 0\},\\ \mathcal{N}_q^0 &= \{u \in \mathcal{N}_q : \psi''(1) = 0\},\\ \mathcal{N}_q^- &= \{u \in \mathcal{N}_q : \psi''(1) < 0\},\end{aligned}$$

as an application of the Implicit Function Theorem one has the following:

Proposition 2.5. *If $\mathcal{N}_q^+, \mathcal{N}_q^- \neq \emptyset$, then $\mathcal{N}_q^+$, $\mathcal{N}_q^-$ are C^1 manifolds of codimension 1 in $H^1(\mathbb{R}^3)$. Moreover, $u \in \mathcal{N}_q^+ \cup \mathcal{N}_q^-$ is a critical point for the functional $\mathcal{J}_q$ if and only if u is a critical point of the constrained functional $(\mathcal{J}_q)_{|\mathcal{N}_q^+\cup\mathcal{N}_q^-} : \mathcal{N}_q^+ \cup \mathcal{N}_q^- \to \mathbb{R}$.*

Note that, for a fixed $u \in H^1(\mathbb{R}^3) \setminus \{0\}$, we have $tu \in \mathcal{N}_q^0$ if and only if $\psi'_{q,tu}(1) = \psi''_{q,tu}(1) = 0$, i.e. the following system of equations is satisfied:

$$\begin{cases} t\|u\|^2 + q^2 t^3 \displaystyle\int \phi_u u^2 - t^{p-1}\|u\|_p^p = 0, \\ \|u\|^2 + 3q^2 t^2 \displaystyle\int \phi_u u^2 - (p-1)t^{p-2}\|u\|_p^p = 0. \end{cases} \tag{2.4}$$

We can solve system (2.4) with respect to the variables q and t to obtain a unique solution given by

$$t(u) = \left(\frac{2\|u\|^2}{(4-p)\|u\|_p^p} \right)^{1/(p-2)}$$

and

$$\begin{aligned} q(u) &= C_p \frac{\|u\|_p^{p/(p-2)}}{\|u\|^{(4-p)/(p-2)}\|\phi_u\|_{\mathcal{D}}}, \\ C_p &= \frac{2(p-2)^{1/2}\pi^{1/2}(4-p)^{(4-p)/2(p-2)}}{2^{1/(p-2)}}. \end{aligned} \tag{2.5}$$

In addition, the solutions $q(u)$ and $t(u)$ are related by

$$t(u) = \left(\frac{2q^2(u)}{4\pi(p-2)} \frac{\|\phi_u\|_{\mathcal{D}}^2}{\|u\|_p^p} \right)^{1/(p-4)}.$$

Define the extremal value (see Il'yasov [**3**])

$$q^* = \sup\{q(u) : u \in H^1(\mathbb{R}^3) \setminus \{0\}\}.$$

Lemma 2.6. *The function $H^1(\mathbb{R}^3) \setminus \{0\} \ni u \mapsto q(u)$ defined in* (2.5) *is* 0*-homogeneous. Moreover, $q^* < \infty$.*

Proof: That $q(u)$ is 0-homogeneous is obvious. Let us prove that $q^* < \infty$. Indeed, since $p \in (2, 3]$ we have from the interpolation inequality that, for all $u \in H^1(\mathbb{R}^3)$,

$$\|u\|_p^p \le \|u\|^{6-2p}\|u\|_3^{3p-6}. \tag{2.6}$$

Combining inequality (2.6) with Proposition 2.2 we conclude that

$$\|u\|_p^p \le C\|u\|^{6-2p}\|u\|^{(3p-6)/3}\|\phi_u\|_{\mathcal{D}}^{(3p-6)/3} = C\|u\|^{4-p}\|\phi_u\|_{\mathcal{D}}^{p-2}, \tag{2.7}$$

for some constant $C > 0$. It follows from (2.7) that

$$q(u) \le C \frac{\|u\|^{(4-p)/(p-2)}\|\phi_u\|_{\mathcal{D}}}{\|u\|^{(4-p)/(p-2)}\|\phi_u\|_{\mathcal{D}}} \le C,$$

completing the proof. □

Another extremal value which will be important for us is the one such that, for larger values of the parameter, the functional is always non-negative. Let us start by fixing $u \in H^1(\mathbb{R}^3) \setminus \{0\}$ and considering the system

$$\begin{cases} \psi_{q_0,u}(t_0) = \dfrac{t_0^2}{2}\|u\|^2 + q_0^2\dfrac{t_0^4}{4}\displaystyle\int \phi_u u^2 - \dfrac{t_0^p}{p}\|u\|_p^p = 0, \\ \psi'_{q_0,u}(t_0) = t_0\|u\|^2 + q_0^2 t_0^3\displaystyle\int \phi_u u^2 - t_0^{p-1}\|u\|_p^p = 0. \end{cases}$$

One can solve this system with respect to the variables t_0 and q_0 to obtain the unique solution given by

$$\begin{aligned} q_0(u) &= C_{0,p}\frac{\|u\|_p^{p/(p-2)}}{\|u\|^{(4-p)/(p-2)}\|\phi_u\|_{\mathcal{D}}}, \\ C_{0,p} &= \frac{2^{3/2}(p-2)^{1/2}\pi^{1/2}(4-p)^{(4-p)/2(p-2)}}{p^{1/(p-2)}}, \end{aligned} \tag{2.8}$$

and $t_0(u)$ is given by

$$t_0(u) = \left(\frac{pq_0^2(u)}{2(p-2)}\frac{\|\phi_u\|_{\mathcal{D}}^2}{\|u\|_p^p}\right)^{1/(p-4)}.$$

Observe that $C_{0,p} < C_p$, where C_p is the one appearing in (2.5). Then $q_0(u) < q(u)$. Define the extremal value as

$$q_0^* = \sup\{q_0(u) : u \in H^1(\mathbb{R}^3) \setminus \{0\}\}.$$

Remark 1. Since $q_0(u)$ is a multiple of $q(u)$, Lemma 2.6 also holds true for the function q_0.

The solutions $q(u)$ and $q_0(u)$ given in (2.5) and (2.8) have the following geometrical interpretation which can be proved starting from Proposition 2.4.

Proposition 2.7. *For each $u \in H^1(\mathbb{R}^3) \setminus \{0\}$ the following holds:*

(i) *$q(u)$ is the unique parameter $q > 0$ for which the fiber map $\psi_{q,u}$ has a critical point with second derivative zero at $t(u)$. Moreover, if $0 < q < q(u)$, then $\psi_{q,u}$ satisfies (i) of Proposition 2.4 while if $q > q(u)$, then $\psi_{q,u}$ satisfies (iii) of Proposition 2.4.*

(ii) *$q_0(u)$ is the unique parameter $q>0$ for which the fiber map $\psi_{q,u}$ has a critical point with zero energy at $t_0(u)$. Moreover, if $0<q<q_0(u)$, then $\inf_{t>0}\psi_{q,u}(t) < 0$ while if $q > q_0(u)$, then $\inf_{t>0}\psi_{q,u}(t) = 0$.*

Moreover, the parameter q_0^* has the geometrical interpretation that for each $q \in (0, q_0^*)$, there exists at least one $u \in H^1(\mathbb{R}^3) \setminus \{0\}$ for which $\mathcal{J}_q(u) < 0$, while if $q \geq q_0^*$, then $\mathcal{J}_q(u) \geq 0$ for all $u \in H^1(\mathbb{R}^3)$. In both papers [**2, 5**] the necessity of small values of q was imposed in order to show that there exists a function where the functional is negative, in such a way that $\mathcal{J}_q$ possesses a Mountain Pass Geometry. Therefore, the argument employed in both papers does not work for $q > q_0^*$.

The above proposition has the following important consequences.

Corollary 2.8. *If $q > q^*$, the functional $\mathcal{J}_q$ has no critical points other than the zero function. Moreover, if $q < q^*$, then $\mathcal{N}_q^- \neq \emptyset$ and $\mathcal{N}_q^+ \neq \emptyset$.*

In particular, item 1 of Theorem 1.3 is proved.

Proof: It is sufficient to show that, for each $u \in H^1(\mathbb{R}^3) \setminus \{0\}$, the function $\psi_{q,u}$ has no critical points for $q > q^*$. Actually, this is a consequence of the inequalities $q(u) \leq q^* < q$ and (i) of Proposition 2.7.

Now, assume that $q < q^*$. From the definition of q^*, there exists $u \in H^1(\mathbb{R}^3) \setminus \{0\}$ such that $q < q(u) < q^*$. Therefore, from (i) of Proposition 2.7 we conclude that $\mathcal{N}_q^- \neq \emptyset$ and $\mathcal{N}_q^+ \neq \emptyset$. □

Corollary 2.9. *For each $q \geq q_0^*$, we have $\mathcal{J}_q(u) \geq 0$ for all $u \in H^1(\mathbb{R}^3)$. Moreover, if $q < q_0^*$, then there exists $u \in H^1(\mathbb{R}^3)$ such that $\mathcal{J}_q(u) < 0$.*

Proof: Indeed, assume that $q \geq q_0^*$. It follows that $q > q_0(u)$ for each $u \in H^1(\mathbb{R}^3)\setminus\{0\}$ and from item (ii) of Proposition 2.7 one has $\inf_{t>0} \psi_{q,u}(t) = 0$. Therefore, $\mathcal{J}_q(u) \geq 0$.

Now, assume that $q < q_0^*$. From the definition of q_0^*, there exists $w \in H^1(\mathbb{R}^3) \setminus \{0\}$ such that $q < q_0(w) < q_0^*$. Therefore, from (ii) of Proposition 2.7 we conclude that $\inf_{t>0} \psi_{q,w}(t) < 0$ and hence there exists $t > 0$ such that if $u := tw$, we have $\mathcal{J}_q(u) < 0$. □

Let us conclude this section with the following important result.

Proposition 2.10. *There exists a positive constant m such that*

$$\mathcal{J}_q(u) \geq m \quad \text{for all } q \in \mathbb{R},\, u \in \mathcal{N}_q^0.$$

Proof: From equations (2.4) with $t = 1$ we have that

$$\begin{cases} \|u\|^2 + q^2 \displaystyle\int \phi_u u^2 - \|u\|_p^p = 0, \\ \|u\|^2 + 3q^2 \displaystyle\int \phi_u u^2 - (p-1)\|u\|_p^p = 0. \end{cases}$$

It follows that

$$q^2\int \phi_u u^2 = \|u\|_p^p - \|u\|^2 \quad \text{and} \quad \|u\|_p^p = \frac{2}{4-p}\|u\|^2,$$

so that $\mathcal{J}_q(u) = \frac{p-2}{4p}\|u\|^2$ for each $u \in \mathcal{N}_q^0$. From (2.3) the proof is completed. □

It is worth to point out that all that we have done in this section does not use the radial setting, and clearly these results also hold in $H_r^1(\mathbb{R}^3)$.

3. Global minima and (PS) sequences for $\mathcal{J}_q$

In this section we prove the result below. It is fundamental here and in the following section to work with radial functions.

Proposition 3.1. *The following conditions hold:*

(i) *For each $q \in (0, q_0^*)$, we have that $-\infty < \inf_{u\in H_r^1(\mathbb{R}^3)} \mathcal{J}_q(u) < 0$.*

(ii) *For each $q > 0$, if $\{u_n\} \subset H_r^1(\mathbb{R}^3)$ is a sequence such that $\mathcal{J}_q'(u_n) \to 0$ as $n \to \infty$, then $\{u_n\}$ is convergent, up to subsequences.*

Proof: Let us show (i). From Corollary 2.9 we know that

$$\inf_{u\in H_r^1(\mathbb{R}^3)} \mathcal{J}_q(u) < 0.$$

We claim that $-\infty < \inf_{u\in H_r^1(\mathbb{R}^3)} \mathcal{J}_q(u)$. In fact, given $\varepsilon > 0$ such that $D := \frac{q^2}{16\pi} - \varepsilon^4 > 0$, by Proposition 2.2 we have that for each $u \in H_r^1(\mathbb{R}^3)$

$$\begin{aligned}\mathcal{J}_q(u) &= \frac{1}{4}\|\nabla u\|_2^2 + \frac{1}{4}\|\nabla u\|_2^2 + \frac{1}{2}\|u\|_2^2 + \frac{q^2}{4}\int \phi_u u^2 - \frac{1}{p}\|u\|_p^p \\ &\geq \frac{1}{4}\|\nabla u\|_2^2 + D\|\phi_u\|_{\mathcal{D}}^2 + \frac{1}{2}\|u\|_2^2 + \frac{\pi\varepsilon^2}{4}\|u\|_3^3 - \frac{1}{p}\|u\|_p^p \qquad (3.1)\\ &= \frac{1}{4}\|u\|^2 + D\|\phi_u\|_{\mathcal{D}}^2 + \int f(u),\end{aligned}$$

where

$$f(t) = \frac{1}{4}t^2 + \frac{\pi\varepsilon^2}{4}t^3 - \frac{1}{p}t^p \quad \text{for all } t > 0.$$

A simple analysis shows that $I := \inf_{t>0} f(t) > -\infty$ and if $f(t) < 0$ for some $t > 0$, then $f^{-1}((-\infty, 0)) = (\alpha, \beta)$, where $0 < \alpha < \beta < \infty$.

If $I \geq 0$, since $D_q > 0$, we conclude from (3.1) that

$$-\infty < \inf_{u\in H_r^1(\mathbb{R}^3)} \mathcal{J}_q(u).$$

If $I < 0$, then

$$\mathcal{J}_q(u) \geq \frac{1}{4}\|u\|^2 + D_q\|\phi_u\|_{\mathcal{D}}^2 + I\,\text{meas}(A), \qquad (3.2)$$

where $A = \{x \in \mathbb{R}^3 : u(x) \in (\alpha, \beta)\}$. If there exists a sequence $\{u_n\} \subset H^1_r(\mathbb{R}^3)$ such that $\mathcal{J}_q(u_n) \to -\infty$ as $n \to \infty$, then $\|u_n\| \to +\infty$. Moreover, by (3.2) we can assume without loss of generality that

$$\frac{1}{4}\|u_n\|^2 < |I| \operatorname{meas}(A_n) \quad \text{for all } n \in \mathbb{N}. \tag{3.3}$$

By the result of Strauss [**8**] we know that there exists a positive constant C such that

$$|u(x)| \le C|x|^{-1}\|u\| \quad \text{for all } u \in H^1_r(\mathbb{R}^3). \tag{3.4}$$

Define $\rho_n = \sup\{|x| : x \in A_n\}$ and observe from inequalities (3.3) and (3.4) that, for every $x \in \mathbb{R}^3$ with $|x| = \rho_n$ we have

$$0 < \alpha \le u_n(x) \le C\rho_n^{-1}\|u_n\| \le 2C\rho_n^{-1}(|I| \operatorname{meas}(A_n))^{1/2}$$

and hence, for some $C' > 0$ we deduce

$$C'\rho_n \le \operatorname{meas}(A_n)^{1/2}. \tag{3.5}$$

Similar to the deduction of (3.3), we can assume without loss of generality that

$$D\|\phi_{u_n}\|^2_{\mathcal{D}} < |I| \operatorname{meas}(A_n) \quad \text{for all } n \in \mathbb{N},$$

and hence, since the function $(0,\infty) \ni t \mapsto (1 - e^{-t/a})/t$ is decreasing, we conclude

$$\begin{aligned} |I| \operatorname{meas}(A_n) &> D\|\phi_{u_n}\|^2_{\mathcal{D}} \\ &= \int\int \frac{1 - e^{-\frac{|x-y|}{a}}}{|x-y|} u_n^2(x)u_n^2(y) \\ &\ge \int_{A_n}\int_{A_n} \frac{1 - e^{-\frac{|x-y|}{a}}}{|x-y|} u_n^2(x)u_n^2(y) \\ &\ge \frac{1 - e^{-\frac{2\rho_n}{a}}}{2\rho_n} \alpha^4 \operatorname{meas}(A_n)^2, \end{aligned}$$

which implies

$$\frac{|I|}{\alpha^4} \ge \frac{1 - e^{-\frac{2\rho_n}{a}}}{2\rho_n} \operatorname{meas}(A_n) \quad \text{for all } n \in \mathbb{N}. \tag{3.6}$$

Observe from (3.5) that $|A_n| \to \infty$ as $n \to \infty$ and from (3.6) that $\rho_n \to \infty$ as $n \to \infty$. Combining (3.5) with (3.6) we obtain

$$C'' \ge (1 - e^{-\frac{2\rho_n}{a}})\rho_n,$$

for some $C'' > 0$, which is clearly a contradiction and therefore (i) is proved.

Let us show (ii). From the convergence $\mathcal{J}_q'(u_n) \to 0$ as $n \to \infty$, we can assume without loss of generality that

$$\mathcal{J}_q'(u_n)[u_n] \le \|u_n\| \quad \text{for all } n \in \mathbb{N}. \tag{3.7}$$

On the other hand, from Proposition 2.2 we have

$$\|u_n\|_3^3 \le \frac{1}{\pi}\left(\frac{1}{\varepsilon^2}\|\nabla u_n\|_2^2 + \varepsilon^2\|\phi_{u_n}\|_{\mathcal{D}}^2\right),$$

where $\varepsilon > 0$ is chosen now such that $\frac{q^2}{4\pi} - \frac{\varepsilon^4}{2} > 0$. It follows that, for all $n \in \mathbb{N}$,

$$\begin{aligned}
\mathcal{J}_q'(u_n)[u_n] &= \|\nabla u_n\|^2 + \|u_n\|_2^2 + \frac{q^2}{4\pi}\|\phi_{u_n}\|_{\mathcal{D}}^2 - \|u_n\|_p^p \\
&\ge \frac{1}{2}\|u_n\|^2 + \frac{1}{2}\|u_n\|_2^2 + \frac{\pi\varepsilon^2}{2}\|u_n\|_3^3 - \frac{\varepsilon^4}{2}\|\phi_{u_n}\|_{\mathcal{D}}^2 \\
&\quad + \frac{q^2}{4\pi}\|\phi_{u_n}\|_{\mathcal{D}}^2 - \|u_n\|_p^p \\
&= \frac{1}{2}\|u_n\|^2 + \left(\frac{q^2}{4\pi} - \frac{\varepsilon^4}{2}\right)\|\phi_{u_n}\|_{\mathcal{D}}^2 + \frac{1}{2}\|u_n\|_2^2 \\
&\quad + \frac{\pi\varepsilon^2}{2}\|u_n\|_3^3 - \|u_n\|_p^p \\
&= \frac{1}{2}\|u_n\|^2 + \left(\frac{q^2}{4\pi} - \frac{\varepsilon^4}{2}\right)\|\phi_{u_n}\|_{\mathcal{D}}^2 + \int g(u_n),
\end{aligned} \tag{3.8}$$

where $g(t) = t^2/2 + \frac{\pi\varepsilon^2}{2}t^3 - t^p$ for $t > 0$. We combine (3.7) with (3.8) to conclude

$$\|u_n\| \ge \frac{1}{2}\|u_n\|^2 + \left(\frac{q^2}{4\pi} - \frac{\varepsilon^4}{2}\right)\|\phi_{u_n}\|_{\mathcal{D}}^2 + \int g(u_n).$$

We conclude as in the proof of (i) that $\{u_n\}$ is bounded. Once we know that $\{u_n\}$ is bounded, standard arguments (observe that the analogous of [**5**, Lemma 2.1] is valid) produce a convergent subsequence. □

Remark 2. Note that (ii) in Proposition 3.1 can be extended in the following way: if $q_n \to q$ and if $\{u_n\} \subset H_r^1(\mathbb{R}^3)$ is a sequence such that $\mathcal{J}_{q_n}'(u_n) \to 0$ as $n \to \infty$, then $\{u_n\}$ is convergent, up to subsequences. This is due to the smooth dependence of $\mathcal{J}_q'$ on q.

4. Existence of two radial solutions

In this section we prove item 2 of Theorem 1.3.

Proposition 4.1. *For each $q \in (0, q_0^*)$ there exists a global minimum u_q such that $\mathcal{J}_q(u_q) < 0$.*

Proof: This follows from Proposition 3.1 and Ekeland's Variational Principle. □

Now we prove the existence of a local minimizer for $\mathcal{J}_q$ when q is near q_0^*. To do so, we first prove the existence of a global minimizer for the functional $\mathcal{J}_{q_0^*}$.

Corollary 4.2. *There exists a global minimizer $u_{q_0^*} \neq 0$ of $\mathcal{J}_{q_0^*}$ such that $\mathcal{J}_{q_0^*}(u_{q_0^*}) = 0$.*

Proof: Indeed, suppose that $q_n \uparrow q_0^*$ as $n \to \infty$. From Proposition 4.1, for each n, there exists $u_n := u_{q_n}$ such that u_n is a global minimum for $\mathcal{J}_{q_n}$ and $\mathcal{J}_{q_n}(u_n) < 0$. It follows that $\mathcal{J}'_{q_n}(u_n) = 0$ for each n and, since all the Nehari manifolds are bounded away from zero uniformly in q, by (2.3) we have $\|u_n\| \geq \widetilde{C}$ for each n. From (ii) in Proposition 3.1 (see Remark 2) we conclude that $u_n \to u \neq 0$. As $\mathcal{J}_{q_n}(u_n) < 0$ for each n, we conclude that $\mathcal{J}_{q_0^*}(u) \leq 0$ and from Corollary 2.9 it follows that $\mathcal{J}_{q_0^*}(u) = 0$. Then it is sufficient to set $u_{q_0^*} := u$ and the proof is completed. □

Remark 3. From the definition of q_0^* and Corollary 4.2 it follows that $q_0(u_{q_0^*}) = q_0^*$. Moreover, $q_0^* < q(u_{q_0^*})$.

For $q > 0$, define

$$\widehat{\mathcal{J}}_q := \inf\{\mathcal{J}_q(u) : u \in \mathcal{N}_q^+ \cup \mathcal{N}_q^0\}.$$

Observe that

$$\widehat{\mathcal{J}}_q = \inf_{u \in H_r^1(\mathbb{R}^3)} \mathcal{J}_q(u) \quad \text{for all } q \in (0, q_0^*], \tag{4.1}$$

and from Corollary 2.9 we have $\widehat{\mathcal{J}}_q \geq 0$ for $q \geq q_0^*$.

Proposition 4.3. *Given $\delta > 0$, there exists $\varepsilon > 0$ such that $\widehat{\mathcal{J}}_q < \delta$ for each $q \in (q_0^*, q_0^* + \varepsilon)$.*

Proof: Indeed, let $u_{q_0^*} \in \mathcal{N}_q^+$ be given as in Corollary 4.2. Observe that if $q \downarrow q_0^*$, then $\mathcal{J}_q(u_{q_0^*}) \to \mathcal{J}_{q_0^*}(u_{q_0^*}) = 0$. Moreover, since $q_0^* < q(u_{q_0^*})$, it follows that there exists $\varepsilon_1 > 0$ such that $q_0^* + \varepsilon_1 < q(u_{q_0^*})$. From Proposition 2.4 and (ii) in Proposition 2.7, for each $q \in (q_0^*, q_0^* + \varepsilon_1)$, there exists $t_q^+(u_{q_0^*})$ such that $t_q^+(u_{q_0^*})u_{q_0^*} \in \mathcal{N}_q^+$. Note that $t_q^+(u_{q_0^*}) \to 1$ as $q \downarrow q_0^*$, and therefore

$$\mathcal{J}_q(u_{q_0^*}) \leq \mathcal{J}_q(t_q^+(u_{q_0^*})u_{q_0^*}) \to \mathcal{J}_{q_0^*}(u_{q_0^*}) = 0, \quad q \downarrow q_0^*.$$

If $\varepsilon_2 > 0$ is chosen in such a way that $\mathcal{J}_q(t_q^+(u_{q_0^*})u_{q_0^*}) < \delta$ for each $q \in (q_0^*, q_0^* + \varepsilon_2)$, then we set $\varepsilon = \min\{\varepsilon_1, \varepsilon_2\}$ and the proof is completed. $\square$

Let us recall that by Proposition 2.3, there exist positive constants ρ, M such that $\mathcal{J}_q(u) \geq M$ for each $\|u\| = \rho$. We can assume without loss of generality that $\rho < \widetilde{C}$, where $\widetilde{C}$ is such that

$$\|u\| \geq \widetilde{C} \quad \text{for all } q \in \mathbb{R} \text{ and } u \in \mathcal{N}_q$$

(see (2.3)).

We choose $\delta > 0$ in Proposition 4.3 in such a way that

$$\delta < \min\{M, m\}, \tag{4.2}$$

where m is the positive constant such that, by Proposition 2.10,

$$\mathcal{J}_q(u) \geq m \quad \text{for all } q \in \mathbb{R} \text{ and } u \in \mathcal{N}_q^0.$$

Let $\varepsilon > 0$ be as in Proposition 4.3 in correspondence of the above fixed $\delta > 0$.

Proposition 4.4. *We have*

$$\inf\{\mathcal{J}_q(u) : \|u\| \geq \rho\} = \widehat{\mathcal{J}}_q, \quad \textit{for all } q \in (q_0^*, q_0^* + \varepsilon).$$

Proof: First, observe from the inequality $\rho < \widetilde{C}$ that $\inf\{\mathcal{J}_q(u) : \|u\| \geq \rho\} \leq \widehat{\mathcal{J}}_q$. We claim that the equality holds. Indeed, on one hand, if the fiber map $\psi_{q,u}$ satisfies (ii) or (iii) of Proposition 2.4, then $\inf_{t>\rho} \psi_{q,u}(t) = M$. On the other hand, if the fiber map $\psi_{q,u}$ satisfies (i) of Proposition 2.4, then $\inf_{t>\rho} \psi_{q,u}(t) \geq \widehat{\mathcal{J}}_q$. As $M > \delta > \widehat{\mathcal{J}}_q$, the proof is completed. $\square$

Corollary 4.5. *For each $q \in (q_0^*, q_0^* + \varepsilon)$ there exists $u_q \in \mathcal{N}_q^+$ such that $\mathcal{J}_q(u_q) = \widehat{\mathcal{J}}_q$. In particular $\mathcal{J}_q(u_q) > 0$ and $\|u_q\| \geq \widetilde{C} > \rho$.*

Proof: Fix $q \in (q_0^*, q_0^* + \varepsilon)$ and let $\{u_n\} \subset \mathcal{N}_q^+ \cup \mathcal{N}_q^0$ be a minimising sequence for $\widehat{\mathcal{J}}_q < \delta$ by Proposition 4.3. Since $m > \delta$ and, by Proposition 2.10, $\mathcal{J}_q(u) \geq m$ on $\mathcal{N}_q^0$, we can assume that $\{u_n\} \subset \mathcal{N}_q^+$ and hence, by Ekeland's Variational Principle, also that $\mathcal{J}_q'(u_n) \to 0$. We conclude from Proposition 3.1 that $u_n \to u$ in $H_r^1(\mathbb{R}^3)$ with $\|u\| \geq \widetilde{C} > \rho$. Setting $u_q := u$ clearly we obtain that $u_q \in \mathcal{N}_q^+$ and $\mathcal{J}_q(u_q) = \widehat{\mathcal{J}}_q$. Due to the definition of q_0^* and the fact that $q > q_0^*$, we conclude that $\mathcal{J}_q(u_q) > 0$. $\square$

We observe two properties of the function $(0, q_0^* + \varepsilon) \ni q \mapsto \widehat{\mathcal{J}}_q$.

Lemma 4.6. *The function $(0, q_0^* + \varepsilon) \ni q \mapsto \widehat{\mathcal{J}}_q$ is increasing and continuous.*

Proof: Suppose that $q < q'$. From Proposition 4.1, Corollary 4.2, and Corollary 4.5 there exists $u_{q'}$ such that $\widehat{\mathcal{J}}_{q'} = \mathcal{J}_{q'}(u_{q'})$.

If $q' \in (q_0^*, q_0^* + \varepsilon)$, from Corollary 4.5 we have $\|u_{q'}\| \geq \widetilde{C} > \rho$ and hence from Proposition 4.4 we obtain

$$\widehat{\mathcal{J}}_q \leq \mathcal{J}_q(u_{q'}) < \mathcal{J}_{q'}(u_{q'}) = \widehat{\mathcal{J}}_{q'}.$$

If $q' \in (0, q_0^*]$, the lemma follows by (4.1).

Now we prove that $(0, q_0^* + \varepsilon) \ni q \mapsto \widehat{\mathcal{J}}_q$ is continuous. Suppose that $q_n \uparrow q \in (0, q_0^* + \varepsilon)$. From Proposition 4.1, Corollary 4.2, and Corollary 4.5, for each n there exists $u_n := u_{q_n}$ such that $\widehat{\mathcal{J}}_{q_n} = \mathcal{J}_{q_n}(u_n)$. Similar to the proof of Corollary 4.2 we may assume that $u_n \to u \neq 0$.

As before, if $q \in (0, q_0^*]$, the lemma follows from (4.1).

If $q \in (q_0^*, q_0^* + \varepsilon)$, observe from Corollary 4.5 that $\|u\| > \rho$. We claim that $\widehat{\mathcal{J}}_{q_n} \to \widehat{\mathcal{J}}_q$ as $n \to \infty$. Indeed, since $(0, q_0^* + \varepsilon) \ni q \mapsto \widehat{\mathcal{J}}_q$ is increasing, we can assume that $\widehat{\mathcal{J}}_{q_n} < \widehat{\mathcal{J}}_q$ for each n and $\widehat{\mathcal{J}}_{q_n} \to \mathcal{J}_q(u) \leq \widehat{\mathcal{J}}_q$ as $n \to \infty$, which implies that $\mathcal{J}_q(u) = \widehat{\mathcal{J}}_q$.

Now, suppose that $q_n \downarrow q \in (0, q_0^* + \varepsilon)$. As $(0, q_0^* + \varepsilon) \ni q \mapsto \widehat{\mathcal{J}}_q$ is increasing, we can assume that $\widehat{\mathcal{J}}_q < \widehat{\mathcal{J}}_{q_n}$ and $\widehat{\mathcal{J}}_q \leq \lim_{n\to\infty} \widehat{\mathcal{J}}_{q_n}$. Choose u_q such that $\widehat{\mathcal{J}}_q = \mathcal{J}_q(u_q)$ ($\|u\| > \rho$ in case $q \in (q_0^*, q_0^* + \varepsilon)$) and observe that $\widehat{\mathcal{J}}_q \leq \lim_{n\to\infty} \widehat{\mathcal{J}}_{q_n} \leq \lim_{n\to\infty} \mathcal{J}_{q_n}(u_q) = \widehat{\mathcal{J}}_q$. □

Now, we turn our attention to the second solution. Let $q \in (0, q_0^*)$. As a consequence of Corollary 2.9 we have that

$$\Gamma_q = \{\gamma \in C([0,1], H_r^1(\mathbb{R}^3)) : \gamma(0) = 0,\ \mathcal{J}_q(\gamma(1)) < 0\}$$

is non-empty. Define the Mountain Pass level

$$c_q := \inf_{\gamma\in\Gamma_q} \max_{t\in[0,1]} \mathcal{J}_q(\gamma(t)) > 0.$$

By Proposition 2.3 and Proposition 3.1 we deduce the following:

Proposition 4.7. *For each $q \in (0, q_0^*)$ there exists $w_q \in H_r^1(\mathbb{R}^3) \setminus \{0\}$ such that $\mathcal{J}_q(w_q) = c_q$ and $\mathcal{J}_q'(w_q) = 0$. In particular, $\mathcal{J}_q(w_q) > \mathcal{J}_q(u_q) \in (-\infty, 0)$.*

Let now $q \in (q_0^*, q_0^* + \varepsilon)$, where $\varepsilon > 0$ is fixed corresponding to δ in (4.2). Let $u_q \in \mathcal{N}_q^+$ (by Corollary 4.5) such that $\mathcal{J}_q(u_q) = \widehat{\mathcal{J}}_q$. Define

$$d_q = \inf_{\gamma\in\Gamma_q} \max_{t\in[0,1]} \mathcal{J}(\gamma(t)),$$

where $\Gamma_q = \{\gamma \in C([0,1], H_r^1(\mathbb{R}^3)) : \gamma(0) = 0,\ \gamma(1) = u_q\}$.

Proposition 4.8. *For each $q \in (q_0^*, q_0^*+\varepsilon)$ there exists $w_q \in H_r^1(\mathbb{R}^3)\setminus\{0\}$ such that $\mathcal{J}_q(w_q) = d_q$ and $\mathcal{J}_q'(w_q) = 0$. In particular, $\mathcal{J}_q(w_q) > \mathcal{J}_q(u_q)$.*

Proof: We combine Proposition 2.3 with the inequality $M > \delta \geq \widehat{\mathcal{J}}_q = \mathcal{J}_q(u_q)$ to obtain a Mountain Pass Geometry for the functional $\mathcal{J}_q$. The proof follows from (ii) in Proposition 3.1. □

Now, we conclude the proof of item 2 of Theorem 1.3. Let ε be given as in Proposition 4.4. The existence of the minimum u_q follows from Proposition 4.1, Corollary 4.2, and Corollary 4.5. The existence of a Mountain Pass critical point w_q satisfying $\mathcal{J}_q(w_q) > \max\{0, \mathcal{J}_q(u_q)\}$ follows by Proposition 4.7 and Proposition 4.8. That u_q and w_q are actually critical points of $\mathcal{J}_q$ follows by Proposition 2.5.

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