Publ. Mat. **64** (2020), 423–452
DOI: 10.5565/PUBLMAT6422003

FOLIATIONS ON PROJECTIVE SPACES ASSOCIATED TO THE AFFINE LIE ALGEBRA

Raphael Constant da Costa

Abstract: In this work we construct some irreducible components of the space of two-dimensional holomorphic foliations on $\mathbb{P}^n$ associated to some algebraic representations of the affine Lie algebra $\mathfrak{aff}(\mathbb{C})$. We give a description of the generalized Kupka components, obtaining a classification of them in terms of the degree of the foliations, in both cases $n=3$ and $n=4$.

2010 Mathematics Subject Classification: 32S65.

Key words: holomorphic foliations, the affine Lie algebra, generalized Kupka components.

1. Introduction

We consider a holomorphic foliation of dimension k and degree d on the projective space $\mathbb{P}^n$, $n \geq 3$. Recall that the degree of such a foliation $\mathcal{F}$ is the degree of the locus of tangency of $\mathcal{F}$ with a generic linear subspace $\mathbb{P}^{n-k} \subset \mathbb{P}^n$. The set of those foliations, which we denote by $\mathcal{F}_k(d,n)$, has a natural structure of quasi-projective variety. In fact, such foliations are defined by an integrable $(n-k)$-form Ω on $\mathbb{C}^{n+1}$, whose coefficients are homogeneous polynomials of degree $d+1$ satisfying $i_{R_{n+1}}\Omega = 0$, where R_{n+1} denotes the radial vector field on $\mathbb{C}^{n+1}$. The $(n-k)$-form Ω is defined up to multiplication by a nonzero scalar, giving rise to a projective space, and the integrability condition imposes polynomial relations on that space. Finally, from the condition $\operatorname{codim}(\operatorname{Sing}\Omega) \geq 2$, where $\operatorname{Sing}(\Omega)$ denotes the singular set of Ω, we identify $\mathcal{F}_k(d,n)$ with a Zariski open subset of a projective variety. A very interesting question is to describe the irreducible components of $\mathcal{F}_k(d,n)$. The known results are mostly concentrated in the codimension one case $(k=n-1)$. Some of the irreducible components of $\mathcal{F}_{n-1}(d,n)$ have been described: linear pull-back [**3**], rational [**11**], logarithmic [**1**], generic pull-back [**5**], associated to the affine Lie algebra [**2**], rigid [**9**], and more recently branched pull-back [**6**]. A complete description of the irreducible components of $\mathcal{F}_{n-1}(d,n)$ is known only in low degrees. In [**12**] it has been shown that $\mathcal{F}_{n-1}(0,n)$ has only one irreducible component, while $\mathcal{F}_{n-1}(1,n)$ consists of two irreducible components. The classification of $\mathcal{F}_{n-1}(2,n)$ was achieved by Cerveau and Lins Neto in [**4**], where they show that $\mathcal{F}_{n-1}(2,n)$ has six irreducible components. The literature on the irreducible components of $\mathcal{F}_k(d,n)$ for $1 \leq k < n-1$ is not as extensive in comparison with the codimension one case. Some results in this direction can be found in [**7**] and [**9**]. The classification of $\mathcal{F}_k(0,n)$ was given in [**10**, Theorem 3.8], while a complete description of $\mathcal{F}_k(1,n)$ was obtained in [**17**, Theorem 6.2 and Corollary 6.3].

In this paper we construct and classify certain components of $\mathcal{F}_2(d,n)$ associated to the affine Lie Algebra $\mathfrak{aff}(\mathbb{C}) = \langle e_1, e_2 \rangle$, where $[e_1, e_2] = e_2$. These components include those described in [**2**]. Let $p_1 > p_2 > \cdots > p_n \geq 1$ be relatively prime positive integers and S the diagonal vector field of $\mathbb{C}^n$ defined by

$$S = p_1 x_1 \frac{\partial}{\partial x_1} + p_2 x_2 \frac{\partial}{\partial x_2} + \cdots + p_n x_n \frac{\partial}{\partial x_n}.$$

Let X be another polynomial vector field on $\mathbb{C}^n$ such that $[S, X] = \lambda X$, for some $\lambda \in \mathbb{Z}$. Note that if $\lambda \neq 0$, S and X give a representation of $\mathfrak{aff}(\mathbb{C})$ in the algebra of polynomial vector fields of $\mathbb{C}^n$. In addition, if $\{p \in \mathbb{C}^n \mid S(p) \wedge X(p) = 0\}$ has codimension at least two, they give rise to a dimension two algebraic foliation $\mathcal{F} = \overline{\mathcal{F}}(S, X)$ on $\mathbb{C}^n$, which is defined by the following integrable $(n-2)$-form

$$\omega = i_S i_X (dx_1 \wedge dx_2 \wedge \cdots \wedge dx_n).$$

Define $\mathfrak{P} = (p_1, p_2, \ldots, p_n)$ and

$$\mathcal{F}(\mathfrak{P}, \lambda, d+1) = \{\mathcal{F} \in \mathcal{F}_2(d+1, n) \mid \mathcal{F} = \overline{\mathcal{F}}(S, X) \text{ in some affine chart}\}.$$

Remark 1.1. The polynomial vector field X on $\mathbb{C}^n$ appearing in the definition of $\mathcal{F} \in \mathcal{F}(\mathfrak{P}, \lambda, d+1)$ extends to a one dimensional foliation on $\mathbb{P}^n$, denoted by $\mathcal{G}_X$. As we show in Lemma 2.5, we can additionally assume that the degree of $\mathcal{G}_X$ is one less than the degree of the foliation $\mathcal{F}$ (see also Remark 2.4). Furthermore, denoting by d the degree of $\mathcal{G}_X$ helps in writing down some parameters that we will need to introduce later on and results in more transparent computations. This is the reason we have adopted the notation $\mathcal{F}(\mathfrak{P}, \lambda, d+1)$, with $d+1$ standing for the degree of the two dimensional foliation.

It turns out that $\overline{\mathcal{F}(\mathfrak{P}, \lambda, d+1)}$, the Zariski closure of $\mathcal{F}(\mathfrak{P}, \lambda, d+1)$, is an irreducible subvariety of $\mathcal{F}_2(d+1, n)$ (see Proposition 2.6). In the cases $n = 3$ and $n = 4$, we use $\mathcal{F}(p, q, r; \lambda, d+1)$ and $\mathcal{F}(p, q, r, s; \lambda, d+1)$, respectively. Next we present some conditions that entail the existence of irreducible components $\overline{\mathcal{F}(\mathfrak{P},\lambda,d+1)}$ of $\mathcal{F}_2(d+1,n)$.

Let ω be a germ of integrable $(n-2)$-form defined at $p \in \mathbb{C}^n$, with $p \in \mathrm{Sing}(\omega)$ and $n \geq 3$.

Definition 1.2. We say that p is a *weakly generalized Kupka* (WGK) singularity of ω if $\mathrm{codim}(\mathrm{Sing}(d\omega)) \geq 3$, where by convention $\mathrm{codim}(\emptyset) = n+1$. In addition, if $\mathrm{codim}(\mathrm{Sing}(d\omega)) \geq n$ we say that p is a *generalized Kupka* (GK) singularity.

Definition 1.3. A dimension two holomorphic foliation $\mathcal{F}$ on $\mathbb{P}^n$ is WGK (respectively GK) if all the singularities of $\mathcal{F}$ are WGK (respectively GK).

The following result was proved in [**2**].

Theorem 1.4. *Suppose that $\mathcal{F}(p,q,r;\lambda,d+1)$ contains some GK foliation, where $\lambda \neq 0$ and $p > q > r$ are relatively prime positive integers. Then $\overline{\mathcal{F}(p,q,r;\lambda,d+1)}$ is an irreducible component of $\mathcal{F}_2(d+1,3)$.*

The irreducible components $\overline{\mathcal{F}(\mathfrak{P},\lambda,d+1)} \subset \mathcal{F}_2(d+1,n)$ containing GK foliations will be called GK components.

Corollary 1.5. *For $d \geq 1$, $\overline{\mathcal{F}(d^2+d+1,d+1,1;-1,d+1)}$ is an irreducible component of $\mathcal{F}_2(d+1,3)$ of dimension N, where $N = 13$ if $d = 1$ and $N = 14$ if $d > 1$. Moreover, this component is the closure of a* $\mathrm{PGL}(4,\mathbb{C})$ *orbit on $\mathcal{F}_2(d+1,3)$.*

Note that these families extend the so-called exceptional component ($d = 1$), that appears originally in [**4**], and they consist of the only general families of GK components provided by Theorem 1.4 that are known so far.

Even when $p_1 \geq p_2 \geq \cdots \geq p_n$, the construction of $\mathcal{F}(\mathfrak{P},\lambda,d+1)$ makes sense. Recently, the latter case was treated in [**16**]. Our first result extends Theorem 1.4 to higher dimensional projective spaces. Thinking of S as defined in an affine chart $(E \cong \mathbb{C}^n, (x_1,\ldots,x_n))$, denote by $q_0 \in \mathbb{P}^n$ the point corresponding to $0 \in E$.

Theorem A. *Let n, d, λ be integers, with $n \geq 3$ and $d \geq 1$. If $\lambda > 0$ and $\mathcal{F}(\mathfrak{P},\lambda,d{+}1)$ contains some WGK foliation $\mathcal{F}$, where q_0 is a GK singularity of $\mathcal{F}$, then $\overline{\mathcal{F}(\mathfrak{P},\lambda,d+1)}$ is an irreducible component of $\mathcal{F}_2(d+1,n)$. In particular, if $\mathcal{F}(\mathfrak{P},\lambda,d+1)$ contains some GK foliation where $\lambda \neq 0$, then $\overline{\mathcal{F}(\mathfrak{P},\lambda,d+1)}$ is an irreducible component of $\mathcal{F}_2(d+1,n)$.*

Remark 1.6. It is worth pointing out that there are irreducible components of $\mathcal{F}_2(d+1,n)$ provided by Theorem A that are not GK. We give an example of this situation in Remark 3.8.

The proof of Theorem A has much in common with the proof of Theorem 1.4. The main difference is related to recent results on quasi-homogeneous singularities, previously restricted to the case of dimension 3.

Next we give a description of the components $\overline{\mathcal{F}(\mathfrak{P},\lambda,d+1)} \subset \mathcal{F}_2(d+1,n)$ provided by the second part of Theorem A. Loosely speaking, $\mathcal{F}(\mathfrak{P},\lambda,d+1)$ contains a GK foliation if and only if q_0 is a GK singularity of some $\mathcal{F} \in \mathcal{F}(\mathfrak{P},\lambda,d+1)$ and $p_1,\dots,p_n$, λ, d satisfy certain arithmetic relations.

Throughout the text, several parameters will appear. We take the opportunity to define most of them now. Given $p_1,\dots,p_n$, λ, d, by convention set $p_{n+1}=0$. Define $\lambda_1,\dots,\lambda_n$, τ, $\tau_1,\dots,\tau_n$, $\overline{p}_1,\dots,\overline{p}_n$ as follows

$$
(1.1)\qquad \begin{cases} \lambda_1 = p_1(d-1)-\lambda,\ \lambda_i = \lambda - p_i(d-1), & i=2,\dots,n,\\ \tau = \lambda + \sum_{k=1}^{n} p_k, & \\ \tau_1 = p_1(n+d)-\tau,\ \tau_i = \tau - p_i(n+d), & i=2,\dots,n,\\ \overline{p}_j = p_1 - p_{n-j+2}, & j=1,\dots,n. \end{cases}
$$

For $i=1,\dots,n-1$, $j=1,\dots,n$ denote by c_{ij} the following condition

$$
c_{ij}:\begin{cases} p_j+\lambda = p_{i+1}d, & \text{if } j\le i,\\ p_{j+1}+\lambda = p_{i+1}d, & \text{if } j>i. \end{cases}
$$

Before stating the next result, it is worth mentioning that $\mathcal{F}(\mathfrak{P},\lambda,d+1) = \mathcal{F}(\overline{\mathfrak{P}},\lambda_1,d+1)$, where $\overline{\mathfrak{P}} = (\overline{p}_1,\dots,\overline{p}_n)$ (see Proposition 2.10).

Theorem B. *Let $l_1 > \cdots > l_n$ be relatively prime positive integers, $\mu \in \mathbb{Z}$ and $d\ge 1$. Then $\overline{\mathcal{F}(\mathfrak{L},\mu,d+1)}$ is a GK component of $\mathcal{F}_2(d+1,n)$ if and only if it can be written in the form $\overline{\mathcal{F}(\mathfrak{L},\mu,d+1)} = \overline{\mathcal{F}(\mathfrak{P},\lambda,d+1)}$, such that $p_1 > \cdots > p_n$ are relatively prime positive integers, $\lambda \in \mathbb{Z}_{>0}$, satisfying*

(a) *q_0 is a GK singularity of some $\mathcal{F} \in \mathcal{F}(\mathfrak{P},\lambda,d+1)$,*

and $p_1,\dots,p_n$, λ, d satisfy either

(b.1)
- *$c_{11}, c_{22},\dots,c_{ii}, c_{i+1,i+2}, c_{i+2,i+3},\dots,c_{n-1,n}$, for some $0\le i\le \lfloor \frac{n-1}{2}\rfloor$,*
- *$\tau_j \ne 0$, $j=2,3,\dots,n$,*

or

(b.2)
- *$c_{11}, c_{22},\dots,c_{i-2,i-2}$, $\lambda = p_i(d-1)$, $c_{i,i+1}, c_{i+1,i+2},\dots,c_{n-1,n}$, for some $2\le i\le \lfloor \frac{n+2}{2}\rfloor$,*
- *$\tau_j\ne 0$, $j\in\{2,3,\dots,n\}\setminus\{i\}$.*

Moreover, if $\mathfrak{L}\ne\mathfrak{P}$ or $\mu\ne\lambda$, then $\mathfrak{L}=\overline{\mathfrak{P}}$ and $\mu=\lambda_1$.

For each $d \geq 1$, we have at least one irreducible component of $\mathcal{F}_2(d+1,n)$ described by Theorem B. As we will see in Corollary 4.5, for

$$\mathfrak{L} = (d^{n-1} + \cdots + 1, d^{n-2} + \cdots + 1, \ldots, d+1, 1),$$

the family $\overline{\mathcal{F}(\mathfrak{L}, -1, d+1)}$ is an irreducible component of $\mathcal{F}_2(d+1,n)$, extending the irreducible components of Corollary 1.5. Moreover, when $d = 1$ it is the only GK irreducible component $\overline{\mathcal{F}(\mathfrak{P}, \lambda, 2)} \subset \mathcal{F}_2(2,n)$. This is the reason we sometimes focus on the case $d \geq 2$. We point out that for $d = 1$ this irreducible component was established in [**9**]. For $d \geq 2$ it is new.

For the cases $n = 3$ and $n = 4$, we can exhibit the GK components in a more explicit way, as follows.

Theorem B.1. *Let $p > q > r$ be relative prime positive integers. Then $\overline{\mathcal{F}(p,q,r;\lambda,d+1)}$ is a GK component of $\mathcal{F}_2(d+1,3)$, $d \geq 2$, if and only if either p, q, r, λ, d, or $\overline{p}$, $\overline{q}$, $\overline{r}$, λ_1, d satisfy one of the following relations:*

(a) $p > q = m(d+1) > r = md$, $\lambda = md^2$, $\gcd(p,m) = 1$, *p divides either d^2 or d^2+d+1.*

(b) $p = d > q = r+1 > r$, $\lambda = dr$.

(c) $p = kd > q = md + k > r = md$, $\lambda = md^2$, $\gcd(k,m) = 1$, *k divides $d+1$.*

(d) $p > q = md > r = m(d-1)$, $\lambda = m(d^2-d)$, $\gcd(p,m) = 1$, *p divides either $d^2 - d$, or d^2, or $d^2 - 1$.*

We make some comments about Theorem B.1. It provides a classification of the irreducible components given by Theorem 1.4 in terms of the degree of foliations. In fact, for each $d \geq 2$, we can find (in an algorithmic fashion) all the GK components $\overline{\mathcal{F}(p,q,r;\lambda,d+1)}$ of $\mathcal{F}_2(d+1,3)$. For example, we do so in Corollary 4.8 for the cases $d = 2$ and $d = 3$, obtaining irreducible components of $\mathcal{F}_2(3,3)$ and $\mathcal{F}_2(4,3)$ which have been unknown until now. Corollary 4.9 gives a negative answer to Problem 1 of [**2**], which asks whether, given three positive integers $p > q > r \geq 1$, we can find (λ, d) such that $\mathcal{F}(p,q,r;\lambda,d+1)$ is a GK family. Finally, we describe in Corollary 4.10 new families of irreducible components like those of Corollary 1.5.

For the case $n = 4$, we have an equivalent result.

Theorem B.2. *Let $p > q > r > s$ be relatively prime positive integers. Then $\overline{\mathcal{F}(p,q,r,s;\lambda,d+1)}$ is a GK component of $\mathcal{F}_2(d+1,4)$, $d \geq 2$, if and only if either p, q, r, s, λ, d or $\overline{p}$, $\overline{q}$, $\overline{r}$, $\overline{s}$, λ_1, d satisfy one of the following relations:*

(a) $p > q = m(d^2+d+1) > r = m(d^2+d) > s = md^2$, $\lambda = md^3$, $\gcd(p,m) = 1$, *p divides either d^3 or d^3+d^2+d+1.*

(b) $p = kd > q = md+k > r = m(d+1) > s = md$, $\lambda = md^2$, $\gcd(k,m) = 1$, *either k divides d, or kd divides $m(d^2+d)+k$ (which implies $k = jd$ where j divides $d+1$), or d divides m and k divides d^2+d+1, or k divides $d+1$ and* $\gcd\big(\frac{m(d+1)}{k}, d\big) = 1$.

(c) $p > q = md^2 > r = m(d^2-1) > s = m(d^2-d)$, $\lambda = m(d^3-d^2)$, $\gcd(p,m) = 1$, *p divides either d^3-d^2, or d^3, or d^3-1.*

(d) $p = kd > q = m(d-1)+k > r = md > s = m(d-1)$, $\lambda = m(d^2-d)$, $\gcd(k,m) = 1$, *either k divides $d-1$, or k divides d, or d divides m and k divides d^2-1.*

The same comments on Theorem B.1 apply to Theorem B.2. We exhibit in Corollary 4.11, for example, new irreducible components of $\mathcal{F}_2(3,4)$.

The paper is organized as follows. In Section 2 we list some basic properties of the foliations in $\mathcal{F}(\mathfrak{P},\lambda,d+1)$ which will be used throughout. For the sake of completeness, we determine the tangent sheaf of their foliations and the dimension of these subvarieties as well. In Section 3 we recall basic facts concerning the stability of quasi-homogeneous singularities, settling a key result to obtain Theorem B. Theorem A is also proved in this section. Finally, Section 4 is dedicated to the proofs of Theorems B, B.1, B.2, and some consequences.

2. Preliminaries

Throughout this paper, given a polynomial vector field Z on $\mathbb{C}^n$, we denote by $Z = \hat{Z}_0 + \hat{Z}_1 + \cdots + \hat{Z}_k$, $\deg(\hat{Z}_i) = i$, $i = 0,\ldots,k$, its decomposition into homogeneous polynomial vector fields. In parallel, we write $\omega = \hat{\omega}_0 + \cdots + \hat{\omega}_k$ for a polynomial $(n-2)$-form ω on $\mathbb{C}^n$.

2.1. Quasi-homogeneous vector fields. Consider the diagonal vector field

$$S = p_1x_1\frac{\partial}{\partial x_1} + p_2x_2\frac{\partial}{\partial x_2} + \cdots + p_nx_n\frac{\partial}{\partial x_n},$$

where $p_1 \geq p_2 \geq \cdots \geq p_n$ are not necessarily positive integers.

The next result is an adapted version of Proposition 4.2.1 of [**14**]. The proof of the original proposition still holds.

Proposition 2.1. *Let $X \neq 0$ be a holomorphic vector field on $\mathbb{C}^n$, where $[S, X] = \lambda X$. Then*

(a) $\lambda \in \mathbb{Z}$.

(b) $\mathrm{Ld}(S, X) = \{z \in \mathbb{C}^n \mid S(z)$ *and* $X(z)$ *are linearly dependent*$\}$ *is a union of orbits of the action induced by the vector field* S.

Additionally, if $p_n \geq 1$, *then*

(c) $\lambda \geq -p_1$ *and* X *is a polynomial vector field.*

(d) *If* $0 \in \mathbb{C}^n$ *is an isolated singularity of* X, *then the Milnor number of* X *at* 0 *is given by*

$$m(X, 0) = \frac{\prod_{j=1}^{n}(p_j + \lambda)}{\prod_{j=1}^{n} p_j}.$$

By Proposition 2.1(a), there is no loss of generality in assuming that $p_1, \ldots, p_n$ are relatively prime in the definition of $\mathcal{F}(\mathfrak{P}, \lambda, d+1)$. The relation $[S, X] = \lambda X$ can be given in some equivalent ways, as follows.

Proposition 2.2. *Let* $X = \sum_{j=1}^{n} X_j(z)\partial/\partial z_j$ *be a holomorphic vector field on* $\mathbb{C}^n$. *Then the following are equivalent:*

(a) $[S, X] = \lambda X$.

(b) $X_j(t^{p_1} z_1, \ldots, t^{p_n} z_n) = t^{p_j+\lambda} X_j(z_1, \ldots, z_n)$, *for all* $1 \leq j \leq n$ *and all* $t \in \mathbb{C}$.

(c) *Write* $X_j = \sum_{j\sigma} a_{j\sigma} z^\sigma$, $1 \leq j \leq n$, *where* $a_{j\sigma} \in \mathbb{C}$ *and for* $\sigma = (\sigma_1, \ldots, \sigma_n)$, $z^\sigma = z_1^{\sigma_1} \cdots z_n^{\sigma_n}$. *If* $a_{j\sigma} \neq 0$, *then* $\sum_{k=1}^{n} p_k \sigma_k = p_j + \lambda$.

For example, if $p_j = 1$, $1 \leq j \leq n$, then $S = R_n$ is the radial vector field on $\mathbb{C}^n$ and the equality $[S, X] = \lambda X$ implies that X is a homogeneous polynomial vector field of degree $\lambda + 1$.

Remark 2.3. Let X be a holomorphic vector field on $\mathbb{C}^n$, satisfying $[S, X] = \lambda X$. Assume that $p_n \geq 1$. If $0 \in \mathbb{C}^n$ is an isolated singularity of X, then $\lambda \geq 0$. If $X(0) \neq 0$, then $\lambda < 0$. This is an immediate consequence of Propositions 2.1 and 2.2 above.

2.2. Some facts about foliations in $\mathcal{F}(\mathfrak{P}, \lambda, d+1)$. Hereafter we assume that

$$p_1 > p_2 > \cdots > p_n \geq 1.$$

Let $\mathcal{F}$ be some foliation of $\mathcal{F}(\mathfrak{P}, \lambda, d+1)$. By definition, $\mathcal{F}$ is given by the following $(n-2)$-form

$$\omega = i_S i_X(\nu_n), \quad [S, X] = \lambda X, \quad \text{and} \quad \nu_n = dx_1 \wedge dx_2 \wedge \cdots \wedge dx_n$$

in some affine coordinate system $(E_0, (x_1, \ldots, x_n))$ that for now we assume

$$E_0 = \{(x_1 : \cdots : x_n : 1) \in \mathbb{P}^n \mid (x_1, \ldots, x_n) \in \mathbb{C}^n\}.$$

As $d\omega$ is a $(n-1)$-form, there exists a vector field Y such that $d\omega = i_Y(\nu_n)$. The latter is called the rotational of ω, and denoted by $Y = \mathrm{rot}(\omega)$. Using Cartan's formulas, we get

$$Y = \mathrm{rot}(\omega) = \tau \cdot X - \mathrm{div}(X) \cdot S,$$

where $\tau = \lambda + \sum_{i=1}^n p_i$ and writing $X = \sum_{i=1}^n X_i \partial/\partial x_i$, we have $\mathrm{div}(X) = \sum_{i=1}^n \frac{\partial X_i}{\partial x_i}$.

By Proposition 2.1(c), we see that $\tau > 0$. Using the above expression for Y, one verifies that

$$(2.1) \qquad [S, Y] = \lambda Y, \quad \omega = \frac{1}{\tau} i_S i_Y(\nu_n), \quad \text{and} \quad \mathrm{div}(Y) = 0.$$

Remark 2.4. Recall that a foliation $\mathcal{F}$ of dimension k and degree l on $\mathbb{P}^n$ is defined, on affine coordinate system $E \cong \mathbb{C}^n$, by an integrable polynomial $(n-k)$-form

$$\omega = \hat{\omega}_0 + \hat{\omega}_1 + \cdots + \hat{\omega}_{l+1},$$

where $i_{R_n}(\hat{\omega}_{l+1}) = 0$, and if $\hat{\omega}_{l+1} = 0$, then $i_{R_n}(\hat{\omega}_l) \neq 0$. On the other hand, a one dimensional foliation $\mathcal{G}$ of degree l on $\mathbb{P}^n$ is defined on E by a polynomial vector field

$$X = \hat{X}_0 + \hat{X}_1 + \cdots + \hat{X}_{l+1},$$

with $\hat{X}_{l+1} = g_l \cdot R_n$, where g_l is a homogeneous polynomial of degree l. If $\hat{X}_{l+1} = 0$, then $\hat{X}_l \neq 0$ and is not of the form $\hat{X}_l = g_{l-1} \cdot R_n$, where g_{l-1} is a homogeneous polynomial of degree $l-1$.

Lemma 2.5. *Let $\mathcal{F}$ be a foliation in $\mathcal{F}(\mathfrak{P}, \lambda, d+1)$, given by $\omega = i_S i_X(\nu_n)$ on E_0. Then X can be chosen in such a way that* $\deg(\mathcal{G}_X) = d$, *where $\mathcal{G}_X$ denotes the one dimensional foliation on $\mathbb{P}^n$ defined by X on E_0.*

Proof: As $\deg(\mathcal{F}) = d+1$, we can write on E_0

$$\omega = i_S i_X(\nu_n) = \frac{1}{\tau} i_S i_Y(\nu_n) = \hat{\omega}_0 + \hat{\omega}_1 + \cdots + \hat{\omega}_{d+2}, \quad i_{R_n}(\hat{\omega}_{d+2}) = 0.$$

Therefore, $Y = \mathrm{rot}(\omega) = \hat{Y}_0 + \hat{Y}_1 + \cdots + \hat{Y}_{d+1}$. The relation $i_{R_n}(\hat{\omega}_{d+2}) = 0$ implies that $i_{R_n} i_S i_{\hat{Y}_{d+1}}(\nu_n) = 0$. Since $\mathrm{Sing}(i_S i_{R_n}(\nu_n))$ is a union of lines, and in particular it has codimension greater than two, it follows from the last equality and Hartog's Theorem that there exist holomorphic functions f and g on E_0 such that $\hat{Y}_{d+1} = f \cdot S + g \cdot R_n$. As $\hat{Y}_{d+1}$, R_n, S are homogeneous, we can assume that $f = f_d$ and $g = g_d$ are homogeneous polynomials of degree d.

Finally, define $\overline{X} = \frac{Y - f_d \cdot S}{\tau}$. Note that $\omega = i_S i_{\overline{X}}(\nu_n)$ and $\deg(\mathcal{G}_{\overline{X}}) = d$. □

Denote by $\mathcal{P}_n$ the space of polynomial vector fields on $\mathbb{C}^n$. Consider the following finite-dimensional vector space over $\mathbb{C}$

$$W_0 = \{Y \in \mathcal{P}_n \mid [S,Y] = \lambda Y,\ \mathrm{div}(Y) \equiv 0,\ \deg(Y) \leq d+1,\\ i_{R_n} i_S i_{\hat{Y}_{d+1}}(\nu_n) \equiv 0\}.$$

From (2.1) and the proof of Lemma 2.5, if $\omega = i_S i_X(\nu_n)$ defines some foliation $\mathcal{F} \in \mathcal{F}(\mathfrak{P}, \lambda, d+1)$ on E_0, then $Y = \mathrm{rot}(\omega) \in W_0$. Reciprocally, given $Y \in W_0$, setting $\omega_Y = \frac{1}{\tau} i_S i_Y(\nu_n)$ we have that $Y = \mathrm{rot}(\omega_Y)$.

In other words, W_0 is nothing more than the ambient space of $Y = \mathrm{rot}(\omega_Y)$ whenever ω_Y defines a foliation $\mathcal{F} \in \mathcal{F}(\mathfrak{P}, \lambda, d+1)$ on E_0. For example, if Theorem B(a) holds true, there exists some $Y \in W_0$ such that 0 is a GK singularity of Y.

Denote by $V_0 = \mathbb{P}(W_0) = \{[Y] \mid Y \in W_0,\ Y \neq 0\}$ the projectivization of W_0 and $\mathrm{Aut}(\mathbb{P}^n)$ the group of automorphisms of $\mathbb{P}^n$. By definition of $\mathcal{F}(\mathfrak{P}, \lambda, d+1)$, there is a rational map

$$\Phi\colon V_0 \times \mathrm{Aut}(\mathbb{P}^n) \dashrightarrow \mathfrak{F}_2(d+1, n) \tag{2.2}$$

given by $\Phi([Y], T) = T^*\mathcal{F}(S,Y)$, where $\mathcal{F}(S,Y) \in \mathfrak{F}_2(d+1,n)$ is the foliation defined by ω_Y on E_0 and $\mathrm{Image}(\Phi) = \mathcal{F}(\mathfrak{P}, \lambda, d+1)$. As the domain of Φ is irreducible, we have the following result:

Proposition 2.6. $\overline{\mathcal{F}(\mathfrak{P}, \lambda, d+1)}$ *is an irreducible subvariety of* $\mathfrak{F}_2(d+1, n)$.

Proposition 2.7. *Given* $\mathcal{F} \in \mathcal{F}(\mathfrak{P}, \lambda, d+1)$*, then the tangent sheaf of* $\mathcal{F}$ *splits as* $\mathcal{T}\mathcal{F} = \mathcal{O} \oplus \mathcal{O}(1-d)$.

Proof: Let Ω be a homogeneous $(n-2)$-form of degree $d+2$ defining $\mathcal{F}$ in homogeneous coordinates, whose restriction to E_0 is $\omega = i_S i_X(\nu_n)$, where $\deg(\mathcal{G}_X) = d$.

Let Z be a homogeneous vector field of degree d on $\mathbb{C}^{n+1}$ defining $\mathcal{G}_X$ in homogeneous coordinates. Set $\Omega_1 = i_{R_{n+1}} i_S i_Z(\nu_{n+1})$, where in the definition of Ω_1 we consider S as a vector field on $\mathbb{C}^{n+1}$. We have $\Omega_1|_{E_0} = \mu\omega$ for some $\mu \in \mathbb{C}^*$ (see Proposition 4.1.2 of [**14**]). Since $\deg(\Omega_1) = d+2$, Ω_1 also defines $\mathcal{F}$ in homogeneous coordinates. This concludes the proof (see §2.2 of [**9**]). □

Next we will obtain expressions for $\mathcal{F} \in \mathcal{F}(\mathfrak{P}, \lambda, d+1)$ in other affine coordinate systems. For example, on

$$E_1 = \{(1 : u_n : u_{n-1} : \cdots : u_1) \mid (u_1, \ldots, u_n) \in \mathbb{C}^n\},$$

S is given by $-S_1 = -\sum_{j=1}^n \overline{p}_j u_j \partial/\partial u_j$, where $\overline{p}_j = p_1 - p_{n-j+2}$, $j = 1, \ldots, n$.

Observe that $\overline{p}_1 = p_1 > \overline{p}_2 > \cdots > \overline{p}_n$. If $\deg(\mathcal{G}_X) = d$, X has a pole of order $d-1$ at $u_1 = 0$ and we can write $X = \frac{X_1}{u_1^{d-1}}$, where X_1 defines $\mathcal{G}_X$ on E_1. The vector field $S_1 = -S$ on E_1 has positive eigenvalues and it will be considered on this chart. Hence

$$[S_1, X_1] = [-S, u_1^{d-1} \cdot X] = \lambda_1 X_1,$$

where $\lambda_1 = p_1(d-1) - \lambda$ and $\omega_1 = i_{S_1} i_{X_1}(du_1 \wedge du_2 \wedge \cdots \wedge du_n)$ defines $\mathcal{F}$ on E_1. If $Y_1 = \mathrm{rot}(\omega_1)$, we can write in E_1 similar expressions as (2.1).

We can proceed equally in other charts, as summarized in the following proposition (recall the parameters (1.1)).

Proposition 2.8. *Given $\mathcal{F} \in \mathcal{F}(\mathfrak{P}, \lambda, d+1)$, there exist affine coordinate systems $(E_i, (x_1, \ldots, x_n))$, $i = 0, \ldots, n$, such that $\mathbb{P}^n = E_0 \cup \cdots \cup E_n$ and*

(a) *On E_i, $i = 0, \ldots, n$, $\mathcal{F}$ is defined by $\omega_i = i_{S_i} i_{X_i}(\nu_n)$, $[S_i, X_i] = \lambda_i X_i$ ($\lambda_0 = \lambda$). If $Y_i = \mathrm{rot}(\omega_i)$, then*

$$Y_i = \tau_i \cdot X_i - \mathrm{div}(X_i) \cdot S_i, \quad [S_i, Y_i] = \lambda_i Y_i, \quad \tau_i \omega_i = i_{S_i} i_{Y_i}(\nu_n).$$

Since $\tau_0 = \tau$, $\tau_1 \neq 0$, it follows that $\mathrm{div}(Y_i) \equiv 0$, $i = 0, 1$.

(b) *$S_0 = S$, $S_1 = \overline{p}_1 x_1 \partial/\partial x_1 + \cdots + \overline{p}_n x_n \partial/\partial x_n$ and for $i = 2, \ldots, n$, writing $S_i = \sum_{j=1}^n \rho_j x_j \partial/\partial x_j$, we have*

$$\rho_1 = p_1 - p_i > \cdots > \rho_{i-1} = p_{i-1} - p_i > 0 > \rho_i = p_{i+1} - p_i > \cdots > \rho_{n-1} = p_n - p_i > \rho_n = -p_i.$$

(c) *Up to a linear automorphism of $\mathbb{P}^n$, we can assume that*

$$E_0 = \{(x_1 : \cdots : x_n : 1) \mid (x_1, \ldots, x_n) \in \mathbb{C}^n\},$$
$$E_1 = \{(1 : x_n : x_{n-1} : \cdots : x_1) \mid (x_1, \ldots, x_n) \in \mathbb{C}^n\}, \text{ and}$$
$$E_i = \{(x_1 : \cdots : x_{i-1} : 1 : x_i : \cdots : x_n) \mid (x_1, \ldots, x_n) \in \mathbb{C}^n\},$$

for $i = 2, 3, \ldots, n$.

Remark 2.9. We sort the singularities $q_0, \ldots, q_n$ of S, thought of as a global vector field on $\mathbb{P}^n$, as the points corresponding to $0 \in E_i$, $i = 0, \ldots, n$, respectively.

From now on, we think of the foliations in $\mathcal{F}(\mathfrak{P}, \lambda, d+1)$ endowed with the parameters of Proposition 2.8.

Proposition 2.10. *Assume that $p_1 > \cdots > p_n$, $l_1 > \cdots > l_n$ are relatively prime positive integers, $\lambda, \mu \in \mathbb{Z}$. Then $\overline{\mathcal{F}(\mathfrak{P}, \lambda, d+1)} = \overline{\mathcal{F}(\mathfrak{L}, \mu, d+1)}$ if and only if either $\mathfrak{L} = \mathfrak{P}$, $\mu = \lambda$ or $\mathfrak{L} = \overline{\mathfrak{P}}$, $\mu = \lambda_1$, where $\overline{\mathfrak{P}} = (\overline{p}_1, \ldots, \overline{p}_n)$.*

Proof: It follows from Proposition 2.8 that $\mathcal{F}(\mathfrak{P},\lambda,d+1)=\mathcal{F}(\overline{\mathfrak{P}},\lambda_1,d+1)$, which immediately ensures the backward direction of the proposition. For the other direction, let $\alpha_0\colon\mathbb{C}^n\to\mathbb{P}^n$ be the affine coordinate chart given by $\alpha_0(x_1,\dots,x_n)=(x_1:\dots:x_n:1)$. Recall that there exists a one-to-one correspondence between $\mathrm{Aut}(\mathbb{P}^n)$ and the set of affine coordinate charts $\mathcal{C}$, which associates $T\in\mathrm{Aut}(\mathbb{P}^n)$ to $T\circ\alpha_0\in\mathcal{C}$. For $\alpha\in\mathcal{C}$, denote by T_α the element of $\mathrm{Aut}(\mathbb{P}^n)$ inducing α.

Given $\mathcal{F}\in\mathcal{F}(\mathfrak{P},\lambda,d+1)$ and Ω a homogeneous form defining $\mathcal{F}$, by definition and Lemma 2.5 there is some $\beta\in\mathcal{C}$ such that $\beta^*\Omega=\omega=i_Si_X(\nu_n)$, $[S,X]=\lambda X$, and $\deg(\mathcal{G}_X)=d$. We use the following results:

(1) Denote by $\mathcal{V}$ the set of holomorphic vector fields on $\mathbb{P}^n$, $\mathcal{D}_+$ the set of diagonal vector fields $W=k_1x_1\partial/\partial x_1+\dots+k_nx_n\partial/\partial x_n$ on $\mathbb{C}^n$, where $k_1>\dots>k_n$ are relatively prime positive integers, and $\mathcal{D}=\mathcal{D}_+\cup\{-W\mid W\in\mathcal{D}_+\}$. Suppose that $Z\in\mathcal{V}$ is such that $\alpha_0^*Z=S$. If $\alpha\in\mathcal{C}$ satisfies $\alpha^*Z=W\in\mathcal{D}$, then either $W=S$ and T_α is given by a diagonal element of $\mathrm{Aut}(\mathbb{P}^n)=\mathrm{PGL}(n+1,\mathbb{C})$, or $W=-S_1$ and T_α is given by a secondary diagonal element of $\mathrm{Aut}(\mathbb{P}^n)$, where S_1 is as in Proposition 2.8(b).

(2) If $\omega=i_Si_{\tilde X}(\nu_n)$, with $[S,\tilde X]=\tilde\lambda\tilde X$, then $\tilde\lambda=\lambda$.

(3) For $\alpha\in\mathcal{C}$ and $W\in\mathcal{D}$, denote by W_α the element of $\mathcal{V}$ such that $\alpha^*W_\alpha=W$. Given $W\in\mathcal{D}_+$ and $\xi\in\mathbb{Z}$, suppose that $\alpha\in\mathcal{C}$ is such that $\alpha^*\Omega=i_Wi_{\tilde X}(\nu_n)$, $[W,\tilde X]=\xi\tilde X$, and $W_\alpha=\pm S_\beta$. If $W_\alpha=S_\beta$, then $W=S$ and $\xi=\lambda$, while if $W_\alpha=-S_\beta$, then $W=S_1$ and $\xi=\lambda_1$. In fact, it suffices to check this for $\beta=\alpha_0$, and it follows from (1) and (2).

As $\overline{\mathcal{F}(\mathfrak{P},\lambda,d+1)}=\overline{\mathcal{F}(\mathfrak{L},\mu,d+1)}$, we can assume that $\mathcal{F}\in\mathcal{F}(\mathfrak{P},\lambda,d+1)$ is generic and $\mathcal{F}\in\mathcal{F}(\mathfrak{L},\mu,d+1)$. By definition there is $\gamma\in\mathcal{C}$ such that $\gamma^*\Omega=i_{\tilde S}i_{\tilde X}(\nu_n)$, $[\tilde S,\tilde X]=\mu\tilde X$, where $\tilde S=l_1x_1\partial/\partial x_1+\dots+l_nx_n\partial/\partial x_n$.

From now on we assume that $\beta=\alpha_0$. We claim that if $d=1$ and $\lambda=0$ or $d\geq 2$, then $\tilde S_\gamma=\pm S_{\alpha_0}$, therefore by (3) the proposition follows in both cases. In fact, if $d=1$ and $\lambda=0$, it follows from Proposition 2.2(c) that $X=c_1x_1\partial/\partial x_1+\dots+c_nx_n\partial/\partial x_n$, where $c_1,\dots,c_n\in\mathbb{C}$. Since $\tilde S_\gamma$ is tangent to $\mathcal{F}$, there are $a,b\in\mathbb{C}$ such that $\alpha_0^*\tilde S_\gamma=aS+bX$ is a diagonal vector field. For X generic, we have that $a=\pm1$ and $b=0$, then $\tilde S_\gamma=\pm S_{\alpha_0}$. If $d\geq2$, we prove that $\mathcal{G}_S$ is the unique foliation by curves of degree one tangent to $\mathcal{F}$, which implies that $\tilde S_\gamma=\pm S_{\alpha_0}$. We assume that $\Omega=i_{R_{n+1}}i_Si_Z(\nu_{n+1})$ is as in the proof of Proposition 2.7. If Z_1 is a homogeneous vector field of degree 1 in $\mathbb{C}^{n+1}$ such that $i_{Z_1}\Omega=0$, there are holomorphic functions f, g, h on $\mathbb{C}^{n+1}$ such that $Z_1=f\cdot R_{n+1}+g\cdot$

$S+h\cdot Z$. Since $i_{R_{n+1}}i_S i_{Z_1}(\nu_{n+1}) = h\Omega$, it follows that $h=0$. In addition, $f=f(0)$ and $g=g(0)$ are constant functions. Then the foliation defined in homogeneous coordinates by $Z_1 = f(0)\cdot R_{n+1} + g(0)\cdot S$ is $\mathcal{G}_S$, and the result follows.

Finally, suppose that $d=1$ and $\lambda\neq 0$. As $\lambda+\lambda_1 = p_1(d-1)=0$ and $\mathcal{F}(\mathfrak{P},\lambda,2) = \mathcal{F}(\mathfrak{P},-\lambda,2)$, we can assume that $\lambda<0$. It follows from Proposition 2.2(c) that $X = \hat{X}_0 + \hat{X}_1$, where $\hat{X}_1$ is a linear vector field given by a strictly upper triangular matrix. Since $\tilde{S}_\gamma$ is tangent to $\mathcal{F}$, there are $a,b\in\mathbb{C}$ such that $\alpha_0^*\tilde{S}_\gamma = V := aS+bX$. Clearly $a\neq 0$. Note that there is a unique $x_0\in\mathbb{C}^n$ such that $V(x_0)=0$. One can show that there is an invertible affine map $\psi\colon\mathbb{C}^n\to\mathbb{C}^n$, $\psi(0)=x_0$, such that $\psi^*V = aS$; hence $a=\pm 1$. If $a=1$, as $\omega = i_V i_X(\nu_n)$, $[V,X]=\lambda X$, we can write $\psi^*\omega = i_S i_U(\nu_n)$, $[S,U]=\lambda U$. This implies the existence of $\eta\in\mathcal{C}$ such that $\eta^*\Omega = i_S i_U(\nu_n)$, $[S,U]=\lambda U$, and $\tilde{S}_\gamma = S_\eta$. From (3), it follows that $\mu=\lambda$ and $\tilde{S}=S$. If $a=-1$, a similar argument shows that $\mu=\lambda_1=-\lambda$ and $\tilde{S}=S_1$. This concludes the proof of the proposition. $\square$

The proposition below provides the dimension of a general family $\overline{\mathcal{F}(\mathfrak{P},\lambda,d+1)}$.

Proposition 2.11. *Assume that $p_1>\cdots>p_n$ are relatively prime positive integers, $\lambda\in\mathbb{Z}$. Then*

$$\dim\left(\overline{\mathcal{F}(\mathfrak{P},\lambda,d+1)}\right) = \begin{cases} \dim V_0 + n^2+n, & d\geq 2,\\ \dim V_0 + n^2+n-1, & d=1 \text{ and } \lambda\neq 0,\\ n^2+2n-2, & d=1 \text{ and } \lambda=0.\end{cases}$$

Proof: We compute the dimension k of a generic fibre of the map (2.2)

$$\Phi\colon V_0\times \mathrm{Aut}(\mathbb{P}^n) \dashrightarrow \overline{\mathcal{F}(\mathfrak{P},\lambda,d+1)}.$$

Then $\dim\left(\overline{\mathcal{F}(\mathfrak{P},\lambda,d+1)}\right) = \dim V_0 + \dim\mathrm{Aut}(\mathbb{P}^n) - k = \dim V_0 + n^2+2n-k$. We use the notation of the previous proposition. Given $\mathcal{F}\in\mathcal{F}(\mathfrak{P},\lambda,d+1)$ defined in homogeneous coordinates by Ω, there is $\beta\in\mathcal{C}$ such that $\beta^*\Omega = \omega_{\overline{Y}} = \frac{1}{\tau} i_S i_{\overline{Y}}(\nu_n)$, $[S,\overline{Y}]=\lambda\overline{Y}$. We can assume that $\beta=\alpha_0$.

For $Y\in W_0$, let Ω_Y be the homogeneous $(n-2)$-form of degree $d+2$ satisfying $\alpha_0^*\Omega_Y = \omega_Y$.

Claim 2.12. *The linear map $Y\in W_0\mapsto \Omega_Y$ is injective.*

Proof: If $\Omega_Y = 0$, then ω_Y also vanishes. So, there exists a polynomial f such that $Y = f \cdot S$. Since $[S, Y] = \lambda Y$, we have $S(f) = \lambda f$. This implies $0 = \operatorname{div}(Y) = \tau f$ and we get $f = 0$. Therefore, $Y = 0$ and this finishes the proof. □

Note that $([Y], T) \in \Phi^{-1}(\mathcal{F})$ if and only if there exists $\xi \in \mathbb{C}^*$ such that $\alpha_0^*(T^*\Omega_Y) = \xi \cdot \omega_{\overline{Y}}$. By taking a scalar multiple of T instead of T, we can assume that $\xi = 1$. Then

$$\alpha_0^*\Omega_Y = \omega_Y \text{ and } (T \circ \alpha_0)^*\Omega_Y = \omega_{\overline{Y}}. \tag{2.3}$$

If $d \geq 2$ or $d = 1$ and $\lambda = 0$, it follows by (2.3) that $S_{T\circ\alpha_0} = \pm S_{\alpha_0}$. By (1) of the previous proposition, if $S_{T\circ\alpha_0} = S_{\alpha_0}$ then T is an element of the subgroup $D(n+1) \subset \operatorname{PGL}(n+1, \mathbb{C})$ of diagonal matrices. On the other hand, if $S_{T\circ\alpha_0} = -S_{\alpha_0}$, then $S = S_1$ and T belongs to the subgroup $A(n+1) \subset \operatorname{PGL}(n+1, \mathbb{C})$ of secondary diagonal matrices.

Assume that $T \in D(n+1)$ and set $\tilde{T} = \alpha_0^{-1} \circ T \circ \alpha_0$. From (2.3) we have $\tilde{T}^*\omega_Y = \omega_{\overline{Y}}$. Since $\tilde{T}^*S = S$ it follows from Claim 2.12 that $[Y] = [\tilde{T}_*\overline{Y}] \in V_0$. Thus $\Phi^{-1}(\mathcal{F}) = \{(\tilde{T}_*\overline{Y}, T) \mid T \in D(n+1)\}$ as long as $\mathfrak{P} \neq \overline{\mathfrak{P}}$ or $\lambda \neq \lambda_1$. If $d \geq 2$, $\mathfrak{P} = \overline{\mathfrak{P}}$, and $\lambda = \lambda_1$, it follows from a similar argument that $\Phi^{-1}(\mathcal{F})$ has two irreducible components

$$\Phi^{-1}(\mathcal{F}) = \{(\tilde{T}_*\overline{Y}, T) \mid T \in D(n+1)\} \cup \{(\hat{T}_*\overline{Y}_1, T) \mid T \in A(n+1)\},$$

where $\alpha_1 \in \mathcal{C}$ is given by $\alpha_1(x_1, \ldots, x_n) = (1 : x_n : \cdots : x_1)$, $\hat{T} = \alpha_0^{-1} \circ T \circ \alpha_1$, and $\overline{Y}_1 = \operatorname{rot}(\alpha_1^*\Omega)$. In any case $k = \dim D(n+1) = \dim A(n+1) = n$. If $d = 1$ and $\lambda = 0$, we have that $W_0 = \{c_1x_1\partial/\partial x_1 + \cdots + c_nx_n\partial/\partial x_n \mid c_1 + \cdots + c_n = 0\}$, hence $\dim V_0 = n - 2$.

Finally, if $d = 1$ and $\lambda \neq 0$, then the dimension k of a generic fibre is one more. This implies the proposition. □

Example 2.13. For $S = \overline{\mathcal{F}(4, 2, 1; 3, 3)}$, $W_0 = \{-2axyz\partial/\partial x + bxz\partial/\partial y + (ayz^2 + c_1x + c_2y^2)\partial/\partial z \mid a, b, c_1, c_2 \in \mathbb{C}\}$. Hence $\dim S = 3 + 3^2 + 3 = 15$.

3. Quasi-homogeneous singularities

In this section we recall a recent result concerning stability of quasi-homogeneous singularities.

Definition 3.1. Let ω be a germ of an integrable $(n-2)$-form defined at $p \in \mathbb{C}^n$, with $p \in \operatorname{Sing}(\omega)$ and $n \geq 3$. We say that $p \in \mathbb{C}^n$ is a *quasi-homogeneous* singularity of ω if p is an isolated singularity of $Y = \operatorname{rot}(\omega)$ and $DY(p)$ is nilpotent.

The next theorem was recently proved ([**15**]). A stronger version for the case $n = 3$ was already known ([**13**]).

Theorem 3.2. *Assume that $0 \in \mathbb{C}^n$ is a quasi-homogeneous singularity of ω. Then there exists a holomorphic coordinate system $w = (w_1, \ldots, w_n)$ around $0 \in \mathbb{C}^n$ where ω has polynomial coefficients. More precisely, there exist two polynomial vector fields Z and Y in $\mathbb{C}^n$ such that*

(a) *$Z = S + N$, where $S = \sum_{j=1}^{n} p_j w_j \partial/\partial w_j$ is linear semi-simple with eigenvalues $p_1, \ldots, p_n \in \mathbb{Z}_{>0}$, $DN(0)$ is linear nilpotent, and $[S, N] = 0$.*
(b) *$[N, Y] = 0$ and $[S, Y] = \lambda Y$, where $\lambda \in \mathbb{Z}_{>0}$. In other words, Y is quasi-homogeneous with respect to S with weight λ.*
(c) *In this coordinate system we have $\omega = \frac{1}{\lambda + tr(S)} i_Z i_Y (dw_1 \wedge \cdots \wedge dw_n)$ and $L_Y(\omega) = (\lambda + tr(S))\omega$.*

Definition 3.3. In the situation of Theorem 3.2, $S = \sum_{j=1}^{n} p_j w_j \partial/\partial w_j$ and $[S, Y] = \lambda Y$, we say that the quasi-homogeneous singularity is of type $(p_1, \ldots, p_n; \lambda)$.

We are mainly interested in the following consequence of Theorem 3.2.

Corollary 3.4. *Assume that $\omega = i_Z i_Y(\nu_n), d\omega = i_Y(\nu_n)$, and $0 \in \mathbb{C}^n$ is a quasi-homogeneous singularity of the integrable $(n-2)$-form ω. Then the eigenvalues of $DZ(0)$ are all positive rational numbers.*

The main ingredient of the proof of Theorem B is Proposition 3.6 below, which in turn is based on the following lemma (see Lemma 4.1 of [**15**]).

Lemma 3.5. *Let A and L be linear vector fields on $\mathbb{C}^n$ such that $[L, A] = \mu A$, where $\mu \neq 0$. Then A is nilpotent.*

Proposition 3.6. *Suppose that $\mathcal{F} \in \mathcal{F}(\mathfrak{P}, \lambda, d+1)$, where $\lambda \in \mathbb{Z}_{>0}$ and $d \geq 1$.*

(a) *If the singularity $q_0 \in E_0$ is GK, then it is quasi-homogeneous.*
(b) *If $q_i \in E_i$, $i = 2, 3, \ldots, n$, is a non-Kupka GK singularity, then $\lambda = p_i(d-1)$.*

Proof: Assume that $\mathcal{F}$ is defined on E_0 by $\omega = i_S i_X(\nu_n)$. As $\lambda > 0$ and $[S, Y] = \lambda Y$, where $Y = \mathrm{rot}(\omega)$, it follows by Remark 2.3 that 0 is an isolated singularity of Y. Also, since $[S, Y] = \lambda Y$ we have that $[S, DY(0)] = \lambda DY(0)$. Then (a) follows from Lemma 3.5 with $L = S$, $A = DY(0)$, $\mu = \lambda > 0$.

For (b), suppose that there is some $i \in \{2, 3, \ldots, n\}$ such that q_i is a non-Kupka singularity and $\lambda \neq p_i(d-1)$, i.e., $\lambda_i = \lambda - p_i(d-1) \neq 0$.

By Proposition 2.8(a), ω_i defines $\mathcal{F}$ on E_i and

$$\tau_i\omega_i = i_{S_i} i_{Y_i}(\nu_n), \quad [S_i, Y_i] = \lambda_i Y_i, Y_i = \mathrm{rot}(\omega_i),$$

which implies $[S_i, DY_i(0)] = \lambda_i DY_i(0)$. It follows from Lemma 3.5, with $L = S_i$, $A = DY_i(0)$, and $\mu = \lambda_i \neq 0$, that $DY_i(0)$ is nilpotent.

If $\tau_i \neq 0$, we get a contradiction from Corollary 3.4, since by Proposition 2.8(b) the eigenvalues of $\frac{S_i}{\tau_i}$ are not all positive. If $\tau_i = 0$, there exists a polynomial f such that $Y_i = f \cdot S_i$. Set $l = f(0)$. Since q_i is GK, we have that $l \neq 0$. Then we obtain a contradiction since $DY_i(0)$ is nilpotent. □

In the next result (see [**15**, Theorem 3]) we will consider the problem of deformation of two dimensional foliations with a quasi-homogeneous singularity. Consider a holomorphic family of $(n-2)$-forms, $(\omega_t)_{t\in U}$, defined on a polydisc Q of $\mathbb{C}^n$, where the space of parameters U is an open set of $\mathbb{C}^k$ with $0 \in U$. Let us assume that

- For each $t \in U$ the form ω_t defines a two dimensional foliation $\mathcal{F}_t$ on Q. Let $(Y_t)_{t\in U}$ be the family of holomorphic vector fields on Q such that $d\omega_t = i_{Y_t}(\nu_n)$.
- $0 \in \mathbb{C}^n$ is a quasi-homogeneous singularity of $\mathcal{F}_0$.

Theorem 3.7. *In the above situation there exist a neighbourhood $0 \in V \subset U$, a polydisc $0 \in P \subset Q$, and a holomorphic map $\mathcal{P}\colon V \to P \subset \mathbb{C}^n$ such that $\mathcal{P}(0) = 0$ and for any $t \in V$ then $\mathcal{P}(t)$ is the unique quasi-homogeneous singularity of $\mathcal{F}_t$ in P. Moreover, $\mathcal{P}(t)$ is of the same type as $\mathcal{P}(0)$, in the sense that if 0 is a quasi-homogeneous singularity of type $(p_1, \ldots, p_n; \lambda)$ of $\mathcal{F}_0$, then $\mathcal{P}(t)$ is a quasi-homogeneous singularity of type $(p_1, \ldots, p_n; \lambda)$ of $\mathcal{F}_t$ for all $t \in V$.*

Proof of Theorem A: Let $\mathcal{F} \in \mathcal{F}(\mathfrak{P}, \lambda, d+1)$ be the required WGK foliation. By Proposition 3.6(a), q_0 is a quasi-homogeneous singularity of $\mathcal{F}$.

Let $(\mathcal{F}_t)_{t\in\Sigma}$ be a holomorphic family of foliations in $\mathfrak{F}_2(d+1, n)$, parametrized in a open set $0 \in \Sigma \subset \mathbb{C}$, where $\mathcal{F}_0 = \mathcal{F}$, and let $(\Omega_t)_{t\in\Sigma}$ be a holomorphic family of respective homogeneous $(n-2)$-forms on $\mathbb{C}^{n+1}$ that defines $\mathcal{F}_t$. It suffices to prove that $\mathcal{F}_t \in \mathcal{F}(\mathfrak{P}, \lambda, d+1)$ for small $|t|$.

Next we show that $\mathcal{F}_t$ is WGK for small $|t|$. Define $\omega_{i,t} = \Omega_t|_{E_i}$, $i = 0, \ldots, n$, where $E_0, \ldots, E_n$ are defined as in Proposition 2.8. Set

$$\mathcal{S}_{i,t} = \{[z] \in E_i \mid \omega_{i,t}(z) = 0\} \text{ and } \mathcal{T}_{i,t} = \{[z] \in E_i \mid d\omega_{i,t}(z) = 0\}.$$

Denote by $\mathcal{Q}_{i,t}$ and $\mathcal{R}_{i,t}$ the union of the components of codimension ≥ 3 and the union of the components of codimension ≤ 2 of $\mathcal{T}_{i,t}$, respectively. By definition, that $\mathcal{F}_t$ is WGK on E_i means $\mathcal{S}_{i,t} \cap \mathcal{R}_{i,t} = \emptyset$.

For each $p \in \mathbb{P}^n$, take an open set $V_p \subset \mathbb{P}^n$ with compact closure such that $p \in V_p \subset \overline{V_p} \subset E_i$, for some $i = i(p) \in \{0, \dots, n\}$. As $\mathcal{F}_0$ is WGK, there exists $\epsilon_p > 0$ such that $\mathcal{S}_{i,t} \cap \mathcal{R}_{i,t} \cap \overline{V_p} = \emptyset$ if $|t| < \epsilon_p$. By the compactness of $\mathbb{P}^n$, we can assume that there exists a finite number of points $p_1, \dots, p_m$ such that

$$\mathbb{P}^n = \bigcup_{j=1}^{m} V_{p_j}.$$

Then $\mathcal{F}_t$ is WGK if $|t| < \epsilon$, where $\epsilon = \min_{j \in \{1,\dots,m\}} \epsilon_{p_j}$.

Hereafter, the proof of Theorem A is close to that of Theorem 1.4 that can be found in [**2**], if we take into account the following three observations:

(1) As in the case of GK singularities, if p_0 is a WGK singularity of a germ of foliation $\mathcal{G}$ defined by the integrable $(n-2)$-form η, the sheaf of germs of vector fields at p_0 tangent to $\mathcal{G}$ is locally free and has two generators. Indeed, let $z = (z_1, \dots, z_n)$, $z(p_0) = 0$, be a coordinate system around p_0, and $Y = \operatorname{rot}(\eta)$. It suffices to show that there exists a holomorphic vector field X such that $\eta = i_X i_Y (dz_1 \wedge \cdots \wedge dz_n)$ (see the proof of Corollary 2 of [**2**]). On the other hand, as $\operatorname{codim}(\operatorname{Sing}(Y)) \geq 3$, the latter is a consequence of Proposition 1 of [**15**].

(2) The tangent sheaf of $\mathcal{F}$ splits as $\mathcal{TF} = \mathcal{O} \oplus \mathcal{O}(1-d)$.

(3) In the proof of Theorem A, Theorem 3.7 plays the same role that Proposition 1 of [**2**] does in the proof of Theorem 1.4. □

Remark 3.8. Let us show that $\overline{\mathcal{F}(10, 8, 6, 1; 2, 3)} \subset \mathfrak{F}_2(3, 4)$ is an irreducible component provided by Theorem A that is not GK. We use the notation of Proposition 2.8. In fact, up to a change of coordinates, a generic foliation $\mathcal{F} \in \mathcal{F}(10, 8, 6, 1; 2, 3)$ can be defined on

$$E_0 = \{(x_1 : x_2 : x_3 : x_4 : 1) \in \mathbb{P}^4 \mid (x_1, x_2, x_3, x_4) \in \mathbb{C}^4\}$$

by $\omega = i_S i_X (\nu_4)$, where

$$S = 10x_1 \frac{\partial}{\partial x_1} + 8x_2 \frac{\partial}{\partial x_2} + 6x_3 \frac{\partial}{\partial x_3} + x_4 \frac{\partial}{\partial x_4}$$

and

$$X = (x_1 x_4^2 + x_3^2) \frac{\partial}{\partial x_1} + (x_2 x_4^2 + x_1) \frac{\partial}{\partial x_2} + (x_3 x_4^2 + x_2) \frac{\partial}{\partial x_3} + x_4^3 \frac{\partial}{\partial x_4},$$

with $[S, X] = 2X$. It is straightforward to check the following facts:

- $0 \in \mathbb{C}^4$ is an isolated singularity of $Y = \mathrm{rot}(\omega)$. Then q_0 is a GK singularity of $\mathcal{F}$.
- The singularities q_1 and q_2 of $\mathcal{F}$ are WGK (but not GK). In fact, the singular set of both $Y_1 = \mathrm{rot}(\omega_1)$ and $Y_2 = \mathrm{rot}(\omega_2)$ consists of a line through the origin of $\mathbb{C}^4$.
- The singularities q_3 and q_4 of $\mathcal{F}$ are Kupka.

Indeed, all the singularities of $\mathcal{F}$ are Kupka, with exception to q_0 and the line of $\mathbb{P}^4$ through q_1 and q_2 that are WGK, where q_0 is the only GK singularity of $\mathcal{F}$.

4. Depicting GK components

In this section we prove Theorems B, B.1, and B.2.

Proof of Theorem B: If $\overline{\mathcal{F}(\mathfrak{L}, \mu, d+1)}$ is a GK component of $\mathfrak{F}_2(d+1, n)$, we begin by showing that there exist $p_1, \dots, p_n$, λ satisfying the conditions of Theorem B such that $\overline{\mathcal{F}(\mathfrak{L}, \mu, d+1)} = \overline{\mathcal{F}(\mathfrak{P}, \lambda, d+1)}$. In this case, by Proposition 2.10, either $\mathfrak{L} = \mathfrak{P}$, $\mu = \lambda$ or $\mathfrak{L} = \overline{\mathfrak{P}}$, $\mu = \lambda_1$.

The main idea of the proof is to look at the singularities of S. This is obviously observed in part (a) of Theorem B. Part (b) of the theorem relates to the fact that the singularities $q_2, q_3, \dots, q_n$ are also GK singularities of $\mathcal{F} \in \mathcal{F}(\mathfrak{P}, \lambda, d+1)$.

We consider that a GK foliation $\mathcal{F} \in \mathcal{F}(\mathfrak{P}, \lambda, d+1)$ is equipped with the parameters of Proposition 2.8. By Proposition 3.6(b), if $d \geq 2$, as $p_1, \dots, p_n$ are pairwise distinct, the singularities $q_2, q_3, \dots, q_n$ are Kupka, with at most one exception. If $d = 1$, since $\lambda_i = \lambda - p_i(d-1) = \lambda \neq 0$, $q_2, \dots, q_n$ are Kupka singularities of $\mathcal{F}$ (the case where $d = 1$ and $\lambda = 0$ will be treated in Corollary 4.5).

We have the following useful observations easily verified by the reader.

(i) For each $i \in \{2, \dots, n\}$, if the j-th entry of $Y_i(0)$ is not 0, then it follows from $[S_i, Y_i] = \lambda_i Y_i$ and Proposition 2.2(c) that condition c_{i-1j} is satisfied.

(ii) If c_{ij} and $c_{i_1 j_1}$ hold at the same time, $1 \leq i, i_1 \leq n-1$, $1 \leq j, j_1 \leq n$, then $i_1 > i$ implies that $j_1 > j$.

(iii) Denote by $\overline{c}_{ij}$ the same condition as c_{ij} substituting p_k by $\overline{p}_k$ and λ by λ_1, $i = 1, \dots, n-1$, $j = 1, \dots, n$. It follows that

$$c_{ij} \text{ holds} \iff \overline{c}_{n-i,n-j+1} \text{ holds}, \quad i = 1, \dots, n-1,\ j = 1, \dots, n.$$

(iv) Assume that $\lambda = p_i(d-1)$, for some $i \in \{2, \dots, n\}$. If $c_{i_1 j_1}$ holds, it follows that $i+1 \leq j_1 \leq n$ if $i_1 > i-1$ and $1 \leq j_1 \leq i-2$ if $i_1 < i-1$.

It is clear that the condition of Theorem B(a) must be satisfied. Suppose first that the singularities $q_2, \dots, q_n$ are all of Kupka type. Thanks to (i) and (ii), $p_1, \dots, p_n$, λ, d must satisfy the conditions

$$c_{11}, c_{22}, \dots, c_{ii}, c_{i+1,i+2}, c_{i+2,i+3}, \dots, c_{n-1,n},$$

where $0 \le i \le n-1$. From Proposition 2.10 and (iii), we can assume that $0 \le i \le \left\lfloor \frac{n-1}{2} \right\rfloor$. In addition, for each $j \in \{2, \dots, n\}$, since $\tau_j \omega_j = i_{S_j} i_{Y_j}(\nu_n)$ and $Y_j(0) \ne 0$, we have that $\tau_j \ne 0$. Thus we are in the situation of Theorem B(b.1).

Now suppose that the singularities $q_2, \dots, q_n$ are of Kupka type, except q_i. Thanks to (i), (ii), (iv), and Proposition 3.6(b), $p_1, \dots, p_n$, λ, d must satisfy

$$c_{11}, c_{22}, \dots, c_{i-2,i-2},\ \lambda = p_i(d-1),\ c_{i,i+1}, c_{i+1,i+2}, \dots, c_{n-1,n}.$$

From Proposition 2.10, (iii), and the equivalence $\lambda = p_i(d-1) \iff \lambda_1 = \overline{p}_{n+2-i}(d-1)$, we can assume that $2 \le i \le \left\lfloor \frac{n+2}{2} \right\rfloor$. As in the previous case we have $\tau_j \ne 0$, $j \in \{2, \dots, n\} \setminus \{i\}$. Thus we are in the situation of Theorem B(b.2).

Therefore the conditions of Theorem B are needed to the existence of GK foliations in $\mathcal{F}(\mathfrak{P}, \lambda, d+1)$. Next we show that the conditions of Theorem B are also sufficient. The proof follows immediately from the next two lemmas.

Lemma 4.1. *A foliation $\mathcal{F} \in \mathcal{F}(\mathfrak{P}, \lambda, d+1)$ is GK if and only if the singularities $q_0, q_2, q_3, \dots, q_n$ of $\mathcal{F}$ are GK.*

Proof: Of course, if $\mathcal{F}$ is GK, then the singularities $q_0, q_2, \dots, q_n$ are GK. Conversely, assume that they are GK singularities of $\mathcal{F}$. Suppose that there exists a singularity p that is not GK.

Assume that $p \ne q_1$. The orbit of the global vector field S through any point $z \notin \mathrm{Sing}(S)$ accumulates at two points of $\mathrm{Sing}(S)$, say q_i, q_j, $i \ne j$ and $i \ne 1$. Since $[S_i, Y_i] = \lambda_i Y_i$, it follows from Proposition 2.1(b) that the orbit of S through p is contained in $\mathrm{Sing}(Y_i)$. We obtain a contradiction, since q_i is GK.

Next, suppose that $p = q_1$. It is not difficult to see that there is a non-GK singularity of $\mathcal{F}$ on E_1 other than q_1. Once again this contradicts $q_0, q_2, \dots, q_n$ being GK. □

Recall the parameters W_0, V_0, ω_Y introduced before Proposition 2.6. For $Y \in W_0$, set Y_i, $i = 1, \dots, n$, as in Proposition 2.8 (computed in the same way as $\omega_0 = \omega_Y$ provides a foliation of degree $d+1$).

Lemma 4.2. *Under any situation of Theorem B, there exists a proper algebraic subset $\Delta \subset V_0$ such that if $[Y] \in V_0 \setminus \Delta$, then the singularities $q_0, q_2, q_3, \dots, q_n$ of the foliation $\mathcal{F}(S, Y) \in \mathcal{F}(\mathfrak{P}, \lambda, d+1)$ defined by ω_Y on E_0 are GK.*

Proof: Consider the following subsets of V_0, where i is given by Theorem B.

$\Gamma = \{[Y] \in V_0 \mid \omega_Y$ does not define a foliation of degree $d+1$ on $\mathbb{P}^n\}$,

$\Sigma = \{[Y] \in V_0 \mid q_0$ is a non-isolated singularity of $Y\}$,

$L_i = \{[Y] \in V_0 \mid \det(DY_i(q_i)) = 0\}$,

$H_j = \{[Y] \in V_0 \mid Y_j(q_j) = 0\}$, $j = 2, 3, \dots, n$.

Note that if $[Y] \in \Gamma$, then either ω_Y gives rise to a foliation on $\mathbb{P}^n$ of degree less than $d+1$ or $\operatorname{codim}(\operatorname{Sing}(\omega_Y)) \leq 1$. We proceed in the following way. We start by observing that Γ, Σ, L_i, and the H_j's are algebraic subsets of V_0. Note that if $Y_i(q_i) = 0$ and $[Y] \notin L_i$, then q_i is an isolated singularity of Y_i. Under any situation of Theorem B, Γ and Σ are proper subsets of V_0. In the situation of Theorem B(b.2), we show additionally that L_i, H_j, $j \in \{2, 3, \dots, n\} \setminus \{i\}$, are proper subsets of V_0, so we can take

$$\Delta = \Gamma \cup \Sigma \cup L_i \cup \bigcup_{\substack{k=2 \\ k \neq i}}^{n} H_k.$$

Analogously, in the situation of Theorem B(b.1), the H_j's are proper subsets of V_0 and then we take

$$\Delta = \Gamma \cup \Sigma \cup \bigcup_{k=2}^{n} H_k.$$

It is easy to see that Γ, L_i, and the H_j's are algebraic. For Σ, by Proposition 2.2(c), the change $(x_1, \dots, x_n) \mapsto (x_1^{p_1}, \dots, x_n^{p_n})$ turns the entries of $Y \in W_0$ into homogeneous polynomials. Thus we can use the multipolynomial resultant for n homogeneous polynomials to conclude that Σ is algebraic (see [**8**] for example).

Since condition $c_{n-1,n}$ holds in any situation of Theorem B, we have $\lambda = p_n d > 0$. It follows from Remark 2.3 and Theorem B(a) that $\Sigma \subset V_0$ is proper. We claim that $\Gamma \subset V_0$ is also proper. In fact, with the exception of very few cases, condition $c_{n-2,n-1}$ is also satisfied, i.e., $p_n + \lambda = p_{n-1} d$. As also $\lambda = p_n d$, we have that

$$\overline{X} = x_n^d \cdot R_n + x_{n-1}^d \frac{\partial}{\partial x_n}$$

is such that $[S, \overline{X}] = \lambda \overline{X}$. Then the singular set of $\overline{\omega} = i_S i_{\overline{X}}(\nu_n)$ has no divisorial components, and $[\overline{Y}] \notin \Gamma$, where $\overline{Y} = \operatorname{rot}(\overline{\omega})$. Likewise one can check that Γ is proper in the other cases.

Next assume that we are in the situation of Theorem B(b.2). We show that $L_i \subset V_0$ is proper. In fact, let $\omega_i = i_{S_i} i_{X_i}(\nu_n)$ be a polynomial $(n-2)$-form defining some foliation of $\mathcal{F}(\mathfrak{P}, \lambda, d+1)$ on E_i, like in Proposition 2.8(a). Since $\lambda_i = 0$, in this case we have $[S_i, X_i] = 0$. By the parametrization Φ (2.2), if $Y_i = \mathrm{rot}(\omega_i) = \tau_i \cdot X_i - \mathrm{div}(X_i) \cdot S_i$ is such that $\det(DY_i(q_i)) \neq 0$, then we are done. Otherwise, let $\epsilon_1, \epsilon_2, \ldots, \epsilon_n$ denote n arbitrary non-real complex numbers satisfying

$$\sum_{k=1}^{n} \epsilon_k \in \mathbb{Z} - \{0\}.$$

Set $\tilde{\omega}_i = i_{S_i} i_{\tilde{X}_i}(\nu_n)$, where $\tilde{X}_i = X_i + \epsilon\big(\sum_{k=1}^{n} \epsilon_k x_k \frac{\partial}{\partial x_k}\big)$, $\epsilon \in \mathbb{C}$. We have $[S_i, \tilde{X}_i] = 0$ and $\tilde{Y}_i = \mathrm{rot}(\tilde{\omega}_i) = Y_i + \epsilon Z$, where

$$Z = \tau_i \left(\sum_{k=1}^{n} \epsilon_k x_k \frac{\partial}{\partial x_k} \right) - \left(\sum_{k=1}^{n} \epsilon_k \right) S_i.$$

Since Z is a diagonal vector field satisfying $\det(DZ(q_i)) \neq 0$, if we take $|\epsilon|$ sufficiently large we have $\det(D\tilde{Y}_i(q_i)) \neq 0$. This finishes the proof that $L_i \subset V_0$ is proper.

Finally, in both situations of Theorem B, $H_j \subset V_0$ is proper, $j = 2, \ldots, n$. In fact, let $\omega_j = i_{S_j} i_{X_j}(\nu_n)$ be a polynomial $(n-2)$-form defining some foliation of $\mathcal{F}(\mathfrak{P}, \lambda, d+1)$ on E_j, where $[S_j, X_j] = \lambda_j X_j$. As $\tau_j \neq 0$ and $c_{j-1,j-1}$ or $c_{j-1,j}$ is verified, if necessary we can redefine ω_j by adding to X_j either $c \cdot \partial/\partial x_j$ in the former case or $c \cdot \partial/\partial x_{j+1}$ in the latter case, where $c \in \mathbb{C}^*$, in order to obtain $Y_j(q_j) \neq 0$. Once again by the parametrization Φ it is sufficient to conclude that $H_j \subset V_0$ is proper. □

Remark 4.3. In any situation of Theorem B, there is some $k \in \{1, \ldots, n\}$ such that p_1 divides $p_k + \lambda$. In fact, if 0 is an isolated singularity of some $Y \in W_0$, there must exist $m \in \mathbb{Z}_{>0}$ and $k \in \{1, \ldots, n\}$ such that $x_1^m \partial/\partial x_k \in W_0$, for otherwise we would have $\{(x_1, \ldots, x_n) \mid x_2 = x_3 = \cdots = x_n = 0\} \subset \mathrm{Sing}(Y)$, a contradiction. By Proposition 2.2(c), $m \cdot p_1 = p_k + \lambda$, which implies the result.

Remark 4.4. If $\lambda = p_n d$, one can show that for $Y \in W_0$ there is $\mu \in \mathbb{C}$ such that $\hat{Y}_{d+1} = \mu x_n^d(\tau R_n - (n+d)S) = \mu x_n^d\big(-\tau_1 x_1 \partial/\partial x_1 + \sum_{k=2}^{n} \tau_k x_k \partial/\partial x_k\big)$. This happens in Theorem B, since condition $c_{n-1,n}$ is always valid.

Corollary 4.5. *Let $\mathfrak{L} = (d^{n-1} + \cdots + 1, d^{n-2} + \cdots + 1, \ldots, d+1, 1)$. Then for every $d \geq 1$, $\overline{\mathcal{F}(\mathfrak{L}, -1, d+1)}$ is an irreducible component of $\mathcal{F}_2(d+1, n)$ of dimension $k(n,d)$, where $k(n,d) = n^2 + 2n - 1$ if $d \geq 2$ and $k(n,1) = n^2 + 2n - 2$. This component is the closure of a* $\mathrm{PGL}(n+1, \mathbb{C})$ *orbit on $\mathcal{F}_2(d+1, n)$. Furthermore, if $d = 1$ this is the unique GK component of $\mathcal{F}_2(2, n)$ of the form $\overline{\mathcal{F}(\mathfrak{P}, \lambda, 2)}$.*

Proof: For $i = 1, \dots, n$, set $r_i = \sum_{j=i-1}^{n-1} d^j$, $\mu = d^{n-1}$, and $\mathfrak{R} = (r_1, \dots, r_n)$. We show first that $\mathcal{F}(\mathfrak{R}, \mu, d+1)$ satisfy the conditions of Theorem B. By Proposition 2.10 we have $\mathcal{F}(\mathfrak{R}, \mu, d+1) = \mathcal{F}(\mathfrak{L}, -1, d+1)$, and the result follows. It is straightforward the verification that the conditions $c_{12}, c_{23}, \dots, c_{n-1,n}$ in Theorem B(b.1), $i = 0$, are verified, i.e., for $p_1 = r_1, \dots, p_n = r_n$, $\lambda = \mu$ we have

(4.1) $$p_3 + \lambda = p_2 d, p_4 + \lambda = p_3 d, \dots, p_n + \lambda = p_{n-1} d, \quad \lambda = p_n d.$$

Since $r_2, \dots, r_n$ are multiple of d and r_1 is not, we have that $\tau_j = \tau - r_j(n+d) \neq 0$, $j = 2, \dots, n$. We assert that

$$W_0 = \left\{ \mu x_n^d(\tau R_n - (n+d)S) + \sum_{k=2}^{n} a_{k-1} x_{k-1}^d \frac{\partial}{\partial x_k};\ \mu, a_1, \dots, a_{n-1} \in \mathbb{C} \right\}.$$

In fact, by Proposition 2.2(c), this is due to Remark 4.4 and the following result.

Claim 4.6. *Given $k \in \{1, \dots, n\}$, the system $b_1 \cdot r_1 + \cdots + b_n \cdot r_n = r_k + \lambda$, $b_1 + \cdots + b_n \leq d$ and $b_1, \dots, b_n \in \mathbb{Z}_{\geq 0}$ has no solution if $k = 1$. If $k \neq 1$, then $b_j = 0$, $j \neq k-1$, $b_{k-1} = d$ is the unique solution.*

Proof: The above equality means

(4.2) $$\sum_{i=0}^{n-1} \left(\sum_{j=1}^{i+1} b_j \right) d^i = d^n + d^{n-1} + \cdots + d^{k-1}.$$

If $b_1 + \cdots + b_n < d$, we have that both sides of (4.2) provide the representation of $r_k + \lambda$ in the base d system, and since the term d^n would not appear on the left-hand side we obtain a contradiction. Assume that $b_1 + \cdots + b_n = d$. If $k = 1$, from (4.2) we have $\sum_{j=1}^{k} b_j = d$, for any $k = 2, \dots, n$ and $b_1 = d+1$, and we get a contradiction. If $k > 1$, a similar argument shows that $b_{k-1} = d$ and $b_j = 0$, $j \neq k-1$. ☐

Setting $\mu = a_1 = \cdots = a_{n-1} = 1$ in the definition of W_0, by Remark 4.4 it follows that 0 is an isolated singularity of

$$Y = -\tau_1 x_1 x_n^d \partial/\partial x_1 + \sum_{k=2}^{n} (\tau_k x_k x_n^d + x_{k-1}^d) \frac{\partial}{\partial x_k},$$

and then Theorem B(a) holds (recall that $\tau_1 \neq 0$). Moreover, since $\dim V_0 = n-1$, it follows from Proposition 2.11 that $\overline{\mathcal{F}(\mathfrak{R}, \mu, d+1)}$ has dimension $k(n,d)$. Of course the action of $\operatorname{Aut}(\mathbb{P}^n) = \operatorname{PGL}(n+1, \mathbb{C})$ on $\mathcal{F}_2(d+1, n)$ leaves $\mathcal{F}(\mathfrak{L}, -1, d+1)$ invariant, and by the description of W_0 it is easy to see that this action is transitive.

Next we show that of all components $\overline{\mathcal{F}(\mathfrak{P},\lambda,d+1)}$ described by Theorem B, $\overline{\mathcal{F}(\mathfrak{R},\mu,d+1)}$ is the only one where $\lambda_1<0$. In fact, with the exception of the situation of Theorem B(b.1), $i=0$, condition c_{11} holds, i.e., $p_1+\lambda=p_2d$, which implies $\lambda_1=\overline{p}_nd>0$. In the situation of Theorem B(b.1), $i=0$, $p_1,\dots,p_n$, λ, d satisfy (4.1) and there exists a positive integer m such that

$$p_2=mr_2, p_3=mr_3,\dots,p_n=mr_n,\quad \lambda=md^{n-1}.$$

Assume, by Remark 4.3, that p_1 divides $p_2+\lambda$. This means that p_1 divides $d^{n-1}+\dots+d+1$. Since $p_1>p_2$, we have that $m\le d$ and

$$\lambda_1=p_1(d-1)-\lambda\ge(p_2+1)(d-1)-md^{n-1}=d-(m+1)\ge-1.$$

Moreover, if $\lambda_1=-1$, then $m=d$ and $p_1=d^{n-1}+d^{n-2}+\dots+d+1$, and we obtain $\overline{\mathcal{F}(\mathfrak{R},\mu,d+1)}$. In an analogous way, one can show that if p_1 divides $p_l+\lambda$, for $l\neq 2$, then $\lambda_1\ge0$.

Finally, we consider $d=1$. In this case $\lambda+\lambda_1=p_1(d-1)=0$. Suppose first that $\lambda,\lambda_1\neq0$. By Proposition 2.10, we can assume that $\lambda_1<0$, and we obtain $\overline{\mathcal{F}(\mathfrak{R},\mu,d+1)}$ with $d=1$. Now suppose that $\lambda=\lambda_1=0$. Take some $\mathcal{F}\in\mathcal{F}(\mathfrak{P},0,2)$ defined by $\omega=\frac{1}{\tau}i_Si_Y(\nu_n)$ on E_0, $Y=\operatorname{rot}(\omega)$. Since q_0 is is an isolated singularity of Y, there are $c_1,\dots,c_n\in\mathbb{C}^*$ such that $Y=\sum_{k=1}^n c_kx_k\frac{\partial}{\partial x_k}$. Thus $\eta=\frac{\omega}{x_1\cdots x_n}$ is a logarithmic form defining $\mathcal{F}$, and one can deform $\mathcal{F}$ to a logarithmic foliation that does not belong to $\mathcal{F}(\mathfrak{P},0,2)$. Consequently, $\overline{\mathcal{F}(\mathfrak{P},0,2)}$ is not an irreducible component of $\mathcal{F}_2(2,n)$. □

Remark 4.7. As a consequence of the proof of Corollary 4.5, $\overline{\mathcal{F}(\mathfrak{L},-1,d+1)}$, $d\ge1$, are the only irreducible components of the form $\overline{\mathcal{F}(\mathfrak{P},\lambda,d+1)}$ whose generic element has exactly one non-Kupka singularity (quasi-homogeneous), namely q_1.

Proof of Theorem B.2: We will give an explicit form to the components described by Theorem B. In the case $n=4$, we have four different situations: Theorem B(b.1), $i=0$, and $i=1$, Theorem B(b.2), $i=2$, and $i=3$. These, combined with the possibilities of Remark 4.3, give rise to what we call cases. There are, therefore, 16 cases to consider. We begin by showing the three cases where the families $\mathcal{F}(p,q,r,s;\lambda,d+1)$ never contain GK foliations.

In the sequel, given two integers a and b, when a divides b we sometimes denote this by $a\mid b$. We also define

$$m_1=\frac{(p+\lambda)(q+\lambda)(r+\lambda)(s+\lambda)}{pqrs}.$$

(1) Theorem B(b.1), $i = 0$, p divides $r + \lambda$.

The conditions c_{12}, c_{23}, and c_{34} are satisfied, i.e., $r+\lambda = qd$, $s+\lambda = rd$, $\lambda = sd$. An easy verification shows that there exists $m \in \mathbb{Z}_{>0}$ such that

$$(4.3) \quad p > q = m(d^2 + d + 1) > r = m(d^2 + d) > s = md^2, \quad \lambda = md^3.$$

Hence $\gcd(p, q, r, s) = 1 \iff \gcd(p, m) = 1$. As $\lambda > 0$, by Remark 2.3 and Proposition 2.1(d) it suffices to show that $m_1 \notin \mathbb{Z}$. Suppose, by contradiction, that $m_1 \in \mathbb{Z}$. By (4.3), this means that $p \mid d^3(d^3 + d^2 + d + 1)$. On the other hand, since $p \mid r + \lambda$ and $p > q$, we have that $\gcd(p, d^2+d+1) \neq 1$. Clearly a common prime factor of p and d^2+d+1 cannot divide neither d^3 nor $d^3 + d^2 + d + 1$, which is a contradiction.

(2) Theorem B(b.1), $i = 0$, p divides $s + \lambda$.

Once again p, q, r, s, λ are given as in (4.3). We cannot proceed as before, because now $m_1 \in \mathbb{Z}$. Let us write

$$Y = A_1\frac{\partial}{\partial x} + A_2\frac{\partial}{\partial y} + A_3\frac{\partial}{\partial z} + A_4\frac{\partial}{\partial w},$$

for $Y \in W_0$. We claim that $A_1(x, y, z, 0) \equiv 0$ and $A_2(x, y, z, 0) \equiv 0$, which clearly implies that 0 is a non-isolated singularity of Y.

We check that $A_1(x, y, z, 0) \equiv 0$. Suppose this is not true. Then a monomial term $x^a y^b z^c$ must appear in the expansion of A_1. By Proposition 2.2(c), $p + \lambda = ap + bq + rc$, that is

$$(4.4) \qquad p(a-1) = m(d^3 - b(d^2 + d + 1) - c(d^2 + d)).$$

As $p \mid s + \lambda$, we have that $p \mid d^3 + d^2 = d^2(d + 1)$, so we can write $p = j_1 j_2$, where $j_1 \mid d^2$ and $j_2 \mid d + 1$. Since j_1 divides the right-hand side of (4.4), $\gcd(j_1, m) = 1$ and j_1 divides d^2, it follows that

$$\begin{aligned} j_1 \mid b(d+1) + cd = d(b+c) + b &\implies j_1 \mid d(d(b+c) + b) \\ &\implies j_1 \mid bd \\ &\implies j_1 \mid (d(b+c) + b) - bd = cd + b \\ &\implies j_1 \mid b^2 = b(cd + b) - cbd \\ &\implies j_1 \mid d^2 - b^2 = (d-b)(d+b). \end{aligned}$$

Since j_2 divides the right-hand side of (4.4), $\gcd(j_2, m) = 1$ and j_2 divides $d + 1$, it follows that

$$\begin{aligned} j_2 \mid d^3 - b = d^3 + 1 - (b+1) &\implies j_2 \mid b + 1 \\ &\implies j_2 \mid d - b = (d+1) - (b+1). \end{aligned}$$

As $\gcd(j_1, j_2) = 1$ and both j_1, j_2 divide $d^2 - b^2 = (d-b)(d+b)$, we conclude that $p = j_1 j_2 \mid d^2 - b^2$. Since $b \leq d$, we have $p \leq d^2 - b^2 \leq d^2$. As $p > q$, we obtain a contradiction. Therefore, $A_1(x, y, z, 0) \equiv 0$.

Analogously, if we suppose $A_2(x, y, z, 0) \not\equiv 0$, writing $p = j_1 j_2$ as above, it is possible to show that $j_1 \mid d^2 - (b-1)^2$ and $j_2 \mid d + 1 - b$, which implies $p \mid d^2 - (b-1)^2$. Once again since $p > q$ we obtain a contradiction.

(3) Theorem B(b.2), $i = 2$, p divides $s + \lambda$.

In this case, $\lambda = q(d-1)$, c_{23}, and c_{34} are satisfied, i.e., $s + \lambda = rd$, $\lambda = q(d-1) = sd$. So

$$(4.5)\quad p > q = md^2 > r = m(d^2-1) > s = m(d^2-d), \quad \lambda = m(d^3-d^2),$$

for some $m \in \mathbb{Z}_{>0}$. Hence $\gcd(p, q, r, s) = 1 \iff \gcd(p, m) = 1$.

We show that $m_1 \notin \mathbb{Z}$. In fact, if $m_1 \in \mathbb{Z}$ by (4.5) $p \mid d^3(d-1)(d^2+d+1)$. On the other hand, $p \mid s + \lambda$ means that $p \mid d^3 - d = d(d+1)(d-1)$. Then $p \mid d(d-1)(d^2+d+1) = d^4 - d$. This implies

$$p \mid d^4 - d - d(d^3 - d) = d^2 - d < q,$$

and we obtain a contradiction.

It remains to consider the cases which provide GK irreducible components, corresponding to the situations of Theorem B.2. In all cases, p, q, r, s, λ, d satisfy certain c_{ij}'s and we show, in a very similar way, that the other conditions of Theorem B also hold. Therefore, we do so only for two cases which contain the main aspects.

(4) Theorem B(b.1), $i = 1$, p divides $s + \lambda$.

The conditions c_{11}, c_{23}, and c_{34} are satisfied, i.e., $p + \lambda = qd$, $s + \lambda = rd$, $\lambda = sd$. Thus

$$p = kd > q = md + k > r = m(d+1) > s = md, \quad \lambda = md^2,$$

for some $m \in \mathbb{Z}_{>0}$. Hence $\gcd(p, q, r, s) = 1 \iff \gcd(k, m) = 1$ and the fact that p divides $s + \lambda$ means that k divides $d + 1$.

- $\tau_2, \tau_3, \tau_4 \neq 0$.

In this case, $\tau_2 = r + s - 3q < 0$ and $\tau_4 = p + q + r - 3s > 0$. Suppose that $\tau_3 = p + q - 3r = 0$. This implies $k(d+1) = m(2d+3)$. Since the pairs k, m and $d+1$, $2d+3$ are relatively prime, it follows that $m = d+1$ and $k = 2d+3$. We obtain a contradiction since k divides $d + 1$.

• Theorem B(a) holds true.

We claim that Theorem B(a) is satisfied if and only if

$$\gcd\left(\frac{m(d+1)}{k}, d\right) = 1. \tag{4.6}$$

Note that we are in the situation of Theorem B.2(b), where additionally k divides $d+1$.

Assume that Theorem B(a) holds, i.e., 0 is an isolated singularity of some $Y \in W_0$. Write $Y \in W_0$ as in case (2). We claim that $A_1(x,0,z,0) \equiv 0$ and $A_3(x,0,z,0) \equiv 0$. Let us check that $A_1(x,0,z,0) \equiv 0$. If this is not true, then a monomial term $x^a z^b$ must appear in the expansion of A_1. It follows that $p+\lambda = ap+br$ or, equivalently, $kd(a-1) = m(d^2 - b(d+1))$.

Hence k divides $d^2 - b(d+1)$, which implies that $k = 1$ since also k divides $d+1$. We get $p = d < q = md + k$, which is a contradiction. Proceeding in an analogous way, we obtain $A_3(x,0,z,0) \equiv 0$.

As both $A_1(x,0,z,0)$ and $A_3(x,0,z,0)$ vanish, it is necessary that a monomial term $x^a z^b$ appears in the expansion of A_2. Thus $q+\lambda = ap+br$, that is, $ad-1 = mj(d-b)$, where $j = \frac{d+1}{k} \in \mathbb{Z}_{>0}$. Hence $\gcd(mj, d) = 1$ and we have (4.6).

Conversely, assume that (4.6) holds and set j as above. As $\gcd(mj,d) = 1$, there exists a integer b such that $d \mid mjb - 1$. We can assume that $0 < b < d$. Thus

$$d \mid mjd - (mjb-1) = mj(d-b) + 1.$$

If we define $a = \frac{mj(d-b)+1}{d} \in \mathbb{Z}_{>0}$, then $q + \lambda = ap + br$. One can check that $a + b \le d$. Set $l = \frac{s+\lambda}{p} < d$ and

$$Y = (-\tau_1 x w^d + y^d)\frac{\partial}{\partial x} + (\tau_2 y w^d + x^a z^b)\frac{\partial}{\partial y} + \tau_3 z w^d \frac{\partial}{\partial z} + (\tau_4 w^{d+1} + x^l + z^d)\frac{\partial}{\partial w}.$$

We have that $Y \in W_0$ and 0 is an isolated singularity of Y. Then Theorem B(a) holds.

(5) Theorem B(b.2), $i = 2$, p divides $p + \lambda$.

Besides $\lambda = q(d-1)$, conditions c_{23} and c_{34} are satisfied, i.e., $s + \lambda = rd$, $\lambda = q(d-1) = sd$. Thus

$$p > q = md^2 > r = m(d^2-1) > s = m(d^2-d), \quad \lambda = m(d^3 - d^2),$$

for some $m \in \mathbb{Z}_{>0}$. Hence $\gcd(p,q,r,s) = 1 \iff \gcd(p,m) = 1$ and p divides $p+\lambda$ means that p divides $d^3 - d^2$. So we are in the situation of Theorem B.2(c), where p divides $d^3 - d^2$.

• $\tau_3, \tau_4 \neq 0$.
In this case $\tau_3 = p+q-3r$ and $\tau_4 = p+q+r-3s > 0$. Suppose that $\tau_3 = 0$. This implies that $p = m(2d^2-3)$. Since $\gcd(p,m) = 1$ it follows that $m = 1$. Then $p = 2d^2-3$, $q = d^2$, $r = d^2-1$, $s = d^2-d$, and $\lambda = d^3-d^2$. Using polynomial division, we have $2(p+\lambda) = p(d+1) + 3(d-1)$. As p divides $p+\lambda$ it follows that $2d^2-3$ divides $3(d-1)$, and we obtain a contradiction since $d \geq 2$.

• Theorem B(a) holds true.
Set $l = \frac{p+\lambda}{p} = 1 + \frac{sd}{p} \in \mathbb{Z}$. We have $1 < l < d+1$. Take

$$Y = x(-\tau_1 w^d + ax^{l-1} + a_1 y^{d-1})\frac{\partial}{\partial x} + y(\tau_2 w^d + bx^{l-1} + b_1 y^{d-1})\frac{\partial}{\partial y} + z(\tau_3 w^d + cx^{l-1} + c_1 y^{d-1})\frac{\partial}{\partial z} + (w(\tau_4 w^d + ex^{l-1} + e_1 y^{d-1}) + z^d)\frac{\partial}{\partial w}.$$

Then $Y \in W_0$ as long as $la + b + c + e = 0$ and $a_1 + db_1 + c_1 + e_1 = 0$. Furthermore, 0 is an isolated singularity of Y if and only if

$$\begin{vmatrix} -\tau_1 & a & a_1 \\ \tau_2 & b & b_1 \\ \tau_3 & c & c_1 \end{vmatrix} \neq 0, \quad \begin{vmatrix} -\tau_1 & a & a_1 \\ \tau_2 & b & b_1 \\ \tau_4 & e & e_1 \end{vmatrix} \neq 0, \quad \begin{vmatrix} -\tau_1 & a \\ \tau_3 & c \end{vmatrix} \neq 0, \quad \begin{vmatrix} -\tau_1 & a \\ \tau_4 & e \end{vmatrix} \neq 0,$$

$$\begin{vmatrix} \tau_2 & b_1 \\ \tau_3 & c_1 \end{vmatrix} \neq 0, \quad \begin{vmatrix} \tau_2 & b_1 \\ \tau_4 & e_1 \end{vmatrix} \neq 0, \quad \begin{vmatrix} a & a_1 \\ b & b_1 \end{vmatrix} \neq 0, \quad a \neq 0, \quad \text{and} \quad b_1 \neq 0,$$

where $|\cdot|$ stands for the determinant. After making the substitutions $e = -(la+b+c)$ and $e_1 = -(a_1 + db_1 + c_1)$, we see that the conditions above are given by a non-empty Zariski open set on $\mathbb{C}^6$ with coordinates (a, a_1, b, b_1, c, c_1), which shows that 0 is an isolated singularity of generic $Y \in W_0$.

The case (4) is the only one where a further condition is required in order to ensure that the families contain GK foliations. In all other cases, the verification that the τ_j's are not zero is either immediate, as $\tau_2 \neq 0$ in case (4), or it can obtained with the aid of polynomial division, as $\tau_3 \neq 0$ in case (5). Moreover, since $\lambda = sd$ in all cases, we have $\tau_4 = p+q+r-3s > 0$. The verification that Theorem B(a) holds is very close to what we did in cases (4) and (5). By symmetry, in Theorem B(b.2), $i = 3$, the families given by the case p divides $q + \lambda$ coincide with those given by the case p divides $s+\lambda$. We summarize in the following table the correspondence between Theorem B and Theorem B.2.

Case	$p \mid p+\lambda$	$p \mid q+\lambda$	$p \mid r+\lambda$	$p \mid s+\lambda$
Theorem B(b.1) $i=0$	Theorem B.2(a) p divides d^3	Theorem B.2(a) p divides d^3+d^2+d+1	Theorem B(a) not satisfied $m_1 \notin \mathbb{Z}$	Theorem B(a) not satisfied
Theorem B(b.1) $i=1$	Theorem B.2(b) k divides d	Theorem B.2(b) kd divides $m(d^2+d)+k$	Theorem B.2(b) d divides m and k divides d^2+d+1	Theorem B.2(b) k divides $d+1$ and $\gcd\left(\frac{m(d+1)}{k}, d\right)=1$
Theorem B(b.2) $i=2$	Theorem B.2(c) p divides d^3-d^2	Theorem B.2(c) p divides d^3	Theorem B.2(c) p divides d^3-1	Theorem B(a) not satisfied $m_1 \notin \mathbb{Z}$
Theorem B(b.2) $i=3$	Theorem B.2(d) k divides $d-1$	Same as case p divides $s+\lambda$	Theorem B.2(d) k divides d	Theorem B.2(d) d divides m and k divides d^2-1

□

Proof of Theorem B.1: The proof is very similar to that of Theorem B.2. In Theorem B, $n=3$, there are three different situations: Theorem B(b.1), $i=0$, $i=1$, and Theorem B(b.2), $i=2$. These, combined with the possibilities of Remark 4.3, give rise to nine cases to consider. There is only one case in which we do not have GK foliations, namely Theorem B(b.1), $i=0$, p divides $r+\lambda$. In fact, the proof that

$$m_1 = \frac{(p+\lambda)(q+\lambda)(r+\lambda)}{pqr} \notin \mathbb{Z}$$

is similar to that of case (1) of the proof of Theorem B.2.

All the other eight cases provide families containing GK foliations, and we can proceed as in the previous proof to verify that the conditions of Theorem B are satisfied. By symmetry, the cases corresponding to Theorem B(b.1), $i=1$, p divides $q+\lambda$ and p divides $r+\lambda$, generate the same families of foliations. We summarize in the following table the correspondence between the two theorems.

Case	$p \mid p+\lambda$	$p \mid q+\lambda$	$p \mid r+\lambda$
Theorem B(b.1) $i=0$	Theorem B.1(a) p divides d^2	Theorem B.1(a) p divides d^2+d+1	Theorem B(a) not satisfied $m_1 \notin \mathbb{Z}$
Theorem B(b.1) $i=1$	Theorem B.1(b)	Same as case p divides $r+\lambda$	Theorem B.1(c)
Theorem B(b.2) $i=2$	Theorem B.1(d) p divides d^2-d	Theorem B.1(d) p divides d^2	Theorem B.1(d) p divides d^2-1

□

From Theorem B.1, for instance, we display the GK components of $\mathcal{F}_2(3,3)$ and $\mathcal{F}_2(4,3)$ given by Theorem 1.4.

Corollary 4.8. *For each p, q, r, λ in the table below, $\overline{\mathcal{F}(p,q,r;\lambda,3)}$ is an irreducible component of $\mathcal{F}_2(3,3)$:*

p	7	7	6	4	4	3
q	6	3	5	3	2	2
r	4	2	2	2	1	1
λ	8	4	4	4	2	2

For each p, q, r, λ in the table below, $\overline{\mathcal{F}(p,q,r;\lambda,4)}$ is an irreducible component of $\mathcal{F}_2(4,3)$:

p	13	13	13	12	9	9	9	9	8	6	6	4	3
q	12	8	4	7	8	6	4	3	3	5	3	3	2
r	9	6	3	3	6	4	3	2	2	3	2	2	1
λ	27	18	9	9	18	12	9	6	6	9	6	6	3

Corollary 4.9. *If $q \geq 3$, there are no $\lambda \neq 0$ and $d \geq 1$ such that*

$$\overline{\mathcal{F}(q+1,q,1;\lambda,d+1)}$$

contains some GK foliation.

Proof: A simple verification shows that $p = q+1$, q, $r = 1$ do not satisfy any of the four relations of Theorem B.1, regardless of the choices for $\lambda \neq 0$ and $d \geq 2$. The result follows, since $\overline{p} = p$, $\overline{q} = q$, and $\overline{r} = r$. □

Corollary 4.10. *For $d \geq 2$, $\overline{\mathcal{F}(p,q,r;\lambda,d+1)}$ is an irreducible component of $\mathcal{F}_2(d+1,3)$ for the following values of p, q, r, λ:*

p	q	r	λ
d^2+d+1	$d+1$	1	-1
d^2+d+1	$d+1$	d	d^2
d^2+d	$2d+1$	d	d^2
d^2	$d+1$	d	d^2
d^2	d	1	0
d^2	d	$d-1$	d^2-d
d^2-1	d	$d-1$	d^2-d

Proof: In Theorem B.1, just make the following substitutions: in (a), $m = d$, $p = d^2+d+1$, and apply Proposition 2.10; in (a), $m = 1$ and $p = d^2+d+1$; in (c), $k = d+1$ and $m = 1$; in (a), $m = 1$ and $p = d^2$; in (a), $m = d-1$, $p = d^2$, and apply Proposition 2.10; in (d), $m = 1$ and $p = d^2$; in (d), $m = 1$ and $p = d^2-1$. □

Similar results can be established from Theorem B.2. For instance, we have:

Corollary 4.11. *For each* p, q, r, s, λ *in the table below,* $\overline{\mathcal{F}(p,q,r,s;\lambda,3)}$ *is an irreducible component of* $\mathcal{F}_2(3,4)$*:*

p	15	15	14	12	8	8	7	6	6	4
q	14	7	11	8	7	4	4	5	5	3
r	12	6	6	3	6	3	3	4	3	2
s	8	4	4	2	4	2	2	2	2	1
λ	16	8	8	4	8	4	4	4	4	2

Acknowledgements. I am deeply grateful to A. Lins Neto for several fruitful discussions. I would like also to thank R. Lizarbe, T. Fassarela, and H. Movasati for suggestions and comments on the manuscript. This work was developed at IMPA (Rio de Janeiro, Brazil) and partially supported by CNPq and CAPES.

References

[1] O. Calvo-Andrade, Irreducible components of the space of holomorphic foliations, *Math. Ann.* **299(4)** (1994), 751–767. DOI: 10.1007/BF01459811.

[2] O. Calvo-Andrade, D. Cerveau, L. Giraldo, and A. Lins Neto, Irreducible components of the space of foliations associated to the affine Lie algebra, *Ergodic Theory Dynam. Systems* **24(4)** (2004), 987–1014. DOI: 10.1017/S0143385703000580.

[3] C. Camacho and A. Lins Neto, The topology of integrable differential forms near a singularity, *Inst. Hautes Études Sci. Publ. Math.* **55** (1982), 5–35. DOI: 10.1007/BF02698693.

[4] D. Cerveau and A. Lins Neto, Irreducible components of the space of holomorphic foliations of degree two in $\mathbf{C}\mathrm{P}(n)$, $n \geq 3$, *Ann. of Math. (2)* **143(3)** (1996), 577–612. DOI: 10.2307/2118537.

[5] D. Cerveau, A. Lins Neto, and S. J. Edixhoven, Pull-back components of the space of holomorphic foliations on $\mathbb{CP}(n)$, $n \geq 3$, *J. Algebraic Geom.* **10(4)** (2001), 695–711.

[6] W. Costa e Silva, Stability of branched pull-back projective foliations, *Bull. Braz. Math. Soc. (N.S.)* **48(1)** (2017), 29–44. DOI: 10.1007/s00574-016-0007-7.

[7] W. Costa e Silva and A. Lins Neto, Pull-back components of the space of foliations of codimension ≥ 2, *Trans. Amer. Math. Soc.* **371(2)** (2019), 949–969. DOI: 10.1090/tran/7245.

[8] D. Cox, J. Little, and D. O'Shea, *"Using Algebraic Geometry"*, Graduate Texts in Mathematics **185**, Springer-Verlag, New York, 1998. DOI: 10.1007/978-1-4757-6911-1.

[9] F. Cukierman and J. V. Pereira, Stability of holomorphic foliations with split tangent sheaf, *Amer. J. Math.* **130(2)** (2008), 413–439. DOI: 10.1353/ajm.2008.0011.

[10] J. Déserti and D. Cerveau, Feuilletages et actions de groupes sur les espaces projectifs, *Mém. Soc. Math. Fr. (N.S.)* **103** (2005), 124 pp.
[11] X. Gómez-Mont and A. Lins Neto, Structural stability of singular holomorphic foliations having a meromorphic first integral, *Topology* **30(3)** (1991), 315–334. DOI: 10.1016/0040-9383(91)90017-X.
[12] J.-P. Jouanolou, *"Équations de Pfaff algébriques"*, Lecture Notes in Mathematics **708**, Springer, Berlin, 1979. DOI: 10.1007/BFb0063393.
[13] A. Lins Neto, Finite determinacy of germs of integrable 1-forms in dimension 3 (a special case), in: *"Geometric Dynamics"* (Rio de Janeiro, 1981), Lecture Notes in Math. **1007**, Springer, Berlin, 1983, pp. 480–497. DOI: 10.1007/BFb0061430.
[14] A. Lins Neto, *"Componentes Irredutíveis dos Espaços de Folheações"*, Publicaçöes Matemáticas do IMPA, 26° Colóquio Brasileiro de Matemática, Instituto Nacional de Matemática Pura e Aplicada (IMPA), Rio de Janeiro, 2007.
[15] A. Lins Neto, Germs of complex two dimensional foliations, *Bull. Braz. Math. Soc. (N.S.)* **46(4)** (2015), 645–680. DOI: 10.1007/s00574-015-0107-9.
[16] R. Lizarbe, New irreducible components of the space of foliations associated to the affine Lie algebra, *Bull. Braz. Math. Soc. (N.S.)* **48(3)** (2017), 377–388. DOI: 10.1007/s00574-017-0029-9.
[17] F. Loray, J. V. Pereira, and F. Touzet, Foliations with trivial canonical bundle on Fano 3-folds, *Math. Nachr.* **286(8–9)** (2013), 921–940. DOI: 10.1002/mana.201100354.

Rio de Janeiro State University (UERJ), R. São Francisco Xavier, 524, Maracanã, 20550-900, Rio de Janeiro, Brazil
E-mail address: raphaelconstant@ime.uerj.br

Primera versió rebuda el 2 d'octubre de 2018,
darrera versió rebuda el 26 de novembre de 2019.