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ON THE JUMPING LINES OF BUNDLES OF LOGARITHMIC VECTOR FIELDS ALONG PLANE CURVES

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Abstract: For a reduced curve $C : f = 0$ in the complex projective plane $\mathbb{P}^2$, we study the set of jumping lines for the rank two vector bundle $T\langle C\rangle$ on $\mathbb{P}^2$ whose sections are the logarithmic vector fields along C. We point out the relations of these jumping lines with the Lefschetz type properties of the Jacobian module of f and with the Bourbaki ideal of the module of Jacobian syzygies of f. In particular, when the vector bundle $T\langle C\rangle$ is unstable, a line is a jumping line if and only if it meets the 0-dimensional subscheme defined by this Bourbaki ideal, a result going back to Schwarzenberger. Other classical general results by Barth, Hartshorne, and Hulek resurface in the study of this special class of rank two vector bundles.

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1. Introduction

Let $C : f = 0$ be a reduced curve of degree d in $X = \mathbb{P}^2$, $S = \mathbb{C}[x, y, z]$ be the polynomial ring with the usual grading, and $AR(f)$ be the graded S-module of Jacobian syzygies of f; see equation (2.1) below. Let E_C be the locally free sheaf on X corresponding to the graded S-module $AR(f)$, and recall that

$$E_C = T\langle C\rangle(-1), \tag{1.1}$$

where $T\langle C\rangle$ is the sheaf of logarithmic vector fields along C as considered for instance in [**1, 13, 25**]. For a line L in X, the pair of integers (d_1^L, d_2^L) such that $d_1^L \le d_2^L$ (resp. without the condition $d_1^L \le d_2^L$), and such that $E_C|_L \simeq \mathcal{O}_L(-d_1^L) \oplus \mathcal{O}_L(-d_2^L)$ is called the (ordered) splitting type (resp. the unordered splitting type) of E_C along L; see for

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instance [**16, 26**]. Unless we say the opposite, in this paper we use the (ordered) splitting type. For a generic line L_0, the corresponding splitting type $(d_1^{L_0}, d_2^{L_0})$ is known to be constant; see [**26**, Definition 2.2.3 and Lemma 3.2.2]. A line L in X is called a jumping line of order $o(L)$ for E_C or, equivalently, for $T\langle C\rangle$, if

$$o(L) := d_1^{L_0} - d_1^L > 0;$$

see for instance [**22**, Section 5].

When the graded S-module $AR(f)$ is free (equivalently, when E_C splits as a direct sum of two line bundles on X), which can be considered as the simplest case, then the corresponding curve is called *free*, a notion going back to K. Saito [**27**]. When the minimal resolution of the graded S-module $AR(f)$ is slightly more complicated, we get the *nearly free* curves considered in [**1, 2, 14, 25**]; see Definition 2.1 below for details. For a free curve C, there are no jumping lines for E_C, while for a nearly free curve C the jumping lines for E_C, if they exist, are of order 1 and form a line $\mathcal{L}$ in the dual projective space $\mathbb{P}(S_1)$, dual to the jumping point $P(C) \in X$ associated to C by S. Marchesi and J. Vallès in [**25**]. In this note we study the set of jumping lines for E_C for *any reduced plane curve* C. As one *motivation* for this study, note that when C is a line arrangement $\mathcal{A}$ in X, the question whether the combinatorics of $\mathcal{A}$ determines the generic splitting type of the corresponding vector bundle $T\langle\mathcal{A}\rangle$ is actively considered; see for instance [**5**, Question 7.12] or the relation to Terao's Conjecture on the freeness of line arrangement explained in [**1**] (see also Remark 2.4).

In Section 2 we start by recalling some basic notions and results, in particular Theorem 2.3 which determines completely the generic splitting type $(d_1^{L_0}, d_2^{L_0})$ in terms of the minimal degree $r = mdr(f)$ of a Jacobian syzygy and the degree d of the curve C. The invariant r also decides whether the vector bundle E_C is stable: the stability holds if and only if $2r \geq d$; see [**29**] or the discussion below in Section 2.

In Section 3 we study the *Hilbert function* $\{k \mapsto h^1(\mathbb{P}^2, E_C(k))\}$. Theorem 3.1, which treats the case E_C stable, is similar to, and can be obtained from, a result by Hartshorne, namely [**21**, Theorem 7.4]. The other main result, Theorem 3.2, which shows that when E_C is unstable, the behavior of the above Hilbert function is very different, is in our opinion completely new. In particular, this gives a *partial strong Lefschetz property of the Jacobian module* $N(f)$ *in the case* E_C unstable; see Corollary 3.3.

In Section 4 we relate the integer d_1^L to some Lefschetz type properties of the multiplication by an equation α_L of the line L, acting on the

Jacobian module $N(f)$; see Proposition 4.1. Then we define and establish the first properties of the k-th *jumping locus* $V_k(C)$ of the curve C, which consists of all lines L in X such that $d_1^L \leq k$; see Theorem 4.4. The main results in this section are Corollaries 4.5 and 4.6. We note that the claim in Corollary 4.6(1) fits perfectly well with a general result of W. Barth about the pure 1-dimensionality of the set of jumping lines of stable rank 2 bundles with even first Chern class; see Remark 4.7. For stable rank 2 bundles with odd first Chern class, to get a similar result, K. Hulek has introduced the notion of jumping line of second kind, and his results in [**23**] lead again to a *partial strong Lefschetz property of the Jacobian module* $N(f)$ in the case E_C stable; see Remark 4.7.

In Section 5 we introduce the main new technical tool, namely the Bourbaki ideal $B(C, \rho_1) \subset S$ associated to the curve C and to a minimal degree Jacobian syzygy ρ_1 for f; see Theorem 5.1. This allows us to present the vector bundle E_C as an extension of the ideal of a codimension 2 locally complete intersection by a line bundle. The general construction of this type goes back to Serre [**30**], and it was widely used to construct rank 2 vector bundles on $\mathbb{P}^n$ for $n \geq 3$; see [**26**, Chapter 1, Section 5] and the many references given there. *The new point in our approach is the very explicit description of the ideal* $B(C, \rho_1)$. When $r \leq d/2$ (resp. $r < d/2$), a line L is not a jumping line if (resp. if and only if) it avoids the support of the subscheme $Z(C, \rho_1)$ of $\mathbb{P}^2$ defined by the ideal $B(C, \rho_1)$; see Theorem 5.4, Theorem 5.10, and Corollary 5.5. This result generalizes the result of S. Marchesi and J. Vallès in [**25**] concerning nearly free curves and in fact goes back to Proposition 10 in Schwarzenberger's paper [**28**]. When $r > d/2$, a line L avoiding the support of the subscheme $Z(C, \rho_1)$ may be a jumping line, but it satisfies $d_1^L \geq d - r - 1$ (see Theorem 5.7) and this lower bound seems to be strict in many cases. The dependence of the ideal $B(C, \rho_1)$ and of the scheme $Z(C, \rho_1)$ on the choice of the syzygy ρ_1 of minimal degree in $AR(f)$ is illustrated in Example 6.6. This is not a surprise, since $Z(C, \rho_1)$ is exactly the zero locus subscheme of the section of the vector bundle $E_C(r)$ associated to the syzygy ρ_1, as explained in Remark 5.2.

We conclude with five examples in Section 6. In the first one we discuss the case of smooth curves, and point out in particular, that for $d = 2d' + 1$ odd, the geometry of the jumping locus curve $V_{d'-1}(C)$ is quite interesting. This is a special case of Barth's result in [**3**, Theorem 2 and Section 7], saying that a rank 2 stable vector bundle on $\mathbb{P}^2$ with even second Chern class is determined by the associated net of quadrics, having the curve $V_{d'-1}(C)$ as its discriminant.

The other four examples discuss singular curves C, satisfying all $d_1^{L_0} = 2$ and hence $V_2(C) = \mathbb{P}(S_1)$, the set of all lines in $\mathbb{P}^2$. A quintic C such that E_C is semistable is considered in Example 6.2. In Example 6.3, C is again a singular quintic, the first jumping locus $V_1(C)$ is a smooth conic, hence the 1-dimensional irreducible components of the jumping loci are not necessarily lines. This is related to another general result by W. Barth on the smoothness of some sets of jumping lines; see Remark 6.4. In Example 6.5, C is a Zariski sextic with 6 cusps on a conic, the first jumping locus $V_1(C)$ is the union of a line $\mathcal{L}$ and two points, and hence it is not pure dimensional. In Example 6.6, C is another singular sextic, the first jumping locus $V_1(C)$ consists of 11 points, and the 0-th jumping locus $V_0(C)$ is one of the points in $V_1(C)$. In the last three examples the corresponding vector bundles E_C are stable, and hence the structure of the jumping loci can be rather subtle even in the class of stable rank two vector bundles of type E_C.

As shown by these examples, the jumping loci $V_k(C)$ for any plane curve C can be determined explicitly using a Computer Algebra software; in our case we have used the package Singular (see [**6**]).

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2. Preliminaries

For the coordinate ring $S = \mathbb{C}[x, y, z]$ and a graded S-module M, let M_k be the homogeneous part of degree k of M and, for an integer m, define the shifted graded S-module $M(m)$ by the condition $M(m)_k = M_{m+k}$ for any k. For $g \in S$, let g_x, g_y, g_z denote the partial derivative of g with respect to x, y, z. Then the graded S-module $AR(f) = AR(C) \subset S^3$ of *all Jacobian relations* for f is defined by

$$AR(f)_k := \{(a, b, c) \in S_k^3 \mid af_x + bf_y + cf_z = 0\}. \tag{2.1}$$

Its sheafification $E_C := \widetilde{AR(f)}$ is a rank two vector bundle on $\mathbb{P}^2$; see [**1, 27, 29**] for details. More precisely, one has $E_C = T\langle C\rangle(-1)$, where $T\langle C\rangle$ is the sheaf of logarithmic vector fields along C as considered for instance in [**1, 13, 25**]. We set

$$ar(f)_m = \dim AR(f)_m = h^0(\mathbb{P}^2, E_C(m)) = h^0(\mathbb{P}^2, T\langle C\rangle(m-1)), \tag{2.2}$$

for any integer m. We have the following (see [**1, 14**]):

Definition 2.1. (1) A curve C is *free* if the graded S-module $AR(f)$ is free, say with a basis ρ_1, ρ_2. If $\deg \rho_i = d_i$ $(i = 1, 2)$, the multiset of integers (d_1, d_2) is called the *exponents* of a free curve C.

(2) A curve C is *nearly free* if the graded S-module $AR(f)$ has a minimal generator system of syzygies ρ_1, ρ_2, ρ_3, such that the degrees $\deg \rho_i$ satisfy $d_1 \leq d_2 = d_3$ and there is a relation

$$h\rho_1 + \ell_2\rho_2 + \ell_3\rho_3 = 0,$$

for $h \in S$ and independent linear forms $\ell_2, \ell_3 \in S$. The multiset (d_1, d_2) is called the *exponents* of a nearly free curve C.

Let $mdr(f) := \min\{k \mid AR(f)_k \neq (0)\}$ be the minimal degree of a Jacobian syzygy for f. In this paper we assume that $mdr(f) \geq 1$, unless otherwise specified. Let $N(f) = \widehat{J}_f/J_f$, with J_f the Jacobian ideal of f in S spanned by the partial derivatives f_x, f_y, f_z of f, and $\widehat{J}_f$ the saturation of the ideal J_f with respect to the maximal ideal $\mathbf{m} = (x, y, z)$ in S. The quotient module $N(f)$ coincides with $H^0_{\mathbf{m}}(S/J_f)$ and is called the *Jacobian module* of f, or of the plane curve C; see [**29**]. The quotient $M(f) = S/J_f$ is called the *Jacobian algebra* of f and we denote

$$m(f)_k = \dim M(f)_k$$

for any integer k. Let $\nu(C) = \dim N(f)_{\lfloor T/2 \rfloor}$, where $T = 3(d-2)$. It is known that the curve $C : f = 0$ is free (resp. nearly free) if and only if $\nu(C) = 0$ (resp. $\nu(C) = 1$); see [**9, 12, 14**]. Recall the definition of the global Tjurina number

$$\tau(C) = \sum_{p \in C} \tau(C, p)$$

of the curve C, where $\tau(C, p)$ is the Tjurina number of the singularity (C, p). Recall also that $\tau(C)$ is the degree of the Jacobian ideal J_f. We have the following result:

Theorem 2.2 ([**11**, Theorem 1.2])**.** *Let $C : f = 0$ be a reduced plane curve of degree d and let $r = mdr(f)$. Then the following hold:*

(1) *If $r < d/2$, then*

$$\nu(C) = \tau_{\max}(d, r) - \tau(C),$$

where $\tau_{\max}(d, r) = (d-1)^2 - r(d-1-r)$.

(2) *If $r \geq (d-2)/2$, then*

$$\nu(C) = \left\lceil \frac{3}{4}(d-1)^2 \right\rceil - \tau(C).$$

Here, for any real number u, $\lceil u \rceil$ denotes the round up of u, namely the smallest integer U such that $U \geq u$. Recall the following formulas for the Chern numbers of the vector bundle $T\langle C\rangle(k) = E_C(k+1)$, namely

$$c_1(T\langle C\rangle(k)) = 3 - d + 2k, \\ c_2(T\langle C\rangle(k)) = d^2 - (k+3)d + k^2 + 3k + 3 - \tau(C); \tag{2.3}$$

see for instance [**13**, equation (3.2)]. Associated to the vector bundle E_C there is the *normalized* vector bundle $\mathcal{E}_C$, which is the twist of E_C such that $c_1(\mathcal{E}_C) \in \{-1, 0\}$. More precisely, when $d = 2d' + 1$ is odd, then

$$\mathcal{E}_C = E_C(d') \text{ and } c_1(\mathcal{E}_C) = 0,\ c_2(\mathcal{E}_C) = 3(d')^2 - \tau(C). \tag{2.4}$$

When $d = 2d'$ is even, then one has

$$\mathcal{E}_C = E_C(d'-1) \text{ and } c_1(\mathcal{E}_C) = -1,\ c_2(\mathcal{E}_C) = 3(d')^2 - 3d' + 1 - \tau(C). \tag{2.5}$$

Recall that the vector bundle E_C is *stable* if and only if $\mathcal{E}_C$ has no sections (see [**26**, Lemma 1.2.5]) and in our case this is equivalent to $r = mdr(f) \geq d/2$; see also [**29**, Proposition 2.4]. Note that for d even, E_C is *semistable* if and only if it is stable, while for $d = 2d' + 1$ odd, E_C is semistable if and only if $r \geq d'$. To see this, use the characterization of semistable rank 2 vector bundles on $\mathbb{P}^n$ given by [**26**, Lemma 1.2.5]. Theorem 2.2(2) and the formulas (2.4) and (2.5) imply that, for a stable bundle E_C, one has

$$c_2(\mathcal{E}_C) = \nu(C). \tag{2.6}$$

The following result was established in [**1**]; see Theorem 1.1, Proposition 3.1, and Proposition 3.2.

Theorem 2.3. *With the above notation, set $r = mdr(f)$. Then the following hold, where the line L_0 is generic and the line L is arbitrary.*

(1) $d_1^L + d_2^L = d - 1$.

(2) $d_1^{L_0} \geq d_1^L$.

(3) $\max(r - \nu(C), 0) \leq d_1^L \leq d_1^{L_0} = \min(r, \lfloor (d-1)/2 \rfloor)$ *and*

$$0 \leq o(L) = d_1^{L_0} - d_1^L \leq \min(r, \nu(C)).$$

(4) $(d-1)^2 - d_1^{L_0} d_2^{L_0} = \tau(C) + \nu(C)$.

Remark 2.4. The above formulas for the Chern classes of E_C imply that, for two plane curves $C : f = 0$ and $C' : f' = 0$ with $\deg C = \deg C'$ and $\tau(C) = \tau(C')$, the associated bundles E_C and $E_{C'}$ are topologically equivalent; see for instance [**26**, Section 6.1]. This applies to the pair of line arrangements C and C' constructed by Ziegler in [**34**] such that

$\deg C = \deg C' = 9$ and $\tau(C) = \tau(C') = 42$; see [**10**, Remark 8.4]. Since $5 = mdr(f) < mdr(f') = 6$ for these line arrangements C and C', it follows that E_C and $E_{C'}$ are non-isomorphic stable vector bundles, even though C and C' have the same combinatorics. However, the bundles E_C and $E_{C'}$ have the same generic splitting type $(d_1^{L_0}, d_2^{L_0})$, as follows from Theorem 2.3(3) above. It seems that *no similar example exists involving unstable vector bundles.*

Let α_L be the defining equation of the line L in X. Then one has an exact sequence

$$0 \longrightarrow \mathcal{O}_X(-1) \xrightarrow{\cdot\alpha_L} \mathcal{O}_X \xrightarrow{\pi_L} \mathcal{O}_L \longrightarrow 0,$$

where the first non-trivial morphism is induced by multiplication by the linear form α_L. Let k be an integer and tensor the above exact sequence by the vector bundle $E_C(k)$. We get

$$0 \longrightarrow E_C(k-1) \xrightarrow{\cdot\alpha_L} E_C(k) \xrightarrow{\pi_L} E_C(k)|_L \longrightarrow 0,$$

with $E_C(k)|_L \simeq \mathcal{O}_L(k-d_1^L) \oplus \mathcal{O}_L(k-d_2^L)$, since we assume as in the introduction that $E_C|_L \simeq \mathcal{O}_L(-d_1^L) \oplus \mathcal{O}_L(-d_2^L)$. Then we have the following:

Proposition 2.5. *The long exact sequence of cohomology groups of the short exact sequence above starts as follows:*

$$\begin{aligned}(2.7)\quad & 0 \longrightarrow AR(f)_{k-1} \xrightarrow{\cdot\alpha_L} AR(f)_k \xrightarrow{\pi_L} H^0(L, \mathcal{O}_L(k-d_1^L) \oplus \mathcal{O}_L(k-d_2^L)) \\ & \longrightarrow N(f)_{k+d-2} \xrightarrow{\cdot\alpha_L} N(f)_{k+d-1} \longrightarrow \dots\end{aligned}$$

Moreover, for $k=-1$, the corresponding morphism $N(f)_{d-3} \xrightarrow{\cdot\alpha_L} N(f)_{d-2}$ is injective and hence $d_1^L \geq 0$ for any line L.

Proof: This is exactly as in the proof of [**13**, Theorem 5.7]. The key point is the identification

$$H^1(X, E_C(k)) = N(f)_{k+d-1}, \tag{2.8}$$

valid for any integer k, for which we refer to [**29**, Proposition 2.1]. For the last claim, note that $N(f)_{d-3} \subset S_{d-3}$ and $N(f)_{d-2} \subset S_{d-2}$, as the Jacobian ideal is generated in degree $d-1$. □

Finally, recall the following result, saying that the Jacobian module $N(f)$ enjoys a weak Lefschetz type property; see [**12**] for this result and [**18, 17, 24**] for Lefschetz properties of Artinian algebras.

Theorem 2.6. *If* $L_0 : \alpha_{L_0} = 0$ *is a generic line in* X*, then the morphism*

$$N(f)_{s-1} \xrightarrow{\cdot \alpha_{L_0}} N(f)_s,$$

induced by the multiplication by α_{L_0}*, is injective for* $s < \lceil T/2 \rceil$ *and surjective for* $s \geq \lceil T/2 \rceil$.

See Corollary 3.3 and the end of Remark 4.7 for partial strong Lefschetz property of the Jacobian module $N(f)$, the second one coming from a result by K. Hulek.

3. On the Hilbert function of the Jacobian module $N(f)$

The study of the dimensions $h^1(X, E_C(k))$ or, equivalently, in view of (2.8), the study of the Hilbert function

$$n(f)_k = \dim N(f)_k$$

of the Jacobian module $N(f)$, is a central question in the study of rank 2 (stable) vector bundles on X; see for instance [**21, 22**]. One has the following result for the vector bundle E_C, in the stable situation, saying that, in the middle range, the points $(j, n(f)_j)$ lie on an *upward pointing parabola*.

Theorem 3.1. *If* $r = mdr(f) \geq d/2$*, then the following hold for*

$$2d - 4 - r \leq j \leq d - 2 + r.$$

(1) *For* $d = 2d' + 1$ *odd, one has* $T = 3(d-2) = 6d' - 3$ *and*

$$n(f)_j = 3(d')^2 - (j - 3d' + 2)(j - 3d' + 1) - \tau(C) = \nu(C) - (j - \lfloor T/2 \rfloor)(j - \lceil T/2 \rceil).$$

(2) *For* $d = 2d'$ *even, one has* $T = 3(d-2) = 6d' - 6$ *and*

$$n(f)_j = 3(d')^2 - 3d' + 1 - (j - 3d' + 3)^2 - \tau(C) = \nu(C) - \left(j - \frac{T}{2}\right)^2.$$

Proof: The equality of the two formulas for $n(f)_j$ in both cases follows from the formulas for $\nu(C)$ given in Theorem 2.2(2). One can derive a proof for the first equality in (1) above using [**21**, Theorem 7.4(a)], in the case $-t-2 \leq l \leq t-1$, and for the first equality in (2) using [**21**, Theorem 7.4(b)], in the case $-t-1 \leq l \leq t-1$. We present below an alternative proof. First we check both formulas for a smooth curve C_F : $f_F = 0$, where $f_F = x^d + y^d + z^d$, when $N(f) = M(f)$, and hence $n(f)_k = m(f)_k$ for all k. The formulas for these dimensions are given for instance in [**33**, Proposition 2.1]; see in particular the explicit form for $n = 2$ given just after the proof. Consider now the general case, and note that

$$n(f)_j = m(f)_j - \dim(S/\widehat{J}_f)_j.$$

By the definition of the coincidence threshold $ct(f)$ (see [**9**, Definition 1.5]), one has $m(f)_j = m(f_F)_j$ for all $j \leq ct(f)$, and

$$ct(f) = d - 2 + mdr'(f) \geq d - 2 + r,$$

where $mdr'(f)$ is the minimal degree of a syzygy in $AR(f)$ which is not in the submodule $KR(f) \subset AR(f)$ generated by the Koszul relations $(f_y, -f_x, 0)$, $(f_z, 0, -f_x)$, and $(0, f_z, -f_y)$; see [**7**, formula (1.3)]. On the other hand, one has

$$\dim(S/\widehat{J}_f)_j = \tau(C)$$

for $j \geq T - ct(f) = 3(d-2) - (d - 2 + mdr'(f)) = 2d - 4 - mdr'(f)$ (see [**7**, Proposition 2]), and hence in particular for $j \geq 2d - 4 - r$. This completes the alternative proof of Theorem 3.1. □

The case of unstable rank 2 vector bundle on X does not seem to have been considered until now. In this case, and assuming C is not free, we have the following result saying that, in the middle range, the points $(j, n(f)_j)$ lie on a *horizontal line segment with a one-unit drop at the extremities.*

Theorem 3.2. *If $r = mdr(f) < d/2$ and e is an integer such that $0 \leq e \leq 2$, then the following holds:*

$$n(f)_{d+r+e-5} = n(f)_{2d-r-e-1} = \nu(C) - \frac{(e-2)(e-3)}{2} + \alpha(C, e),$$

where $\alpha(C, e) \geq 0$. In particular, we have the following:

(1) *For $e = 2$, we get $\alpha(C, e) = 0$ and*

$$n(f)_j = \nu(C) \text{ for any } j \in [d + r - 3, 2d - r - 3].$$

(2) *For $e = 1$, either $\alpha(C, e) = 1$, and then C is free and*

$$n(f)_{d+r-4} = n(f)_{2d-r-2} = \nu(C) = 0,$$

or else $\alpha(C, e) = 0$, and then C is not free and $n(f)_{d+r-4} = n(f)_{2d-r-2} = \nu(C) - 1$.

Proof: The exact sequence

$$0 \longrightarrow T\langle C\rangle(k) \longrightarrow \mathcal{O}^3_{\mathbb{P}^2}(k+1) \longrightarrow \mathcal{O}_{\mathbb{P}^2}(k+d) \longrightarrow \mathcal{O}_\Sigma(k+d) \longrightarrow 0,$$

where Σ denotes the singular subscheme of C, implies the equalities

$$\chi(T\langle C\rangle(k)) = \chi(\mathcal{O}^3_{\mathbb{P}^2}(k+1)) - \chi(\mathcal{O}_{\mathbb{P}^2}(k+d)) + \chi(\mathcal{O}_\Sigma(k+d))$$
$$= 3\binom{k+3}{2} - \binom{d+k+2}{2} + \tau(C).$$

On the other hand, we have

$$\chi(T\langle C\rangle(k)) = h^0(\mathbb{P}^2, T\langle C\rangle(k)) - h^1(\mathbb{P}^2, T\langle C\rangle(k)) + h^2(\mathbb{P}^2, T\langle C\rangle(k)),$$

where $h^0(\mathbb{P}^2, T\langle C\rangle(k)) = ar(f)_{k+1}$ using (2.2), $h^1(\mathbb{P}^2, T\langle C\rangle(k)) = n(f)_{d+k}$ using (2.8), and $h^2(\mathbb{P}^2, T\langle C\rangle(k)) = ar(f)_{d-5-k}$ using Serre's Duality, as explained in [**13**, Section 3]. These two formulas for $\chi(T\langle C\rangle(k))$ imply the equality

$$(3.1) \quad ar(f)_{k+1} + ar(f)_{d-5-k} + \binom{d+k+2}{2} - 3\binom{k+3}{2} = n(f)_{d+k} + \tau(C),$$

for any integer k. We set $k+1 = d-r-e$, for an integer $e \geq 0$, and we note that

$$(3.2) \quad ar(f)_{d-r-e} = \dim S_{d-2r-e}\rho_1 + \alpha(C,e) = \binom{d-2r-e+2}{2} + \alpha(C,e),$$

for some integer $\alpha(C,e) \geq 0$, if $r = mdr(f)$ and we assume that $2r \leq d$, $e \leq 2$. Since $d-5-k = r+e-4 \leq r-2$, we see that $ar(f)_{d-5-k} = 0$ and a direct computation transforms equation (3.1) into

$$(3.3) \quad n(f)_{2d-r-e-1} + \tau(C) - \alpha(C,e) = (d-1)^2 - r(d-r-1) - \frac{(e-2)(e-3)}{2}.$$

Use the formula for $\nu(C)$ in Theorem 2.2(1) and the well known duality result for $N(f)$ implying that $n(f)_j = n(f)_{T-j}$ for any integer j; see [**29**]. □

The combination of Theorem 2.6 and Theorem 3.2 yields the following *partial strong Lefschetz property holds for the Jacobian module $N(f)$.*

Corollary 3.3. *If* $r = mdr(f) < d/2$ *and* $L_0 : \alpha_{L_0} = 0$ *is a generic line in* X*, then the morphism*

$$N(f)_p \xrightarrow{\cdot \alpha_{L_0}^{q-p}} N(f)_q,$$

induced by the multiplication by $\alpha_{L_0}^{q-p}$*, is an isomorphism for any*

$$d + r - 3 \leq p < q \leq 2d - r - 3.$$

4. Jumping lines and Lefschetz type properties for the Jacobian module

The following result relates the splitting type of E_C along a line $L : \alpha_L = 0$ to the Lefschetz properties of the Jacobian module $N(f)$ with respect to the multiplication by α_L.

Proposition 4.1. *For any line* $L : \alpha_L = 0$ *in* X*, we have* $d_1^L = \min\{mdr(f), k(f, L)\}$*, where*

$$k(f, L) = \min\{k \in \mathbb{N} : N(f)_{k+d-2} \xrightarrow{\cdot \alpha_L} N(f)_{k+d-1} \textit{ is not injective}\}.$$

Proof: If $k < \min\{mdr(f), k(f, L)\}$, the exact sequence (2.7) implies

$$H^0(L, \mathcal{O}_L(k - d_1^L) \oplus \mathcal{O}_L(k - d_2^L)) = 0,$$

and hence $k < d_1^L$. If $\min\{mdr(f), k(f, L)\} > d_1^L$, then choosing $k = d_1^L$ yields a contradiction. Hence $\min\{mdr(f), k(f, L)\} \leq d_1^L$. If $k = mdr(f)$ or if $k = k(f, L)$, the same exact sequence implies $k \geq d_1^L$. Hence $d_1^L \leq \min\{mdr(f), k(f, L)\}$, which proves our claim. □

The above proof also implies the following:

Corollary 4.2. *Let* $C : f = 0$ *be a reduced plane curve of degree* d *and set* $r = mdr(f)$*. Then the following hold:*

1. *If* $d_1^L = r$*, then* L *is not a jumping line,* $ar(f)_r \leq 2$*, and the equality is possible only when* C *is free with exponents* (d_1, d_1)*, and* $d = 2d_1 + 1$ *is odd.*
2. *If* $d_1^L < r < d_2^L$*, then* $ar(f)_r \leq r - d_1^L + 1$*.*
3. *If* $d_2^L \leq r$*, then* $ar(f)_r \leq 2r - d + 3$*.*

The equality $ar(f)_r = 2r - d + 3$ occurs when C is a nearly free curve with $d = 2d_1$ even and exponents (d_1, d_1), and in many other cases (see Examples 6.3 and 6.6 below).

Proof: For the first claim note that $d_1^{L_0} \leq r$ by Theorem 2.3(4) or by Proposition 4.1, and hence L is not a jumping line. The inequality $ar(f)_r \leq 2$ follows from the exact sequence (2.7) since $AR(f)_{r-1} = 0$. If equality $ar(f)_r = 2$ holds, it follows that f has two linearly independent Jacobian syzygies, both of degree r. Hence the sum of their degrees is $2r = 2d_1^L \leq d_1^L + d_2^L = d - 1$. This is possible only when there are equalities everywhere and the curve C is free with exponents (r, r) by [**31**, Lemma (1.1)]. The remaining claims follow along the same lines. □

Remark 4.3. If the morphism $N(f)_{s-1} \xrightarrow{\cdot\alpha_L} N(f)_s$ is not injective and $s \leq \lceil T/2 \rceil$, then the morphism $N(f)_s \xrightarrow{\cdot\alpha_L} N(f)_{s+1}$ is also not injective. Indeed, let $u \in N(f)_{s-1}$ be a non-zero element such that $u \cdot \alpha_L = 0$. Then, for a generic line $L_0 = \alpha_{L_0}$, the element $u_0 = u \cdot \alpha_{L_0} \in N(f)_s$ is non-zero by Theorem 2.6. On the other hand, it is clear that

$$u_0 \cdot \alpha_L = u \cdot \alpha_{L_0} \cdot \alpha_L = u \cdot \alpha_L \cdot \alpha_{L_0} = 0.$$

In other words, the injective morphism $N(f)_{s-1} \xrightarrow{\cdot\alpha_{L_0}} N(f)_s$ sends $K(\alpha_L)_{s-1}$ into $K(\alpha_L)_s$, where

$$K(\alpha_L)_m = \ker\{N(f)_m \xrightarrow{\cdot\alpha_L} N(f)_{m+1}\}.$$

Now we investigate the jumping lines of E_C, namely the lines L in X such that $d_1^L < d_1^{L_0}$. Any line L in X corresponds clearly to a point in $\mathbb{P}(S_1)$, corresponding to a defining linear form α_L. For any integer $k < mdr(f)$, consider the linear map

$$\lambda_k \colon S_1 \longrightarrow \mathrm{Hom}(N(f)_{d-2+k}, N(f)_{d-1+k}), \tag{4.1}$$

sending a linear form $\alpha_L \in S_1$ to the morphism of multiplication by α_L. We assume that $d - 2 + k < T/2$, i.e. $k < (d-2)/2$, and hence

$$n(f)_{d-2+k} \leq n(f)_{d-1+k},$$

by Theorem 2.6. Let

$$\Sigma_k \subset \mathrm{Hom}(N(f)_{d-2+k}, N(f)_{d-1+k}) \tag{4.2}$$

denote the affine variety of linear maps which are not of maximal rank. Recall that

$$\mathrm{codim}\, \Sigma_k = n(f)_{d-1+k} - n(f)_{d-2+k} + 1 \tag{4.3}$$

when $n(f)_{d-2+k} > 0$, and $\Sigma_k = \emptyset$ when $n(f)_{d-2+k} = 0$.

We define the k-th jumping locus of the curve $C : f = 0$ to be the set

$$V_k(C) = \{L \in \mathbb{P}(S_1) : d_1^L \leq k\}. \tag{4.4}$$

Theorem 4.4. *If $k \geq mdr(f)$, then $V_k(C) = \mathbb{P}(S_1)$. On the other hand, for $k < mdr(f)$, the following hold:*

(1) *If $n(f)_{d-2+k} = 0$, then $V_k(C) = \emptyset$.*

(2) *If $n(f)_{d-2+k} > 0$, then $V_k(C) = (\lambda_k^{-1}(\Sigma_k) \setminus \{0\})/\mathbb{C}^*$ is a determinantal subvariety in $\mathbb{P}(S_1) = (S_1 \setminus \{0\})/\mathbb{C}^*$.*

(3) $\emptyset = V_{-1}(C) \subset V_0(C) \subset \cdots \subset V_{d_1^{L_0}-1}(C) \subset V_{d_1^{L_0}}(C) = \mathbb{P}(S_1) = \mathbb{P}^2$.

(4) *If $\delta_k = n(f)_{d-1+k} = n(f)_{d-2+k} > 0$, then $V_k(C)$ is a curve of degree at most δ_k.*

(5) *If $\delta_k = n(f)_{d-1+k} = n(f)_{d-2+k} + 1 > 1$, then $V_k(C)$ is either 1-dimensional, or 0-dimensional and $|V_k(C)| \leq \delta_k(\delta_k - 1)/2$ in this latter case.*

Proof: Theorem 2.3(3) implies that $d_1^L \leq mdr(f)$ for any line L. Hence if $k \geq mdr(f)$, then $d_1^L \leq k$ for any line L, that is, $V_k(C) = \mathbb{P}(S_1)$.

Assume from now on that $k < mdr(f)$. To prove claim (1), note that $n(f)_{d-2+k} = 0$ implies $n(f)_m = 0$ for any $m \leq d - 2 + k$, which in turn implies $k(f, L) > k$ for any line L. Using Proposition 4.1, this implies that $d_1^L = \min\{mdr(f), k(f, L)\} > k$, and hence $V_k(C) = \emptyset$.

To prove claim (2), recall that λ_k is a linear map, and that the set Σ_k is defined by the vanishing of all the maximal minors in a matrix of size $n(f)_{d-2+k} \times n(f)_{d-1+k}$.

The third claim follows from the inequality $d_1^L \leq d_1^{L_0}$; see Theorem 2.3(2). To prove (4), note that in this case Σ_k is a hypersurface of degree δ_k given by the vanishing of the determinant of a square matrix of size $\delta_k \times \delta_k$, and $0 \in \Sigma_k$. Note that $\Lambda_k = \operatorname{im} \lambda_k$ is a linear space not contained in Σ_k by Theorem 2.6. It follows that $\lambda_k^{-1}(\Sigma_k)$ is a (possibly non-reduced) surface in $S_1 = \mathbb{C}^3$ defined by a homogeneous polynomial of degree δ_k. The proof of the last claim is similar. In this case Σ_k has codimension 2, and hence $\lambda_k^{-1}(\Sigma_k)$ has codimension either 1 or 2, i.e. it cannot consist only of the origin 0. When $\lambda_k^{-1}(\Sigma_k)$ has codimension 1, it consists of a number of lines bounded by the degree of the determinantal variety Σ_k. This degree is known to be $\delta_k(\delta_k - 1)/2$; see [**19**, Example 19.10]. □

Corollary 4.5. *Let $C : f = 0$ be a reduced plane curve of degree d which is neither free nor nearly free, and assume that $r = mdr(f)$ satisfies $r < d/2$. Then the vector bundle E_C is not stable, and it is semistable exactly when $d = 2d' + 1$ is odd and $r = d'$. Moreover, the following hold:*

(1) $d_1^{L_0} = r$ *and hence* $V_r(C) = \mathbb{P}(S_1) = \mathbb{P}^2$.

(2) *The set of jumping lines $V_{r-1}(C)$ is a curve of degree at most $\nu(C)$ in $\mathbb{P}(S_1)$.*

(3) *The set of jumping lines of order at least two* $V_{r-2}(C)$ *is either* 1*-dimensional, or* 0*-dimensional and in this latter case* $|V_{r-2}(C)| \leq \nu(C)(\nu(C)-1)/2$ *.*

In Example 6.2 we have $d = 5 > 4 = 2r$, $\nu(C) = 3$, and $V_0(C)$ consists of 3 points, hence the bound in Corollary 4.5(3) is sharp in this case. The curve $V_{r-1}(C)$ is in fact a line arrangement in this case, as shown in Theorem 5.10 below.

Proof: The first claim in Corollary 4.5 follows from Theorem 2.3(3), the second claim from Theorem 3.2(1) and Theorem 4.4(4) for $k = r-1$, and the final claim from Theorem 3.2(2) and Theorem 4.4(5) for $k = r-2$. □

Corollary 4.6. *Let* $C : f = 0$ *be a reduced plane curve of degree d which is not nearly free, and assume that* $r = mdr(f)$ *satisfies* $r \geq d/2$*. Then the vector bundle* E_C *is stable and the following hold:*

(1) *For* $d = 2d'+1$*, one has* $d_1^{L_0} = d'$ *and set of jumping lines* $V_{d'-1}(C)$ *is a curve of degree at most* $\nu(C)$ *in* $\mathbb{P}(S_1)$*.*
(2) *For* $d = 2d'$*, one has* $d_1^{L_0} = d'-1$ *and set of jumping lines* $V_{d'-2}(C)$ *is either* 1*-dimensional, or* 0*-dimensional and in this latter case* $|V_{d'-2}(C)| \leq \nu(C)(\nu(C)-1)/2$ *.*

In Example 6.1, for the Fermat quartic we have $d = 4 < 6 = 2r$ and set of jumping lines $V_0(C)$ is the union of 3 lines, hence a pure 1-dimensional variety. In Example 6.5, we have $d = 6 = 2r$ and set of jumping lines $V_1(C)$ is the union of a line and a point, hence it is 1-dimensional, but not pure 1-dimensional. On the other hand, in Example 6.6 we have $d = 6 < 8 = 2r$, $\nu(C) = 7$, and the set of jumping lines $V_1(C)$ consists of 11 points.

Proof: The first claim follows from Theorem 3.1(1) and Theorem 4.4(4) for $k = d' - 1$, and the final claim from Theorem 3.1(2) and Theorem 4.4(5) for $k = d' - 2$. □

Remark 4.7. The claims above saying that some jumping sets $V_k(C)$ are pure 1-dimensional are related to Barth's Theorem (applied to our setting); see [**3**], [**26**, Theorem 2.2.4] as well as [**26**, pp. 118–119], saying that if $\mathcal{E}$ is a rank 2 vector bundle on $\mathbb{P}^2$, which is semistable and has an even Chern class $c_1(\mathcal{E})$, then the set of jumping lines of $\mathcal{E}$ is pure 1-dimensional. In this situation, the equation of the curve $V_{d'-1}$ is given by the determinant of the mapping $N(f)_{3d'-2} \xrightarrow{\cdot \alpha_L} N(f)_{3d'-1}$.

For semistable rank 2 vector bundles $\mathcal{E}$ with odd Chern class, i.e. $d = 2d'$, the corresponding result to Barth's Theorem fails. An example of this situation for our bundles E_C is given below in Example 6.5. In this case, K. Hulek has introduced in [**23**] the notion of a *jumping line L of the second kind*, which means that the mapping $N(f)_{3d'-4} \xrightarrow{\cdot\alpha_L^2} N(f)_{3d'-2}$ has not maximal rank. Theorem 4.4(2) implies that the set of jumping lines of the second kind is defined by the vanishing of the determinant $\Delta(a,b,c)$ of this latter mapping, regarded as a polynomial in the coefficients a, b, c of the linear form α_L. Hence the corresponding jumping set $V_k(C)$ is a (possibly non-reduced) curve $C(E_C)$ of degree $2(\nu(C)-1)$, since this determinant $\Delta(a,b,c)$ is not identically zero by [**23**, Theorem 3.2.2]. See Example 6.1 below, the case when C is the Fermat quartic, for a situation where the curve $C(E_C)$, considered with reduced structure, has degree $< 2(\nu(C)-1)$.

Note also that the non-vanishing of $\Delta(a,b,c)$ for a generic line L implies that

$$N(f)_{3d'-4} \xrightarrow{\cdot\alpha_L^2} N(f)_{3d'-2}$$

is an isomorphism in this case, i.e. a *partial strong Lefschetz property holds for the Jacobian module $N(f)$.*

Example 4.8. Let $C : f = 0$ be a nearly free curve of degree d, with exponents $d_1 \leq d_2$. When $d_1 = d_2$, it is known that there are no jumping lines and the generic splitting type is $(d_1^{L_0}, d_2^{L_0}) = (d_1 - 1, d_1)$. The corresponding vector bundles E_C is isomorphic to $T_X(-d_1 - 1)$, the shifted tangent bundle of X; see [**1, 25**] for details. Consider now the case $d_1 < d_2$, when it is known that the generic splitting type is $(d_1^{L_0}, d_2^{L_0}) = (d_1, d_2 - 1)$ and a jumping line L has a splitting type $(d_1^L, d_2^L) = (d_1 - 1, d_2)$; see [**1, 25**]. Apply now Theorem 4.4 to this situation. By [**14**, Corollary 2.17], we know that $n(f)_m = 1$ if $d + d_1 - 3 \leq m \leq d + d_2 - 3$ and $n(f)_m = 0$ otherwise. If we apply Theorem 4.4(1) for $k = d_1 - 2 < mdr(f) = d_1$, we have $V_k(C) = \emptyset$, i.e. the only possible splitting type is indeed $(d_1^L, d_2^L) = (d_1 - 1, d_2)$. Apply now Theorem 4.4(4) for $k = d_1 - 1 < mdr(f) = d_1$, and conclude that $V_{d_1-1}(C)$ is a line since $\delta_k = 1$. A geometric description of this line was given in [**25**], and a generalization of this result is discussed in our next section; see Theorem 5.4.

5. Jumping lines and the Bourbaki ideal of the syzygy module

Let $C : f = 0$ be a reduced plane curve of degree d. For any choice of a nonzero syzygy $\rho_1 = (a_1, b_1, c_1) \in AR(f)_r$, where $r = mdr(f)$, we get a morphism of graded S-modules

(5.1) $$S(-r) \xrightarrow{u} AR(f), \quad u(h) = h \cdot \rho_1.$$

For any syzygy $\rho = (a,b,c) \in AR(f)_m$, consider the determinant $\Delta(\rho) = \det M(\rho)$ of the 3×3 matrix $M(\rho)$ which has as first row x, y, z, as second row a_1, b_1, c_1, and as third row a, b, c. Then it turns out that $\Delta(\rho)$ is divisible by f (see [**9**]) and we define thus a new morphism of graded S-modules

$$AR(f) \xrightarrow{v} S(r-d+1), \quad v(\rho) = \Delta(\rho)/f, \tag{5.2}$$

and a homogeneous ideal $B(C,\rho_1) \subset S$ such that $\operatorname{im} v = B(C,\rho_1)(r-d+1)$. The following result, except claim (2), was stated for line arrangements in [**15**, Proposition 2.1].

Theorem 5.1. *Let $C : f = 0$ be a reduced plane curve of degree d and set $r = mdr(f)$. Then, for any choice of a nonzero syzygy $\rho_1 \in AR(f)_r$, there is an exact sequence*

$$0 \longrightarrow S(-r) \xrightarrow{u} AR(f) \xrightarrow{v} B(C,\rho_1)(r-d+1) \longrightarrow 0,$$

and the following hold:

(1) *The ideal $B(C,\rho_1)$ is saturated, defines a subscheme $Z(C,\rho_1) = V(B(C,\rho_1))$ of $\mathbb{P}^2$ of dimension at most 0, and its degree is given by*

$$\deg B(C,\rho_1) = (d-1)^2 - r(d-r-1) - \tau(C) = \tau_{\max}(d,r) - \tau(C).$$

(2) *The ideal $B(C,\rho_1)$ and the codimension 2 subscheme $Z(C,\rho_1)$ are locally complete intersections.*

(3) *The ideal $B(C,\rho_1)$ and the subscheme $Z(C,\rho_1)$ do not depend on the choice of ρ_1 when $\dim AR(f)_r = 1$.*

(4) *The curve C is free if and only if $B(C,\rho_1) = S$.*

(5) *The curve C is nearly free if and only if the subscheme $Z(C,\rho_1)$ is a reduced point $P(C,\rho_1)$ in $\mathbb{P}^2$. The exact sequence and the point $P(C,\rho_1)$ are independent of ρ_1 when $2r < d$, i.e. when the exponents of the nearly free curve C satisfy $r = d_1 < d_2 = d-r$.*

It follows that $B(C,\rho_1)$ is a Bourbaki ideal for the syzygy module $AR(f)$; see [**4**, Chapitre 7, §4, Théorème 6] as well as Section 3 in [**32**]. A similar construction for surfaces in $\mathbb{P}^3$ was given in [**8**]. The dependence of the ideal $B(C,\rho_1)$ and of the scheme $Z(C,\rho_1)$ of the choice of the syzygy ρ_1 is illustrated in Example 6.6. For $2r < d$, it follows from Theorem 2.2(1) that

$$\deg B(C,\rho_1) = \deg Z(C,\rho_1) = \nu(C).$$

Proof: We let the reader check that the proof given for [**15**, Proposition 2.1] works as well in this more general setting. As for the new claim (2), we proceed as follows. If we sheafify the exact sequence of graded S-modules from Theorem 5.1, we get an exact sequence

$$0 \longrightarrow \mathcal{O}_X(-r) \xrightarrow{\tilde{u}} E_C \xrightarrow{\tilde{v}} \mathcal{I}(r-d+1) \longrightarrow 0. \tag{5.3}$$

Here $\mathcal{I}$ is the sheaf ideal in $\mathcal{O}_X$ associated to the Bourbaki ideal $B(C,\rho_1)$, and hence the support of $\mathcal{O}_X/\mathcal{I}$ coincides with the support of the scheme $Z(C,\rho_1)$. If p belongs to this support, the surjectivity of $\tilde{v}_p$ implies that the corresponding ideal $\mathcal{I}_p$ is generated by at most two elements. Indeed, $E_{C,p}$ is a free $\mathcal{O}_{X,p}$-module of rank 2. Since the scheme $Z(C,\rho_1)$ is 0-dimensional, this yields the claim (2). □

Remark 5.2. Note that the syzygy ρ_1 determines a section of the bundle $E_C(r)$, whose scheme of zeroes is exactly $Z(C,\rho_1)$. In particular, one has

$$\deg B(C,\rho_1) = \deg Z(C,\rho_1) = c_2(E_C(r)).$$

However, the *explicit construction* of the ideal $B(C,\rho_1)$ given above is useful, since it provides a simple method to obtain a minimal set of generators for this ideal $B(C,\rho_1)$. Let $I(\rho_1)$ be the ideal in S generated by the components a_1, b_1, c_1 of the syzygy ρ_1 and let $Z(I(\rho_1))$ be the corresponding subscheme in $\mathbb{P}^2$. Then it is easy to see that the support $|Z(I(\rho_1))|$ of $Z(I(\rho_1))$ coincides with the support $|Z(C,\rho_1)|$ of $Z(C,\rho_1)$ outside C. The example $C : f = x^5y^2z^2 + x^9 + y^9 = 0$, where $\rho_1 = (-2xy^2z, 0, 9x^4 + 5y^2z^2)$, $|Z(I(\rho_1))| = \{(0:1:0),(0:0:1)\}$, and $|Z(C,\rho_1)| = \{(0:1:0)\}$, shows that these two supports do not coincide in general. Note that in this example $r = mdr(f) = 4$ and $ar(f)_4 = 1$, so the choice of ρ_1 is unique (up to a nonzero factor).

5.3. On lines avoiding the support of the jumping subscheme. We discuss first the lines disjoint from the support of the jumping subscheme $Z(C,\rho_1)$.

Theorem 5.4. *Let $C : f = 0$ be a reduced plane curve of degree d, set $r = mdr(f)$, and consider the subscheme $Z(C,\rho_1)$ introduced above. Any line L in $\mathbb{P}^2$ which avoids the support of $Z(C,\rho_1)$ is not a jumping line if $2r \le d$. More precisely, the (unordered) splitting type of E_C along L is $(r, d-1-r)$.*

Proof: If we tensor the exact sequence (5.3) by $\mathcal{O}_L$, for L a line disjoint from the support of $Z(C,\rho_1)$, we get the following exact sequence

$$0 \longrightarrow \mathcal{O}_L(-r) \xrightarrow{\alpha} E_C|_L \xrightarrow{\beta} \mathcal{O}_L(r-d+1) \longrightarrow 0. \tag{5.4}$$

The isomorphism classes of such extensions of $\mathcal{O}_L(r-d+1)$ by $\mathcal{O}_L(-r)$ are classified by

$$\begin{aligned}\mathrm{Ext}^1(\mathcal{O}_L(r-d+1),\mathcal{O}_L(-r)) &= \mathrm{Ext}^1(\mathcal{O}_L,\mathcal{O}_L(d-1-2r))\\ &= H^1(L,\mathcal{O}_L(d-1-2r)) = 0\end{aligned}$$

(see [**20**, Section III.6]), which proves our claim. □

Corollary 5.5. *Let $C: f=0$ be a reduced plane curve of degree d, such that $r=mdr(f)\leq d/2$. Then the set of jumping lines for the vector bundle E_C is contained in a union of at most $(d-1)^2-r(d-r-1)-\tau(C)$ lines in $\mathbb{P}(S_1)$.*

Remark 5.6. The condition $2r\leq d$ in Theorem 5.4 is necessary, as Example 6.3 below shows.

Theorem 5.7. *Let $C: f=0$ be a reduced plane curve of degree d and consider the subscheme $Z(C,\rho_1)$ introduced above. Then, if $r = mdr(f) > d/2$, the splitting type (d_1^L, d_2^L) along any line L in $\mathbb{P}^2$ which avoids the support of $Z(C,\rho_1)$ satisfies $d_1^L \geq d-1-r$. In particular, if $2r-d\in\{1,2\}$, then $d_1^L\in\{d_1^{L_0}-1, d_1^{L_0}\}$.*

Examples in the next section show that this lower bound is sharp in many cases, e.g. in the situation of the last claim, both values for d_1^L are obtained; see the final parts of Examples 6.3 and 6.6.

Proof: We use the same notation as in the proof of Theorem 5.4. In the exact sequence (5.4) we have $E_C|_L = \mathcal{O}_L(-d_1^L)\oplus\mathcal{O}_L(-d_2^L)$. The surjective morphism β is induced by a pair of homogeneous polynomials $(A_1,A_2)\in S_{a_1}\times S_{a_2}$, where $a_i = r-d+1+d_i^L$ for $i=1,2$, satisfying the condition $\gcd(A_1,A_2)=1$. Indeed, at the level of sections, the morphism β is given by

$$(s_1,s_2)\longmapsto A_1s_1+A_2s_2.$$

Note that $a_1\leq a_2$. If $A_1\neq 0$, then $a_1\geq 0$, and this yields the claim of our theorem. If $A_1=0$, it follows that A_2 is a non-zero constant, and hence $a_2=0$. This implies

$$d_2^L = d-1-r < \frac{d-1}{2},$$

which is a contradiction. Indeed, $d_2^L\geq d_1^L$ implies

$$d_2^L \geq \frac{d-1}{2}.$$

The last claim follows by checking that, in these two situations, one has

$$d-r-1 = d_1^{L_0}-1.$$ □

5.8. On lines meeting the support of the jumping subscheme $Z(C,\rho_1)$. Let L be a line in $\mathbb{P}^2$ such that $L\cap|Z(C,\rho_1)| = \{p_1,\dots,p_s\}$. For each such point p_k we define its multiplicity as follows. Consider a system of local coordinates (u,v) centered at p_k such the equation of the line L is given by $u=0$. The localized ideal $\mathcal{I}_{p_k}\subset\mathcal{O}_{X,p_k}=\mathbb{C}\{u,v\}$, being a complete intersection, is generated by two analytic germs, say $g(u,v)$ and $h(u,v)$. Then we set

$$m_k=\dim_{\mathbb{C}}\frac{\mathbb{C}\{u,v\}}{(u,g(u,v),h(u,v))}=\dim_{\mathbb{C}}\frac{\mathbb{C}\{v\}}{(g(0,v),h(0,v))}.$$

Then clearly $1\le m_k<+\infty$ and one has

$$\frac{\mathbb{C}\{u,v\}}{(u)}\otimes_{\mathbb{C}\{u,v\}}\frac{\mathbb{C}\{u,v\}}{(g(u,v),h(u,v))}=\frac{\mathbb{C}\{u,v\}}{(u,g(u,v),h(u,v))},$$

and hence the latter ring can be regarded as the local ring of the point p_k in the scheme theoretic intersection $Z(C,\rho_1)\cap L$. The ideal $\mathcal{I}_{p_k}=(g(u,v),h(u,v))\subset\mathcal{O}_{X,p_k}$ is a complete intersection, and hence we have a free resolution

$$0\longrightarrow\mathcal{O}_{X,p_k}\longrightarrow\mathcal{O}^2_{X,p_k}\longrightarrow\mathcal{I}_{p_k}\longrightarrow 0,$$

where the non-trivial morphisms are given by the pair $(g(u,v),h(u,v))$. When we tensor by $\mathcal{O}_{L,p_k}$ we get the following exact sequence

$$\mathcal{O}_{L,p_k}\longrightarrow\mathcal{O}^2_{L,p_k}\longrightarrow\mathcal{I}_{p_k}\otimes\mathcal{O}_{L,p_k}\longrightarrow 0,$$

and the corresponding morphisms are given by the pair $(g(0,v),h(0,v))\neq(0,0)$. It follows that the first morphism is injective and, up-to a change of basis in $\mathcal{O}^2_{L,p_k}=\mathbb{C}\{v\}^2$, is given by the pair $(v^{m_k},0)$. It follows that

$$\mathcal{I}_{p_k}\otimes\mathcal{O}_{L,p_k}=\mathbb{C}\{v\}\oplus\frac{\mathbb{C}\{v\}}{(v^{m_k})}.$$

If we tensor now the exact sequence (5.3) by $\mathcal{O}_L$ we get, keeping track of the twists and using the above local computations, the following result. When the points $p_k\in Z(C,\rho_1)\cap L$ are all simple points, then this result is already in [**16**]; see equation (7).

Proposition 5.9. *With the above notation, there is an exact sequence*

$$0\longrightarrow\mathcal{O}_L(-r)\longrightarrow E_C|_L\longrightarrow\mathcal{O}_L(r-d+1-m_L)\oplus\left(\oplus_{k=1,s}\frac{\mathcal{O}_{L,p_k}}{M_{p_k}^{m_k}}\right)\longrightarrow 0,$$

where $m_L=\sum_{k=1,s}m_k$ and $M_{p_k}\subset\mathcal{O}_{L,p_k}$ denotes the corresponding maximal ideal.

Using this proposition and its notation, we can prove the following result.

Theorem 5.10. *Let $C : f = 0$ be a reduced plane curve of degree d, set $r = mdr(f)$, and consider the subscheme $Z(C, \rho_1)$ introduced above. Any line L in $\mathbb{P}^2$ which meets the support of $Z(C, \rho_1)$ is a jumping line if $2r \leq d - 1$. More precisely, the splitting type of E_C along L is $(r - m_L, d - 1 - r + m_L)$ or, equivalently, the order of the jumping line L is given by $o(L) = m_L \leq r$. Moreover, the set of jumping lines $V_{r-1}(C)$ is a line arrangement consisting of at most $\nu(C)$ lines, dual to the support of the subscheme $Z(C, \rho_1)$.*

Proof: It is clear that the splitting type of E_C along L is $(r - h, d - 1 - r + h)$ for some $0 \leq h \leq r$. If $0 \leq h < m_L$, then we have $-r + h \geq r - d + 1 - h > r - d + 1 - m_L$ and hence there is no surjective morphism from $E_C|_L$ to $\mathcal{O}_L(r - d + 1 - m_L)$, which is a contradiction in view of Proposition 5.9. It follows that $h \geq m_L$. Assume now that $h > m_L$. Then $-r > r - d + 1 - h$, and hence the first nontrivial morphism in the exact sequence from Proposition 5.9 is given by a pair $(H, 0)$, where H is a homogeneous polynomial of degree $h > m_L$. This implies that the torsion part of the cokernel of this morphism has dimension equal to $h > m_L$, a contradiction.

Since the degree of the subscheme $Z(C, \rho_1)$ is $\nu(C)$ for $2r \leq d - 1$ by Theorem 5.1(1) and Theorem 2.3(3) and (5), the last claim follows as well. □

A computation of the splitting type using this approach can be seen in Example 6.2.

Remark 5.11. (i) Note that the unstable rank 2 vector bundles on $X = \mathbb{P}^2$ have been studied by Schwarzenberger in [**28**] under the name of *almost decomposable* vector bundles. The equivalence of the two notions follows for instance from [**26**, Theorems 1.2.9 and 1.2.10]. Schwarzenberger has shown that for such a vector bundle, the set of jumping lines is a union of *pencils*, that is, lines in the dual projective plane; see [**28**, Proposition 10]. Since E_C is unstable exactly when $2r < d$, our Theorem 5.10 can be regarded as a refinement of Schwarzenberger's result for the bundles E_C.

(ii) The example of a nearly free curve C with exponents (d_1, d_1) discussed in Example 4.8, when there are no jumping lines but the scheme $Z(C, \rho_1)$ consists of a simple point, shows that a line L meeting the support of $Z(C, \rho_1)$ may not be a jumping line if $r = mdr(f) \geq d/2$. A similar situation is described in Examples 6.3 and 6.5 below. Note that Example 6.5 shows that the set of jumping lines described in Corollary 5.5 is not necessarily pure 1-dimensional, i.e. it may consist of lines and isolated points when $r = d/2$, unlike the case $r < d/2$ covered by Theorem 5.10.

6. Some examples

First we consider the smooth curves.

Example 6.1. Let $C : f = 0$ be a smooth curve of degree $d \geq 3$. Then $r = mdr(f) = d - 1$ and the graded S-module $AR(f)$ is generated by the Koszul type syzygies

$$\rho_1 = (f_y, -f_x, 0),\ \rho_2 = (f_z, 0, -f_x),\ \text{and } \rho_3 = (0, f_z, -f_y).$$

With this choice, the Bourbaki ideal $B(C, \rho_1)$ is spanned by $v(\rho_2) = d \cdot f_x$ and $v(\rho_3) = d \cdot f_y$, hence it is a global complete intersection. For the Fermat type curve

$$C : f_F = x^d + y^d + z^d = 0,$$

the support of the scheme $Z(C, \rho_1)$ is the multiple point $p = (0 : 0 : 1)$. The line $L : z = 0$ does not pass through this point and Proposition 4.1 implies that $d_1^L = 0 = d - r - 1$. It follows that for this line L we get equality in the inequality $d_1^L \geq d - r - 1$ in Theorem 5.7, hence this result is sharp.

Case $d = 2d' + 1$ odd. In this case Corollary 4.6 implies that the set of jumping lines $V_{d'-1}(C)$ is a curve in $\mathbb{P}(S_1)$. The geometry of these curves $V_{d'-1}(C)$ depends on the equation f. For instance, in the case of a plane cubic

$$C : f = x^3 + y^3 + z^3 + 3txyz = 0,\ \text{where } t \in \mathbb{C},\ t^3 \neq -1,$$

an easy direct computation shows that

$$V_{d'-1}(C) : t(a^3 + b^3 + c^3) + (2 - t^3)abc = 0, \tag{6.1}$$

where $(a : b : c)$ are the coordinates on $\mathbb{P}(S_1)$. Using the classification of smooth cubics by the j-invariant, see for instance [**20**, Chapter IV, §4], it follows that the jumping variety $V_{d'-1}(C)$ determines the complex structure of C up to finite indeterminacy in this case. This is related to Barth's result in [**3**, Theorem 2 and Section 7], saying that a rank 2 stable vector bundle on $\mathbb{P}^2$ with even second Chern class is determined by the associated net of quadrics, having the curve $V_{d'-1}(C)$ as its discriminant.

Case $d = 2d'$ even. In this case Corollary 4.6 implies that the set of jumping lines $V_{d'-2}(C)$ is nonempty. For $f = x^4 + y^4 + z^4$ and using the usual monomial bases for $N(f) = M(f)$, we get $V_0(C) : abc = 0$, hence the union of 3 lines. In particular, $V_0(C)$ is pure 1-dimensional in this case.

Note that the determinant of the mapping $N(f)_2 \xrightarrow{\cdot \alpha_L^2} N(f)_4$, where $\alpha_L = ax + by + cz$, is given by $a^4 b^4 c^4$. Hence the curve of jumping lines of second order $C(E_C)$ is given by the equation $a^4b^4c^4 = 0$, and hence its support coincides with $V_0(C)$ in this case. In other words, we have equality in [**23**, Proposition 9.1].

The computations in the following examples were all done using the computer algebra software Singular (see [**6**]). The Chern classes of E_C can be computed in each case using (2.3) above, since we give in each example the corresponding global Tjurina number $\tau(C)$.

Example 6.2. Let $C : f = 0$, where $f = x^5 + y^5 + (x^4 + y^4)z$. Then $d = 5$, $\tau(C) = 9$, and $r = mdr(f) = 2$. Therefore, the bundle E_C is semistable. Theorem 2.3(3) implies that the corresponding generic splitting type of E_C is $(d_1^{L_0}, d_2^{L_0}) = (2, 2)$. The Jacobian ideal J_f is spanned by f_x, f_y, f_z, and its saturation $\widehat{J_f}$ is spanned by x^3, y^3. The only non-zero dimensions $n(f)_m$ are in this case $n(f)_4 = n(f)_5 = 3$ and $n(f)_3 = n(f)_6 = 2$. Moreover, a vector space basis of $N(f)_3$ (resp. of $N(f)_4$) is given by x^3, y^3 (resp. x^4, x^3y, xy^3). With respect to these bases, the multiplication $\{N(f)_3 \xrightarrow{\cdot \alpha_L} N(f)_4\}$, where $\alpha_L = ax+by+cz$, is given by

$$(ax + by + cz) \cdot x^3 = \left(a - \frac{5c}{4}\right) x^4 + bx^3y$$

and

$$(ax + by + cz) \cdot y^3 = \left(\frac{5c}{4} - b\right) x^4 + axy^3.$$

It follows that $V_0(C)$ consists of 3 points, namely $(0 : 0 : 1)$, $(0 : 5 : 4)$, $(5 : 0 : 4)$. Since $\nu(C) = 3$, it follows that we have equality in Corollary 4.5(3), hence the bound is sharp in this situation. Similarly, a basis for $N(f)_5$ is given by x^5, x^3y^2, x^2y^3, and the multiplication $\{N(f)_4 \xrightarrow{\cdot \alpha_L} N(f)_5\}$ is given by $(ax + by + cz) \cdot x^4 = \big(a + b - \frac{5c}{4}\big)x^5$, $(ax + by + cz) \cdot x^3y = \big(a - \frac{5c}{4} - b\big)x^5 + bx^3y^2$, and $(ax + by + cz) \cdot xy^3 = -\big(b - \frac{5c}{4}\big)x^5 + ax^2y^3$. It follows that $V_1(C)$ consists of 3 lines, namely $\mathcal{L}_1 : a = 0$, $\mathcal{L}_2 : b = 0$, and $\mathcal{L}_3 : 4(a + b) - 5c = 0$. The S-module $AR(f)$ has 4 generating syzygies, of degrees 2, 4, 4, 4, and a direct computation shows that the scheme $Z(C, \rho_1)$, which does not depend on the choice of the syzygy ρ_1, consists of the simple points $P_1 = (1 : 0 : 0)$, $P_2 = (0 : 1 : 0)$, and $P_3 = (4 : 4 : -5)$. It follows that the line $\mathcal{L}_j \subset \mathbb{P}(S_1)$ above consists of all the lines in $\mathbb{P}^2$ passing through the point P_j, for $j = 1, 2, 3$. Note that the corresponding lines $L = L_{i,j}$ in $V_0(C)$ pass through the points P_i, P_j

in the support of $Z(C, \rho_1) = \{P_1, P_2, P_3\}$, and one has $m_L = r = 2$ in this case, as predicted by Theorem 5.10. More precisely, one has $L_{1,2} : z = 0$, $L_{1,3} : 5y + 4z = 0$, and $L_{2,3} : 5x + 4z = 0$.

Example 6.3. Let $C : f = 0$, where $f = 2x^5 + 2y^5 + 5x^2y^2z$. Then $d = 5$, $\tau(C) = 10$, and we see that the S-module $AR(f)$ is generated by 4 syzygies ρ_i, $i = 1, \dots, 4$, all of degree $r = mdr(f) = 3$. Hence Theorem 2.3(3) implies that the corresponding generic splitting type of E_C is $(d_1^{L_0}, d_2^{L_0}) = (2, 2)$. The Jacobian ideal J_f is spanned by f_x, f_y, f_z, and its saturation $\widehat{J}_f$ is spanned by f_x, f_y, f_z, x^3y, xy^3. The only non-zero dimensions $n(f)_m$ are in this case $n(f)_4 = n(f)_5 = 2$. Moreover, a vector space basis of $N(f)_4$ (resp. of $N(f)_5$) is given by x^3y, xy^3 (resp. x^4y, xy^4). With respect to these bases, the multiplication $\{N(f)_4 \xrightarrow{\cdot \alpha_L} N(f)_5\}$, where $\alpha_L = ax + by + cz$, is given by

$$(ax + by + cz) \cdot x^3y = ax^4y - cxy^4$$

and

$$(ax + by + cz) \cdot xy^3 = -cx^4y + bxy^4.$$

Using Theorem 4.4(4) for $k = 1$, we get that $V_1(C)$, the set of jumping lines for E_C, is the smooth conic $Q : ab - c^2 = 0$ in $\mathbb{P}(S_1)$.

Hence in this case we have

$$\emptyset = V_{-1}(C) = V_0(C) \subset V_1(C) = Q \subset V_2(C) = \mathbb{P}(S_1).$$

Indeed, Theorem 4.4(1) implies that $V_0(C) = \emptyset$. If we choose

$$\rho_1 = (0, x^2y, -2(y^3 + x^2z)) \in AR(f)_3,$$

then the corresponding Bourbaki ideal $B(C, \rho_1)$ is (xz, y^2, xy), and hence the scheme $Z(C, \rho_1)$ consists of two points: a simple one at $(1 : 0 : 0)$, given in local coordinates by an ideal (u, v), and a double point at $(0 : 0 : 1)$, given in local coordinates by an ideal (u, v^2).

Among the lines on Q, only the lines $x = 0$ and $y = 0$ meet the support of $Z(C, \rho_1)$. For the other lines in Q, the bound given by Theorem 5.7 is $d_1^L \geq d - r - 1 = 1$. In fact, we have equality, hence this bound is sharp in this situation.

Remark 6.4. The smooth conic Q above is one of the smooth degree n curves occurring as jumping loci predicted by Barth for stable rank 2 vector bundles $\mathcal{E}$ on $\mathbb{P}^2$, with $c_1(\mathcal{E}) = 0$ and $c_2(\mathcal{E}) = n$; see [**3**, Application 1, Section 5.4]. Indeed, note that the normalization of our vector bundle E_C is $\mathcal{E}_C = E_C(2)$ and it satisfies $c_1(\mathcal{E}_C) = 0$ and $c_2(\mathcal{E}_C) = 2$. Similar remarks apply for the cubic curve in (6.1), which is smooth for $t^3 \notin \{-1, 0, 8\}$.

Example 6.5. Let $C : f = 0$, where $f = (x^2 + y^2)^3 + (y^3 + z^3)^2$, i.e. C is a Zariski sextic with 6 cusps on a conic. Then $d = 6$, $\tau(C) = 12$, and we see that the S-module $AR(f)$ is generated by 4 syzygies ρ_i, $i = 1, \ldots, 4$, of degrees $r = mdr(f) = 3 = d_1 < d_2 = d_3 = d_4 = 5$. Hence Theorem 2.3(4) implies that the corresponding generic splitting type of E_C is $(d_1^{L_0}, d_2^{L_0}) = (2, 3)$. The Jacobian ideal J_f is spanned by f_x, f_y, f_z, and its saturation $\widehat{J_f}$ is spanned by $g = y^3 + z^3$ and $h = (x^2 + y^2)^2$. The only non-zero dimensions $n(f)_m$ are in this case $n(f)_3 = n(f)_9 = 1$, $n(f)_4 = n(f)_8 = 4$, $n(f)_5 = n(f)_7 = 6$, and $n(f)_6 = 7$. Moreover, a vector space basis of $N(f)_5$ (resp. of $N(f)_6$) is given by

$$x^2g,\ y^2g,\ xyg,\ xzg,\ yzg,\ zh,$$

and respectively by

$$x^3g,\ x^2yg,\ y^3g,\ x^2zg,\ y^2zg,\ xyzg,\ z^2h.$$

With respect to these bases, the multiplication $\{N(f)_5 \xrightarrow{\cdot\alpha_L} N(f)_6\}$, where $\alpha_L = ax + by + cz$, is given by the matrix

$$M(L) = \begin{pmatrix} a & 0 & 0 & 0 & 0 & 0 \\ b & 0 & a & 0 & 0 & 0 \\ 0 & b & 0 & 0 & 0 & 0 \\ c & 0 & 0 & a & 0 & 0 \\ 0 & c & 0 & 0 & b & -b \\ 0 & 0 & c & b & a & 0 \\ 0 & 0 & 0 & 0 & 0 & c \end{pmatrix}.$$

Using Theorem 4.4(5) for $k = 1$, we get that $V_1(C)$, the set of jumping lines for E_C, is the set of lines L such that $\operatorname{rank} M(L) < 6$. A direct computation shows that $V_1(C)$ consists of the line $\mathcal{L} : a = 0$ and one point, namely $P_1 = (1 : 0 : 0)$. A vector basis for $N(f)_4$ is given by xg, yg, zg, h, and using the given bases, the multiplication $\{N(f)_4 \xrightarrow{\cdot\alpha_L} N(f)_5\}$, where $\alpha_L = ax + by + cz$, is given by the matrix

$$M'(L) = \begin{pmatrix} a & 0 & 0 & 0 \\ 0 & b & 0 & -b \\ b & a & 0 & 0 \\ c & 0 & a & 0 \\ 0 & c & b & 0 \\ 0 & 0 & 0 & c \end{pmatrix}.$$

Using Proposition 4.1, it follows that $V_0(C)$ is the set of lines L such that $\operatorname{rank} M'(L) < 4$, which implies that $V_0(C) = \{P_1, P_2, P_3\}$, where P_1 is as above, $P_2 = (0 : 1 : 0)$, and $P_3 = (0 : 0 : 1)$. Hence in this case we have

$$\emptyset = V_{-1}(C) \subset V_0(C) = \{P_1, P_2, P_3\} \subset V_1(C) = \{P_1\} \cup \mathcal{L} \subset V_2(C) = \mathbb{P}(S_1).$$

Since $ar(f)_3 = 1$, there is essentially a unique choice

$$\rho_1 = (yz^2, -xz^2, xy^2) \in AR(f)_3.$$

The corresponding Bourbaki ideal $B(C, \rho_1)$ is the ideal (xy^2, xz^2, yz^2), and hence the scheme $Z(C, \rho_1)$ consists of three points, say p_1, p_2, and p_3. Two of them are non-reduced, namely the point $p_1 = (1 : 0 : 0)$, given in local coordinates by an ideal (u^2, v^2), and the point $p_2 = (0 : 1 : 0)$, given in local coordinates by an ideal (u, v^2). The third point $p_3 = (0 : 0 : 1)$ is reduced, hence it is given by the ideal (u, v). Note further that the line $\mathcal{L}$ consists of all the lines passing through the point p_1. The line $L_1 : x = 0$, corresponding to the point P_1, is the line $\overline{p_2 p_3}$ determined by the points p_2 and p_3. Similarly, the line $L_2 : y = 0$, corresponding to the point P_2, is the line $\overline{p_1 p_3}$ and the line $L_3 : z = 0$, corresponding to the point P_3, is the line $\overline{p_1 p_2}$. None of the points p_i is situated on the sextic C.

Example 6.6. Let $C : f = 0$, where $f = x^6 + y^6 + 3x^2y^2z^2$. Then $d = 6$, $\tau(C) = 12$, and we see that the S-module $AR(f)$ is generated by 5 syzygies ρ_i', $i = 1, \ldots, 5$, of degrees $r = mdr(f) = 4 = d_1 = d_2 < d_3 = d_4 = d_5 = 5$ (see their expressions given below). Hence Theorem 2.3(4) implies that the corresponding generic splitting type of E_C is $(d_1^{L_0}, d_2^{L_0}) = (2, 3)$. The Jacobian ideal J_f is spanned by f_x, f_y, f_z, and its saturation $\widehat{J}_f$ is spanned by f_x, f_y, f_z, x^3y, x^2y^2, xy^3. The only non-zero dimensions $n(f)_m$ are in this case $n(f)_4 = n(f)_8 = 3$, $n(f)_5 = n(f)_7 = 6$, and $n(f)_6 = 7$. Moreover, a vector space basis of $N(f)_5$ (resp. of $N(f)_6$) is given by

$$xy^4,\ x^2y^3,\ x^3y^2,\ x^4y,\ xy^3z,\ x^3yz,$$

and respectively by

$$xy^5,\ x^2y^4,\ x^3y^3,\ x^4y^2,\ x^5y,\ xy^4z,\ x^4yz.$$

With respect to these bases, the multiplication $\{N(f)_5 \xrightarrow{\cdot\alpha_L} N(f)_6\}$, where $\alpha_L = ax + by + cz$, is given by the matrix

$$M(L) = \begin{pmatrix} b & 0 & 0 & 0 & 0 & -c \\ a & b & 0 & 0 & 0 & 0 \\ 0 & a & b & 0 & 0 & 0 \\ 0 & 0 & a & b & 0 & 0 \\ 0 & 0 & 0 & a & -c & 0 \\ c & 0 & 0 & 0 & b & 0 \\ 0 & 0 & 0 & c & 0 & a \end{pmatrix}.$$

Using Theorem 4.4(5) for $k = 1$, we get that $V_1(C)$, the set of jumping lines for E_C, is the set of lines L such that $\operatorname{rank} M(L) < 6$. A direct computation shows that $V_1(C)$ consists of the following 11 points in $\mathbb{P}(S_1)$:

$$P_1=(1:1:1),\quad P_2=(1:1:-1),\quad P_3=(1:-1:1),\quad P_4=(-1:1:1),$$
$$P_5=(1:0:0),\quad P_6=(0:1:0),\quad P_7=(0:0:1),\quad P_8=(\alpha^2:\alpha:1),$$
$$P_9=(\alpha:\alpha^2:1),\quad P_{10}=(\beta^2:\beta:1),\quad P_{11}=(\beta:\beta^2:1),$$

where $\alpha^2+\alpha+1=0$ and $\beta^2-\beta+1=0$. A vector space basis for $N(f)_4$ is given by xy^3, x^2y^2, x^3y, and using the given bases, the multiplication $\{N(f)_4 \xrightarrow{\cdot\alpha_L} \to N(f)_5\}$, where $\alpha_L = ax + by + cz$, is given by the matrix

$$M'(L) = \begin{pmatrix} b & 0 & 0 \\ a & b & 0 \\ 0 & a & b \\ 0 & 0 & a \\ c & 0 & 0 \\ 0 & 0 & c \end{pmatrix}.$$

Using Proposition 4.1, it follows that $V_0(C)$ is the set of lines L such that $\operatorname{rank} M'(L) < 3$, which implies that $V_0(C) = P_7 = (0:0:1)$. Hence in this case we have

$$\begin{aligned}\emptyset = V_{-1}(C) \subset V_0(C) = \{P_7\} \subset V_1(C) \\ = \{P_j : j = 1, \ldots, 11\} \subset V_2(C) = \mathbb{P}(S_1).\end{aligned}$$

The software SINGULAR gives the following minimal system of generators for the graded S-module $AR(f)$:

$$\rho'_1 = (0, -x^2yz, y^4 + x^2z^2),\quad \rho'_2 = (-xy^2z, 0, x^4 + y^2z^2),$$
$$\rho'_3 = (xyz^3, -x^4z, x^2y^3 - yz^4),\quad \rho'_4 = (-y^4z, xyz^3, x^3y^2 - xz^4),$$
$$\text{and } \rho'_5 = (-y^5 - x^2yz^2, x^5 + xy^2z^2, 0).$$

Since now $ar(f)_4 = 2$, there are several choices for the syzygy ρ_1 in Theorem 5.1. We discuss three choices.

Choice 1. If we choose $\rho_1 = \rho'_1$, then the corresponding Bourbaki ideal $B(C, \rho_1)$ is spanned by $g_2 = v(\rho'_2) = -xyz$, $g_3 = v(\rho'_3) = xz^3$, $g_4 = v(\rho'_4) = -y^3z$, and $g_5 = v(\rho'_5) = -y^4 - x^2z^2$, where v is the morphism defined in (5.2). Hence the scheme $Z(C, \rho_1)$ consists of two points, both nonreduced: one at $p_1 = (1 : 0 : 0)$, given in local coordinates u, v by an ideal $(uv, v^2 + u^4)$, and another at $p_2 = (0 : 0 : 1)$, given in local coordinates by an ideal (u, v^3).

Choice 2. If we choose $\rho_1 = \rho'_2$, then the corresponding Bourbaki ideal $B(C, \rho_1)$ is spanned by $h_1 = v(\rho'_1) = xyz$, $h_3 = v(\rho'_3) = x^3z$, $h_4 = v(\rho'_4) = -yz^3$, and $h_5 = v(\rho'_5) = -x^4 - y^2z^2$. Hence the support of the scheme $Z(C, \rho_1)$ consists of two points: one at $q_1 = (0 : 1 : 0)$, and another at $p_2 = (0 : 0 : 1)$, the same point as in Choice 1.

Choice 3. If we choose $\rho_1 = \rho'_1 + t\rho'_2$, where $t \in \mathbb{C}^*$, then the corresponding Bourbaki ideal $B(C, \rho_1)$ is spanned by $k_1 = v(\rho'_1) = txyz$, $k_2 = v(\rho'_2) = -xyz$, $k_3 = v(\rho'_3) = xz(z^2 + tx^2)$, $k_4 = v(\rho'_4) = -yz(y^2 + tz^2)$, and $k_5 = v(\rho'_5) = -y^4 - x^2z^2 - t(x^4 + y^2z^2) = -y^2(y^2 + tz^2) - x^2(tx^2 + z^2)$. If we take $t = -s^4$ for $s \in \mathbb{C}^*$, then the support of the scheme $Z(C, \rho_1)$ consists of the following 9 points:

(i) $z_j(s) = (\epsilon_j : s : 0)$ for $j = 1, 2, 3, 4$, where ϵ_j are the four roots of $\epsilon^4 = 1$;

(ii) $z_j(s) = (0 : s^2 : (-1)^j)$, where $j = 5, 6$;

(iii) $z_j(s) = ((-1)^j : 0 : s^2)$, where $j = 7, 8$, and

(iv) $z_9(s) = p_2 = (0 : 0 : 1)$.

Theorem 5.1(1) implies that $\deg B(C, \rho_1) = 9$, and hence all these points $z_j(s)$ are simple points. When $s \to 0$, we see that the 6 points $z_j(s)$ for $j \in \{1, 2, 3, 4, 7, 8\}$ converge to the point p_1, and the 2 points $z_j(s)$ for $j \in \{5, 6\}$ converge to the point $p_2 = z_9(s)$. Similarly, when $|s| \to +\infty$, the 6 points $z_j(s)$ for $j \in \{1, 2, 3, 4, 5, 6\}$ converge to the point q_1, and the 2 points $z_j(s)$ for $j \in \{7, 8\}$ converge to the point $p_2 = z_9(s)$. Moreover, the line $L_7 : z = 0$, corresponding to the point P_7, contains the 4 points $z_j(s)$ for $j \in \{1, 2, 3, 4\}$ for any s, the maximal number of collinear points among the points $z_j(s)$. Note that the line $L_1 : x+y+z = 0$ is disjoint from the support of the scheme $Z(C, \rho_1)$ for most choices of ρ_1, and the bound given by Theorem 5.7 is $d_1^L \geq d - r - 1 = 1$. In fact, we have equality, hence this bound is sharp in this situation as well.

Remark 6.7. (i) In Example 6.6, the stable vector bundle E_C admits a *unique jumping line* P_7 *of maximal order* $o(P_7) = 2$. Note that condition (a) in [**22**, Theorem 6.2] is not fulfilled, hence we cannot use Hartshorne's result to deduce the unicity of a jumping line of maximal order.

(ii) A twist of the stable vector bundle E_C in Example 6.6 admits a section with 9 simple zeros $z_j(s)$ as explained in the third choice for ρ_1. However, the set of jumping lines does not coincide with the set of all lines passing through these points, and the line $P_7 : z = 0$ of maximal order 2 contains 4 of these points $z_j(s)$. This should be compared with [**26**, Theorem 2.2.5] and the previous discussion.

References

[1] T. Abe and A. Dimca, Splitting types of bundles of logarithmic vector fields along plane curves, *Internat. J. Math.* **29(8)** (2018), 1850055, 20 pp. DOI: 10.1142/S0129167X18500556.

[2] E. Artal Bartolo, L. Gorrochategui, I. Luengo, and A. Melle-Hernández, On some conjectures about free and nearly free divisors, in: *"Singularities and Computer Algebra"*, Springer, Cham, 2017, pp. 1–19. DOI: 10.1007/978-3-319-28829-1_1.

[3] W. Barth, Moduli of vector bundles on the projective plane, *Invent. Math.* **42** (1977), 63–91. DOI: 10.1007/BF01389784.

[4] N. Bourbaki, *"Algèbre commutative"*, Chapitres I–IX, Hermann & Cie, Paris, 1961–1983.

[5] D. Cook II, B. Harbourne, J. Migliore, and U. Nagel, Line arrangements and configurations of points with an unexpected geometric property, *Compos. Math.* **154(10)** (2018), 2150–2194. DOI: 10.1112/s0010437x18007376.

[6] W. Decker, G.-M. Greuel, G. Pfister, and H. Schönemann, Singular 4-0-1 — A computer algebra system for polynomial computations (2014). Available at http://www.singular.uni-kl.de.

[7] A. Dimca, Syzygies of Jacobian ideals and defects of linear systems, *Bull. Math. Soc. Sci. Math. Roumanie (N.S.)* **56(104)**, no. 2 (2013), 191–203.

[8] A. Dimca, Freeness versus maximal degree of the singular subscheme for surfaces in $\mathbb{P}^3$, *Geom. Dedicata* **183** (2016), 101–112. DOI: 10.1007/s10711-016-0148-2.

[9] A. Dimca, Freeness versus maximal global Tjurina number for plane curves, *Math. Proc. Cambridge Philos. Soc.* **163(1)** (2017), 161–172. DOI: 10.1017/S0305004116000803.

[10] A. Dimca, *"Hyperplane Arrangements. An Introduction"*, Universitext, Springer, Cham, 2017. DOI: 10.1007/978-3-319-56221-6.

[11] A. Dimca, On rational cuspidal plane curves and the local cohomology of Jacobian rings, *Comment. Math. Helv.* **94(4)** (2019), 689–700. DOI: 10.4171/cmh/471.

[12] A. Dimca and D. Popescu, Hilbert series and Lefschetz properties of dimension one almost complete intersections, *Comm. Algebra* **44(10)** (2016), 4467–4482. DOI: 10.1080/00927872.2015.1087535.

[13] A. DIMCA AND E. SERNESI, Syzygies and logarithmic vector fields along plane curves, *J. Éc. polytech. Math.* **1** (2014), 247–267. DOI: 10.5802/jep.10.
[14] A. DIMCA AND G. STICLARU, Free and nearly free curves vs. rational cuspidal plane curves, *Publ. Res. Inst. Math. Sci.* **54(1)** (2018), 163–179. DOI: 10.4171/PRIMS/54-1-6.
[15] A. DIMCA AND G. STICLARU, On supersolvable and nearly supersolvable line arrangements, *J. Algebraic Combin.* **50(4)** (2019), 363–378. DOI: 10.1007/s10801-018-0859-6.
[16] D. FAENZI AND J. VALLÈS, Logarithmic bundles and line arrangements, an approach via the standard construction, *J. Lond. Math. Soc. (2)* **90(3)** (2014), 675–694. DOI: 10.1112/jlms/jdu046.
[17] T. HARIMA, T. MAENO, H. MORITA, Y. NUMATA, A. WACHI, AND J. WATANABE, *"The Lefschetz Properties"*, Lecture Notes in Mathematics **2080**, Springer, Heidelberg, 2013. DOI: 10.1007/978-3-642-38206-2.
[18] T. HARIMA, J. C. MIGLIORE, U. NAGEL, AND J. WATANABE, The weak and strong Lefschetz properties for Artinian K-algebras, *J. Algebra* **262(1)** (2003), 99–126. DOI: 10.1016/S0021-8693(03)00038-3.
[19] J. HARRIS, *"Algebraic Geometry. A First Course"*, Graduate Texts in Mathematics **133**, Springer-Verlag, New York, 1992. DOI: 10.1007/978-1-4757-2189-8.
[20] R. HARTSHORNE, *"Algebraic Geometry"*, Graduate Texts in Mathematics **52**, Springer-Verlag, New York-Heidelberg, 1977. DOI: 10.1007/978-1-4757-3849-0.
[21] R. HARTSHORNE, Stable vector bundles of rank 2 on $\mathbf{P}^3$, *Math. Ann.* **238(3)** (1978), 229–280. DOI: 10.1007/BF01420250.
[22] R. HARTSHORNE, Stable reflexive sheaves, *Math. Ann.* **254(2)** (1980), 121–176. DOI: 10.1007/BF01467074.
[23] K. HULEK, Stable rank-2 vector bundles on $\mathbb{P}_2$ with c_1 odd, *Math. Ann.* **242(3)** (1979), 241–266. DOI: 10.1007/BF01420729.
[24] G. ILARDI, Jacobian ideals, arrangements and the Lefschetz properties, *J. Algebra* **508** (2018), 418–430. DOI: 10.1016/j.jalgebra.2018.04.029.
[25] S. MARCHESI AND J. VALLÈS, Nearly free curves and arrangements: A vector bundle point of view, *Math. Proc. Cambridge Philos. Soc.*, Published online (2019), 1–24. DOI: 10.1017/S0305004119000318.
[26] C. OKONEK, M. SCHNEIDER, AND H. SPINDLER, *"Vector Bundles on Complex Projective Spaces"*, Progress in Mathematics **3**, Birkhäuser, Boston, Mass., 1980. DOI: 10.1007/978-3-0348-0151-5.
[27] K. SAITO, Theory of logarithmic differential forms and logarithmic vector fields, *J. Fac. Sci. Univ. Tokyo Sect. IA Math.* **27(2)** (1980), 265–291.
[28] R. L. E. SCHWARZENBERGER, Vector bundles on algebraic surfaces, *Proc. London Math. Soc. (3)* **11(1)** (1961), 601–622. DOI: 10.1112/plms/s3-11.1.601.
[29] E. SERNESI, The local cohomology of the Jacobian ring, *Doc. Math.* **19** (2014), 541–565.
[30] J.-P. SERRE, Modules projectifs et espaces fibrés à fibre vectorielle, in: *"Séminaire P. Dubreil, M.-L. Dubreil-Jacotin et C. Pisot, 1957/58", Fasc. 2, Exposé 23"*, Secrétariat mathématique, Paris, 1958, 18 pp.
[31] A. SIMIS AND Ş. O. TOHĂNEANU, Homology of homogeneous divisors, *Israel J. Math.* **200(1)** (2014), 449–487. DOI: 10.1007/s11856-014-0025-3.

[32] A. Simis, B. Ulrich, and W. V. Vasconcelos, Rees algebras of modules, *Proc. London Math. Soc. (3)* **87(3)** (2003), 610–646. DOI: 10.1112/S0024611502014144.

[33] G. Sticlaru, Some criteria to check if a projective hypersurface is smooth or singular, *British Journal of Mathematics & Computer Science* **4(7)** (2014), 924–932.

[34] G. M. Ziegler, Combinatorial construction of logarithmic differential forms, *Adv. Math.* **76(1)** (1989), 116–154. DOI: 10.1016/0001-8708(89)90045-5.

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