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THE ASCENT-DESCENT PROPERTY FOR 2-TERM SILTING COMPLEXES

Simion Breaz

Abstract: We will prove that over commutative rings the silting property of 2-term complexes induced by morphisms between projective modules is preserved and reflected by faithfully flat extensions.

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1. Introduction

Given $\lambda\colon R \to S$ a homomorphism of unital rings, we let $-\otimes_R S\colon$ Mod-$R \to$ Mod-S be the extension of scalars functor. We will say that a property $\mathcal{P}$ associated to a complex of modules *ascends* along λ if the functor $-\otimes_R S$ preserves the property $\mathcal{P}$, i.e. for every complex $\mathbf{C}$ of right R-modules which satisfies $\mathcal{P}$, the complex $\mathbf{C}\otimes_R S$ satisfies $\mathcal{P}$ in Mod-S. The property $\mathcal{P}$ *descends* along λ if a complex $\mathbf{C}$ in Mod-R satisfies $\mathcal{P}$ provided that $\mathbf{C}\otimes_R S$ satisfies $\mathcal{P}$ as a complex of right S-modules. The above definitions are natural extension of the corresponding ascent/descent notions associated to module properties; see [**16**, Definition 3.5]. The properties of modules which ascend along flat ring homomorphisms and descend along faithful flat ring homomorphisms (called *ascent-descent* properties) play an important role in commutative algebra since the corresponding properties associated to quasi-coherent sheaves have a local character [**16**, Lemma 3.4]. For instance, for modules over commutative rings the properties "projective" ([**27**] and [**30**, Section 058B]) and "1-tilting" ([**19**, Theorem 3.13]) are ascent-descent. We mention here that the ascending property of tilting also plays an important role in the non-commutative case since it is used to characterize derived equivalences [**28**]. We refer to [**24**] for a general approach to this case.

In this paper we will study the ascent and descent properties for 2-*term silting complexes*. These are complexes $\cdots \to 0 \to P^{-1} \overset{\sigma}{\to} P^0 \to 0 \to \cdots$ concentrated in -1 and 0 which are silting objects in the unbounded derived category $\mathbf{D}(R)$ of R. In order to simplify the presentation, we will identify, as in [**5**], every 2-term complex $\cdots \to 0 \to P^{-1} \overset{\sigma}{\to} P^0 \to 0 \to \cdots$ with the homomorphism $\sigma\colon P^{-1} \to P^0$. If σ is a 2-term silting complex we will say that $\mathrm{Coker}(\sigma)$ is a silting module (with respect to the homomorphism σ). These notions were introduced in [**5**] as non-compact versions of the τ-tilting modules ([**1**]) and of the two term silting complexes ([**21**]). The role of silting modules in the study of module categories is described in [**2**]. For the finitely presented case we mention here the results proved in [**1**] and [**15**, Section 5]. If R is a ring, then (bounded) silting complexes play in the derived category $\mathbf{D}(R)$ a similar role to that of tilting modules in module categories [**31**]. In spite of this, correspondences or similarities between the influences of tilting modules and silting homomorphisms on the module category can be established only in particular instances (e.g. [**14**]). We refer to [**3**] for the general theory of silting objects in triangulated categories.

In Theorem 2.2 we provide, in the general case, a characterization for the ascending property of 2-term silting complexes. In the commutative case the ascending property of 2-term silting complexes is valid for all ring homomorphisms (see Theorem 2.7). Moreover, we will prove in Theorem 3.16 that the silting property associated to 2-term complexes descends along faithfully flat ring homomorphisms of commutative rings.

In this paper all rings and all ring homomorphisms are unital. If $\lambda\colon R \to S$ is a homomorphism of rings, then the extension of scalars functor is denoted by $- \otimes_R S\colon \text{Mod-}R \to \text{Mod-}S$. The restriction of scalars functor is denoted by $\lambda^\star = \mathrm{Hom}_S(S, -)\colon \text{Mod-}S \to \text{Mod-}R$. When there is no danger of confusion, we will consider every class of objects in Mod-S as a subclass of Mod-R. Therefore, if $\mathcal{X}$ is a class of R-modules and $\mathcal{Y}$ is a class of S-modules, then $\mathcal{X} \cap \mathcal{Y}$ means the class of all modules from $N \in \mathcal{Y}$ such that $\lambda^\star(N) \in \mathcal{X}$. In particular, we will identify a right S-module N with its image $\lambda^\star(N)$.

2. The ascent property

Let R be a unital ring. We consider a homomorphism of projective right R-modules $\sigma\colon P^{-1} \to P^0$, and we denote by T the cokernel of σ.

Then we can associate to σ the class

$$\mathcal{D}_\sigma = \{X \in \text{Mod-}R \mid \mathrm{Hom}_R(\sigma, X) \text{ is an epimorphism}\}.$$

Since $\mathcal{D}_\sigma$ is the kernel of the functor $\mathrm{Coker}(\mathrm{Hom}_R(\sigma, -))$, by using the properties of this functor (e.g. [**13**, Proposition 4]), it follows that $\mathcal{D}_\sigma$ is closed with respect to epimorphic images, extensions, and direct products.

Following [**5**], we will say that σ is *partial silting* or that the module $T = \operatorname{Coker}(\sigma)$ is *partial silting with respect to* σ if $\mathcal{D}_\sigma$ is a torsion class (i.e. $\mathcal{D}_\sigma$ is also closed under direct sums) and $T \in \mathcal{D}_\sigma$. Then $\operatorname{Gen}(T) \subseteq \mathcal{D}_\sigma \subseteq T^\perp$ and $(\operatorname{Gen}(T), T^\circ)$ is a torsion pair, where

$$\operatorname{Gen}(T) = \{M \in \text{Mod-}R \mid \text{there exists an epimorphism } T^{(I)} \to M\}$$

is the class of all T-generated right R-modules, $T^\perp = \operatorname{Ker}\operatorname{Ext}^1_R(T, -)$, and $T^\circ = \operatorname{Ker}\operatorname{Hom}_R(T, -)$. If $\mathcal{D}_\sigma = \operatorname{Gen}(T)$, then we will say that σ is *a 2-term silting complex* and T is called a *silting module with respect to* σ (we recall that it is proved in [**5**, Theorem 4.9] that a homomorphism σ satisfies the above condition if and only if it represents a silting complex in the associated unbounded derived category).

If $\lambda\colon R \to S$ is a ring homomorphism and $\sigma\colon P^{-1} \to P^0$ is a homomorphism of projective right R-modules, then we will denote $\sigma \otimes_R S = \sigma \otimes_R 1_S$ the induced homomorphism of projective right S-modules.

Lemma 2.1. *Let $\lambda\colon R \to S$ be a ring homomorphism. If $\sigma\colon P^{-1} \to P^0$ is a homomorphism between projective right R-modules, then $\mathcal{D}_{\sigma\otimes_R S} = \mathcal{D}_\sigma \cap \text{Mod-}S$.*

Proof: Let M be a right S-module. We have the commutative diagram

$$\begin{array}{ccc}
\operatorname{Hom}_R(P^0, \operatorname{Hom}_S(S, M)) & \xrightarrow{\operatorname{Hom}_R(\sigma, \operatorname{Hom}_S(S,M))} & \operatorname{Hom}_R(P^{-1}, \operatorname{Hom}_S(S, M)) \\
\downarrow \cong & & \downarrow \cong \\
\operatorname{Hom}_S(P^0 \otimes_R S, M) & \xrightarrow{\operatorname{Hom}_S(\sigma\otimes_R S, M)} & \operatorname{Hom}_S(P^{-1} \otimes_R S, M)
\end{array}$$

It follows that a right S-module M belongs to $\mathcal{D}_{\sigma\otimes_R S}$ if and only if the right R-module $\lambda^\star(M) = \operatorname{Hom}_S(S, M)$ is in $\mathcal{D}_\sigma$. □

Theorem 2.2. *Suppose that σ represents a 2-term silting complex. If $\lambda\colon R \to S$ is a ring homomorphism, then the following are equivalent:*

(1) *$\sigma \otimes_R S$ is a 2-term silting complex in* Mod-*S;*
(2) *$\lambda^\star(T \otimes_R S)$ is T-generated.*

Proof: (1) ⇒ (2) Since $\sigma \otimes_R S$ represents a 2-term silting complex, it follows that $T \otimes_R S \in \mathcal{D}_{\sigma\otimes_R S}$. Then (2) is a consequence of Lemma 2.1.

(2) ⇒ (1) Since $\mathcal{D}_\sigma$ is closed under direct sums, it is obvious, by Lemma 2.1, that $\mathcal{D}_{\sigma\otimes_R S}$ has the same property.

For every $M \in \mathcal{D}_{\sigma\otimes_R S}$ we have $\lambda^\star(M) \in \mathcal{D}_\sigma$. Since σ is a 2-term silting complex, it follows that $\lambda^\star(M)$ is T-generated. Then there exists an R-epimorphism $T^{(I)} \to \lambda^\star(M)$ which induces an S-epimorphism $T \otimes_R S^{(I)} \to \lambda^\star(M) \otimes_R S$. But it is well known that the canonical S-homomorphism $\lambda^\star(M) \otimes_R S \to M$, given by $x \otimes s \mapsto x(s)$, is an epimorphism. It follows that $\mathcal{D}_{\sigma\otimes_R S} \subseteq \mathrm{Gen}(T \otimes_R S)$.

Therefore, it is enough to prove that $T\otimes_R S \in \mathcal{D}_{\sigma\otimes_R S}$. By Lemma 2.1, this is equivalent to $\lambda^\star(T\otimes_R S) \in \mathcal{D}_\sigma$. Since σ is silting, we have $\lambda^\star(T\otimes_R S) \in \mathcal{D}_\sigma$ if and only if $\lambda^\star(T \otimes_R S)$ is T-generated, and this last property is assumed in (2). □

Remark 2.3. A similar proof, using the isomorphism

$$\mathrm{Hom}_S(X, \mathrm{Hom}_R(S, I)) \cong \mathrm{Hom}_R(X \otimes_S S, I),$$

can be used to obtain the dual of Theorem 2.2 for cosilting modules (we refer to [**12**] and [**33**] for the basic properties of these modules). Therefore, if $\lambda\colon R \to S$ is a unital ring homomorphism and $C \in \text{Mod-}R$ is a cosilting module with respect to the injective copresentation $\beta\colon I^0 \to I^1$, then the coinduced S-module $\mathrm{Hom}_R(S, C)$ is a cosilting module with respect to $\mathrm{Hom}_R(S, \beta)$ if and only if $\mathrm{Hom}_R(S, C)$ is C-cogenerated.

In the case of surjective ring homomorphisms, the extension of scalars functor always preserves the 2-term silting complexes.

Corollary 2.4. *Let $\lambda\colon R \to S$ be a surjective ring homomorphism. Then for every 2-term silting complex σ the induced complex $\sigma\otimes_R S$ is a 2-term silting complex.*

Proof: If $I = \mathrm{Ker}(\lambda)$, then for every right R-module M we have a natural R-isomorphism $M \otimes_R S \cong M/IM$ (see [**32**, 12.11]). □

However, even in the case of non-surjective ring epimorphisms the above corollary is not true. The following example was communicated to me by Lidia Angeleri-Hügel.

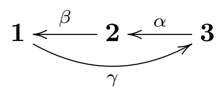
Example 2.5. Let R be the (Kronecker) path algebra associated to the graph $\mathbf{2} \leftleftarrows \mathbf{1}$ over a field K. The silting modules over Kronecker algebras are described in [**6**, Examples 5.10 and 5.18] and in [**20**, Example 2.20]. If $\mathbf{1}$ is the simple injective R-module and $\mathbf{2}$ is the simple projective module in Mod-R, we denote by P an indecomposable preprojective module which is not isomorphic to $\mathbf{2}$. By [**6**, Example 5.18] there exists a universal localization $\lambda\colon R \to S$ such that $\lambda^\star(\text{Mod-}S) = \mathrm{Add}(P)$. Since λ is a universal localization, the natural homomorphism $\mathbf{2} \otimes_R \lambda\colon \mathbf{2} \to \mathbf{2} \otimes_R S$ is an $\mathrm{Add}(P)$-reflection of $\mathbf{2}$, i.e. for every $X \in \mathrm{Add}(P)$, every homomorphism $\mathbf{2} \to X$ factorizes through $\mathbf{2} \otimes_R \lambda$. But $\mathrm{Hom}_R(\mathbf{2}, \mathrm{Add}(P)) \neq 0$, so it follows that $\mathbf{2} \otimes_R S \neq 0$. Moreover, it is obvious that $\mathbf{2}$ cannot generate the modules from $\mathrm{Add}(P)$. Note that $\mathbf{2}$ is a silting module with respect to a homomorphism σ. By the above remarks it follows that the complex $\sigma \otimes_R S$ is not silting.

In the following we will see an example of a tilting module such that the induced module with respect to a ring homomorphism is silting, but it is not 1-tilting. This is based on the example presented in [**7**, Example 4.2].

Example 2.6. Let R be the algebra associated to the quiver

$$\mathbf{1} \xleftarrow{\ \beta\ } \mathbf{2} \xleftarrow{\ \alpha\ } \mathbf{3},$$

and S the algebra associated to the quiver



such that $\alpha\beta\gamma = 0$. Then S is a split by nilpotent extension of R induced by the module generated by γ. Let $\lambda\colon R \to S$ be the corresponding ring homomorphism. Then $T = \mathbf{1} \oplus \mathbf{2} \oplus \begin{smallmatrix}\mathbf{3}\\ \mathbf{2}\\ \mathbf{1}\end{smallmatrix}$ is a tilting R-module. If $0 \to P^{-1} \overset{\sigma}{\to} P^0 \to T \to 0$ is a minimal projective presentation for T, it is proved in [**7**, Example 4.2], by using [**7**, Lemma 2.2], that $T \otimes S$ is not of projective dimension at most 1, so it is not a 1-tilting S-module. However, $\lambda^\star(T \otimes S) \in \mathrm{Gen}(T)$, hence $\sigma \otimes S$ is a 2-term silting complex.

In the commutative case the silting property associated to 2-term silting complexes ascends along all ring homomorphisms.

Theorem 2.7. *If R and S are commutative rings, $\lambda\colon R \to S$ is a unital ring homomorphism, and $\sigma\colon P^{-1} \to P^0$ is a 2-term silting complex, then the complex $\sigma \otimes_R S$ is silting.*

Proof: Let $T = \mathrm{Coker}(\sigma)$. In order to apply Theorem 2.2, we will prove that $\lambda^\star(T \otimes_R S)$ is T-generated. Let $\alpha\colon R^{(I)} \to S \to 0$ be an epimorphism of R-modules. Since R and S are commutative, α is a homomorphism of R-R-bimodules, and it follows that $1_T \otimes_R \alpha\colon T \otimes_R R^{(I)} \to T \otimes_R S$ is an R-epimorphism. □

3. The descent property for commutative rings

The main aim of this section is to prove that in the commutative case the property "2-term silting complex" descends along faithfully flat ring homomorphisms. We note that the restriction to faithfully flat ring homomorphisms is natural.

Example 3.1. Let $\lambda\colon \mathbb{Z} \to \mathbb{Q}$ be the canonical embedding. Therefore, λ is a ring epimorphism but it is not faithfully flat. Let $0 \to F^{-1} \stackrel{\sigma}{\to} F^0 \to \mathbb{Q} \to 0$ be a projective presentation in Mod-$\mathbb{Z}$ for the group of rational numbers. Then $\mathcal{D}_\sigma = \operatorname{Ker}\operatorname{Ext}^1_{\mathbb{Z}}(\mathbb{Q}, -)$ contains all finite abelian groups [**17**, Property 52 (D)]. But for every finite group G we have $\operatorname{Hom}_{\mathbb{Z}}(\mathbb{Q}, G) = 0$, and hence $\mathcal{D}_\sigma \neq \operatorname{Gen}(\mathbb{Q})$. It follows that σ is not a 2-term silting complex. However, it is easy to see that $\sigma \otimes_{\mathbb{Z}} \mathbb{Q}$ is a 2-term silting complex of $\mathbb{Q}$-modules.

In this section all rings are commutative. If R is a commutative ring, then $\operatorname{Spec}(R)$ will be the spectrum of R and for every $\mathfrak{p} \in \operatorname{Spec}(R)$ we will denote by $\kappa(\mathfrak{p})$ the field of fractions of $R/\mathfrak{p}$. If I is an ideal of R, then $V(I) = \{\mathfrak{p} \in \operatorname{Spec}(R) \mid I \subseteq \mathfrak{p}\}$. If $\lambda\colon R \to S$ is a faithfully flat ring homomorphism, then it is injective. Therefore, in order to simplify the presentation, we will often view R as a subring of S. For instance, if J is a subset of S we will write $R \cap J$ instead of $R \cap \lambda^{-1}(J)$. We refer to [**23**] for other notations and for the basic properties which will be used here.

If $\sigma\colon P^{-1} \to P^0$ is a homomorphism between projective R-modules, then we will associate to σ, as in [**4**], the class

$$\mathcal{T}_\sigma = \{M \in \text{Mod-}R \mid \sigma \otimes_R M \text{ is a monomorphism}\}.$$

Moreover, we also use the class

$$V_\sigma = \{\mathfrak{p} \in \operatorname{Spec}(R) \mid \sigma \otimes_R \kappa(\mathfrak{p}) \text{ is not a monomorphism}\}.$$

The class $\mathcal{T}_\sigma$ is closed under submodules and extensions (e.g. [**11**, Lemma 2.2.2]). Moreover, using [**4**, Lemma 3.3 and Lemma 4.2], we observe that if σ is a 2-term silting complex, then $\mathcal{T}_\sigma$ is the torsion-free class associated to a *hereditary* torsion theory of *finite type* $(\mathcal{A}_\sigma, \mathcal{T}_\sigma)$, i.e. $\mathcal{T}_\sigma$ is also closed under direct products, injective envelopes, and direct limits.

A class $\mathcal{D}$ of R-modules is called a *silting class* if there exists a 2-term silting complex σ such that $\mathcal{D} = \mathcal{D}_\sigma$ (i.e. there exists a silting module T such that $\mathcal{D} = \operatorname{Gen}(T)$). It is proved in [**4**, Theorem 4.7] that there exists a bijective correspondence between the silting classes of a commutative ring and the Gabriel filters of finite type. We recall from [**18**, Theorem 2.2] that if R is a commutative ring, then there exist $1-1$ correspondences between the class of Gabriel filters of finite type defined on R, the class of hereditary torsion theories of finite type on Mod-R, and the set of *Thomason subsets* of $\operatorname{Spec}(R)$, i.e. unions of families of subsets of the form $V(I)$ with I finitely generated ideals (these are the open sets associated to Hochster's topology defined on $\operatorname{Spec}(R)$). Therefore, we can state the following:

Theorem 3.2 ([**4**, Theorem 4.7], [**18**, Theorem 2.2])**.** *If R is a commutative ring, then there exist bijections between the following classes:*

1. *the class of hereditary torsion theories of finite type in* Mod-R*;*
2. *the class of Gabriel filters of finite type;*
3. *the class of Thomason subsets in* $\mathrm{Spec}(R)$*;*
4. *the class of silting classes in* Mod-R*.*

These bijections are described in the above mentioned papers. For the reader's convenience we list here the correspondences which will used in the following. We recall that for every finitely presented R-module M we have, by Nakayama's Lemma, $\mathrm{supp}(M) = \{\mathfrak{p} \mid M \otimes_R \kappa(\mathfrak{p}) \neq 0\}$ (see the proof of [**30**, Lemma 10.39.8]). We refer to [**30**, Section 10.39] for the basic properties of the support.

If $\mathcal{G}$ is a Gabriel filter of finite type on R and $\mathcal{B} \subseteq \mathcal{G}$ is a cofinal set of finitely generated ideals, then

- The torsion free class associated to $\mathcal{G}$ is
$$\mathcal{F}_{\mathcal{G}} = \mathrm{Ker}\,\mathrm{Hom}_R\left(\bigoplus_{I\in\mathcal{B}} R/I, -\right).$$
- The Thomason subset associated to $\mathcal{G}$ is
$$V_{\mathcal{G}} = \{\mathfrak{p} \in \mathrm{Spec}(R) \mid \exists I \in \mathcal{G} \text{ such that } \mathfrak{p} \in \mathrm{supp}\, R/I\}.$$
We can use [**30**, Lemma 10.39.9] to conclude that
$$V_{\mathcal{G}} = \{\mathfrak{p} \in \mathrm{Spec}(R) \mid \exists I \in \mathcal{B} \text{ such that } R/I \otimes_R \kappa(\mathfrak{p}) \neq 0\}.$$
- The silting class induced by $\mathcal{G}$ is
$$\mathcal{D}_{\mathcal{G}} = \bigcap_{I\in\mathcal{G}} \mathrm{Ker}(- \otimes_R R/I) = \bigcap_{I\in\mathcal{B}} \mathrm{Ker}(- \otimes_R R/I).$$

If σ is a 2-term silting complex, then the Gabriel filter of finite type induced by $\mathcal{T}_\sigma$ via [**18**, Theorem 2.2] is
$$\mathcal{G}_\sigma = \{I \leq R \mid \mathrm{Hom}_R(R/I, \mathcal{T}_\sigma) = 0\}.$$
In the bijective correspondence constructed in [**4**, Theorem 4.7] the Gabriel filter associated to each silting class $\mathcal{D}$ is defined as
$$\mathcal{G}_{\mathcal{D}} = \{I \leq R \mid M = IM \text{ for all } M \in \mathcal{D}\}.$$

Lemma 3.3. *If $\mathcal{D}$ is a silting class and σ is a 2-term silting complex such that $\mathcal{D} = \mathcal{D}_\sigma$, then $\mathcal{G}_{\mathcal{D}} = \mathcal{G}_\sigma$.*

Proof: Since in [**4**, Lemma 3.3(2)] it is proved that the definable classes $\mathcal{D} = \mathcal{D}_\sigma$ and $\mathcal{T}_\sigma$ are dual, it follows from [**26**, Corollary 3.4.21] that a module X belongs to $\mathcal{T}_\sigma$ if and only if $X^+ \in \mathcal{D}$, where $X^+ = \mathrm{Hom}_{\mathbb{Z}}(X, \mathbb{Q}/\mathbb{Z})$. Therefore, the correspondence $\mathcal{D} \mapsto \mathcal{G}_\sigma$ is independent of the choice of σ. Moreover, a module M is in $\mathcal{D}$ if and only if $X^+ \in \mathcal{T}_\sigma$. By [**18**, Theorem 2.2] this is equivalent to $\mathrm{Hom}(R/I, X^+) = 0$ for all $I \in \mathcal{G}_\sigma$. Using the natural isomorphism induced by Hom and $\otimes$, we obtain $M \in \mathcal{D}$ if and only if $R/I \otimes_R M = 0$ for all $I \in \mathcal{G}_\sigma$. This means that $\mathcal{G}_\sigma$ is the Gabriel filter of finite type constructed in [**4**, Proposition 4.4], and the proof is complete. □

In the following we present some properties of the classes involved in Theorem 3.2. We start with a well-known lemma.

Lemma 3.4. *If $\mathfrak{p} \in \mathrm{Spec}(R)$ and $E(R/\mathfrak{p})$ is the injective envelope for $R/\mathfrak{p}$, then*

(1) *$E(R/\mathfrak{p})$ is an injective cogenerator for* Mod-$R_{\mathfrak{p}}$.
(2) *There is a natural isomorphism* $\mathrm{Hom}_R(-, \kappa(\mathfrak{p})) \cong \mathrm{Hom}_{R_{\mathfrak{p}}}(- \otimes_R \kappa(\mathfrak{p}), E(R/\mathfrak{p}))$.

Proof: The first statement is well known. For the the second statement, we observe that there are the natural isomorphisms

$$\begin{aligned}\mathrm{Hom}_R(-, \kappa(\mathfrak{p})) &\cong \mathrm{Hom}_R(-, \mathrm{Hom}_{R_{\mathfrak{p}}}(\kappa(\mathfrak{p}), E(R/\mathfrak{p})))\\ &\cong \mathrm{Hom}_{R_{\mathfrak{p}}}(- \otimes_R \kappa(\mathfrak{p}), E(R/\mathfrak{p})),\end{aligned}$$

so we have the required isomorphism. □

Corollary 3.5. *Let $\mathcal{G}$ be a Gabriel filter of finite type, and $\mathfrak{p} \in \mathrm{Spec}(R)$. Then the following are equivalent:*

(1) $\mathfrak{p} \notin V_{\mathcal{G}}$;
(2) $\kappa(\mathfrak{p}) \in \mathcal{F}_{\mathcal{G}}$;
(3) $R/\mathfrak{p} \in \mathcal{F}_{\mathcal{G}}$.

In particular, if σ is a 2-term silting complex, and $\mathcal{G}$ is the induced Gabriel filter, then $\mathcal{F}_{\mathcal{G}} = \mathcal{T}_\sigma$, and $V_{\mathcal{G}} = V_\sigma$.

Proof: The equivalence (1) $\Leftrightarrow$ (2) follows from Lemma 3.4, while (2) $\Leftrightarrow$ (3) is true since $\mathcal{F}_{\mathcal{G}}$ is closed under submodules and injective envelopes.

If σ is a 2-term silting complex, the equality $\mathcal{F}_{\mathcal{G}} = \mathcal{T}_\sigma$ is true since $\mathcal{G}$ is the Gabriel filter associated to $\mathcal{T}_\sigma$. The second equality follows from the equivalence (1) $\Leftrightarrow$ (2). □

Lemma 3.6. *Let $\lambda\colon R \to S$ be a homomorphism of commutative rings. If P^{-1} and P^0 are projective R-modules and $\sigma\colon P^{-1} \to P^0$ is a homomorphism, we denote by $\mathcal{T}_{\sigma\otimes_R S} \subseteq$ Mod-S the class associated to the S-homomorphism $\sigma \otimes_R S$. The following hold:*

(1) $\mathcal{T}_{\sigma\otimes_R S} = \mathcal{T}_\sigma \cap$ Mod-S.
(2) *Suppose that λ is faithfully flat. For a module $M \in$ Mod-R we have $M \in \mathcal{T}_\sigma$ if and only if $M \otimes_R S \in \mathcal{T}_{\sigma\otimes_R S}$.*

Proof: (1) This follows by using the natural isomorphisms $\sigma\otimes_R S\otimes_S M \cong \sigma \otimes_R M$ for all $M \in$ Mod-S.

(2) Suppose that $M \in \mathcal{T}_\sigma$. Then $\sigma \otimes_R M$ is monic. Since S is flat, it follows that $\sigma \otimes_R M \otimes_R S$ is monic, hence $\lambda^\star(M \otimes_R S) \in \mathcal{T}_\sigma$. By (1) we obtain $M \otimes_R S \in \mathcal{T}_{\sigma\otimes_R S}$.

Conversely, suppose that $M \otimes_R S \in \mathcal{T}_{\sigma\otimes_R S}$. Then $\lambda^\star(M \otimes_R S) \in \mathcal{T}_\sigma$. Since S is faithfully flat we know that M can be embedded as a submodule of $\lambda^\star(M \otimes S)$, and hence $M \in \mathcal{T}_\sigma$. □

In the following we will use, as in [**13**], the notation

$$\mathrm{Def}_\sigma = \mathrm{Coker}(\mathrm{Hom}_R(\sigma, -))\colon \text{Mod-}R \longrightarrow Ab.$$

Therefore, for every homomorphism σ we have $\mathcal{D}_\sigma = \{M \in \text{Mod-}R \mid \mathrm{Def}_\sigma(M) = 0\}$.

Lemma 3.7. *Let $\lambda\colon R \to S$ be a homomorphism of commutative rings. If L^{-1} and L^0 are finitely generated and projective R-modules and $\sigma\colon L^{-1} \to L^0$ is an R-morphism, then we have a natural isomorphism*

$$\mathrm{Def}_{\sigma\otimes_R S}(- \otimes_R S) \cong \mathrm{Def}_\sigma(-) \otimes_R S.$$

Proof: This is a consequence of the fact that for every finitely presented R-module L, there exists a natural isomorphism between the functors $\mathrm{Hom}_S(L \otimes_R S, - \otimes_R S)$ and $\mathrm{Hom}_R(L, -) \otimes_R S$ (see [**23**, Theorem 7.11]). □

Lemma 3.8. *Suppose that $\lambda\colon R \to S$ is a homomorphism of commutative rings and that σ is a 2-term silting complex. Then*

(1) $\mathcal{D}_\sigma \otimes_R S \subseteq \mathcal{D}_{\sigma\otimes_R S}$.
(2) *If λ is faithfully flat and $N \in$ Mod-R, then $N \in \mathcal{D}_\sigma$ if and only if $N \otimes_R S \in \mathcal{D}_{\sigma\otimes_R S}$.*

Proof: (1) If $N \in \mathcal{D}_\sigma$, then $N \otimes S$ is T-generated. Since it is an S-module we obtain $N \otimes_R S \in \mathcal{D}_{\sigma \otimes_R S}$ by Theorem 2.2.

(2) Suppose that $N \otimes_R S \in \mathcal{D}_{\sigma \otimes_R S}$. It was proved in [**4**, Theorem 2.3] and in [**22**, Theorem 6.3] that every silting class is of finite type. Therefore, there exists a family σ_i, $i \in I$, of homomorphisms between finitely generated projective R-modules such that $\mathcal{D}_\sigma = \cap_{i \in I} \mathcal{D}_{\sigma_i}$. By Lemma 2.1 it follows that $N \otimes_R S \in \mathcal{D}_\sigma$. Using the proof of Lemma 2.1 and Lemma 3.7, we observe that for every $i \in I$ we have isomorphisms

$$0 = \mathrm{Def}_{\sigma_i}(N \otimes S) \cong \mathrm{Def}_{\sigma_i \otimes_R S}(N \otimes_R S) \cong \mathrm{Def}_{\sigma_i}(N) \otimes_R S.$$

Since S is faithfully flat, it follows that $N \in \mathcal{D}_{\sigma_i}$ for all $i \in I$, so $N \in \mathcal{D}_\sigma$. □

Remark 3.9. We recall from [**5**] that if σ is 2-term silting complex, then it is a *generator* in $\mathbf{D}(R)$, i.e. the smallest triangulated subcategory which contains σ and is closed under direct sums is $\mathbf{D}(R)$.

Moreover, it was proved in [**3**, Theorem 3.14] (see also [**5**, Remark 2.7]) that an object X in $\mathbf{D}(R)$ is a generator if and only if X has the following property: for every $Y \in \mathbf{D}(R)$, from $\mathrm{Hom}(X, Y[i]) = 0$ for all $i \in \mathbb{Z}$ it follows that $Y = 0$.

For the proof of the following lemma we use the same techniques as those used in the proof of [**19**, Lemma 4.12].

Lemma 3.10. *Let $\sigma\colon P^{-1} \to P^0$ be a homomorphism between projective R-modules with* $\mathrm{Coker}(\sigma) = T$. *If the complex (concentrated in -1 and 0) which is induced by σ is a generator for $\mathbf{D}(R)$, then for every $\mathfrak{p} \in \mathrm{Spec}(R)$ we have $T \otimes_R \kappa(\mathfrak{p}) \neq 0$ or* $\mathrm{Ker}(\sigma \otimes_R \kappa(\mathfrak{p})) \neq 0$.

Proof: Since σ is a generator for $\mathbf{D}(R)$, for every R-module M there exists $i \in \{0,1\}$ such that $\mathrm{Hom}_{\mathbf{D(R)}}(\sigma, M[i]) = \mathrm{Hom}_{\mathbf{K(R)}}(\sigma, M[i]) \neq 0$. It follows that $\mathrm{Hom}(\sigma, M)$ is not an isomorphism for all R-modules M.

If $\mathfrak{p} \in \mathrm{Spec}(R)$, we take $M = \kappa(\mathfrak{p}) = \mathrm{Hom}_{R_\mathfrak{p}}(\kappa(\mathfrak{p}), E(R/\mathfrak{p}))$, where $E(R/\mathfrak{p})$ is the injective envelope of $\kappa(\mathfrak{p})$ in Mod-$R_\mathfrak{p}$. Using Lemma 3.4 we obtain that $\mathrm{Hom}_R(\sigma, \kappa(\mathfrak{p}))$ is not an isomorphism if and only if $\sigma \otimes_R \kappa(\mathfrak{p})$ is not an isomorphism. □

We recall from [**29**] and [**30**] some basic facts about ring epimorphisms.

Lemma 3.11. *Let R be a commutative ring and $\delta\colon R \to B$ a ring epimorphism. Then*

(1) *B is commutative.*
(2) *The canonical map $\delta^\star\colon \mathrm{Spec}(B) \to \mathrm{Spec}(R)$ is injective.*
(3) *For every $\mathfrak{q} \in \mathrm{Spec}(B)$, $\kappa(\mathfrak{q}) = \kappa(\delta^\star(\mathfrak{q}))$.*

Proposition 3.12. *Let $\sigma\colon P^{-1} \to P^0$ be a homomorphism between projective R-modules. The following are equivalent:*

(1) *σ is a 2-term silting complex;*

(2) (i) *σ is partial silting,*

(ii) *for every $\mathfrak{p} \in \operatorname{Spec}(R)$ we have $T \otimes_R \kappa(\mathfrak{p}) \neq 0$ or $\operatorname{Ker}(\sigma \otimes_R \kappa(\mathfrak{p})) \neq 0$.*

Proof: (1) $\Rightarrow$ (2) This follows from Lemma 3.10.

(2) $\Rightarrow$ (1) Since σ is partial silting, then $\operatorname{Gen}(T) \subseteq \mathcal{D}_\sigma$ are torsion classes and the torsion-free class corresponding to $\operatorname{Gen}(T)$ is $\operatorname{Ker}\operatorname{Hom}_R(T,-)$ [**5**, Remark 3.8]. Therefore, in order to obtain $\operatorname{Gen}(T) = \mathcal{D}_\sigma$ it is enough to prove that $\mathcal{D}_\sigma \cap \operatorname{Ker}\operatorname{Hom}_R(T,-) = 0$.

Let $\mathcal{Y} = \mathcal{D}_\sigma \cap \operatorname{Ker}\operatorname{Hom}_R(T,-)$. By [**6**, Proposition 3.3] the class $\mathcal{Y}$ is bireflective and extension closed. Therefore, it induces an epimorphism of rings $\delta\colon R \to B$ and $\mathcal{Y}$ is the essential image of the restriction of scalars $\delta^\star\colon \text{Mod-}B \to \text{Mod-}R$.

Suppose that $B \neq 0$. Then $\operatorname{Spec}(B) \neq \varnothing$. By Lemma 3.11 it follows that for every $\mathfrak{q} \in \operatorname{Spec}(B)$ we can identify $\kappa(\mathfrak{q}) = \kappa(\mathfrak{p})$ for some $\mathfrak{p} \in \operatorname{Spec}(R)$. Therefore, there exists $\mathfrak{p} \in \operatorname{Spec}(R)$ such that $\kappa(\mathfrak{p}) \in \mathcal{Y}$. If $\mathfrak{p}$ is such an ideal, we obtain that $\operatorname{Hom}_R(\sigma, \kappa(\mathfrak{p}))$ is an isomorphism. Using Lemma 3.4, it follows that $\sigma \otimes_R \kappa(\mathfrak{p})$ is an isomorphism and this contradicts (ii). Then $B = 0$ and the proof is complete. □

The following properties are well known.

Lemma 3.13. *Let $\lambda\colon R \to S$ be a faithfully flat homomorphism of commutative rings. If $\mathfrak{q} \in \operatorname{Spec}(S)$ and $\mathfrak{p} = \mathfrak{q} \cap R$, then there is a natural isomorphism*

$$- \otimes_R S \otimes_S \kappa(\mathfrak{q}) \cong - \otimes_R \kappa(\mathfrak{p}) \otimes_{\kappa(\mathfrak{p})} \kappa(\mathfrak{q}).$$

Moreover, $\kappa(\mathfrak{q})$ is faithfully flat as a $\kappa(\mathfrak{p})$-module.

We recall that we use the notation

$$V_\sigma = \{\mathfrak{p} \in \operatorname{Spec} R \mid \sigma \otimes_R \kappa(\mathfrak{p}) \text{ is not a monomorphism}\}$$

for every homomorphism σ between projective modules.

Lemma 3.14. *Suppose that $\lambda\colon R \to S$ is faithfully flat, and $\sigma\colon P^{-1} \to P^0$ is a homomorphism between projective R-modules such that $\sigma \otimes_R S$ is a 2-term silting complex. Let $\lambda^\star\colon \operatorname{Spec}(S) \to \operatorname{Spec}(R)$, given by $\lambda^\star(\mathfrak{q}) = \mathfrak{q} \cap R$, be the canonical map. Then*

(1) *For an ideal* $\mathfrak{q} \in \operatorname{Spec}(S)$*, we have* $\kappa(\mathfrak{q}) \in \mathcal{T}_{\sigma\otimes S}$ *if and only if* $\kappa(\lambda^\star(\mathfrak{q})) \in \mathcal{T}_\sigma$.
(2) $\lambda^\star(\operatorname{Spec}(S) \setminus V_{\sigma\otimes_R S}) = \operatorname{Spec}(R) \setminus V_\sigma$.
(3) $\lambda^\star(V_{\sigma\otimes_R S}) = V_\sigma$.

Proof: (1) If $\mathfrak{q} \in \operatorname{Spec}(S)$ and $\mathfrak{p} = \mathfrak{q} \cap R$, it follows from Lemma 3.13 that $\sigma \otimes_R \kappa(\mathfrak{p})$ is injective if and only if $\sigma \otimes_R \kappa(\mathfrak{p}) \otimes_{\kappa(\mathfrak{p})} \kappa(\mathfrak{q})$ is injective, and this is equivalent to $\sigma \otimes_R S \otimes_S \kappa(\mathfrak{q})$ being injective.

(2) We observe that

$V_\sigma = \{\mathfrak{p} \in \operatorname{Spec}(S) \mid \kappa(\mathfrak{p}) \notin \mathcal{T}_\sigma\}$ and $V_{\sigma\otimes_R S} = \{\mathfrak{q} \in \operatorname{Spec}(S) \mid \kappa(\mathfrak{q}) \notin \mathcal{T}_{\sigma\otimes_R S}\}$.

By (1) it follows that for an ideal $\mathfrak{q} \in \operatorname{Spec}(S)$ we have $\mathfrak{q} \notin V_{\sigma\otimes_R S}$ if and only if $\lambda^\star(\mathfrak{q}) \notin V_\sigma$.

Since $\lambda^\star$ is surjective, it follows that $\lambda^\star(\operatorname{Spec}(S) \setminus V_{\sigma\otimes_R S}) = \operatorname{Spec}(R) \setminus V_\sigma$.

(3) This follows by using (2) and the surjectivity of $\lambda^\star$. □

To conclude these preliminary considerations we recall the results obtained in [**19**] for the study of the descent of 1-tilting modules.

Proposition 3.15 ([**19**, Section 4])**.** *Let* $\overline{\lambda}\colon \overline{R} \to \overline{S}$ *be a faithfully flat homomorphism of rings. If* T *and* V *are* $\overline{R}$*-modules such that*
(1) V *is tilting,*
(2) $T \otimes_{\overline{R}} \overline{S}$ *is a tilting* $\overline{S}$*-module, and*
(3) $\operatorname{Gen}(T \otimes_{\overline{R}} \overline{S}) = \operatorname{Gen}(V \otimes_{\overline{R}} \overline{S})$,
then T *is a tilting* $\overline{R}$*-module and* $\operatorname{Gen}(T) = \operatorname{Gen}(V)$.

We are ready to prove the descent property for 2-term silting complexes.

Theorem 3.16. *Suppose that* $\lambda\colon R \to S$ *is a faithfully flat ring homomorphism. If* $\sigma\colon P^{-1} \to P^0$ *is a homomorphism in* Mod-R *such that* $\sigma \otimes_R S$ *is a 2-term silting complex of* S*-modules, then* σ *is a 2-term silting complex of* R*-modules.*

Proof: Since $\sigma \otimes_R S$ is a 2-term silting complex it follows that $P^{-1} \otimes_R S$ and $P^0 \otimes_R S$ are projective S-modules. Using the descent property of projective modules [**27**], it follows that the R-modules P^{-1} and P^0 are projective.

Let $\lambda^\star\colon \operatorname{Spec}(S) \to \operatorname{Spec}(R)$, $\lambda^\star(\mathfrak{q}) = \mathfrak{q} \cap R$, be the canonical map. If $V_{\sigma\otimes_R S} \subseteq \operatorname{Spec}(S)$ is the Thomason set associated to $\sigma \otimes S$, then we use [**19**, Lemma 3.15] together with Lemma 3.14(2) to conclude that

$\lambda^\star(V_{\sigma\otimes_R S})$ is a Thomason subset in $\operatorname{Spec}(R)$. By Theorem 3.2 and Corollary 3.5 there exists a 2-term silting complex $\rho\colon L^{-1}\to L^0$ in Mod-R such that $V_\rho=\lambda^\star(V_{\sigma\otimes_R S})$. Using the ascent property proved in Theorem 2.7 we conclude that $\rho\otimes_R S$ is silting in Mod-S.

We apply Lemma 3.14(3) to ρ and σ, and we obtain

$$\lambda^\star(V_{\rho\otimes_R S})=V_\rho=\lambda^\star(V_{\sigma\otimes_R S})=V_\sigma.$$

We use Lemma 3.14(1) and we obtain the equality $V_{\rho\otimes_R S}=V_{\sigma\otimes_R S}$. It follows that the homomorphisms $\rho\otimes_R S$ and $\sigma\otimes_R S$ induce the same silting class, i.e. $\mathcal{D}_{\sigma\otimes_R S}=\mathcal{D}_{\rho\otimes_R S}$.

Let $T=\operatorname{Coker}(\sigma)$ and let $\mathcal{G}$ be the Gabriel filter associated to ρ as in Theorem 3.2 and Lemma 3.3. It follows that

$$\mathcal{D}_\rho=\bigcap_{I\in\mathcal{G}}\operatorname{Ker}(-\otimes_R R/I).$$

Since $T\otimes S\in\mathcal{D}_{\rho\otimes_R S}$, we can use Lemma 3.8 to conclude that $T\in\mathcal{D}_\rho$. Therefore, for every $I\in\mathcal{G}$ we have $T\otimes_R R/I=0$.

Claim 1. *For every $I\in\mathcal{G}$ there exists a set K and a pushout diagram*

$$\begin{array}{ccccccc} (P^{-1})^{(K)} & \xrightarrow{\sigma^{(K)}} & (P^0)^{(K)} & \longrightarrow & T^{(K)} & \longrightarrow & 0 \\ \downarrow{\scriptstyle\delta} & & \downarrow & & \| & & \\ R & \xrightarrow{\alpha} & E & \longrightarrow & T^{(K)} & \longrightarrow & 0 \end{array}$$

such that $E\otimes_R R/I=0$.

Let $I\in\mathcal{G}$ be a fixed ideal. We will use the notation $\sigma\otimes_R R/I=\widehat{\sigma}_{R/I}$.

Applying $-\otimes_R R/I$ to σ we obtain a short exact sequence of R/I-modules

$$0\longrightarrow \operatorname{Ker}(\widehat{\sigma}_{R/I})\xrightarrow{u} P^{-1}\otimes_R R/I\xrightarrow{\widehat{\sigma}_{R/I}} P^0\otimes_R R/I\longrightarrow 0,$$

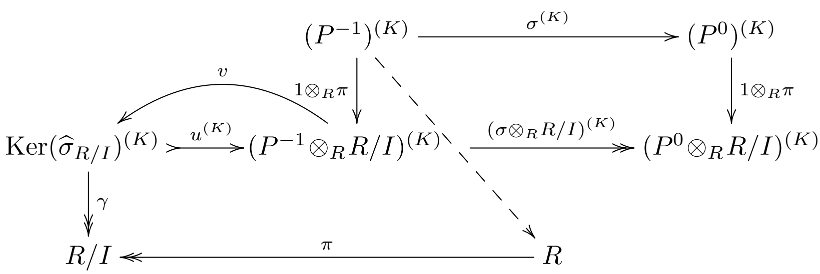
which splits since $P^0\otimes_R R/I$ is projective. Therefore $\operatorname{Ker}(\widehat{\sigma}_{R/I})$ is projective.

Moreover, since this exact sequence splits, for every $\mathfrak{p}\in\operatorname{Spec}(R/I)$ we obtain a commutative diagram

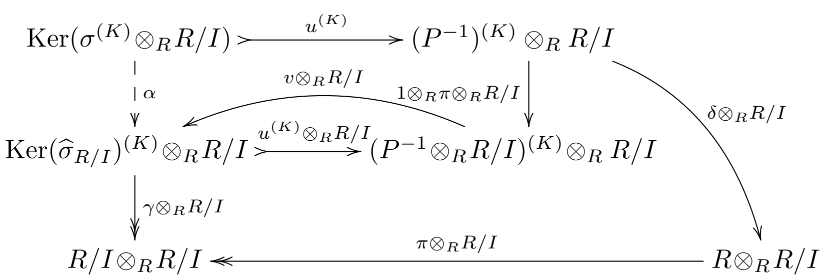
$$\begin{array}{ccccc} \operatorname{Ker}(\widehat{\sigma}_{R/I})\otimes_{R/I}\kappa(\mathfrak{p}) & \xrightarrow{u\otimes_{R/I}\kappa(\mathfrak{p})} & P^{-1}\otimes_R R/I\otimes_{R/I}\kappa(\mathfrak{p}) & \twoheadrightarrow & P^0\otimes_R R/I\otimes_{R/I}\kappa(\mathfrak{p}) \\ \downarrow & & \downarrow & & \downarrow \\ \operatorname{Ker}(\sigma\otimes_R\kappa(\mathfrak{p})) & \rightarrowtail & P^{-1}\otimes_R\kappa(\mathfrak{p}) & \twoheadrightarrow & P^0\otimes_R\kappa(\mathfrak{p}) \end{array}$$

such that the horizontal rows are (split) short exact sequences and the vertical arrows are isomorphisms. Since V_σ is the Thomason set corresponding to $\mathcal{G}$, we have $V(I) \subseteq V_\sigma$. It follows that $\mathrm{Ker}(\widehat{\sigma}_{R/I}) \otimes_{R/I} \kappa(\mathfrak{p}) \neq 0$ for all $\mathfrak{p} \in \mathrm{Spec}(R/I)$ and hence $\mathrm{Ker}(\widehat{\sigma}_{R/I})$ is a projective generator for Mod-R/I.

Thus, there exist a set K and an R/I-epimorphism $\gamma\colon \mathrm{Ker}(\widehat{\sigma}_{R/I})^{(K)} \to R/I$. We also fix a homomorphism $v\colon (P^{-1})^{(K)} \otimes_R R/I \to \mathrm{Ker}(\widehat{\sigma}_{R/I})^{(K)}$ such that $vu^{(K)} = 1$. If $\pi\colon R \to R/I$ is the canonical surjective homomorphism, then there exists a homomorphism $\delta\colon (P^{-1})^{(K)} \otimes_R R \to R$ such that $\pi\delta = \gamma v(1 \otimes_R \pi)$. In order to simplify the presentation we identify $(P^{-1})^{(K)} \otimes_R R$ with $(P^{-1})^{(K)}$. All these data are represented in the following commutative diagram:



where the dashed arrow is δ. We apply the functor $-\otimes_R R/I$ to this diagram, and we obtain the commutative diagram



where α is the canonical map. Using the obvious identifications and the natural isomorphisms $R/I \otimes_R R/I \cong R/I \otimes_{R/I} R/I \cong R/I$ ([**10**, Proposition II.2]) it follows that α and $\pi \otimes_R R/I$ are isomorphisms. It is not hard to conclude that $(\delta \otimes_R R/I)u^{(K)}$ is an epimorphism.

We construct the pushout diagram

$$\begin{array}{ccccccc}
(P^{-1})^{(K)} & \xrightarrow{\sigma^{(K)}} & (P^0)^{(K)} & \longrightarrow & T^{(K)} & \longrightarrow & 0 \\
\downarrow{\scriptstyle \delta} & & \downarrow & & \| & & \\
R & \xrightarrow{\alpha} & E & \longrightarrow & T^{(K)} & \longrightarrow & 0
\end{array}$$

Applying the tensor product $-\otimes_R R/I$, and using the commuting property of the tensor product with respect to direct sums together with the well-known fact that the direct sums preserve exact sequences we obtain the commutative diagram

$$\begin{array}{ccccc}
\mathrm{Ker}(\widehat{\sigma}_{R/I})^{(K)} & \xrightarrow{u^{(K)}} & (P^{-1}\otimes_R R/I)^{(K)} & \xrightarrow{(\widehat{\sigma}_{R/I})^{(K)}} & (P^0\otimes_R R/I)^{(K)} \\
\downarrow & & \downarrow{\scriptstyle \delta\otimes_R R/I} & & \downarrow \\
\mathrm{Ker}(\alpha\otimes_R R/I) & \longrightarrow & R\otimes_R R/I & \xrightarrow{\alpha\otimes_R R/I} & E\otimes_R R/I
\end{array}$$

Since $(\delta\otimes_R R/I)u^{(K)}$ is an epimorphism, it follows that $E\otimes_R R/I=0$.

Claim 2. *$\mathcal{D}_\sigma\subseteq\mathcal{D}_\rho$.*

In order to prove this, let us fix a module $M\in\mathcal{D}_\sigma$. For every $I\in\mathcal{G}$ we construct a pushout diagram as in Claim 1.

We consider an epimorphism $f\colon R^{(L)}\to M$. Then $f\delta^{(L)}\colon[(P^{-1})^{(K)}]^{(L)}\to M$ can be extended to a homomorphism $[(P^0)^{(K)}]^{(L)}\to M$ through $[\sigma^{(K)}]^{(L)}$. It follows that there exists a homomorphism $g\colon E^{(L)}\to M$ f such that $f=g\alpha(L)$. Since f is an epimorphism, we also have that g is an epimorphism. Using the property $E\otimes_R R/I=0$ we obtain $M\otimes_R R/I=0$.

It follows that $M\otimes_R R/I=0$ for all $I\in\mathcal{G}$. Therefore, $M\in\mathcal{D}_\rho$ and the proof for the inclusion $\mathcal{D}_\sigma\subseteq\mathcal{D}_\rho$ is complete.

Claim 3. *If $V=\mathrm{Coker}(\rho)$, then $\mathrm{Ann}(T)=\mathrm{Ann}(V)$.*

Let $U=\mathrm{Ann}(V)$. We have $US\subseteq\mathrm{Ann}(V\otimes_R S)=\mathrm{Ann}(T\otimes_R S)$. But S is faithfully flat, hence $U\subseteq US$. We can view T as a submodule of $T\otimes_R S$. Therefore, $UT=0$ and it follows that $U\subseteq\mathrm{Ann}(T)$.

Since in this proof we only used the equality $\mathrm{Ann}(V\otimes_R S)=\mathrm{Ann}(T\otimes_R S)$, it follows that the converse inclusion is valid, so the proof for Claim 3 is complete.

In the following we will use the notations $U=\mathrm{Ann}(V)=\mathrm{Ann}(T)$, $\overline{R}=R/U$, and $\overline{S}=S/US$.

Claim 4. *The $\overline{R}$-modules V and T are tilting and* $\mathrm{Gen}(T) = \mathrm{Gen}(V)$.

We view Mod-$\overline{R}$ as a full subcategory of Mod-R via the canonical homomorphism $R \to R/U$. Then for every $M \in$ Mod-$\overline{R}$ we have $UM = 0$, and it follows that the left-hand side homomorphism in the exact sequence

$$M \otimes_R US \longrightarrow M \otimes_R S \longrightarrow M \otimes_R \overline{S} \longrightarrow 0$$

is zero. Therefore, the restrictions of the functors $- \otimes_R S$ and $- \otimes_R \overline{S}$ to Mod-$\overline{R}$ are naturally isomorphic. Since $\overline{S}$ is also an $\overline{R}$-module, we also have a natural isomorphism $- \otimes_R \overline{S} \cong - \otimes_{\overline{R}} \overline{S}$ for the restrictions of these functors to Mod-$\overline{R}$ [**10**, Proposition II.2]. This shows that the canonical homomorphism $\overline{\lambda}\colon \overline{R} \to \overline{S}$, induced by λ, is faithfully flat.

From [**5**, Proposition 3.2 and Proposition 3.10] it follows that V is a tilting $\overline{R}$-module. By [**19**, Lemma 2.4] we have that $V \otimes_R S = V \otimes_{\overline{R}} \overline{S}$ is a tilting $\overline{S}$-module. This implies that the annihilator of the $\overline{S}$-module $V \otimes_{\overline{R}} \overline{S}$ is zero, and it follows that $\mathrm{Ann}_{\overline{S}}(T \otimes_{\overline{R}} \overline{S}) = 0$. Since $\sigma \otimes_R \overline{S}$ is a 2-term silting complex, we can use [**5**, Proposition 3.2 and Proposition 3.10] one more time to obtain that $T \otimes_R \overline{S} = T \otimes_{\overline{R}} \overline{S}$ is a tilting $\overline{S}$-module. By Proposition 3.15, we obtain that T is tilting as an $\overline{R}$-module and $\mathrm{Gen}(T) = \mathrm{Gen}(V)$.

Claim 5. *We have $\mathcal{D}_\rho \subseteq \mathcal{D}_\sigma$. In particular, $T^{(I)} \in \mathcal{D}_\sigma$ for all sets I.*

In Mod-$\overline{R}$ we have the projective resolution

$$P^{-1} \otimes_R \overline{R} \xrightarrow{\sigma \otimes_R \overline{R}} P^0 \otimes_R \overline{R} \longrightarrow T \longrightarrow 0.$$

But T is tilting as an $\overline{R}$-module, hence it is of projective dimension at most 1. It follows that we can find in Mod-$\overline{R}$ a direct decomposition $P^{-1} \otimes_R \overline{R} = \overline{P} \oplus \overline{K}$, where $\overline{K} = \mathrm{Im}(\sigma \otimes_R \overline{R})$ and $\overline{P} = \mathrm{Ker}(\sigma \otimes_R \overline{R})$.

Since the complex $\sigma \otimes_R S$ is silting in Mod-S, we can apply Theorem 2.7 to conclude that

$$\sigma \otimes_R \overline{S} \cong \sigma \otimes_R \overline{R} \otimes_{\overline{R}} \overline{S} \cong \sigma \otimes_R S \otimes_{\overline{S}} \overline{S}$$

is 2-term silting complex in Mod-$\overline{S}$. Moreover, since the induced ring homomorphism $\overline{\lambda}\colon \overline{R} \to \overline{S}$ is faithfully flat and T is a tilting $\overline{R}$-module, we obtain by using [**19**, Lemma 2.4] that $T \otimes_{\overline{R}} \overline{S}$ is tilting. It follows that

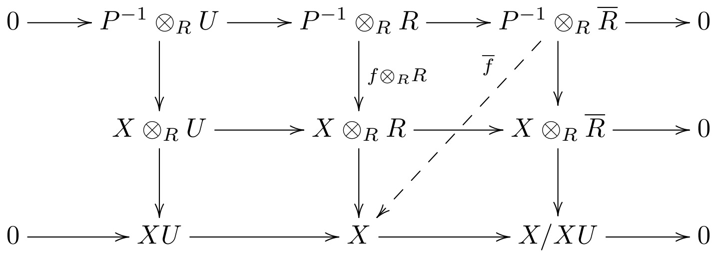
$$\mathcal{D}_{\sigma \otimes_{\overline{R}} \overline{S}} = \mathrm{Gen}(T \otimes_{\overline{R}} \overline{S}) = \mathrm{Ker}\,\mathrm{Ext}^1_{\overline{S}}(T \otimes_{\overline{R}} \overline{S}, -),$$

hence

$$\mathrm{Hom}_{\overline{S}}(\overline{P} \otimes_{\overline{R}} \overline{S}, \mathrm{Gen}(T \otimes_{\overline{R}} \overline{S})) = 0.$$

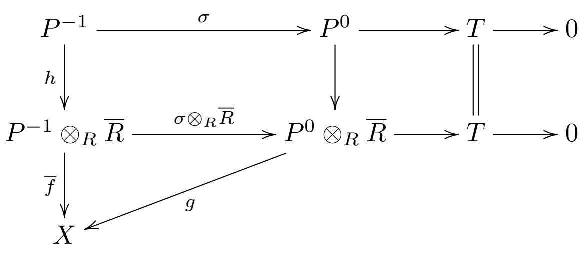
Since $\overline{S}$ is faithfully flat, the last equality implies that $\mathrm{Hom}_{\overline{R}}(\overline{P}, \mathrm{Gen}(T))= 0$.

Let $X \in \mathcal{D}_\rho = \mathrm{Gen}(V) = \mathrm{Gen}(T)$. Then $U \leq \mathrm{Ann}(X)$. If $f\colon P^{-1} \to X$ is a homomorphism, we obtain the commutative diagram



where the vertical arrows in the bottom rectangle are the natural ones. Since $XU = 0$, it follows that f factorizes through $P^{-1} \otimes_R \overline{R}$. Therefore, there exists $\overline{f}\colon P^{-1} \otimes_R \overline{R} \to X$ such that $f = \overline{f}h$, where $h\colon P^{-1} \overset{\cong}{\to} P^{-1} \otimes_R R \to P^{-1} \otimes_R \overline{R}$ is the canonical map.

But T is tilting as an $\overline{R}$-module. It follows that there exists a homomorphism $g\colon P^0 \otimes_R \overline{R} \to X$ such that $\overline{f}_{|\overline{K}} = g(\sigma \otimes_R \overline{R})_{|\overline{K}}$. Since $\mathrm{Hom}(\overline{P}, X) = 0$, we have $\overline{f}(\overline{P}) = 0$. Then $\overline{f} = g(\sigma \otimes_R \overline{R})$, and we obtain the commutative diagram



where the composition of the vertical left-hand side arrows is f. Then f factorizes through σ and hence $X \in \mathcal{D}_\sigma$.

We conclude that $\mathcal{D}_\rho \subseteq \mathcal{D}_\sigma$.

Using Claims 2 and 5 we obtain $\mathcal{D}_\sigma = \mathcal{D}_\rho$ and $T \in \mathcal{D}_\sigma$. It follows that σ is partial silting. By Proposition 3.12, in order to complete the proof it is enough to prove

Claim 6. *For every* $\mathfrak{p} \in \mathrm{Spec}(R)$ *we have* $T \otimes_R \kappa(\mathfrak{p}) \neq 0$ *or* $\mathrm{Ker}(\sigma \otimes_R \kappa(\mathfrak{p})) \neq 0$.

Let $\mathfrak{p} \in \operatorname{Spec}(R)$. Since λ is faithfully flat, the induced map of spectra

$$\lambda^\star\colon \operatorname{Spec}(S) \longrightarrow \operatorname{Spec}(R)\ , \quad \lambda^\star(\mathfrak{q}) = \mathfrak{q} \cap R,$$

is surjective, so there exists $\mathfrak{q} \in \operatorname{Spec}(S)$ such that $\mathfrak{q} \cap R = \mathfrak{p}$. By Lemma 3.13 we observe that $\kappa(\mathfrak{q})$ is faithfully flat as a $\kappa(\mathfrak{p})$-module (see also the proof of [**19**, Proposition 3.16]). Applying the functors from Lemma 3.13 to σ it follows that

$$\operatorname{Ker}(\sigma \otimes_R S \otimes_S \kappa(\mathfrak{q})) \cong \operatorname{Ker}(\sigma \otimes_R \kappa(\mathfrak{p})) \otimes_{\kappa(\mathfrak{p})} \kappa(\mathfrak{q})$$

and

$$T \otimes_R S \otimes_S \kappa(\mathfrak{q}) \cong T \otimes_R \kappa(\mathfrak{p}) \otimes_{\kappa(\mathfrak{p})} \kappa(\mathfrak{q}).$$

By Proposition 3.12, for every $\mathfrak{q} \in \operatorname{Spec}(S)$ we have $T \otimes_R S \otimes_S \kappa(\mathfrak{q}) \neq 0$ or $\operatorname{Ker}(\sigma \otimes_R S \otimes_S \kappa(\mathfrak{q})) \neq 0$. Since $\kappa(\mathfrak{q})$ is faithfully flat as a $\kappa(\mathfrak{p})$-module, we obtain Claim 6, and the proof is complete. □

We close the paper with some comments on the proof of Theorem 3.16.

Remark 3.17. (a) From Claim 3 and Claim 4 it follows that T is tilting as an $R/\operatorname{Ann}(T)$-module. This is equivalent to the fact that the R-module T is a finendo quasi-tilting module by [**5**, Proposition 3.2]. However, this is not enough to conclude that T is a silting R-module, as proved in [**4**, Example 5.4] and [**8**, Example 5.12].

(b) In the noetherian case the proof can be done using Claims 3–6 (i.e. the inclusion $\mathcal{D}_\rho \subseteq \mathcal{D}_\sigma$ and Claim 6) in the following way. We view σ as a complex concentrated in -1 and 0. It is easy to see that $\operatorname{Hom}_{\mathbf{D}(R)}(\sigma, \sigma^{(I)}[1]) = 0$ if and only if $T^{(I)} \in \mathcal{D}_\sigma$. Therefore, we can use [**5**, Theorem 4.9] together with the inclusion $\mathcal{D}_\rho \subseteq \mathcal{D}_\sigma$ to observe that σ is a 2-term silting complex if and only if it is a generator in $\mathbf{D}(R)$. In the noetherian case the converse of Lemma 3.10 is also valid by using [**9**, Theorem 9.5] or [**25**, Theorem 2.8]. It follows that σ is a generator if and only if $\sigma \otimes_R \kappa(\mathfrak{p})$ is not an isomorphism for all $\mathfrak{p} \in \operatorname{Spec}(R)$. Therefore, by using Claim 6, we obtain that σ is a generator and the proof is complete.

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References

[1] T. Adachi, O. Iyama, and I. Reiten, τ-tilting theory, *Compos. Math.* **150(3)** (2014), 415–452. DOI: 10.1112/S0010437X13007422.

[2] L. Angeleri Hügel, On the abundance of silting modules, in: *"Surveys in Representation Theory of Algebras"*, Contemp. Math. **716**, Amer. Math. Soc., Providence, RI, 2018, pp. 1–23. DOI: 10.1090/conm/716/14424.

[3] L. Angeleri Hügel, Silting objects, *Bull. Lond. Math. Soc.* **51(4)** (2019), 658–690. DOI: 10.1112/blms.12264.

[4] L. Angeleri Hügel and M. Hrbek, Silting modules over commutative rings, *Int. Math. Res. Not. IMRN* **2017(13)** (2017), 4131–4151. DOI: 10.1093/imrn/rnw147.

[5] L. Angeleri Hügel, F. Marks, and J. Vitória, Silting modules, *Int. Math. Res. Not. IMRN* **2016(4)** (2016), 1251–1284. DOI: 10.1093/imrn/rnv191.

[6] L. Angeleri Hügel, F. Marks, and J. Vitória, Silting modules and ring epimorphisms, *Adv. Math.* **303** (2016), 1044–1076. DOI: 10.1016/j.aim.2016.08.035.

[7] I. Assem and N. Marmaridis, Tilting modules over split-by-nilpotent extensions, *Comm. Algebra* **26(5)** (1998), 1547–1555. DOI: 10.1080/00927879808826219.

[8] S. Bazzoni, I. Herzog, P Příhoda, J. Šaroch, and J. Trlifaj, Pure projective tilting modules, Preprint (2017). arXiv:1703.04745.

[9] D. J. Benson, S. B. Iyengar, and H. Krause, Colocalizing subcategories and cosupport, *J. Reine Angew. Math.* **2012(673)** (2012), 161–207. DOI: 10.1515/CRELLE.2011.180.

[10] N. Bourbaki, *"Elements of Mathematics. Algebra, Part I: Chapters 1–3"*, Translated from the French. Hermann, Paris; Addison-Wesley Publishing Co., Reading Mass., 1974.

[11] S. Breaz and G. C. Modoi, Equivalences induced by infinitely generated silting modules, *Algebr. Represent. Theory* (2019). DOI: 10.1007/s10468-019-09930-3.

[12] S. Breaz and F. Pop, Cosilting modules, *Algebr. Represent. Theory* **20(5)** (2017), 1305–1321. DOI: 10.1007/s10468-017-9688-x.

[13] S. Breaz and J. Žemlička, The defect functor of a homomorphism and direct unions, *Algebr. Represent. Theory* **19(1)** (2016), 181–208. DOI: 10.1007/s10468-015-9569-0.

[14] S. Breaz and J. Žemlička, Torsion classes generated by silting modules, *Ark. Mat.* **56(1)** (2018), 15–32. DOI: 10.4310/ARKIV.2018.v56.n1.a2.

[15] H. Derksen and J. Fei, General presentations of algebras, *Adv. Math.* **278** (2015), 210–237. DOI: 10.1016/j.aim.2015.03.012.

[16] S. Estrada, P. Guil Asensio, and J. Trlifaj, Descent of restricted flat Mittag-Leffler modules and generalized vector bundles, *Proc. Amer. Math. Soc.* **142(9)** (2014), 2973–2981. DOI: 10.1090/S0002-9939-2014-12056-9.

[17] L. Fuchs, *"Infinite Abelian Groups"*, Vol. I, Pure and Applied Mathematics **36**, Academic Press, New York-London, 1970.

[18] G. Garkusha and M. Prest, Torsion classes of finite type and spectra, in: *"K-Theory and Noncommutative Geometry"*, EMS Ser. Congr. Rep., Eur. Math. Soc., Zürich, 2008, pp. 393–412. DOI: 10.4171/060-1/12.

[19] M. Hrbek, J. Šťovíček, and J. Trlifaj, Zariski locality of quasi-coherent sheaves associated with tilting, *Indiana Univ. Math. J.* (to appear).

[20] G. Jasso, Reduction of τ-tilting modules and torsion pairs, *Int. Math. Res. Not. IMRN* **2015(16)** (2015), 7190–7237. DOI: 10.1093/imrn/rnu163.

[21] B. Keller and D. Vossieck, Aisles in derived categories, Deuxième Contact Franco-Belge en Algèbre (Faulx-les-Tombes, 1987), *Bull. Soc. Math. Belg. Sér. A* **40(2)** (1988), 239–253.

[22] F. Marks and J. Šťovíček, Universal localizations via silting, *Proc. Roy. Soc. Edinburgh Sect. A* **149(2)** (2019), 511–532. DOI: 10.1017/prm.2018.37.
[23] H. Matsumura, *"Commutative Ring Theory"*, Translated from the Japanese by M. Reid, Cambridge Studies in Advanced Mathematics **8**, Cambridge University Press, Cambridge, 1987. DOI: 10.1017/CBO9781139171762.
[24] J.-i. Miyachi, Extensions of rings and tilting complexes, *J. Pure Appl. Algebra* **105(2)** (1995), 183–194. DOI: 10.1016/0022-4049(94)00145-6.
[25] A. Neeman, The chromatic tower for $D(R)$, With an appendix by Marcel Bökstedt, *Topology* **31(3)** (1992), 519–532. DOI: 10.1016/0040-9383(92)90047-L.
[26] M. Prest, *"Purity, Spectra and Localisation"*, Encyclopedia of Mathematics and its Applications **121**, Cambridge University Press, Cambridge, 2009. DOI: 10.1017/CBO9781139644242.
[27] M. Raynaud and L. Gruson, Critères de platitude et de projectivité. Techniques de «platification» d'un module, *Invent. Math.* **13** (1971), 1–89. DOI: 10.1007/BF01390094.
[28] J. Rickard, Derived equivalences as derived functors, *J. London Math. Soc. (2)* **43(1)** (1991), 37–48. DOI: 10.1112/jlms/s2-43.1.37.
[29] L. Silver, Noncommutative localizations and applications, *J. Algebra* **7(1)** (1967), 44–76. DOI: 10.1016/0021-8693(67)90067-1.
[30] The Stacks project authors, The Stacks project, https://stacks.math.columbia.edu/tag/04VM.
[31] J. Wei, Semi-tilting complexes, *Israel J. Math.* **194(2)** (2013), 871–893. DOI: 10.1007/s11856-012-0093-1.
[32] R. Wisbauer, *"Foundations of Module and Ring Theory"*, A handbook for study and research, Revised and translated from the 1988 German edition, Algebra, Logic and Applications **3**, Gordon and Breach Science Publishers, Philadelphia, PA, 1991.
[33] P. Zhang and J. Wei, Cosilting complexes and AIR-cotilting modules, *J. Algebra* **491** (2017), 1–31. DOI: 10.1016/j.jalgebra.2017.07.022.

"Babeş-Bolyai" University, Faculty of Mathematics and Computer Science, Str. Mihail Kogălniceanu 1, 400084, Cluj-Napoca, Romania
E-mail address: bodo@math.ubbcluj.ro

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