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FINITE C^0-DETERMINACY OF REAL ANALYTIC MAP GERMS WITH ISOLATED INSTABILITY

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Abstract: Let $f\colon(\mathbb{R}^n,0)\to(\mathbb{R}^p,0)$ be a real analytic map germ with isolated instability. We prove that if $n=2$ and $p=2,3$, then f is finitely C^0-determined. This result can be seen as a weaker real counterpart of Mather–Gaffney finite determinacy criterion.

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1. Introduction

Given an analytic map germ $f\colon(\mathbb{K}^n,0)\to(\mathbb{K}^p,0)$ with $\mathbb{K}=\mathbb{R}$ or $\mathbb{C}$, f is finitely determined if there exists $k\in\mathbb{N}$ such that for any map germ g with the same k-order Taylor expansion $j^kg(0)=j^kf(0)$, g is $\mathcal{A}$-equivalent to f (i.e., equivalent by coordinate changes in the source and target). In the complex case, the Mather–Gaffney finite determinacy criterion says that a map germ is finitely determined if and only if it has isolated instability. In the real case, finite determinacy implies isolated instability but the converse is not true (see Examples 2.4 and 2.5).

When $p=1$, a function $f\colon(\mathbb{R}^n,0)\to(\mathbb{R},0)$ has isolated instability if and only if it has an isolated singularity. A celebrated theorem of Kuo ([**3**]) states that if f has an isolated singularity, then it is finitely C^0-determined, that is, there exists $k\in\mathbb{N}$ such that for any g with $j^kf(0)=j^kg(0)$, g is C^0-$\mathcal{A}$-equivalent to f (i.e. the coordinate changes are homeomorphisms). Instead of C^0-$\mathcal{A}$-equivalence, we can also consider C^0-$\mathcal{K}$-equivalence, where $\mathcal{K}$ is the contact Mather's group. Then it is well known that if $f\colon(\mathbb{R}^n,0)\to(\mathbb{R}^p,0)$ has isolated $\mathcal{K}$-instability, then f is finitely C^0-$\mathcal{K}$-determined (see [**10**, Theorem 6.1]). In [**1**], the authors consider C^0-$\mathcal{A}$-equivalence and give sufficient conditions for finite C^0-determinacy of C^∞-map germs $f\colon(\mathbb{R}^n,0)\to(\mathbb{R}^p,0)$ in terms of some Łojasiewicz inequalities in the multi-jet spaces. But it is not known if

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isolated instability implies finite C^0-determinacy in general for analytic map germs $f\colon (\mathbb{R}^n,0)\to(\mathbb{R}^p,0)$.

In this paper we consider the case of analytic map germs $f\colon(\mathbb{R}^2,0)\to(\mathbb{R}^p,0)$, with $p=2,3$ and we prove that isolated instability implies finite C^0-determinacy in these cases. The procedure to prove this (Sections 4 and 5) will be to construct a family of map germs $F=f+th$, with h being a map germ of higher order terms and prove that F is topologically trivial in both cases. We will use different results, including the characterization of topologically trivial families previously studied by both authors (see [**8, 9**]). In fact, when $p=2$, a family $f_t\colon(\mathbb{R}^2,0)\to(\mathbb{R}^2,0)$ is topologically trivial if it is excellent (in the sense of Gaffney [**2**]) and the critical point space $S(f_t)$ is a topologically trivial family of plane curves. For $p=3$, a family $f_t\colon(\mathbb{R}^2,0)\to(\mathbb{R}^3,0)$ is topologically trivial if it is excellent and the double point space $D^2(f_t)$ is a topologically trivial family of curves in $(\mathbb{R}^2\times\mathbb{R}^2,0)$. Then an intensive use of the Łojasiewicz inequalities will allow to control the degree of h in order to have such conditions.

We adopt the notation and basic definitions that are usual in singularity theory (e.g., $\mathcal{A}$-equivalence, stability, finite determinacy, etc.), as the reader can find in Wall's survey paper [**10**].

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2. Stability and finite determinacy

In this section we state the results and basic definitions that we will need during the paper, including the definition of stable map, finite determinacy, and Mather–Gaffney finite determinacy criterion.

Two smooth map germs $f,g\colon(\mathbb{K}^n,S)\to(\mathbb{K}^p,0)$, with $\mathbb{K}=\mathbb{R}$ or $\mathbb{C}$, are $\mathcal{A}$-equivalent if there exist diffeomorphism germs $\phi\colon(\mathbb{K}^n,S)\to(\mathbb{K}^n,S)$, $\psi\colon(\mathbb{K}^p,0)\to(\mathbb{K}^p,0)$ such that $g=\psi\circ f\circ\phi^{-1}$. If ϕ, ψ are homeomorphisms instead of diffeomorphisms, then f, g are C^0-$\mathcal{A}$-equivalent or topologically equivalent.

We denote by $\Sigma(f)=\{x\in\mathbb{K}^n : f \text{ is not submersive at } x\}$ the set of critical points of f and by $S(f)=\{x\in\mathbb{K}^n : \operatorname{rank} Jf(x)<\min\{n,p\}\}$ the set of singular points of f. It is obvious that if $n\geqslant p$, then $\Sigma(f)=S(f)$.

Let F be a 1-parameter unfolding of a smooth map germ $f\colon(\mathbb{K}^n,S)\to(\mathbb{K}^p,0)$, with $F(x,t)=(f_t(x),t)$.

- F is *trivial* (respectively, *topologically trivial*) if there are diffeomorphism germs (respectively, homeomorphism germs) $\Phi\colon(\mathbb{K}^n\times\mathbb{K},S\times\{0\})\to(\mathbb{K}^n\times\mathbb{K},S\times\{0\})$, $\psi\colon(\mathbb{K}^p\times\mathbb{K},0)\to(\mathbb{K}^p\times\mathbb{K},0)$ such that they are unfoldings of the identity and $F=\Psi\circ(f\times\mathrm{id})\circ\Phi^{-1}$.

- The map germ $f\colon (\mathbb{K}^n, S) \to (\mathbb{K}^p, 0)$ is *stable* if any unfolding of f is trivial.

In the case that F is origin preserving (i.e., $f_t(x) = 0$, for all $x \in S$ and for all t), we can consider $f_t\colon (\mathbb{K}^n, S) \to (\mathbb{K}^p, 0)$ as a family of germs.

Let $(X, 0) \subset (\mathbb{K}^n, 0)$ be an analytic set germ. A *deformation* of $(X, 0)$ is another analytic set germ $(\mathcal{X}, 0) \subset (\mathbb{K}^n \times \mathbb{K}, 0)$ such that $\pi^{-1}(0) = X$, where $\pi\colon (\mathcal{X}, 0) \to (\mathbb{K}, 0)$ is the projection $\pi(x, t) = t$. We say that $(\mathcal{X}, 0)$ is a *topologically trivial* deformation if there exists a homeomorphism $\Phi\colon (\mathcal{X}, 0) \to (X \times \mathbb{K}, 0)$ such that $\pi' \circ \Phi = \pi$, where $\pi'\colon (X \times \mathbb{K}, 0) \to (\mathbb{K}, 0)$ is also the projection onto the second factor.

The map germ $f\colon (\mathbb{K}^n, S) \to (\mathbb{K}^p, 0)$ is k-determined (respectively, $k-C^0$-determined) if, for any map germ g with the same k-jet, we have that g is $\mathcal{A}$-equivalent to f (respectively, topologically equivalent to f). We say that f is finitely determined (respectively, finitely C^0-determined) if it is k-determined (respectively, $k-C^0$-determined) for some k. We term order of determinacy (respectively, order of C^0-determinacy) of f the smallest k such that f is k-determined (respectively, $k-C^0$-determined).

In the complex case, we have the following Mather–Gaffney finite determinacy criterion (see [**10**, Theorem 2.1]).

Theorem 2.1. *Let $f\colon (\mathbb{C}^n, 0) \to (\mathbb{C}^p, 0)$ be a holomorphic map germ. Then f is finitely determined if and only if there is a representative $f\colon U \to V$, where U and V are open neighbourhoods of the origin in $\mathbb{C}^n$ and $\mathbb{C}^p$ respectively, such that*

1. $f^{-1}(0) \cap \Sigma(f) = \{0\}$,
2. $f\colon \Sigma(f) \to f(\Sigma(f))$ *is finite (i.e., finite-to-one and proper),*
3. $f\colon U \setminus f^{-1}(0) \to V \setminus \{0\}$ *is locally stable (i.e., for each $y \in V \setminus \{0\}$, $f\colon (\mathbb{C}^n, S) \to (\mathbb{C}^p, y)$ is stable, where $S = f^{-1}(y) \cap \Sigma(f)$).*

Coming back to the real case, if $f\colon (\mathbb{R}^n, 0) \to (\mathbb{R}^p, 0)$ is real analytic, then we can consider its complexification $\hat{f}\colon (\mathbb{C}^n, 0) \to (\mathbb{C}^p, 0)$. It is well known that f is finitely determined if and only if $\hat{f}$ is finitely determined (see [**10**, Proposition 1.7]). Thus, we have the following immediate consequence of the Mather–Gaffney finite determinacy criterion.

Corollary 2.2. *Let $: (\mathbb{R}^n, 0) \to (\mathbb{R}^p, 0)$ be a finitely determined map germ. Then there is a representative $f\colon U \to V$, where U, V are open neighbourhoods of the origin in $\mathbb{R}^n$ and $\mathbb{R}^p$ respectively, such that*

1. $f^{-1}(0) \cap \Sigma(f) = \{0\}$,
2. $f\colon \Sigma(f) \to f(\Sigma(f))$ *is finite,*
3. $f\colon U \setminus f^{-1}(0) \to V \setminus \{0\}$ *is a locally stable mapping.*

Given a stable multi-germ $f\colon (\mathbb{R}^n, S) \to (\mathbb{R}^p, y)$, we denote by $B \subset (\mathbb{R}^p, y)$ the germ of all points $y' \in f(\Sigma(f))$ such that the multi-germ of f at $S' = f^{-1}(y') \cap \Sigma(f)$ is $\mathcal{A}$-equivalent to the multi-germ of f at S. We call B the *analytic stratum of f in the target*.

If $\dim B = d$, then the multi-germ f represents a d-dimensional stable type. In particular, when $d = 0$ we say that f is a 0-stable type.

Let $f\colon U \to V$ be a representative of a map germ satisfying conditions (1), (2), and (3) of Corollary 2.2. Then the 0-stable points of f in $V \setminus \{0\}$ are isolated. Moreover, given a 0-stable type, the analytic stratum at that point will be analytic in $V \setminus \{0\}$ and thus cannot accumulate at the origin, by the Curve Selection Lemma (see [**6**]). After shrinking the neighbourhoods U and V if necessary, we can assume that f has no 0-stable points in $V \setminus \{0\}$.

Definition 2.3. A real analytic map germ $f\colon (\mathbb{R}^n, 0) \to (\mathbb{R}^p, 0)$ has *isolated instability* if there exists a representative $f\colon U \to V$, where U, V are open neighbourhoods of the origin in $\mathbb{R}^n$ and $\mathbb{R}^p$ respectively, such that

(1) $f^{-1}(0) \cap \Sigma(f) = \{0\}$,
(2) $f\colon \Sigma(f) \to f(\Sigma(f))$ is finite,
(3) the restriction $f\colon U \setminus f^{-1}(0) \to V \setminus \{0\}$ is a locally stable mapping with no 0-stable points.

In such a case, we say that $f\colon U \to V$ is a *good representative* of the map germ.

By Corollary 2.2 finite determinacy implies isolated instability, but the converse is not true in general in the real case as we can see in the examples below.

Example 2.4. If $n = p = 2$, the 0-stable points are the double fold points and the cusps. Thus, f has isolated instability if $f^{-1}(0) = \{0\}$, $f\colon U \to V$ is finite, and $f : U \setminus \{0\} \to V \setminus \{0\}$ is stable with only simple folds. Consider the following examples:

(1) The real analytic map germ $f(x, y) = (xy, x^2 - y^2)$ has singular set $S(f) = \{(x, y) : x^2 + y^2 = 0\} = \{(0, 0)\}$. Therefore, f is regular outside of the origin and, since $f^{-1}(0) = \{0\}$, we conclude that f has isolated instability. However, when looking at its complexification $\hat{f}$, we have that $S(\hat{f})$ has two branches $x = \pm iy$ and the restriction of $\hat{f}$ to each branch is $f(\pm iy, y) = (-iy^2, -2y^2)$, giving a branch of double fold points in the target. It follows that $\hat{f}$ and as a consequence f are not finitely determined.

(2) Let $f(x,y) = (x,(x^2+y^2)^2)$, which has isolated instability. In fact, $f^{-1}(0) = \{0\}$, $S(f)$ is the line $y = 0$, and the restriction of f to this line gives $x \mapsto (x, x^4)$, so f has no cusps or double folds outside of the origin. However, $\hat{f}(S(\hat{f}))$ has a branch of double folds and thus f is not finitely determined.

Example 2.5. If $n = 2$ and $p = 3$, the 0-stable points are the cross caps and the triple points. We have that f has isolated instability if $f^{-1}(0) = \{0\}$, $f\colon U \to V$ is finite, and $f\colon U \setminus \{0\} \to V \setminus \{0\}$ is an immersion with only transverse double points.

For instance, consider the real analytic map germ $f(x,y) = (x, y^2, y(x^2+y^2)^2)$. We have $f^{-1}(0) = \{0\}$ and its double point curve in $\mathbb{R}^2$ is $D(f) = \{(x^2+y^2)^2 = 0\} = \{0\}$. Therefore, f has isolated instability. However, its Mond number $\mu(D(f))$ is not finite, so f is not finitely determined (see [**5**] for details).

We will prove in the following sections that for $n = 2$ and $p = 2, 3$, if $f\colon (\mathbb{R}^n, 0) \to (\mathbb{R}^p, 0)$ has isolated instability, then f is finitely C^0-determined. These results can be seen as a real counterpart of the Mather–Gaffney finite determinacy criterion.

3. The Łojasiewicz inequality

The main tool to prove our result will be the Łojasiewicz inequality [**4**], which is strongly connected with the concept of finite C^0-determinacy of a map germ.

Definition 3.1. Let us denote by $\mathcal{O}_n$ the set of real analytic function germs $f\colon (\mathbb{R}^n, 0) \to \mathbb{R}$. Let $I \subseteq \mathcal{O}_n$ be an ideal and $h \in \mathcal{O}_n$ be such that $V(I) \subseteq V(h)$. Let $g_1, \dots, g_s$ be a system of generators of I. By [**4**, p. 136] we can consider the greatest lower bound of those $\theta > 0$ such that there exists an open neighbourhood U of 0 in $\mathbb{R}^n$ and a constant $C > 0$ such that

$$|h(x)|^\theta \leqslant C \sup_i |g_i(x)|$$

for all $x \in U$. We call this number the *Łojasiewicz exponent of h with respect to I* and we denote it by $l(h, I)$. If $J = \langle h_1, \dots, h_t \rangle \subseteq \mathcal{O}_n$ is any other ideal such that $V(I) \subseteq V(J)$, then we set $l(J, I) = \max_i l(h_i, I)$. Note that everything works properly if we consider in each side of the inequality the sum of the squares of the respective system of generators.

Using this definition we can prove the following result:

Proposition 3.2. *Let* $g\colon(\mathbb{R}^n,0)\to(\mathbb{R}^N,0)$ *be analytic such that* $g^{-1}(0)=\{0\}$*. Let* $G\colon(\mathbb{R}^n\times\mathbb{R},0)\to(\mathbb{R}^N,0)$ *be analytic with* $G(x,t)=g(x)+th(x,t)$*. If* $k>l(\mathfrak{m}_n,g)$ *and* $h\in\mathfrak{m}_n^k\mathcal{O}_{n+1}^N$*, then there exist* U*,* I *open neighborhoods of the origin in* $\mathbb{R}^n$ *and* $\mathbb{R}$ *respectively such that, for all* $g_t\colon U\to\mathbb{R}^N$*,* $t\in I$*, we have* $g_t^{-1}(0)=\{0\}$*.*

Proof: Since g is analytic and $g^{-1}(0)=\{0\}$, by Łojasiewicz inequality, there exists $U\subset\mathbb{R}^n$ open neighborhood of 0 and constants $C,\theta>0$ such that $\|g(x)\|\geqslant C\|x\|^\theta$. On the other hand, since $h(x)\in\mathfrak{m}_n^k\mathcal{O}_{n+1}^N$, after shrinking U and taking $I\subset\mathbb{R}$ an open neighborhood of 0 small enough, there exists $D>0$ such that $\|h(x)\|\leqslant D\|x\|^k$. Let us take k big enough with $k>\theta=l(\mathfrak{m}_n,g)$. Then, for all $x\in U$ and $|t|<\frac{C}{2D}$, we have

$$\begin{aligned}\|g_t(x)\|&\geqslant\|g(x)\|-|t|\|h(x)\|\geqslant C\|x\|^\theta-|t|D\|x\|^k\\&\geqslant(C-|t|D\|x\|^{k-\theta})\|x\|^\theta\geqslant(C-|t|D)\|x\|^\theta\geqslant\frac{1}{2}C\|x\|^\theta.\end{aligned}$$

This implies the result. □

4. Finite C^0-determinacy of map germs from $\mathbb{R}^2$ to $\mathbb{R}^2$

The purpose of this section is to prove that any real analytic map germ $f\colon(\mathbb{R}^2,0)\to(\mathbb{R}^2,0)$ with isolated instability is finitely C^0-determined (Theorem 5.7).

For each $n\in\mathbb{N}$ we denote by $\mathcal{O}_n$ the local ring of analytic functions $g\colon(\mathbb{R}^n,0)\to\mathbb{R}$ and by $\mathfrak{m}_n$ its maximal ideal.

To prove the finite C^0-determinacy of $f\colon(\mathbb{R}^2,0)\to(\mathbb{R}^2,0)$, we will consider a 1-parameter unfolding $F\colon(\mathbb{R}^2\times\mathbb{R},0)\to(\mathbb{R}^2\times\mathbb{R},0)$ of f with $F(x,t)=(f_t(x),x)$, $f_t(x)=f(x)+th(x)$, $h(x)\in\mathfrak{m}_2^k\mathcal{O}_2^2$, and $k\in\mathbb{N}$ big enough and we will see that F is topologically trivial (Theorem 4.7).

The notion of excellent unfolding is due to Gaffney [**2**].

Definition 4.1. An origin preserving unfolding $F\colon(\mathbb{R}^n\times\mathbb{R},0)\to(\mathbb{R}^p\times\mathbb{R},0)$ given gy $F(x,t)=(f_t(x),t)$ is called *excellent* if there exists a representative $F\colon U\to V\times I$, where U, V, I are open neighbourhoods of the origin in $\mathbb{R}^{n+1}$, $\mathbb{R}^p$, $\mathbb{R}$ respectively such that $f_t\colon U_t\to V$ is a good representative of the map germ f_t for all $t\in I$, where $U_t=\{x\in\mathbb{R}^n : (x,t)\in U\}$.

Lemma 4.2. *Let* F *be an excellent unfolding of a map germ* $f\colon(\mathbb{R}^2,0)\to(\mathbb{R}^2,0)$ *with isolated instability. If* $S(F)$ *is topologically trivial, then* F *is topologically trivial.*

Proof: This is just a reformulation of [**8**, Theorem 3.6], where we show that F is topologically trivial if it is excellent and the discriminant

$\Delta(F) = F(S(F))$ is a topologically trivial family of plane curves. Since F is excellent, the restriction $F\colon S(F) \to \Delta(F)$ is a homeomorphism which preserves the parameter. Hence, $\Delta(F)$ is topologically trivial if so is $S(F)$. □

In our case, we fix a representative of F of the form $F\colon U \times I \to \mathbb{R}^3$, where U, I are open neighbourhoods of the origin in $\mathbb{R}^2$, $\mathbb{R}$ respectively. To prove that F is excellent and $S(F)$ is topologically trivial, we need to check the following conditions for all $f_t\colon U \to \mathbb{R}^2$, with $t \in I$:

(1) The Jacobian determinant J_t is regular on $U \setminus \{0\}$.

(2) $f_t^{-1}(0) = \{0\}$.

(3) The restriction $f_t\colon S(f_t) \to \mathbb{R}^2$ is an immersion on $U \setminus \{0\}$.

(4) The restriction $f_t\colon S(f_t) \to \mathbb{R}^2$ is injective.

In fact, conditions (1), (2), (3), and (4) imply that f_t has only simple fold singularities on $U\setminus\{0\}$, so F is excellent (see [**8**]). On the other hand, condition (1) also implies that $S(F)$ is topologically trivial by Kuo's Theorem [**3**]. We prove each one of these conditions in the following lemmas.

Lemma 4.3. *For $k \in \mathbb{N}$ big enough and after shrinking the neighbourhoods U and I if necessary, the Jacobian determinant J_t is regular on $U \setminus \{0\}$ for all $t \in I$.*

Proof: Since $f_t = f + th$ with $h \in \mathfrak{m}_2^k\mathcal{O}_2^2$, we have $J_t(x) = J_0(x) + tB_t(x)$ for some $B_t \in \mathfrak{m}_2^{k-1}\mathcal{O}_3$. We consider the gradient of J_t:

$$\nabla J_t = \nabla J_0 + t\nabla B_t = \left(\frac{\partial J_0}{\partial x_1} + t\frac{\partial B_t}{\partial x_1}, \frac{\partial J_0}{\partial x_2} + t\frac{\partial B_t}{\partial x_2}\right),$$

with $\frac{\partial B_t}{\partial x_1}, \frac{\partial B_t}{\partial x_2} \in \mathfrak{m}_2^{k-2}\mathcal{O}_3$.

Since f has isolated instability, we can assume $V(\nabla J_0) = \{0\}$. Therefore, applying Proposition 3.2 for $k_1 = k-2 > \theta_1 = l(\mathfrak{m}_2, \nabla J_0)$, after shrinking U and I if necessary, we have that J_t is regular on $U \setminus \{0\}$ for all $t \in I$. □

Lemma 4.4. *For $k \in \mathbb{N}$ big enough and after shrinking the neighbourhoods U and I if necessary, $f_t^{-1}(0) = \{0\}$ for all $t \in I$.*

Proof: We assume $f^{-1}(0) = \{0\}$ and $f_t = f + th$ with $h \in \mathfrak{m}_2^k\mathcal{O}_2^2$. By Proposition 3.2 for $k_2 = k > \theta_2 = l(\mathfrak{m}_2, f)$ we get the desired result. □

Lemma 4.5. *For $k \in \mathbb{N}$ big enough and after shrinking the neighbourhoods U and I if necessary, $f_t = (f_{1t}, f_{2t})\colon S(f_t) \to \mathbb{R}^2$ is an immersion on $U \setminus \{0\}$ for all $t \in I$.*

Proof: Let C_t be the analytic subset given by the vanishing of the three 2×2-minors of the Jacobian matrix of (f_{1t}, f_{2t}, J_t). We have that C_t contains the non-immersive points of $f_t\colon S(f_t)\setminus\{0\} \to \mathbb{R}^2$, so it is enough to show that $C_t \subset \{0\}$. We denote the three minors by M_{1t}, M_{2t}, M_{3t}, and $M_t = (M_{1t}, M_{2t}, M_{3t})$. A simple computation shows that if $f_t = f + th$ with $h \in \mathfrak{m}_2^k \mathcal{O}_2^2$, then $M_t = M_0 + tQ_t$ for some $Q_t \in \mathfrak{m}_2^{k-2}\mathcal{O}_3^3$.

Since f has isolated instability, we can assume that $C_0 \subset \{0\}$. If $C_0 = \emptyset$, then $C_t = \emptyset$ for t small enough and we are done. Otherwise, if $C_0 = \{0\}$, as a consequence of Proposition 3.2 for $k_3 = k - 2 > \theta_3 = l(\mathfrak{m}_2, M_0)$ we get the desired result. □

Before stating the last lemma, we recall here the construction of the double point space due to Mond [**7**]. Let $f\colon (\mathbb{R}^n, 0) \to (\mathbb{R}^p, 0)$ be an analytic map germ with $n \le p$. For each $i = 1, \dots, p$ we can write

$$f_i(y) - f_i(x) = \sum_{j=1}^{n} \alpha_{ij}(x, y)(y_j - x_j)$$

for some functions $\alpha_{ij} \in \mathcal{O}_{2n}$. In matrix notation, we can write

$$f(y) - f(x) = \alpha(x, y)(y - x),$$

where $\alpha = (\alpha_{ij})$. The ideal $I_2(f)$ is the ideal in $\mathcal{O}_{2n}$ generated by the $n \times n$-minors of α and by the functions $f_i(y) - f_i(x)$ with $i = 1, \dots, p$. The double point space $D^2(f)$ is the analytic subset of $(\mathbb{R}^n \times \mathbb{R}^n, 0)$ given by the vanishing of $I_2(f)$. This is equal to the set of pairs (x, y) such that either $x \ne y$ and $f(x) = f(y)$ or $x = y$ is a non-immersive point of f.

Lemma 4.6. *For $k \in \mathbb{N}$ big enough and after shrinking the neighbourhoods U and I if necessary, $f_t\colon S(f_t) \to \mathbb{R}^2$ is injective for all $t \in I$.*

Proof: Let D_t be the analytic subset of $U \times U$ given by the vanishing of the ideal $I_2(f_t)$ together with $J_t(x) = J_t(y) = 0$. This includes the double points of $f_t\colon S(f_t) \to \mathbb{R}^2$, so it is enough to prove $D_t = \{0\}$.

In our case, $f_t = f + th$ with $h \in \mathfrak{m}_2^k \mathcal{O}_2^2$, which gives

$$f_t(y) - f_t(x) = (\alpha(x, y) + t\beta(x, y))(y - x),$$

with

$$\alpha + t\beta = \begin{pmatrix} \alpha_{11} + t\beta_{11} & \alpha_{12} + t\beta_{12} \\ \alpha_{21} + t\beta_{21} & \alpha_{22} + t\beta_{22} \end{pmatrix}$$

and $\beta_{ij} \in \mathfrak{m}_4^{k-1}$. For simplicity, we denote the defining equations of D_t by $E_{1t}, \dots, E_{5t}$, where $E_{1t} = \det(\alpha + t\beta)$, $E_{2t} = f_{1t}(y) - f_{1t}(x)$, $E_{3t} = f_{2t}(y) - f_{2t}(x)$, $E_{4t} = J_t(x)$, $E_{5t} = J_t(y)$, and $E_t = (E_{1t}, \dots, E_{5t})$.

Since f has isolated instability, we can assume $D_0 = \{0\}$. On the other hand, we have $E = E_0 + tF_t$ for some $F_t \in \mathfrak{m}_4^{k-1}\mathcal{O}_5^5$. Hence, applying Proposition 3.2 for $k_4 = k - 1 > \theta_4 = l(\mathfrak{m}_2, E_0)$, we get the desired result. □

Theorem 4.7. *Let* $F\colon (\mathbb{R}^2\times\mathbb{R}, 0) \to (\mathbb{R}^2\times\mathbb{R}, 0)$ *be a* 1*-parameter unfolding of an analytic map germ with isolated instability* $f\colon (\mathbb{R}^2, 0) \to (\mathbb{R}^2, 0)$*, with* $F(x,t) = (f_t(x), x)$*,* $f_t(x) = f(x) + th(x)$*,* $h(x) \in \mathfrak{m}_2^k\mathcal{O}_2^2$*, and* $k \in \mathbb{N}$ *big enough. Then* F *is topologically trivial.*

Proof: If we take k big enough such that $k - 2 = \min\{k_1, k_2, k_3, k_4\} > \theta = \max\{\theta_1, \theta_2, \theta_3, \theta_4\}$, where k_1, k_2, k_3, k_4 and θ_1, θ_2, θ_3, θ_4 are taken as in the proofs of Lemmas 4.3, 4.4, 4.5, and 4.6, we have that F is topologically trivial. □

Finally, we can state and prove the main result of this section.

Theorem 4.8. *Let* $f\colon (\mathbb{R}^2, 0) \to (\mathbb{R}^2, 0)$ *be a real analytic map germ with isolated instability. Then* f *is finitely* C^0*-determined.*

Proof: Let $g\colon (\mathbb{R}^2, 0) \to (\mathbb{R}^2, 0)$ be a real analytic map germ such that $j^k f(0) = j^k g(0)$ for $k \in \mathbb{N}$ big enough. This implies that $g - f \in \mathfrak{m}_2^{k+1}\mathcal{O}_2^2$. Let us consider the unfolding $F(x,t) = (f(x) + th(x), x)$ as in Theorem 4.7, with $h = g - f$. For each $t_0 \in \mathbb{R}$, the germ of F at $(0, t_0)$ is topologically trivial, so f_t is topologically equivalent to f'_t for any t, t' near t_0. Since $\mathbb{R}$ is connected, we have that f_t is topologically equivalent to f'_t for any $t, t' \in \mathbb{R}$. In particular, if we take $t = 0$ and $t = 1$ we have that f and g are topologically equivalent and the result is proved. □

Remark 4.9. The order of C^0-determinacy of $f\colon (\mathbb{R}^2, 0) \to (\mathbb{R}^2, 0)$ is given by the integral part of the Łojasiewicz exponent θ plus two.

Example 4.10. If we consider the map germs studied in Example 2.4, $f(x,y) = (xy, x^2 - y^2)$ and $f(x,y) = (x, (x^2 + y^2)^2)$, applying Theorem 4.8 we have that both germs are C^0-finitely determined. In particular, following the arguments of the proof of Theorem 4.7 we can take as Łojasiewicz exponents $\theta = 2$ and $\theta = 3$, and conclude that they are $4 - C^0$-determined and $5 - C^0$-determined respectively.

5. Finite C^0-determinacy of map germs from $\mathbb{R}^2$ to $\mathbb{R}^3$

In this last section we study the case of analytic map germs $f\colon (\mathbb{R}^2, 0) \to (\mathbb{R}^3, 0)$ with isolated instability. We want to proof analogous results of Theorems 4.7 and 4.8 for this case, concluding that f is finitely C^0-determined (Theorem 5.7).

We proceed as in Section 3. We consider an unfolding $F\colon (\mathbb{R}^2\times\mathbb{R},0)\to(\mathbb{R}^3\times\mathbb{R},0)$ of f with $F(x)=(f_t(x),x)$, $f_t(x)=f(x)+th(x)$, $h(x)\in\mathfrak{m}_2^k\mathcal{O}_2^3$, and $k\in\mathbb{N}$ big enough. We will see that F is topologically trivial (Theorem 5.6). To prove this, we will use the following result:

Lemma 5.1. *Let F be an excellent unfolding of a map germ $f\colon(\mathbb{R}^2,0)\to(\mathbb{R}^3,0)$ with isolated instability. If $D^2(F)$ is topologically trivial, then F is topologically trivial.*

Proof: We showed in [**9**, Theorem 7.4] that F is topologically trivial if F is excellent and $D(f_t)=p_1(D^2(f_t))$ is a topologically trivial family of plane curves, where $p_1\colon\mathbb{R}^2\times\mathbb{R}^2\to\mathbb{R}^2$ is the projection onto the first factor. Since F is excellent, f_t has no triple points and hence the restriction $p_1\colon D^2(f_t)\to D(f_t)$ is a homeomorphism. Thus $D(f_t)$ is topologically trivial if so is $D^2(f_t)$. □

Given $f\colon(\mathbb{R}^2,0)\to(\mathbb{R}^3,0)$, the ideal $I_2(f)$ which defines the double point space $D^2(f)$ has six generators in $\mathcal{O}_4$: the three 2×2-minors of the matrix α together with $f_i(y)-f_i(x)$, $i=1,2,3$. For simplicity, we denote these six generators by $M_1,\dots,M_6$ and $M=(M_1,\dots,M_6)$. When f is stable and $(x,y)\in D^2(f)$, the ideal $I_2(f)$ is in fact generated by three equations and M has rank three, so that $D^2(f)$ is a regular curve. Conversely, if M has rank three at $(x,y)\in D^2(f)$, then either $x=y$ and the germ of f at x is $\mathcal{A}$-equivalent to the Whitney umbrella or $x\neq y$ and the bi-germ of f at $\{x,y\}$ is $\mathcal{A}$-equivalent to a transverse double point (see [**5**] for details).

Let $F(x,t)=(f(x)+th(x),t)$ and fix a representative of F of the form $F\colon U\times I\to\mathbb{R}^4$, where U, I are open neighbourhoods of the origin in $\mathbb{R}^2$, $\mathbb{R}$ respectively. For each $t\in I$, we have a map $f_t\colon U\to\mathbb{R}^3$. To prove that F is excellent, we need to check the following conditions for all $f_t\colon U\to\mathbb{R}^3$ with $t\in I$:

(1) $M_t\colon U\times U\to\mathbb{R}^6$ has rank 3 on $U\times U\setminus\{0\}$.

(2) $f_t^{-1}(0)=\{0\}$.

(3) f_t is an immersion on $U\setminus\{0\}$.

(4) f_t has at most double points.

By the above discussion, conditions (1), (2), (3), and (4) imply that f_t has only transverse double points on $U\setminus\{0\}$ and hence F is excellent.

We prove at the same time condition (1) together with the fact that $D^2(F)$ is topologically trivial.

Lemma 5.2. *For $k \in \mathbb{N}$ big enough and after shrinking the neighbourhoods U and I if necessary, $M_t\colon U \times U \to \mathbb{R}^6$ has rank 3 on $U \times U \setminus \{0\}$ for all $t \in I$. Moreover, $D^2(F)$ is topologically trivial.*

Proof: We prove the two assertions at the same time. Instead of M_t, we consider the map $\tilde{M}_t = (M_t, \rho)\colon U \times U \to \mathbb{R}^7$, where $\rho(x,y) = \|(x,y)\|^2$. We claim that it is enough to show that $\tilde{M}_t$ has rank 4 on $U \times U \setminus \{0\}$. In fact, this implies

(1) M_t has rank 3 on $U \times U \setminus \{0\}$, so that $D^2(f_t) \setminus \{0\}$ is a regular curve.
(2) The restriction $\rho\colon D^2(f_t) \setminus \{0\} \to \mathbb{R}$ is regular. Hence, there exists $\delta_0 > 0$ such that $D^2(f_t)$ is transverse to S^3_δ for all $0 < \delta \leq \delta_0$. But this implies that $D^2(F)$ is topologically trivial by the cone structure of $D^2(f_t)$ on the closed disk $D^4_{\delta_0}$.

We prove the claim. Let $C_{1t}, \dots, C_{\ell,t}$ be the 4×4-minors of the Jacobian matrix of $\tilde{M}_t$ and let $C = (C_{1t}, \dots, C_{\ell,t})$. For $t = 0$ we know that f has isolated instability and after shrinking U if necessary we can assume that M_0 has rank 3 on $U \times U \setminus \{0\}$, so that $D^2(f) \setminus \{0\}$ is a regular curve. By the Curve Selection Lemma, the critical values of $\rho\colon D^2(f) \setminus \{0\} \to \mathbb{R}$ cannot accumulate at the origin. There exists $\delta_0 > 0$ such that δ is not a critical value for all $0 < \delta \leq \delta_0$. Again, after shrinking U if necessary we can assume that $\rho\colon D^2(f_0) \setminus \{0\} \to \mathbb{R}$ has no critical points and hence $\tilde{M}_0$ has rank 4 on $U \times U \setminus \{0\}$. In other words, we have $C_0^{-1}(0) = \{0\}$.

On the other hand, we have $f_t = f + th$ with $h \in \mathfrak{m}_2^k \mathcal{O}_2^3$, which implies that $C_t = C_0 + tD_t$ with $D_t \in \mathfrak{m}_4^{k-1} \mathcal{O}_5^\ell$. Thus, applying Proposition 3.2 for $k_1 = k - 1 > \theta_1 = l(\mathfrak{m}_2, C_0)$ we get the desired result. □

Lemma 5.3. *For $k \in \mathbb{N}$ big enough and after shrinking the neighbourhoods U and I if necessary, $f_t^{-1}(0) = \{0\}$ for all $t \in I$.*

Proof: We follow exactly the same argument as in the proof of Lemma 4.4. By Proposition 3.2 we get the desired result for $k_2 = k > \theta_2 = l(\mathfrak{m}_2, f)$. □

Lemma 5.4. *For $k \in \mathbb{N}$ big enough and after shrinking the neighbourhoods U and I if necessary, f_t is an immersion on $U \setminus \{0\}$ for all $t \in I$.*

Proof: Let S_t be the set of non-immersive points of f_t. This set is analytic in U and is defined by the vanishing of the 2×2-minors of the Jacobian matrix of f_t. We proceed as in the previous lemmas. Denote by E_{1t}, E_{2t}, and E_{3t} the three minors and let $E_t = (E_{1t}, E_{2t}, E_{3t})$. For $t = 0$ we can assume that $S_0 = \{0\}$.

On the other hand, $E_t = E_0 + tF_t$ with $F_t \in \mathfrak{m}_2^{k-1}\mathcal{O}_3^3$. Therefore, by Proposition 3.2 we have the desired result for $k_3 = k - 1 > \theta_3 = l(\mathfrak{m}_2, E_0)$. □

Lemma 5.5. *For $k \in \mathbb{N}$ big enough and after shrinking the neighbourhoods U and I if necessary, f_t has at most double points for all $t \in I$.*

Proof: Here we work on $U \times U \times U$. We consider the analytic subset T_t defined by the vanishing of the functions $M_{it}(y) - M_{it}(x)$, $M_{it}(z) - M_{it}(y)$, and $M_{it}(x) - M_{it}(z)$ with $i = 1, \dots, 6$, where M_{it} are the generators of the double point ideal $I_2(f_t)$. It is clear that T_t contains the triple points of f_t, so it suffices to prove that $T_t = \{0\}$ for all $t \in I$.

For simplicity we rewrite the equations of T_t as $G_{1t}, \dots, G_{18,t}$ and $G_t = (G_{1t}, \dots, G_{18,t})$ and proceed as above. For $t = 0$ we may assume $T_0 = \{0\}$. On the other hand, $G_t = G_0 + tH_t$ with $H_t \in \mathfrak{m}_6^{k-1}\mathcal{O}_7^{18}$. Then, as a consequence of Proposition 3.2, we have the desired result for $k_4 = k - 1 > \theta_4 = l(\mathfrak{m}_2, G_0)$. □

Theorem 5.6. *Let $F\colon (\mathbb{R}^2 \times \mathbb{R}, 0) \to (\mathbb{R}^3 \times \mathbb{R}, 0)$ be a 1-parameter unfolding of an analytic map germ with isolated instability $f\colon (\mathbb{R}^2, 0) \to (\mathbb{R}^3, 0)$ with $F(x) = (f_t(x), x)$, $f_t(x) = f(x) + th(x)$, $h(x) \in \mathfrak{m}_2^k\mathcal{O}_2^3$, and $k \in \mathbb{N}$ big enough. Then F is topologically trivial.*

Proof: We take k big enough such that $k - 1 = \min\{k_1, k_2, k_3, k_4\} > \theta = \max\{\theta_1, \theta_2, \theta_3, \theta_4\}$ as in Lemmas 5.2, 5.3, 5.4, and 5.5, and we get the desired result. □

Finally, by using the same arguments as in Theorem 4.8, we get the main result of this section.

Theorem 5.7. *Let $f\colon (\mathbb{R}^2, 0) \to (\mathbb{R}^3, 0)$ be a real analytic map germ with isolated instability. Then f is finitely C^0-determined.*

Remark 5.8. The order of $\mathcal{C}^0$-determinacy of $f\colon (\mathbb{R}^2, 0) \to (\mathbb{R}^3, 0)$ is given by the integral part of the Łojasiewicz exponent θ plus one.

Example 5.9. If we consider the germ $f(x, y) = (x, y^2, y(x^2 + y^2)^2)$ of Example 2.5, applying Theorem 5.7 we have that f is finitely C^0-determined. In particular, using the arguments of the proof of Theorem 5.6, we can take as Łojasiewicz exponent $\theta = 4$ and conclude that f is C^0-5-determined.

Remark 5.10. No proof or counterexample are known for the converse of Theorems 4.8 and 5.7, that is, of the statement that *any finitely C^0-determined map germ $f\colon (\mathbb{R}^n, 0) \to (\mathbb{R}^p, 0)$ has isolated instability*, even for particular cases. This problem remains open.

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