STRUCTURE MONOIDS OF SET-THEORETIC SOLUTIONS OF THE YANG–BAXTER EQUATION

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Abstract: Given a set-theoretic solution \((X,r)\) of the Yang–Baxter equation, we denote by \(M = M(X,r)\) the structure monoid and by \(A = A(X,r)\), respectively \(A' = A'(X,r)\), the left, respectively right, derived structure monoid of \((X,r)\). It is shown that there exist a left action of \(M\) on \(A\) and a right action of \(M\) on \(A'\) and 1-cocycles \(\pi\) and \(\pi'\) of \(M\) with coefficients in \(A\) and in \(A'\) with respect to these actions, respectively. We investigate when the 1-cocycles are injective, surjective, or bijective. In case \(X\) is finite, it turns out that \(\pi\) is bijective if and only if \((X,r)\) is left non-degenerate, and \(\pi'\) is bijective if and only if \((X,r)\) is right non-degenerate. In case \((X,r)\) is left non-degenerate, in particular \(\pi\) is bijective, we define a semi-truss structure on \(M(X,r)\) and then we show that this naturally induces a set-theoretic solution \((\tilde{M},\tilde{r})\) on the least cancellative image \(\tilde{M} = M(X,r)/\eta\) of \(M(X,r)\). In case \(X\) is naturally embedded in \(M(X,r)/\eta\), for example when \((X,r)\) is irretractable, then \(\tilde{r}\) is an extension of \(r\). It is also shown that non-degenerate irretractable solutions necessarily are bijective.

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1. Introduction

Let \(V\) be a vector space over a field \(K\). Solutions \(R: V \otimes V \to V \otimes V\) of the linear braid or Yang–Baxter equation (abbreviated YBE)

\[
(R \otimes \text{id}_V) \circ (\text{id}_V \otimes R) \circ (R \otimes \text{id}_V) = (\text{id}_V \otimes R) \circ (R \otimes \text{id}_V) \circ (\text{id}_V \otimes R)
\]

on the vector space \(V \otimes V \otimes V\) have led to several algebraic structures, including some classes of bialgebras, quantum groups, and Hopf algebras. Because the variety of solutions remains elusive, Drinfeld ([11]) in 1992 proposed to consider solutions that are linearizations of solutions on a basis of \(V\). These are the so called set-theoretic solutions of the YBE.

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Thus a pair \((X, r)\), where \(X\) is a non-empty set and \(r : X \times X \to X \times X\) is a map, is called a set-theoretic solution of the YBE if
\[
(r \times \text{id}_X) \circ (\text{id}_X \times r) \circ (r \times \text{id}_X) = (\text{id}_X \times r) \circ (r \times \text{id}_X) \circ (\text{id}_X \times r).
\]
For \(x, y \in X\), write \(r(x, y) = (\sigma_x(y), \gamma_y(x))\). The solution \((X, r)\) is said to be left (resp. right) non-degenerate if each map \(\sigma_x\) (resp. \(\gamma_y\)) is bijective. A left and right non-degenerate solution is simply called a non-degenerate solution. The solution \((X, r)\) is said to be involutive if \(r^2 = \text{id}_{X \times X}\), and in particular such a solution is bijective.

This study started in the seminal papers of Etingof, Schedler, and Soloviev [12] and Gateva-Ivanova and Van den Bergh [17]. Since then, different aspects of this combinatorial problem have been developed [14, 17, 25, 26, 31] and several interesting connections have been found, such as braid and Garside groups [7, 10], (semi)groups of I-type [17, 21], matched pairs of groups [25, 32], Artin–Schelter regular algebras [13], Jacobson radical rings and generalizations [5, 27], regular subgroups and Hopf–Galois extensions [30], affine manifolds [28], orderability [3, 8], and factorizable groups [33].

It is now well-known that all non-degenerate involutive set-theoretic solutions \((X, r)\) are restrictions of a set-theoretic solution on the structure monoid
\[
M(X, r) = \langle x \in X \mid xy = \sigma_x(y)\gamma_y(x) \text{ for all } x, y \in X \rangle.
\]
Furthermore, in this case, the structure group
\[
G(X, r) = \text{gr}(x \in X \mid xy = \sigma_x(y)\gamma_y(x) \text{ for all } x, y \in X)
\]
and the permutation group \(G(X, r) = \text{gr}(\sigma_x \mid x \in X)\) have a brace structure, an algebraic structure introduced by Rump in [27]. Moreover, in [2], it is shown that all finite non-degenerate involutive set-theoretic solutions with a given permutation group, as a brace, can be explicitly constructed. For this case of finite solutions \((X, r)\), Etingof, Schedler, and Soloviev ([12]) proved that \(G(X, r)\) is a finitely generated, solvable abelian-by-finite group and independently Gateva-Ivanova and Van den Bergh ([17]) have shown that \(G(X, r)\) is a Bieberbach group, i.e. \(G(X, r)\) is an abelian-by-finite group, torsion-free, and finitely generated. To deal with arbitrary finite bijective non-degenerate solutions Guarnieri and Vendramin ([18]) introduced the algebraic structure called a skew brace. Bachiller ([1]) then also showed that all such solutions can be described from finite skew braces. Lu, Yan, and Zhu ([25]) and Soloviev ([31]) showed that for such solutions the structure group \(G(X, r)\) is abelian-by-finite (see also Lebed and Vendramin – [24] – for another
proof), and Jespers, Kubat, and Van Antwerpen ([19]) showed that the structure monoid $M(X, r)$ is also abelian-by-finite. Note that, if $(X, r)$ is not involutive, then the canonical map $i: X \to G(X, r)$ is not necessarily injective and thus one cannot recover $r$ from the associated solution on $G(X, r)$. However, it can be recovered from the solution associated to $M(X, r)$.

The associated structure algebras, i.e. the monoid algebra $KM(X, r)$ and the group algebra $KG(X, r)$, where $K$ is any field, have also been studied by Jespers and Okniński [21], Gateva-Ivanova and Van den Bergh [17], and Jespers, Kubat, and Van Antwerpen [19]. In the latter it is shown that if $(X, r)$ is a left non-degenerate bijective finite set-theoretic solution, then the algebra $KM(X, r)$ (and $KG(X, r)$) is a module-finite normal extension of a commutative affine subalgebra. In particular, these algebras are Noetherian PI-algebras of finite Gelfand–Kirillov dimension. Furthermore, it was shown that many properties, such as being a domain or prime, of the algebra $KM(X, r)$ are equivalent with the solution $(X, r)$ being involutive.

A crucial fact to prove the above results is (see [12, 19, 25]) that if $(X, r)$ is a left non-degenerate solution, then the structure monoid $M(X, r)$ is a regular submonoid of the semidirect product

$$A(X, r) \rtimes G(X, r),$$

where

$$A(X, r) = \langle x \in X \mid x\sigma_x(y) = \sigma_x(y)\sigma_{\sigma_x(y)}(\gamma_y(x)) \text{ for all } x, y \in X \rangle,$$

i.e. for any element $a \in A(X, r)$ there is a unique $\phi(a) \in G(X, r)$ such that $(a, \phi(a)) \in M(X, r)$. In particular, one has a bijective 1-cocycle $M(X, r) \to A(X, r)$, determined by the natural action of $G(X, r)$ on $A(X, r)$. Here, the derived monoid $A(X, r)$ “encodes” the relations determined by the map $r^2: X^2 \to X^2$. If, furthermore, the left non-degenerate solution $(X, r)$ is bijective, then the monoid $A = A(X, r)$ is such that $aA = Aa$ for all $a \in A$. So $A(X, r)$ consists of normal elements and thus $A$ has a much richer structure than $M(X, r)$. For example, if $(X, r)$ is involutive, then $A$ is a free abelian monoid of rank $|X|$. It is this “richer structure” that has been exploited in several papers to obtain information on the structure monoid $M(X, r)$ and the structure algebra $KM(X, r)$.

In this paper we continue these investigations for arbitrary set-theoretic solutions $(X, r)$. So, $r$ is not necessarily bijective and $X$ is any set. In the first section we recall the important result of Gateva-Ivanova and Majid [16]: there exists a unique set-theoretic solution $(M, r_M)$
associated to the structure monoid $M = M(X, r)$ such that the restriction of $r_M$ to $X^2$ equals $r$. In the second section we introduce two derived monoids $A(X, r)$ and $A'(X, r)$ and we prove that there is a unique 1-cocycle $\pi: M(X, r) \to A(X, r)$, with respect to the natural left action $\lambda: M(X, r) \to \text{End}(A(X, r))$, such that $\pi(x) = x$, and a unique 1-cocycle $\pi': M(X, r) \to A'(X, r)$, with respect to the natural right action $\rho': M(X, r) \to \text{End}(A'(X, r))$ such that $\pi'(x) = x$. Hence one gets a monoid homomorphism $f: M(X, r) \to A(X, r) \rtimes \text{Im}(\lambda)$: $a \mapsto (\pi(a), \lambda'_a)$ and a monoid anti-homomorphism $f': M(X, r) \to A'(X, r)^\text{op} \rtimes \text{Im}(\rho')$: $a \mapsto (\pi'(a), \rho'_a)$, where $\lambda'_a(y) = \sigma_x(y)$ and $\rho'_a(y) = \gamma_x(y)$ for all $x, y \in X$. In general these 1-cocycles are not bijective. But we investigate when they are injective, respectively surjective. In case $(X, r)$ is finite, the bijectiveness of $\pi$ (respectively $\pi'$) is equivalent to the solution being left (respectively right) non-degenerate. In Section 4 we prove the surprising result that any non-degenerate irretractable solution is necessarily bijective. In Section 5 we link the algebraic structure of $M(X, r)$ to that of semi-trusses as introduced by Brzeziński [4].

We determine the left cancellative (additive) congruence $\eta$ on $M(X, r)$ for $(X, r)$ a left non-degenerate solution, and we show that we obtain a solution $(M/\eta, \tau)$ determined by a semi-truss structure on $M/\eta$.

2. Solution associated with the structure monoid

In this section we recall a result of Gateva-Ivanova and Majid in [16, Section 3.2], stating that any set-theoretic solution $(X, r)$ of the YBE can be extended to a set-theoretic solution on its structure monoid $M(X, r)$. The result in [16] is stated for bijective solutions but the proof remains valid without this assumption.

We recall this construction. Let $(X, r)$ be a set-theoretic solution of the YBE which is not necessarily bijective. We write $r(x, y) = (\sigma_x(y), \gamma_y(x))$ for all $x, y \in X$. It is known that $(X, r)$ is a set-theoretic solution of the YBE if and only if the following conditions hold:

1. $\sigma_x \sigma_y = \sigma_{\sigma_x(y)} \sigma_{\gamma_y(x)}$,
2. $\sigma_{\sigma_x(y)}(z)(\gamma_y(x)) = \gamma_{\sigma_x(z)}(\sigma_z(x))$,
3. $\gamma_x \gamma_y = \gamma_{\gamma_x(y)} \gamma_x(\sigma_y)$,

for all $x, y, z \in X$.

Let $M = M(X, r)$ be the structure monoid of $(X, r)$, that is, the multiplicative monoid with operation $\circ$ and with presentation

$$M(X, r) = \langle X \mid x \circ y = \sigma_x(y) \circ \gamma_y(x) \text{ for all } x, y \in X \rangle.$$
One defines the following “left action” on $M$:

$$\lambda: M \rightarrow \text{Map}(M, M): a \mapsto \lambda_a,$$

with $\lambda_1 = \text{id}_M$, and for $x_1, \ldots, x_m, y_1, \ldots, y_n \in X$ and $n > 1$, $\lambda_{x_1}(1) = 1$,

$$\lambda_{x_1}(y_1) = \sigma_{x_1}(y_1), \quad \lambda_{x_1}(y_1 \circ \cdots \circ y_n) = \sigma_{x_1}(y_1) \circ \lambda_{y_1}(y_2 \circ \cdots \circ y_n),$$

and for $m > 1$,

$$\lambda_{x_1 \circ \cdots \circ x_m} = \lambda_{x_1} \circ \cdots \circ \lambda_{x_m}.$$

One also defines a “right action” on $M$:

$$\rho: M \rightarrow \text{Map}(M, M): a \mapsto \rho_a,$$

with $\rho_1 = \text{id}_M$, and for $x_1, \ldots, x_m, y_1, \ldots, y_n \in X$ and $n > 1$,

$$\rho_{x_1}(y_1) = \gamma_{x_1}(y_1), \quad \rho_{x_1}(y_1 \circ \cdots \circ y_n) = \rho_{y_1}(x_1) \circ \gamma_{y_1}(y_2 \circ \cdots \circ y_{n-1}) \circ \sigma_{x_1}(y_n),$$

and for $m > 1$,

$$\rho_{x_1 \circ \cdots \circ x_m} = \rho_{x_m} \circ \cdots \circ \rho_{x_1}.$$

In [16] it is proved that $\lambda$ and $\rho$ are well defined. Furthermore, it is then shown that every set-theoretic solution $(X, r)$ of the YBE is the restriction of a set-theoretic solution defined on the structure monoid $M(X, r)$.

**Theorem 2.1** (Gateva-Ivanova and Majid [16, Theorem 3.6]). Let $(X, r)$ be a set-theoretic solution of the YBE. Then the mapping $\lambda$ is a monoid homomorphism and the mapping $\rho$ is monoid anti-homomorphism such that

$$\rho_b(c \circ a) = \rho_{\lambda_a(b)}(c) \circ \rho_b(a), \quad (8)$$

$$\lambda_b(a \circ c) = \lambda_b(a) \circ \lambda_{\rho_a(b)}(c), \quad (9)$$

for all $a, b, c \in M$. Furthermore, for $a, b \in M = M(X, r)$,

$$a \circ b = \lambda_a(b) \circ \rho_b(a). \quad (10)$$

Let $r_M: M \times M \rightarrow M \times M$ be defined by $r_M(a, b) = (\lambda_a(b), \rho_b(a))$ for all $a, b \in M$. Then, $(M, r_M)$ is a set-theoretic solution of the YBE. Obviously, $r_M$ extends the solution $r$.

Note that if the solution $(X, r)$ is bijective and left and right non-degenerate, i.e. all $\sigma_x$ and $\gamma_x$ are bijective maps, then as in the proof of the above result one can show that the mappings $\sigma_x$ and $\gamma_x$ induce actually left and right actions on $G = G(X, r)$, say $\lambda^e: G \rightarrow \text{Sym}(G)$ and $\rho^e: G \rightarrow \text{Sym}(G)$; this is Theorem 4 in [25]. Furthermore, the mapping $r_G(a, b) = (\lambda^e_a(b), \rho^e_b(a))$, for $a, b \in G$, defines a set-theoretic solution on $G$. Note that, in general, the natural map $i: X \rightarrow G$ is not
injective. One obtains that \( r_G \) is an extension of the induced set-theoretic solution \((i(X), r_{i(X)}) = (i(X), (r_G)_{i(X)})\).

A natural question is whether one can extend a solution \((X, r)\), via the actions induced from \(\sigma_x\) and \(\gamma_y\), to a solution on the structure group. This however is not possible in general as shown by the following example. Consider the set-theoretic solution \((X, \text{id}_{X^2})\) on a set \(X\) with more than one element. Obviously, each \(\sigma_x\) and \(\gamma_x\) is constant with image \(\{x\}\). Hence, \(M = M(X, \text{id}_{X^2})\) is the free monoid on the set \(X\) and \(G = G(X, \text{id}_{X^2})\) is the free group on \(X\). However, because the maps \(\sigma_x\) are not injective one cannot extend the maps \(\sigma_x\) to a monoid homomorphism \(\lambda: G \rightarrow \text{Map}(G, G)\) with \(\lambda_y(y) = \sigma_x(y)\) for \(y \in G\).

A remarkable fact shown by Lu, Yan, and Zhu in \([25]\) is that if one can extend the mappings \(\sigma_x\) and \(\gamma_x\) to left and right actions on the structure group, then the induced set-theoretic solution is bijective.

### 3. Derived monoids

Let \((X, r)\) be a set-theoretic solution of the YBE. Write \(r(x, y) = (\sigma_x(y), \gamma_y(x))\) for all \(x, y \in X\). If \((X, r)\) is left non-degenerate, then Soloviev defined in \([31]\) its derived solution \((X, r')\) by

\[
r'(x, y) = (y, \sigma_y \gamma_{\sigma_x^{-1}(y)}(x))
\]

for all \(x, y \in X\). For general solutions one cannot define such a derived solution. But in \([19]\) one defines the derived monoids of \((X, r)\) as

\[
A(X, r) = \langle X \mid x + \sigma_x(y) = \sigma_x(y) + \sigma_x(y) \gamma_y(x) \text{ for all } x, y \in X \rangle
\]

and

\[
A'(X, r) = \langle X \mid \gamma_y(x) \oplus y = \gamma_y(x) \sigma_x(y) \oplus \gamma_y(x) \text{ for all } x, y \in X \rangle.
\]

The zero element of \(A(X, r)\) is denoted \(0\) and the zero element of \(A'(X, r)\) is denoted \(0'\). We will say that \(A(X, r)\) is the \textit{left derived} structure monoid of \((X, r)\) and \(A'(X, r)\) is the \textit{right derived} structure monoid of \((X, r)\).

Note that \(X \subseteq M(X, r), X \subseteq A(X, r), \text{ and } X \subseteq A'(X, r)\), because the defining relations of these three monoids are homogeneous of degree 2.

**Proposition 3.1.** Let \((X, r)\) be a set-theoretic solution of the YBE, where \(r(x, y) = (\sigma_x(y), \gamma_y(x))\) for all \(x, y \in X\). Then there exists a unique monoid homomorphism \(\lambda': M(X, r) \rightarrow \text{End}(A(X, r))\) such that \(\lambda'(x)(y) = \sigma_x(y)\) for all \(x, y \in X\) and there exists a unique anti-homomorphism \(\rho': M(X, r) \rightarrow \text{End}(A'(X, r))\) such that \(\rho'(x)(y) = \gamma_x(y)\) for all \(x, y \in X\). Furthermore, if \((X, r)\) is left (right) non-degenerate, then \(\text{Im}(\lambda') \subseteq \text{Aut}(A(X, r))\) \((\text{Im}(\rho') \subseteq \text{Aut}(A'(X, r)))\).
**Proof:** We will write $\lambda'(a) = \lambda'_a$ and $\rho'(a) = \rho'_a$ for all $a \in M(X, r)$.

Let $x_1, \ldots, x_m, y_1, \ldots, y_n \in X$. We denote by $1, 0, 0'$ the identity elements of the monoids $M(X, r), A(X, r), A'(X, r)$, respectively. We define $\lambda'_1 = \text{id}_{A(X, r)}$, $\rho'_1 = \text{id}_{A'(X, r)}$, $\lambda'_a(0) = 0$, $\rho'_a(0') = 0'$, for all $a \in M(X, r)$, and

$$
\lambda'_{x_10 \cdots o x_m}(y_1 + \cdots + y_n) = \sigma_{x_1} \cdots \sigma_{x_m}(y_1) + \cdots + \sigma_{x_1} \cdots \sigma_{x_m}(y_n),
$$

and

$$
\rho'_{x_10 \cdots o x_m}(y_1 \oplus \cdots \oplus y_n) = \gamma_{x_m} \cdots \gamma_{x_1}(y_1) \oplus \cdots \oplus \gamma_{x_m} \cdots \gamma_{x_1}(y_n).
$$

First we shall prove that $\lambda'$ and $\rho'$ are well-defined. To do so it is enough to prove that the following equalities hold:

\begin{align}
(11) & \quad \lambda'_{x_10x_2}(y_1 + \cdots + y_n) = \lambda'_{\sigma_{x_1}(x_2) \circ \gamma_{x_2}(x_1)}(y_1 + \cdots + y_n), \\
(12) & \quad \lambda'_1(y_1 + \sigma_{y_1}(y_2)) = \lambda'_{x_10 \cdots o x_m}(\sigma_{y_1}(y_2) + \sigma_{y_1}(y_2)(\gamma_{y_2}(y_1))), \\
(13) & \quad \rho'_{x_10x_2}(y_1 \oplus \cdots \oplus y_n) = \rho'_{\sigma_{x_1}(x_2) \circ \gamma_{x_2}(x_1)}(y_1 \oplus \cdots \oplus y_n), \\
(14) & \quad \rho'_{x_10 \cdots o x_m}(\gamma_{y_2}(y_1) \oplus y_2) = \rho'_{x_10 \cdots o x_m}(\gamma_{y_2}(y_1)(\sigma_{y_1}(y_2)) \oplus \gamma_{y_2}(y_1)).
\end{align}

Using relations (1) and (3), equations (11) and (13) are easily checked:

\begin{align}
\lambda'_{x_10x_2}(y_1 + \cdots + y_n) & = \sigma_{x_1} \sigma_{x_2}(y_1) + \cdots + \sigma_{x_1} \sigma_{x_2}(y_n) \\
& = \sigma_{\sigma_{x_1}(x_2)} \sigma_{\gamma_{x_2}(x_1)}(y_1) + \cdots + \sigma_{\sigma_{x_1}(x_2)} \sigma_{\gamma_{x_2}(x_1)}(y_n) \\
& = \lambda'_{\sigma_{x_1}(x_2) \circ \gamma_{x_2}(x_1)}(y_1 + \cdots + y_n), \\
\rho'_{x_10x_2}(y_1 \oplus \cdots \oplus y_n) & = \gamma_{x_2} \gamma_{x_1}(y_1) \oplus \cdots \oplus \gamma_{x_2} \gamma_{x_1}(y_n) \\
& = \gamma_{\gamma_{x_2}(x_1)} \gamma_{\sigma_{x_1}(x_2)}(y_1) \oplus \cdots \oplus \gamma_{\gamma_{x_2}(x_1)} \gamma_{\sigma_{x_1}(x_2)}(y_n) \\
& = \rho'_{\sigma_{x_1}(x_2) \circ \gamma_{x_2}(x_1)}(y_1 \oplus \cdots \oplus y_n).
\end{align}

Using relations (1), (2), and (3) we shall prove equations (12) and (14) by induction on $m$. For $m = 0$, (12) and (14) follows by the defining relations of $A(X, r)$ and $A'(X, r)$. Suppose that $m > 0$. Assume
that \( \lambda'_{x_1 \cdots x_k}(y_1 + \sigma y_1(y_2)) = \lambda'_{x_1 \cdots x_k}(\sigma y_1(y_2) + \sigma \sigma y_1(y_2)(\gamma y_2(y_1))) \) and \( \rho'_{x_1 \cdots x_k}(\gamma y_2(y_1) \oplus y_2) = \rho'_{x_1 \cdots x_k}(\gamma \gamma y_2(y_1)(\sigma y_1(y_2)) \oplus \gamma y_2(y_1)) \), for \( k < m \), then

\[
\begin{align*}
\lambda'_{x_1 \cdots x_m}(y_1 + \sigma y_1(y_2)) &= \sigma_{x_1} \cdots \sigma_{x_m}(y_1) + \sigma_{x_1} \cdots \sigma_{x_m}(\sigma y_1(y_2)) \\
&= \lambda'_{x_1 \cdots x_m-1}(\sigma_{x_m}(y_1) + \sigma_{x_m}(\sigma y_1(y_2))) \\
&= \lambda'_{x_1 \cdots x_m-1}(\sigma_{x_m}(y_1) + \sigma_{x_m}(\sigma y_1(x_m)(y_2))) \\
&= \lambda'_{x_1 \cdots x_m-1}(\sigma_{x_m}(y_1)(\sigma y_1(x_m)(y_2))) \\
&\quad + \sigma_{x_m}(\gamma y_1(x_m)(y_2))(\gamma y_1(x_m)(y_2)) (\sigma_{x_m}(y_1))) \\
&= \lambda'_{x_1 \cdots x_m-1}(\sigma_{x_m}(\sigma y_1(y_2)) + \sigma_{x_m}(\sigma y_1(y_2))(\gamma y_1(x_m)(y_2)) (\sigma_{x_m}(y_1))) \\
&= \lambda'_{x_1 \cdots x_m-1}(\sigma_{x_m}(\sigma y_1(y_2)) + \sigma_{x_m}(\sigma y_1(y_2))(\gamma y_2(y_1))) \\
&= \sigma_{x_1} \cdots \sigma_{x_m}(\sigma y_1(y_2)) + \sigma_{x_1} \cdots \sigma_{x_m}(\sigma y_1(y_2)(\gamma y_2(y_1))) \\
&= \lambda'_{x_1 \cdots x_m}(\sigma y_1(y_2) + \sigma y_1(y_2)(\gamma y_2(y_1))),
\end{align*}
\]

and

\[
\begin{align*}
\rho'_{x_1 \cdots x_m}(\gamma y_2(y_1) \oplus y_2) &= \rho'_{x_2 \cdots x_m}(\gamma y_2(y_1) \oplus \gamma x_1(y_2)) \\
&= \rho'_{x_2 \cdots x_m}(\gamma y_2(y_1) \oplus \gamma y_2(y_1)) \oplus \gamma x_1(y_2) \\
&= \rho'_{x_2 \cdots x_m}(\gamma y_2(y_1) \oplus \gamma y_2(y_1)) \oplus \gamma y_2(y_1) \\
&= \rho'_{x_2 \cdots x_m}(\gamma y_2(y_1) \oplus \gamma y_2(y_1)) \\
&= \gamma x_m \cdots \gamma x_1(y_2) \oplus \gamma y_2(y_1) \\
&= \rho'_{x_1 \cdots x_m}(\gamma y_2(y_1) \oplus y_2) \\
&= \lambda'_{x_1 \cdots x_m}(y_1 + \sigma y_1(y_2)),
\end{align*}
\]

This proves that \( \lambda'_a \) and \( \rho'_a \) are well-defined and clearly \( \lambda'_a \in \text{End}(A(X, r)) \) and \( \rho'_a \in \text{End}(A'(X, r)) \) for all \( a \in M(X, r) \). Thus \( \lambda' \) and \( \rho' \) are well-defined. It is clear that \( \lambda' \) is a monoid homomorphism and that it is unique with respect to the condition \( \lambda'_{x}(y) = \sigma x(y) \) for all \( x, y \in X \). It is also clear that \( \rho' \) is a monoid anti-homomorphism and that it is unique for the condition \( \rho'_{x}(y) = \gamma x(y) \) for all \( x, y \in X \).
Assume now that \((X, r)\) is left non-degenerate. Let \(x, y_1, \ldots, y_n \in X\). We define \(f_x \in \text{End}(A(X, r))\) by

\[
f_x(y_1 + \cdots + y_n) = \sigma_{x}^{-1}(y_1) + \cdots + \sigma_{x}^{-1}(y_n).
\]

To see that \(f_x\) is well-defined it is enough to prove that

\[
f_x(y_1 + \sigma y_1(y_2)) = f_x(\sigma y_1(y_2) + \sigma \sigma y_1(y_2)(\gamma y_2(y_1))).
\]

Note that, from (1),

\[(15) \quad \sigma_{x}^{-1} \sigma y_1(y_2) = \sigma_{x}^{-1}(y_1) \sigma_{y_1}(x)(y_2)
\]

and thus, also using (2), we get that

\[
(16) \quad \sigma_{x}^{-1} \sigma y_1(y_2) \gamma_{\sigma_{x}^{-1}(y_1)(x)}(y_2)(\sigma_{x}^{-1}(y_1))
\]

\[
= \gamma_{\sigma_{x}^{-1}(y_1)(x)}(\sigma_{x}^{-1}(y_1)(y_2)) \gamma_{\sigma_{x}^{-1}(y_1)(x)}(y_2)(\sigma_{x}^{-1}(y_1))
\]

\[
= \gamma_{\sigma_{x}^{-1}(y_1)(x)}(\gamma_{\sigma_{x}^{-1}(y_1)(x)}(y_2))(\sigma_{x}^{-1}(y_1))
\]

\[
= \gamma_{y_2}(y_1).
\]

We have that

\[
f_x(y_1 + \sigma y_1(y_2)) = \sigma_{x}^{-1}(y_1) + \sigma_{x}^{-1}(\sigma y_1(y_2))
\]

\[
= \sigma_{x}^{-1}(y_1) + \sigma_{x}^{-1}(\sigma_{x}^{-1}(y_1)(y_2))
\]

\[
= \sigma_{x}^{-1}(y_1)(\sigma_{x}^{-1}(y_1)(y_2)) + \sigma_{x}^{-1}(\sigma_{x}^{-1}(y_1)(y_2))(\sigma_{x}^{-1}(y_1))
\]

\[
= \sigma_{x}^{-1}(y_1)(\sigma y_1(y_2)) + \sigma_{x}^{-1}(\sigma y_1(y_2))(\gamma_{\sigma_{x}^{-1}(y_1)(x)}(y_2)(\sigma_{x}^{-1}(y_1)))
\]

\[
= \sigma_{x}^{-1}(y_1)(\sigma y_1(y_2)) + \sigma_{x}^{-1}(\sigma y_1(y_2))(\gamma_{\sigma_{x}^{-1}(y_1)(x)}(y_2)(\sigma_{x}^{-1}(y_1)))
\]

\[
= \sigma_{x}^{-1}(y_1)(\sigma y_1(y_2)) + \sigma_{x}^{-1}(\sigma y_1(y_2))(\gamma_{y_2}(y_1))
\]

\[
= f_x(\sigma y_1(y_2) + \sigma y_1(y_2)(\gamma_{y_2}(y_1))
\]

where the third equality follows from the defining relations in \(A(X, r)\). Hence \(f_x\) is well-defined. Note that \(f_x \lambda_{x}' = \lambda_{x}' f_x = \text{id}\). Thus \(\lambda_{x}' \in \text{Aut}(A(X, r))\) for all \(x \in X\). Therefore \(\text{Im}(\lambda') \subseteq \text{Aut}(A(X, r))\).

Similarly one can prove that if \((X, r)\) is right non-degenerate, then \(\text{Im}(\rho') \subseteq \text{Aut}(A'(X, r))\). \(\square\)
Proposition 3.2. Let \((X, r)\) be a set-theoretic solution of the YBE. Then

(i) There is a unique 1-cocycle \(\pi : M(X, r) \to A(X, r)\) with respect to the left action \(\lambda'\) such that \(\pi(x) = x\) for all \(x \in X\).

(ii) There is a unique 1-cocycle \(\pi' : M(X, r) \to A'(X, r)\) with respect to the right action \(\rho'\) such that \(\pi'(x) = x\) for all \(x \in X\).

Furthermore, the mapping

\[
f : M(X, r) \longrightarrow A(X, r) \rtimes \text{Im}(\lambda') : a \mapsto (\pi(a), \lambda'_a)
\]

is a monoid homomorphism and the mapping

\[
f' : M(X, r) \longrightarrow A'(X, r)^{\text{op}} \rtimes \text{Im}(\rho') : a \mapsto (\pi'(a), \rho'_a)
\]

is a monoid anti-homomorphism.

Proof: We define for \(x_1, \ldots, x_m \in X\),

\[
\pi(1) = 0,
\]
\[
\pi(x_1) = x_1, \quad \text{and for } m > 1,
\]
\[
\pi(x_1 \circ \cdots \circ x_m) = x_1 + \lambda'_{x_1}(\pi(x_2 \circ \cdots \circ x_m)),
\]
\[
\pi'(1) = 0',
\]
\[
\pi'(x_1) = x_1, \quad \text{and for } m > 1,
\]
\[
\pi'(x_1 \circ \cdots \circ x_m) = \rho'_{x_m}(\pi'(x_1 \circ \cdots \circ x_{m-1})) \oplus x_m.
\]

We prove that \(\pi(x_1 \circ \cdots \circ x_m)\) and \(\pi'(x_1 \circ \cdots \circ x_m)\) are well-defined by induction on \(m\). For \(m = 1\) it is clear. Suppose that \(m > 1\) and that \(\pi(x_1 \circ \cdots \circ x_{m-1})\) and \(\pi'(x_1 \circ \cdots \circ x_{m-1})\) are well-defined.

By the induction hypothesis, it is enough to show that

\[
(17) \quad x_1 + \lambda'_{x_1}(\pi(x_2 \circ \cdots \circ x_m)) = \sigma_{x_1}(x_2) + \lambda'_{\sigma_{x_1}(x_2)}(\pi(\gamma_{x_2}(x_1) \circ x_3 \circ \cdots \circ x_m))
\]

and

\[
(18) \quad \rho'_{x_m}(\pi'(x_1 \circ \cdots \circ x_{m-1})) \oplus x_m
\]
\[
= \rho'_{\gamma_{x_m}(x_{m-1})}(\pi'(x_1 \circ \cdots \circ x_{m-2} \circ \sigma_{x_{m-1}}(x_m))) \oplus \gamma_{x_m}(x_{m-1}).
\]
Thus, indeed, such that deg($\pi$) = $\rho$ and that

$\pi \in \langle \lambda_a' \rangle$.

By (11) and (13) we get that

$\sigma_{x_1}(x_2) + \lambda'_{\sigma_{x_1}(x_2)}(\pi(x_2(x_1) \circ x_3 \circ \cdots \circ x_m)))
= \sigma_{x_1}(x_2) + \lambda'_{\sigma_{x_1}(x_2)}(\gamma_{x_2}(x_1) + \lambda'_{\gamma_{x_2}(x_1)}(\pi(x_3 \circ \cdots \circ x_m)))
= x_1 + \sigma_{x_1}(x_2) + \lambda'_{\sigma_{x_1}(x_2)\circ\gamma_{x_2}(x_1)}(\pi(x_3 \circ \cdots \circ x_m))
= x_1 + \sigma_{x_1}(x_2) + \lambda'_{\sigma_{x_1}(x_2)}(\pi(x_3 \circ \cdots \circ x_m))
= x_1 + \sigma_{x_1}(x_2) + \lambda'_{\sigma_{x_1}(x_2)}(\pi(x_3 \circ \cdots \circ x_m))
= x_1 + \lambda'_{\sigma_{x_1}(x_2)}(\pi(x_3 \circ \cdots \circ x_m))
= x_1 + \lambda'_{\sigma_{x_1}(x_2)}(\pi(x_3 \circ \cdots \circ x_m))$

and

$\rho'_{\gamma_{x_m}(x_{m-1})}(\pi'(x_1 \circ \cdots \circ x_{m-2} \circ \sigma_{x_{m-1}}(x_m))) \oplus \gamma_{x_m}(x_{m-1})$
$= \rho'_{\gamma_{x_m}(x_{m-1})}(\rho_{\sigma_{x_{m-1}}(x_m)}(\pi'(x_1 \circ \cdots \circ x_{m-2})) \oplus \gamma_{x_m}(x_{m-1})$
$= \rho'_{\gamma_{x_m}(x_{m-1})}(\rho_{\sigma_{x_{m-1}}(x_m)}(\pi'(x_1 \circ \cdots \circ x_{m-2}))) \oplus \gamma_{x_m}(x_{m-1})$
$= \rho'_{\gamma_{x_m}(x_{m-1})}(\pi'(x_1 \circ \cdots \circ x_{m-2})) \oplus \gamma_{x_m}(x_{m-1}) \oplus x_m$
$= x_1 + \lambda'_{\sigma_{x_1}(x_2)}(\pi(x_3 \circ \cdots \circ x_m)))$

Thus, indeed, $\pi$ and $\pi'$ are well-defined.

For all $a, b \in M(X, r)$, we shall prove by induction on deg$(a) +$ deg$(b)$ that

(19) $\pi(a \circ b) = \pi(a) + \lambda_a' (\pi(b))$

and

(20) $\pi'(a \circ b) = \rho'_b (\pi'(a)) \oplus \pi'(b)$.

If deg$(a) =$ deg$(b) =$ 1, then (19) and (20) follow by definition. Hence, we may suppose that deg$(a) +$ deg$(b) >$ 2 and that $\pi(a' \circ b') = \pi(a') + \lambda_{a'}' (\pi(b'))$ and $\pi'(a' \circ b') = \rho_{b'}' (\pi'(a')) \oplus \pi'(b')$ for all $a', b' \in M(X, r)$ such that deg$(a') +$ deg$(b') <$ deg$(a) +$ deg$(b)$. 
Write $a = x \circ a'$ and $b = b' \circ y$ for some $x, y \in X$ and $a', b' \in M(X, r)$. By the induction hypothesis we have
\[
\pi(a \circ b) = \pi(x \circ a' \circ b)
\]
\[
= x + \lambda'_x(\pi(a' \circ b))
\]
\[
= x + \lambda'_x(\pi(a') + \lambda'_{a'}(\pi(b)));
\]
\[
= x + \lambda'_x(\pi(a')) + \lambda'_x(\lambda'_{a'}(\pi(b)));
\]
\[
= \pi(x \circ a') + \lambda'_{x \circ a'}(\pi(b));
\]
\[
= \pi(a) + \lambda'_a(\pi(b))
\]
and
\[
\pi'(a \circ b) = \pi'(a \circ b' \circ y)
\]
\[
= \rho'_y(\pi'(a \circ b')) \oplus y
\]
\[
= \rho'_y(\rho'_{b'}(\pi'(a)) \oplus \pi'(b')) \oplus y
\]
\[
= \rho'_y(\rho'_{b'}(\pi'(a))) \oplus \rho'_y(\pi'(b')) \oplus y
\]
\[
= \rho'_{b' \circ y}(\pi'(a)) \oplus \pi'(b' \circ y)
\]
\[
= \rho'_b(\pi'(a)) \oplus \pi'(b).
\]
Thus (19) and (20) follow by induction. It is clear that $\pi$ and $\pi'$ are the unique 1-cocycles satisfying the hypothesis. Therefore the result follows.

\[\Box\]

A natural question is the following.

**Question 3.3.** When are the 1-cocycles $\pi$ and $\pi'$ bijective?

In general, these 1-cocycles are not bijective. We provide two examples. The first one is an example where $\pi$ is injective but not surjective, and the second one where $\pi$ and $\pi'$ are neither injective nor surjective.

**Example 3.4.** Let $(X, r)$ be a set-theoretic solution of the YBE, where $X$ is set of cardinality greater than 1 and $r : X \times X \to X \times X$ is a map defined by $r(x, y) = (x, x)$ for all $x, y \in X$. The associated monoids are
\[
M(X, r) = \langle X \mid x \circ y = x \circ x \text{ for all } x, y \in X \rangle,
\]
\[
A(X, r) = \langle X \mid x + x = x + x \text{ for all } x, y \in X \rangle,
\]
\[
A'(X, r) = \langle X \mid x \oplus y = x \oplus x \text{ for all } x, y \in X \rangle.
\]
The 1-cocycle $\pi'$ is bijective, but it is clear that the 1-cocycle $\pi$ is not. The latter is not surjective. For example, the element $x + y$, where $x \neq y \in X$ is not in the image of $\pi$. Note that $\pi$ is still injective. Similarly, $(X, r)$ with $r : X \times X \to X \times X$ defined by $r(x, y) = (y, y)$ is
an example of a set-theoretic solution of the YBE where \( \pi' \) is injective but not surjective.

**Example 3.5.** Let \( S = \{0, 1, 2\} \) and define the skew lattice \((S, \wedge, \vee)\) by

<table>
<thead>
<tr>
<th></th>
<th>0</th>
<th>1</th>
<th>2</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \wedge )</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>( \vee )</td>
<td>0</td>
<td>0</td>
<td>1</td>
</tr>
</tbody>
</table>

The skew lattice \((S, \wedge, \vee)\) is an example of a distributive and cancellative skew lattice that is not a co-strongly distributive skew lattice; see ([9, Example 4.4]). By [9, Theorem 5.7], \((S, \wedge, \vee)\) is a left distributive solution, i.e. \((S, r)\) is a set-theoretic solution of the YBE, where \(r: S \times S \to S \times S\) is defined by \(r(x, y) = (x \wedge y, y \vee x)\) for all \(x, y \in S\). The associated monoids are

\[
M(X, r) = \langle 0, 1, 2 \mid 1 \circ 0 = 0 \circ 1, 2 \circ 0 = 0 \circ 2, 1 \circ 2 = 2 \circ 2, 2 \circ 1 = 1 \circ 1 \rangle,
\]

\[
A(X, r) = \langle 0, 1, 2 \mid 1 + 0 = 0 + 0, 2 + 0 = 0 + 0, 1 + 2 = 2 + 2, 2 + 1 = 1 + 1 \rangle,
\]

\[
A'(X, r) = \langle 0, 1, 2 \mid 1 \oplus 0 = 1 \oplus 1, 2 \oplus 0 = 2 \oplus 2 \rangle.
\]

Both \(\pi\) and \(\pi'\) are not injective, as \(\pi(1 \circ 0) = 1 + 0 = 0 + 0 = \pi(0 \circ 0)\) and \(\pi'(1 \circ 0) = 1 \oplus 0 = 1 \oplus 1 = \pi'(1 \circ 1)\), but \(1 \circ 0 \neq 0 \circ 0\) and \(1 \circ 0 \neq 1 \circ 1\) in \(M(X, r)\). Both \(\pi\) and \(\pi'\) are not surjective as \(0 + 1\) (resp. \(0 \oplus 1\)) is not in the image of \(\pi\) (resp. \(\pi'\)).

**Proposition 3.6.** Let \((X, r)\) be a set-theoretic solution of the YBE. Write \(r(x, y) = (\sigma_x(y), \gamma_y(x))\). Let \(\pi: M(X, r) \to A(X, r)\) and \(\pi': M(X, r) \to A'(X, r)\) be the 1-cocycles of Proposition 3.2. Then

(i) \(\pi\) is surjective if and only if \(\sigma_x\) is surjective for all \(x \in X\).

(ii) \(\pi'\) is surjective if and only if \(\gamma_x\) is surjective for all \(x \in X\).

**Proof:** Suppose that \(\sigma_x\) is surjective for all \(x \in X\). First, we claim that \(\sigma_x\) being surjective implies that \(\lambda'_x\) is surjective. Take an arbitrary positive integer. Let \(x_1, \ldots, x_n \in X\) such that \(x_1 + \cdots + x_n \in A(X, r)\). As \(\sigma_x\) is surjective, there exist \(y_1, \ldots, y_n \in X\) such that \(\sigma_x(y_i) = x_i\) for all \(i \in \{1, \ldots, n\}\). Then, \(\lambda'_x(y_1 + \cdots + y_n) = \sigma_x(y_1) + \cdots + \sigma_x(y_n) = x_1 + \cdots + x_n\), which proves that \(\lambda'_x\) is surjective.

Next, we prove that \(\pi\) is surjective by induction on the length of the elements in \(A(X, r)\). As \(\pi(x) = x\) for all \(x \in X\), \(\pi\) is surjective on elements of length 1. Assume now that for a fixed positive integer \(n\) and for any \(x_1, \ldots, x_n \in X\), there exist \(y_1, \ldots, y_n \in X\) such that \(\pi(y_1 \circ \cdots \circ y_n) = x_1 + \cdots + x_n\). Take \(x_1, \ldots, x_{n+1} \in X\). Since \(\lambda'_{x_1}\) is surjective, there exist \(z_2, \ldots, z_{n+1} \in X\) such that \(\lambda'_{x_1}(z_2 + \cdots + z_{n+1}) = x_2 + \cdots + x_{n+1}\).
Using the induction hypothesis, there exist $y_2, \ldots, y_{n+1} \in X$ such that 

$$\pi(y_2 \circ \cdots \circ y_{n+1}) = z_2 + \cdots + z_{n+1}.$$ 

Thus, we obtain

$$x_1 + \cdots + x_{n+1} = x_1 + \lambda'_{x_1}(z_2 + \cdots + z_{n+1})$$

$$= x_1 + \lambda'_{x_1}(\pi(y_2 \circ \cdots \circ y_{n+1}))$$

$$= \pi(x_1 \circ y_2 \circ \cdots \circ y_{n+1}),$$

and $\pi$ is surjective.

Suppose now that $\pi$ is surjective. Let $x, y \in X$ and consider $x + y \in A(X, r)$. Since $\pi$ is surjective (and it preserves the degree), there exist $z, t \in X$ such that $\pi(z \circ t) = x + y$. Thus $z + \sigma_z(t) = x + y$ in $A(X, r)$. By the defining relations of $A(X, r)$, this equality implies that there exists $y' \in X$ such that $\sigma_x(y') = y$. Therefore $\sigma_x$ is surjective for all $x \in X$.

The proof for $\pi'$ is similar. \qed

**Proposition 3.7.** Let $(X, r)$ be a set-theoretic solution of the YBE. Write $r(x, y) = (\sigma_x(y), \gamma_y(x))$. Let $\pi : M(X, r) \to A(X, r)$ and $\pi' : M(X, r) \to A'(X, r)$ be the 1-cocycles of Proposition 3.2.

(i) If $\sigma_x$ is injective for all $x \in X$, then $\pi$ is injective.

(ii) If $\gamma_x$ is injective for all $x \in X$, then $\pi'$ is injective.

**Proof:** We shall prove (i). The proof of (ii) is similar. Let $FM(X)$ be the (multiplicative) free monoid on $X$. Suppose that $\sigma_x$ is injective for all $x \in X$. Since $\pi(x) = x$ for all $x \in X$, the restriction of $\pi$ to elements of degree one in $M(X, r)$ is injective. Let $n$ be an integer greater than 1. Let $x_1, \ldots, x_n, y_1, \ldots, y_n \in X$ be elements such that $\pi(x_1 \circ \cdots \circ x_n) = \pi(y_1 \circ \cdots \circ y_n)$. Thus, in $A(X, r)$, we have that

$$x_1 + \sigma_{x_1}(x_2) + \cdots + \sigma_{x_{n-1}}(x_n) = y_1 + \sigma_{y_1}(y_2) + \cdots + \sigma_{y_{n-1}}(y_n).$$

Let $w_1, w_2 \in FM(X)$ be two elements of degree $n$. Suppose that $w_1 = z_1 \cdots z_n$ and $w_2 = t_1 \cdots t_n$, for some $z_i, t_i \in X$. We say that $w_1 \sim w_2$ if there exist $1 \leq i \leq n-1$ and $z \in X$ such that $z_j = t_j$ for all $j \in \{1, 2, \ldots, n\} \setminus \{i, i+1\}$ and, either $z_{i+1} = \sigma_{z_i}(z) = t_i$ and $t_{i+1} = \sigma_{t_i}(z_i)$, or $t_{i+1} = \sigma_{t_i}(z) = z_i$ and $z_{i+1} = \sigma_{z_i}(z_i)$. Note that $z_1 + \cdots + z_n = t_1 + \cdots + t_n$ in $A(X, r)$ if and only if there exist $w'_1, \ldots, w'_m \in FM(X)$ of degree $n$ such that

$$w_1 = w'_1 \sim w'_2 \sim \cdots \sim w'_m = w_2.$$ 

Hence, to prove that $x_1 \circ \cdots \circ x_n = y_1 \circ \cdots \circ y_n$, we may assume that there exist $1 \leq i \leq n-1$ and $z \in X$ such that $\sigma_{x_1} \cdots \sigma_{x_{i-1}}(x_i) = \sigma_{y_1} \cdots \sigma_{y_{i-1}}(y_i)$ for all $j \in \{1, 2, \ldots, n\} \setminus \{i, i+1\}$, and also $\sigma_{x_1} \cdots \sigma_{x_{i-1}}(x_i+1) = \sigma_{y_1} \cdots \sigma_{y_{i-1}}(y_i)$, as well as $\sigma_{y_1} \cdots \sigma_{y_{i}}(y_{i+1}) = \sigma_{y_1} \cdots \sigma_{y_{i}}(y_i)$.
Since $\sigma_{x_1} \ldots \sigma_{x_{i-1}}(x_j) = \sigma_{y_1} \ldots \sigma_{y_{j-1}}(y_j)$ for all $j \in \{1, 2, \ldots, n\} \setminus \{i, i + 1\}$, and $\sigma_x$ is injective for all $x \in X$, we have that $x_j = y_j$ for all $j \in \{1, \ldots, i - 1\}$. Hence, since $\sigma_{x_1} \ldots \sigma_{x_i}(x_{i+1}) = \sigma_{y_1} \ldots \sigma_{y_{i-1}}(y_i)$, and $\sigma_x$ is injective for all $x \in X$, we have that $y_i = \sigma_{x_i}(x_{i+1})$. Now we have that

$$\sigma_{x_1} \ldots \sigma_{x_{i-1}}(x_i)(z) = \sigma_{x_1} \ldots \sigma_{x_i}(x_{i+1})$$

$$= \lambda_{x_1 \circ \ldots \circ x_{i-1}} \sigma_{x_i}(x_{i+1})$$

$$= \lambda_{x_1 \circ \ldots \circ x_{i-1}}(x_i) \lambda_{\rho_{x_i}}(x_{i+1})$$

where the third equality follows Theorem 2.1.

Hence, since $\sigma_x$ is injective for all $x \in X$, we have that

$$z = \lambda_{\rho_{x_i}}(x_{i+1}).$$

By Theorem 2.1,

$$\sigma_{y_1} \ldots \sigma_{y_i}(y_{i+1}) = \sigma_{\sigma_{y_1} \ldots \sigma_{y_{i-1}}(y_i)} \gamma_{z}(\sigma_{x_1} \ldots \sigma_{x_{i-1}}(x_i))$$

By Theorem 2.1,

$$= \sigma_{x_1} \ldots \sigma_{x_{i-1}}(x_i)(y_{i+1})$$

$$= \lambda_{x_1 \circ \ldots \circ x_{i-1}}(x_i) \lambda_{\rho_{x_i}}(x_{i+1})$$

$$= \lambda_{x_1 \circ \ldots \circ x_{i-1}}(x_i) \lambda_{\rho_{x_i}}(x_{i+1})$$

$$= \lambda_{y_1 \circ \ldots \circ y_{i-1}} \lambda_{y_i}(\rho_{x_{i+1}}(x_i))$$

$$= \sigma_{y_1} \ldots \sigma_{y_{i-1}} \lambda_{y_i}(\rho_{x_{i+1}}(x_i))$$

Since $\sigma_x$ is injective for all $x \in X$, we have that $y_{i+1} = \gamma_{x_{i+1}}(x_i)$. Thus,

$$y_i \circ y_{i+1} = \sigma_{x_i}(x_{i+1}) \circ \gamma_{x_{i+1}}(x_i) = x_i \circ x_{i+1}.$$
If \( \pi \) (resp. \( \pi' \)) is injective, then it is clear that the map \( f \) (resp. \( f' \)) defined in Proposition 3.2 is an embedding. The latter was proved in [19] under the assumption that \((X, r)\) is a left non-degenerate solution. In this case \( \pi \) is bijective and \( M(X, r) \) is a regular submonoid of the semidirect product \( A(X, r) \rtimes \text{gr}(\sigma_x \mid x \in X) \).

The following result answers Question 3.3 for finite solutions.

**Corollary 3.9** (Jespers, Kubat, and Van Antwerpen [19]). Let \((X, r)\) be a set-theoretic solution of the YBE, \( \lambda' \) (resp. \( \rho' \)) the left (resp. right) action as defined before, \( \pi \) (resp. \( \pi' \)) the unique 1-cocycle with respect to \( \lambda' \) (resp. \( \rho' \)). Then, \( \pi \) (resp. \( \pi' \)) is bijective if \((X, r)\) is left non-degenerate (resp. right non-degenerate). The converse holds if \( X \) is finite.

**Proof:** Assume first that \((X, r)\) is a left non-degenerate set-theoretic solution of the YBE. Then, by Propositions 3.6 and 3.7, \( \pi \) is bijective. Similarly, one can prove that \((X, r)\) being a right non-degenerate solution implies that \( \pi' \) is bijective.

Assume now that \( \pi : M(X, r) \to A(X, r) \) is bijective and \( X \) is finite. By Proposition 3.6, \( \sigma_x \) is surjective for all \( x \in X \). Since \( X \) is finite, \( \sigma_x \) is bijective for all \( x \in X \), that is, \((X, r)\) is left non-degenerate.

The next example shows the difficulty of Question 3.3 for infinite solutions.

**Example 3.10.** Consider the set \( \mathbb{N} \) of the non-negative integers. Let \( r : \mathbb{N} \times \mathbb{N} \to \mathbb{N} \times \mathbb{N} \) be the map defined by \( r(x, y) = (\xi(y), \xi(x)) \) for all \( x, y \in \mathbb{N} \), where \( \xi(x) = \max\{0, x - 1\} \) for all \( x \in \mathbb{N} \). Then \((\mathbb{N}, r)\) is a set-theoretic solution of the YBE, such that the associated 1-cocycles \( \pi \) and \( \pi' \) are bijective but, for every \( x \in \mathbb{N} \), \( \sigma_x = \gamma_x = \xi \) is not injective because \( \xi(0) = \xi(1) \).

**Proof:** It is easy to check that \((\mathbb{N}, r)\) is a set-theoretic solution of the YBE. Note that, for every \( x \in \mathbb{N} \), \( \xi^x(x) = 0 \). Hence

\[
M(\mathbb{N}, r) = \langle \mathbb{N} \mid x \circ y = 0 \circ 0 \rangle,
\]

\[
A(\mathbb{N}, r) = \langle \mathbb{N} \mid x + y = 0 + 0 \rangle,
\]

and

\[
A'(\mathbb{N}, r) = \langle \mathbb{N} \mid x \oplus y = 0 \oplus 0 \rangle.
\]

Therefore, for every integer \( n > 1 \), the monoids \( M(\mathbb{N}, r) \), \( A(\mathbb{N}, r) \), and \( A'(\mathbb{N}, r) \) have only one element of degree \( n \). Since \( \pi \) and \( \pi' \) preserve the degree and \( \pi(x) = x \) and \( \pi'(x) = x \), for all \( x \in \mathbb{N} \), we have that \( \pi \) and \( \pi' \) are bijective. Thus the result follows. \( \square \)
4. Non-degenerate irretractable solutions

In [26, Theorem 2] (and independently in [20, Corollary 2.3]) it is proven that any finite involutive left non-degenerate set-theoretic solution of the YBE also is right non-degenerate. In the infinite case, the latter is no longer true. The following example from [26] shows this.

**Example 4.1.** Let $X$ be the set of the integers, and define $r: X^2 \to X^2$ by
\[
r(x, y) = (\lambda_x(y), \lambda^{-1}_{\lambda_x(y)}(x)),
\]
where $\lambda_x(y) = y + \min(x, 0)$ for all $x, y \in X$. Note that $\lambda_x$ is bijective and $\lambda^{-1}_x(y) = y - \min(x, 0)$. It is easy to check that $(X, r)$ is an involutive solution. Note that it is not right non-degenerate. In fact, if $a < 0$, we have that
\[
\rho_a(b) = \lambda^{-1}_{\lambda_a(b)}(b) = b - \min(a \lor b, 0) = b - (a + b) = -a
\]
for all $b < 0$. Hence $\rho_a$ is not bijective if $a < 0$.

It is unclear whether the above holds for arbitrary bijective solutions. Hence the following question is pertinent.

**Question 4.2.** Is any finite bijective left non-degenerate set-theoretic solution of the YBE right non-degenerate?

A natural question is the converse:

**Question 4.3.** Are non-degenerate solutions of the YBE always bijective?

We will give a positive answer to this question in case the solution $(X, r)$ is irretractable, i.e. $\sigma_x = \sigma_y$ implies $x = y$ for all $x, y \in X$. Note that Example 4.1 is a retractable involutive solution. To our knowledge it is unknown whether there exist infinite involutive irretractable solutions that are left but not right non-degenerate. Note that irretractability has been defined with respect to the maps $\sigma_x$. One could equally well define retractability with respect to the maps $\gamma_x$. However, this makes no difference since any solution $r(x, y) = (\sigma_x(y), \gamma_y(x))$ has a dual solution $r'(y, x) = (\gamma_y(x), \sigma_x(y))$. Clearly $r$ (bijective) non-degenerate if and only if $r'$ is (bijective) non-degenerate.

To prove the result we will make use of the following result of Rump [29, Proposition 1]: Let $X$ be a non-empty set and let $r: X \times X \to X \times X$ be a map, with $r(x, y) = (\sigma_x(y), \gamma_y(x))$, such that $\gamma_y: X \to X$ is bijective for all $y \in X$. Then $(X, r)$ is a solution of the YBE if and only if the following conditions hold for all $x, y, z \in X$:
Let $h : (x : y) : (x : z) = (y : x) : (y : z)$, where $x \cdot y = \gamma_x^{-1}(y)$, $x : y = \sigma_{\gamma_y^{-1}(x)}(y)$. Furthermore, $r$ is a bijective solution if the map $X \to X$ defined by $z \mapsto x : z$ is bijective. The use of this result has been proposed by the referee to avoid the arboresque sub- and superscripts in the original proof.

We also will make use of a lemma that was proved by Lebed and Vendramin in [24] for finite non-degenerate bijective solutions.

**Lemma 4.4.** Let $(X, r)$ be a non-degenerate set-theoretic solution of the YBE. Let $h : X \rightarrow X$ be the map defined by $h(x) = \sigma_x^{-1}(x)$ for all $x \in X$. If $(X, r)$ is irretractable, then $h$ is bijective and $h^{-1}(x) = \gamma_x^{-1}(x)$ for all $x \in X$.

**Proof:** As commented above, we may assume that $(X, r)$ is a non-degenerate set-theoretic solution of the YBE such that $\gamma_x = \gamma_y$ implies that $x = y$. Thus conditions (R1), (R2), (R3) hold. Then, by (R1),

$$\gamma_x^{-1}(x \cdot z) = \gamma_x^{-1}(x \cdot z)$$

for all $x, z \in X$. Hence,

$$x \cdot x = x : x.$$

Now $x : x = \sigma_{x,x}(x)$ and thus $\sigma_{x,x}^{-1}(x \cdot x) = x$. This shows that the map $x \mapsto x \cdot x = \gamma_x^{-1}(x)$ is injective. For $x, y \in X$, put

$$\sigma_x^{-1}(y) = x \cdot y.$$

For $y = x \cdot x$, we have $x = \sigma_x(y) = \sigma_{\gamma_y^{-1}(\gamma_y(x))}(y) = \gamma_y(x) : y$. Hence, by (R1),

$$x \cdot (\gamma_y(x) \cdot z) = (\gamma_y(x) : y) \cdot (\gamma_y(x) \cdot z) = (y \cdot \gamma_y(x)) \cdot (y \cdot z) = x \cdot (y \cdot z).$$

Therefore $\gamma_y(x) \cdot z = y \cdot z$, which yields $\gamma_y(x) = y$. Hence

$$(21)\quad x = y \cdot y = (x \cdot x) \cdot (x \cdot x),$$

which shows that the map $x \mapsto x \cdot x = \gamma_x^{-1}(x)$ is bijective. Furthermore the inverse of this map is the map $x \mapsto x \cdot x = \sigma_x^{-1}(x) = h(x)$. 

**Theorem 4.5.** Let $(X, r)$ be an irretractable non-degenerate set-theoretic solution of the YBE. Then $r$ is bijective.

**Proof:** Again we may assume that $(X, r)$ is a non-degenerate set-theoretic solution of the YBE such that $\gamma_x = \gamma_y$ implies that $x = y$. From (R3) we get that

$$x : (z \cdot z) = (z \cdot \gamma_z(x)) : (z \cdot z) = (\gamma_z(x) : z) \cdot (\gamma_z(x) : z) = \sigma_x(z) \cdot \sigma_x(z).$$
From Lemma 4.4 we then get that (see equation (21))

$$x : z = \sigma(x(z * z)) \cdot \sigma(z * z).$$

Since $x * x = \sigma^{-1}(x) = h(x)$, we get from Lemma 4.4 that the map $z \mapsto x : z = \sigma x^{-1}(x)(z)$ is bijective. Hence, by Rump's earlier mentioned result, $r$ is bijective.

Note that if $(X, r)$ is an irretractable non-degenerate solution, then for every $x \in X$ there is a unique $y \in X$ such that $r(x, y) = (x, y)$ and there is a unique $z \in X$ such that $r(z, x) = (z, x)$. Because $(X, r)$ is left non-degenerate, to prove the former, it is sufficient to show that $\sigma_x(y) = x$ implies $\gamma_y(x) = y$. Now, because $(X, r)$ is a solution we obtain from (1) that $\sigma_x \sigma_y = \sigma_x \sigma_y \sigma_x = \sigma_x \sigma_y \sigma_x$ and thus $\sigma_y = \sigma_y(x)$. The irretractable assumption yields that $y = \gamma_y(x)$, as claimed. Similarly one proves the other claim. Hence there are at least $|X|!$ defining relations for the structure monoid. Furthermore, there are precisely $\binom{|X|}{2}$ defining relations if $r$ also is involutive and thus, in this case, $M(X, r)$ is a monoid with a presentation of the type $(x_1, \ldots, x_n | R)$, where $R$ is a set consisting of $\binom{n}{2}$ relations of the type $x_i x_j = x_k x_l$ with $(x_i, x_j) \neq (x_k, x_l)$ and every word $x_i x_j$ appears in at most one relation. Note that such a presentation has associated a map $r: X \times X \to X \times X$, where $X = \{x_1, \ldots, x_n\}$, $r^2 = id_{X^2}$, and $r(x_i, x_j) = (x_k, x_l)$ if and only if either $x_i x_j = x_k x_l$ is one of the relations in $R$ or $x_i x_j$ does not appear in any relation in $R$ and $(x_k, x_l) = (x_i, x_j)$ in this case. Monoids with this type of presentation and their algebras have a rich algebraic structure when $r$ is non-degenerate, even if $(X, r)$ is not a solution of the YBE. Such monoids arc said to be of quadratic type, and if $x_i x_i$ does not appear in any defining relation, then they are said to be of skew type. We refer the reader to [6, 15, 23, 22]. In [23] it has been shown that for such a monoid $r$ is a non-degenerate solution of the YBE if and only if the monoid is cancellative and $r$ is non-degenerate and satisfies the cyclic condition, i.e. if for every $x_1, y \in X$ there exist $x_2, y_1, y_2, z_1, z_2 \in X$ such that $x_1 y = y_1 z_1$ and $x_2 y_1 = y_2 z_2$ with $r(x_2, x_1) = (x_2, x_1)$ and $r(z_2, z_1) = (z_2, z_1)$. The latter monoids were first investigated by Gateva-Ivanova and Van den Bergh in [17].

5. The structure left semi-truss

Braces and skew braces were introduced to deal with bijective non-degenerate solutions $(X, r)$ of the YBE. In order to translate such solutions to associative structures the structure group $G(X, r)$ and the structure monoid $M(X, r)$ were introduced. The group $G(X, r)$ turns
out to be a skew brace, however a structure monoid does not fit in this context. Recently, Brzeziński introduced the algebraic notion of a semi-truss which is built on two semigroup structures on a given set. We show that structure monoids of left non-degenerate solutions of the YBE fit in this context: they turn out to be left semi-trusses with additive structure that is close to being a normal monoid. We then show that also the least left cancellative epimorphic image of \( M(X, r) \) inherits a left non-degenerate solution of the YBE that restricts to the original solution \( r \) for some interesting classes, in particular if \((X, r)\) is irretractable.

We first recall the definition of a left semi-truss.

**Definition 5.1 ([4])**. A left semi-truss is a quadruple \((A, +, \circ, \phi)\) such that \((A, +)\) and \((A, \circ)\) are non-empty semigroups and \(\phi: A \times A \to A\) is a function such that
\[
a \circ (b + c) = (a \circ b) + \phi(a, c)
\]
for all \(a, b, c \in A\).

**Example 5.2.** Let \((X, r)\) be a left non-degenerate set-theoretic solution of the YBE (not necessarily bijective). As stated in Section 3, and with the same notation, the map \(r'(x, y) = (y, \sigma_y \gamma_{\sigma_x^{-1}(y)}(x))\) defines the left derived solution on \(X\). Let \(M = M(X, r)\) and \(M' = A(X, r) = M(X, r')\) be the structure monoids of the solutions \((X, r)\) and \((X, r')\) respectively. From Corollary 3.9 and Proposition 3.1 we obtain a left action \(\lambda': (M, \circ) \to \text{Aut}(M', +)\) and a bijective 1-cocycle \(\pi: M \to M'\) with respect to \(\lambda'\) satisfying \(\lambda'(x)(y) = \sigma_x(y)\) and \(\pi(x) = x\) for all \(x, y \in X\). We identify \(M\) and \(M'\) via \(\pi\), that is, \(a = \pi(a)\) for all \(a \in M\). With this identification, we obtain the operation + on \(M\), and \(a \circ b = a + \lambda'_a(b)\) for all \(a, b \in M\). Put \(\phi(a, b) = \lambda'_a(b)\) for all \(a, b \in M\). Then
\[
a \circ (b + c) = a + \lambda'_a(b + c) = a + \lambda'_a(b) + \lambda'_a(c) = (a \circ b) + \phi(a, c).
\]
Furthermore \(M + a \subseteq a + M\) for all \(a \in M\). Hence \((M, +, \circ, \phi)\) is a left semi-truss. Note that, if \(r\) is furthermore bijective, then it can easily be verified that \((X, r')\) is a right non-degenerate solution and thus \(M + a = a + M\) for all \(a \in M\); that is, \((M, +)\) consists of normal elements. As shown in [19], this property is fundamental in the study of the associated structure algebra \(KM(X, r)\).

In the remainder of this section we show that if \((M, +, \circ, \phi)\) is a left semi-truss such that for every \(a, b \in M\) there exists a unique \(c(a, b) \in M\) such that \(a + b = b + c(a, b)\), then there exists a set-theoretical solution of the YBE on \(M\), say \((M, r')\). In the case that \(M = M(X, r)/\eta\), the least cancellative epimorphic image of \(M(X, r)\), it follows that \(r'\) is the (unique) extension of \(r\) to \(M\).
Lemma 5.3. Let \((A, +)\) be a non-empty semigroup such that, for each \((a, b) \in A \times A\) there exists a unique \(c(a, b) \in A\) such that
\[
a + b = b + c(a, b).
\]
Then \((A, r')\), where
\[
r'(a, b) = (b, c(a, b)),
\]
for all \(a, b \in A\), is a set-theoretic solution of the YBE.

Proof: Let \((a, b, d) \in A^3\). We have
\[
a + b + d = b + c(a, b) + d
\]
and also
\[
a + b + d = a + d + c(b, d)
\]
\[
= d + c(a, d) + c(b, d)
\]
\[
= d + c(b, d) + c(c(a, d), c(b, d))
\]
\[
= b + d + c(c(a, d), c(b, d)).
\]
Hence, by the uniqueness assumption,
\[
(22) \quad c(a, b + d) = c(c(a, b), d) = c(c(a, d), c(b, d)).
\]
Now we have
\[
r'_1r'_2r'_1(a, b, d) = r'_1r'_2(b, c(a, b), d) = r'_1(b, d, c(a, b), d)
\]
\[
= (d, c(b, d), c(c(a, b), d))
\]
and
\[
r'_2r'_1r'_2(a, b, d) = r'_2r'_1(a, d, c(b, d)) = r'_2(d, c(a, d), c(b, d))
\]
\[
= (d, c(b, d), c(c(a, d), c(b, d))).
\]
Therefore, by (22), \(r'_1r'_2r'_1 = r'_2r'_1r'_2\), and the result follows. \(\square\)

Proposition 5.4. Let \((A, +)\) and \((A, \circ)\) be non-empty semigroups. Let \(\lambda: (A, \circ) \to \text{Aut}(A, +)\) be a homomorphism such that \(a \circ b = a + \lambda_a(b)\) for all \(a, b \in A\), where \(\lambda(a) = \lambda_a\). In particular, \((A, +, \circ, \phi)\) is a left semi-truss with \(\phi(a, b) = \lambda_a(b)\) for all \(a, b \in A\). Suppose that for each \((a, b) \in A \times A\) there exists a unique \(c(a, b) \in A\) such that
\[
a + b = b + c(a, b).
\]
Then \((A, r)\), where
\[
r(a, b) = (\lambda_a(b), \lambda_a^{-1}(c(a, \lambda_a(b))))
\]
for all \(a, b \in A\), is a left non-degenerate set-theoretic solution of the YBE.
Proof: Let \( J: A^3 \to A^3 \) be the map defined by \( J(a, b, d) = (a, \lambda_a(b), \lambda_a \lambda_b(d)) \). Clearly \( J \) is bijective and \( J^{-1}(a, b, d) = (a, \lambda_a^{-1}(b), \lambda_a^{-1} \lambda_a^{-1} \lambda_a^{-1}(d)) \) for all \( a, b, d \in A \). We have

\[
J^{-1}r'_1J(a, b, d) = J^{-1}r'_1(a, \lambda_a(b), \lambda_a \lambda_b(d)) = J^{-1}(a, \lambda_a(b), \lambda_a \lambda_b(d)) = (\lambda_a(b), \lambda_{\lambda_a(b)}(c(a, \lambda_a(b))), \lambda_{\lambda_{\lambda_a(b)}(c(a, \lambda_a(b)))}^{-1} \lambda_a(b) \lambda_a \lambda_b(d)),
\]

where \( r' \) is defined as in Lemma 5.3. Since \( a \circ b = a + \lambda_a(b) = \lambda_a(b) + c(a, \lambda_a(b)) = \lambda_a(b) \circ \lambda_{\lambda_a(b)}^{-1}(c(a, \lambda_a(b))) \), it follows that \( J^{-1}r'_1J = r_1 \). Similarly

\[
J^{-1}r'_2J(a, b, d) = J^{-1}r'_2(a, \lambda_a(b), \lambda_a \lambda_b(d)) = J^{-1}(a, \lambda_a \lambda_b(d), c(\lambda_a(b), \lambda_a \lambda_b(d))) = (a, \lambda_b(d), \lambda_{\lambda_b(d)}^{-1}(c(\lambda_a(b), \lambda_a \lambda_b(d)))).
\]

Note that

\[
\lambda_a^{-1}(d) + \lambda_a^{-1}(c(b, d)) = \lambda_a^{-1}(d + c(b, d)) = \lambda_a^{-1}(b + d) = \lambda_a^{-1}(b) + \lambda_a^{-1}(d)
\]

for all \( a, b, d \in A \). Hence, by the uniqueness assumption, \( \lambda_a^{-1}(c(b, d)) = c(\lambda_a^{-1}(b), \lambda_a^{-1}(d)) \). Since each \( \lambda_a \) is bijective it follows that

\[
J^{-1}r'_2J(a, b, d) = (a, \lambda_b(d), \lambda_{\lambda_b(d)}^{-1} \lambda_a^{-1}(c(\lambda_a(b), \lambda_a \lambda_b(d)))) = (a, \lambda_b(d), \lambda_{\lambda_b(d)}^{-1}(c(b, \lambda_b(d))))).
\]

Thus \( J^{-1}r'_2J = r_2 \). By Lemma 5.3, \((A, r')\) is a set-theoretic solution of the YBE. Therefore also \((A, r)\) is a set-theoretic solution of the YBE, and the result follows. \(\square\)

Let \((X, r)\) be a left non-degenerate set-theoretic solution of the YBE. We will write \( r(x, y) = (\sigma_x(y), \gamma_y(x)) \) for all \( x, y \in X \). Thus the \( \sigma_x \) are bijective maps. The derived solution of \((X, r)\) is \((X, r')\), where

\[
r'(x, y) = (y, \sigma_y(\gamma_{\sigma_x^{-1}(y)}(x)))
\]

for all \( x, y \in X \). We will use the notation of Example 5.2. Thus we have \( M = M(X, r) \) and the left semi-truss \((M, +, \circ, \phi)\), where \( \phi(a, b) = \lambda'_a(b) \) for all \( a, b \in M \). Recall that \( \lambda': (M, \circ) \to \text{Aut}(M, +) \) is an homomorphism, that is, an action of \((M, \circ)\) on \((M, +)\), and \( \text{id}: M \to M \) is a bijective 1-cocycle with respect to \( \lambda' \) (because \( a \circ b = a + \lambda'_a(b) \)).
Let \( \eta \) be the left cancellative congruence on \((M, +)\), that is, \( \eta \) is the smallest congruence such that \( M = (M, +)/\eta \) is a left cancellative monoid.

We shall see a description of the elements in \( \eta \). Let

\[
\eta_0 = \{(a, b) \in M^2 \mid \exists c \in M \text{ such that } c + a = c + b\}.
\]

Note that \( \eta_0 \) is a reflexive and symmetric binary relation on \( M \). Let \( \eta_1 \) be its transitive closure, that is,

\[
\eta_1 = \{(a, b) \in M^2 \mid \exists a_1, \ldots, a_n \in M \text{ such that } (a, a_1), (a_1, a_2), \ldots, (a_n, b) \in \eta_0\}.
\]

Thus \( \eta_1 \) is an equivalence relation on \( M \). Let

\[
\eta_2 = \{(c + a, c + b) \in M^2 \mid c \in M \text{ such that } (a, b) \in \eta_1\} \\
\quad \cup \{(a, b) \in M^2 \mid \exists c \in M \text{ such that } (c + a, c + b) \in \eta_1\},
\]

and for every \( m \geq 1 \) we define

\[
\eta_{2m+1} = \{(a, b) \in M^2 \mid \exists a_1, \ldots, a_n \in M \text{ such that } (a, a_1), (a_1, a_2), \ldots, (a_n, b) \in \eta_{2m}\}
\]

and

\[
\eta_{2m+2} = \{(c + a, c + b) \in M^2 \mid c \in M \text{ such that } (a, b) \in \eta_{2m+1}\} \\
\quad \cup \{(a, b) \in M^2 \mid \exists c \in M \text{ such that } (c + a, c + b) \in \eta_{2m+1}\}.
\]

Note that \( \eta_n \subseteq \eta_{n+1} \subseteq \eta \) for all \( n \geq 0 \). Let \( \eta' = \cup_{n=0}^{\infty} \eta_n \).

**Lemma 5.5.** With the above notation we have \( \eta' = \eta \) and \( \lambda'_a = \lambda'_b \) for all \( (a, b) \in \eta \). Furthermore, for all \( z \in M \),

\[
\eta = \{(\lambda'_z(a), \lambda'_z(b)) \mid (a, b) \in \eta\} = \{(\lambda'_z)^{-1}(a), (\lambda'_z)^{-1}(b)) \mid (a, b) \in \eta\},
\]

and \( \eta \) also is a congruence on \((M, \circ)\).

**Proof:** First we shall prove that \( \eta' \) is a congruence on \((M, +)\). Clearly \( \eta' \) is reflexive and symmetric because so is each \( \eta_n \). Let \( a, b, c \in M \) such that \( (a, b), (b, c) \in \eta' \). There exists a positive integer \( m \) such that \( (a, b), (b, c) \in \eta_{2m+1} \). Since \( \eta_{2m+1} \) is the transitive closure of \( \eta_{2m} \), we have \( (a, c) \in \eta_{2m+1} \subseteq \eta' \). Hence \( \eta' \) is an equivalence relation. Note that every \( \eta_n \) satisfies \( (x + z, y + z) \in \eta_n \) for all \( (x, y) \in \eta_n \). Hence \( (a + c, b + c) \in \eta_{2m+1} \subseteq \eta' \). Since \( (a, b) \in \eta_{2m+1} \), we have \( (c + a, c + b) \in \eta_{2m+2} \subseteq \eta' \). Therefore \( \eta' \) is a congruence.
Let \( a, b, c \in M \) be elements such that \((c+a, c+b) \in \eta'\). There exists a positive integer \( t \) such that \((c+a, c+b) \in \eta_{2t+1}\). Thus \((a, b) \in \eta_{2t+2} \subseteq \eta'\). Hence \((M,+) / \eta'\) is a left cancellative monoid. Since \( \eta' \subseteq \eta \), we have \( \eta' = \eta \) by the definition of \( \eta \).

Let \((a, b) \in \eta_0\). Then there exists \( c \in M \) such that \( c+a = c+b \). Let \( z \in M \). We have

\[
(\lambda'_z)^\varepsilon(c) + (\lambda'_z)^\varepsilon(a) = (\lambda'_z)^\varepsilon(c+a) = (\lambda'_z)^\varepsilon(c) + (\lambda'_z)^\varepsilon(b),
\]

for \( \varepsilon = \pm 1 \). Therefore \( \eta_0 = \{(\lambda'_z(a), \lambda'_z(b)) \mid (a, b) \in \eta_0\} = \{((\lambda'_z)^{-1}(a), (\lambda'_z)^{-1}(b)) \mid (a, b) \in \eta_0\} \).

Let \((a, b) \in \eta_2\). Then, either there exist \( c, a', b' \in M \) such that \((a', b') \in \eta_1 \) and \((a, b) = (c+a', c+b')\), or there exists \( d \in M \) such that \((d+a, d+b) \in \eta_1\). In the first case, we have

\[
(((\lambda'_z)^\varepsilon(a), (\lambda'_z)^\varepsilon(b)) = (((\lambda'_z)^\varepsilon(c) + (\lambda'_z)^\varepsilon(a'), (\lambda'_z)^\varepsilon(c) + (\lambda'_z)^\varepsilon(b'))
\]

for \( \varepsilon = \pm 1 \). Since \( (((\lambda'_z)^\varepsilon(a'), (\lambda'_z)^\varepsilon(b')) \in \eta_1 \), we get that \( (((\lambda'_z)^\varepsilon(a), (\lambda'_z)^\varepsilon(b)) \in \eta_2 \), in this case. In the second case, since \((d+a, d+b) \in \eta_1 \), we have \( (((\lambda'_z)^\varepsilon(d) + (\lambda'_z)^\varepsilon(a), (\lambda'_z)^\varepsilon(d) + (\lambda'_z)^\varepsilon(b)) \in \eta_1 \). Thus also in this case we have \( (((\lambda'_z)^\varepsilon(a), (\lambda'_z)^\varepsilon(b)) \in \eta_2 \). Therefore

\[
\eta_2 = \{(\lambda'_z(a), \lambda'_z(b)) \mid (a, b) \in \eta_2\} = \{((\lambda'_z)^{-1}(a), (\lambda'_z)^{-1}(b)) \mid (a, b) \in \eta_2\}.
\]

Now it is easy to show by induction on \( n \) that

\[
\eta_n = \{(\lambda'_z(a), \lambda'_z(b)) \mid (a, b) \in \eta_n\} = \{((\lambda'_z)^{-1}(a), (\lambda'_z)^{-1}(b)) \mid (a, b) \in \eta_n\},
\]

for all non-negative integer \( n \). Hence

\[
\eta = \{(\lambda'_z(a), \lambda'_z(b)) \mid (a, b) \in \eta\} = \{((\lambda'_z)^{-1}(a), (\lambda'_z)^{-1}(b)) \mid (a, b) \in \eta\}.
\]

Let \((a, b) \in \eta_0\). Then there exists \( c \in M \) such that \( c+a = c+b \). Hence \( c \circ (\lambda'_z)^{-1}(a) = c+a = c+b = c \circ (\lambda'_z)^{-1}(b) \). Hence,

\[
\lambda'_c \lambda'_z (\lambda'_z)^{-1}(a) = \lambda'_c (\lambda'_z)^{-1}(a) = \lambda'_c (\lambda'_z)^{-1}(b) = \lambda'_c \lambda'_z (\lambda'_z)^{-1}(b)
\]

and thus

\[
\lambda'_z (\lambda'_z)^{-1}(a) = \lambda'_z (\lambda'_z)^{-1}(b).
\]

Since \( \eta_0 = \{(\lambda'_c(a), \lambda'_c(b)) \mid (a, b) \in \eta_0\} \), we have \( \lambda'_a = \lambda'_b \) for all \((a, b) \in \eta_0\). Because

\[
\eta_n = \{(\lambda'_z(a), \lambda'_z(b)) \mid (a, b) \in \eta_n\},
\]

for all non-negative integers \( n \), it is easy to prove, by induction on \( n \), that \( \lambda'_a = \lambda'_b \) for all \((a, b) \in \eta_n\). Hence \( \lambda'_a = \lambda'_b \) for all \((a, b) \in \eta\).
Let \((a, b) \in \eta\). Then \((\lambda'_c(a), \lambda'_c(b)) \in \eta\). Thus \((c \circ a, c \circ b) = (c + \lambda'_c(a), c + \lambda'_c(b)) \in \eta\). Since \(\lambda'_a = \lambda'_b\), we have
\[
(a \circ c, b \circ c) = (a + \lambda'_c(a), b + \lambda'_c(b)) = (a + \lambda'_a(c), b + \lambda'_a(c)) \in \eta.
\]
Hence \(\eta\) is a congruence on \((M, \circ)\), and the result follows. \(\square\)

With the assumptions and notations as in Example 5.2, \(\overline{M} = M/\eta\). Let \(M \to \overline{M}: a \mapsto \overline{a}\) be the natural projection. Let \(\overline{\lambda}: (\overline{M}, \circ) \to \text{Aut}(\overline{M}, +)\) be the map defined by \(\overline{\lambda}(\overline{a}) = \overline{\lambda_c(a)}\) and \(\overline{\lambda}(\overline{b}) = \overline{\lambda'_a(b)}\) for all \(a, b \in M\).

Note that \(\overline{\lambda}\) is well-defined, because if \(\overline{c} = \overline{a}\) and \(\overline{d} = \overline{b}\), then, by Lemma 5.5, \(\overline{\lambda}(\overline{a}b) = \overline{\lambda}(\overline{d})\) and \(\overline{\lambda}(\overline{c}) = \overline{\lambda'_c}\), and
\[
\overline{\lambda}(\overline{a}b) = \overline{\lambda}(\overline{d}) = \overline{\lambda'_c}(\overline{d}).
\]

Now it is easy to check that \(\overline{\lambda}\) is a homomorphism such that \(\overline{\lambda}(a) \circ \overline{\lambda}(b) = \overline{\lambda}(a) + \overline{\lambda}(b)\) for all \(a, b \in M\).

Remark 5.6. If, furthermore, the left non-degenerate set-theoretic solution \((X, r)\) is finite and bijective then one can say more. To do so, it is convenient to keep the notation \(M = M(X, r)\) and \(A = A(X, r)\). So \(M \subseteq A \rtimes \text{Im} \lambda'\). Jespers, Kubat, and Van Antwerpen ([19, Proposition 2.9]) proved that there exists \(t \geq 1\) and a central element \((z, 1) \in M\), with \(z \in Z(A)\) and \(g(z) = z\) for all \(g \in \text{Aut}(\lambda')\), such that the least cancellative congruence on \((A, +)\) is
\[
\eta = \{(a, b) \in A \times A \mid a + z + \cdots + z = b + z + \cdots + z, \text{ for all } i \geq t\}
\]
\[
= \{(a, b) \in A \times A \mid c + a = c + b \text{ for some } c \in A\}
= \eta_0.
\]

Note that \((a, b) \in \eta\) implies that \(\lambda'_a = \lambda'_b\). Hence, it follows from Proposition 4.2 in [19] that the (least) cancellative congruence on \((M, \circ)\) is
\[
\eta_M = \{((a, \lambda'_a), (b, \lambda'_b)) \mid (a, b) \in \eta\}.
\]

It follows that the natural map
\[
M/\eta_M \longrightarrow (A/\eta) \rtimes \text{Im}(\lambda'),
\]
i.e. \((a, \lambda'_a) \mapsto (\overline{a}, \lambda'_a)\), is an injective monoid homomorphism and \(M/\eta_M\) is a regular submonoid of \((A/\eta) \rtimes \text{Im}(\lambda')\). So we obtain a bijective 1-cocycle \((M/\eta_M, \circ) \to (A/\eta, +)\), with respect to \(\overline{\lambda}\), that extends the mapping \((a, \lambda'_a) \mapsto \overline{a}\). Because \(r\) is bijective we know (see explanation in Example 5.2) that \((A, +)\) consists of normal elements and thus \((A/\eta, +)\) is a left and right Ore monoid and also \((M/\eta_M, \circ)\) is a left and right Ore
monoid. Hence they both have a group of fractions, denoted $\text{gr}(A/\eta)$ and $\text{gr}(M/\eta M)$ respectively. It is easily verified that $\text{gr}(M/\eta M) = G(X, r)$, the structure group of $(X, r)$, $\text{gr}(A/\eta) = G(X, r')$, the structure group of the derived solution $(X, r')$, and $\text{gr}(M/\eta M) \subseteq \text{gr}(A/\eta) \rtimes \text{Im}(\lambda')$ where, by abuse of notation, $\lambda': \text{gr}(M/\eta M) \to \text{Aut}(A/\eta)$ is the natural extension of the mapping $\lambda$ and also $\text{gr}(M/\eta M)$ is a regular subgroup of $\text{gr}(A/\eta) \rtimes \text{Im}(\lambda')$. The latter was proven by Lebed and Vendramin in [24, Theorem 3.4.] in case $(X, r)$ is bijective, (left and right) non-degenerate, and finite.

**Question 5.7.** If $(X, r)$ is a left non-degenerate solution of the YBE, does there exist a bijective 1-cocycle $(M/\eta M, \circ) \to (A/\eta, +)$, with respect to $\lambda$, that extends the mapping $(a, \lambda'_a) \mapsto \lambda a$? In other words, can one avoid the bijective assumption in Remark 5.6?

Let $\overline{\phi}: M \times M \to M$ be the map defined by $\overline{\phi}(a, b) = \lambda^{-1}(b)$ for all $a, b \in M$. Then $(M, +, \circ, \overline{\phi})$ is a left semi-truss.

**Lemma 5.8.** Let $a, b \in M = M(X, r)$. Then there exists $c \in M$ such that $a + b = b + c$.

**Proof:** There exist non-negative integers $n, m$, and $x_1, \ldots, x_n, y_1, \ldots, y_m \in X$ such that $a = x_1 + \cdots + x_n$ and $b = y_1 + \cdots + y_m$. Clearly we may assume that $n, m$ are positive integers. We shall prove the result by induction on $n + m$. If $n = m = 1$, then $x_1 + y_1 = y_1 + \sigma y_1 (\gamma_{x_1}^{-1}(y_1))(x_1))$, by the defining relations of $(M, +)$. Suppose that $m + n > 2$, and that the result is true for $m + n - 1$. If $n > 1$, then by the induction hypothesis there exists $c' \in M$ such that $a + b = x_1 + b + c'$, and by the induction hypothesis again there exists $c'' \in M$ such that $x_1 + b + c' = x_1 + b + c''$. Hence $a + b = b + c'' + c', in this case. Suppose that $n = 1$. In this case $m > 1$ and

$$a + b = x_1 + b = y_1 + \sigma y_1 (\gamma_{x_1}^{-1}(y_1))(x_1)) + y_2 + \cdots + y_m.$$ 

Hence, by the induction hypothesis, there exists $c \in M$ such that

$$\sigma y_1 (\gamma_{x_1}^{-1}(y_1))(x_1)) + y_2 + \cdots + y_m = y_2 + \cdots + y_m + c.$$ 

Thus $a + b = b + c$ in this case. Therefore the result follows by induction.

By Lemma 5.8, the left cancellative monoid $(M, +)$ satisfies that, for all $\overline{a}, \overline{b} \in M$, there exists a unique $\overline{c} \in M$ such that $\overline{a} + \overline{b} = \overline{b} + \overline{c}$. So, the multiplicative monoid $(M, \circ)$ is left cancellative. Hence, we have the following corollary.
Corollary 5.9. Let \((X, r)\) be a left non-degenerate set-theoretic solution of the YBE. Let \(\eta\) be the left cancellative congruence on \((M(X, r'), +)\). Then \((\overline{M}, +, \circ, \overline{\phi})\) is a left semi-truss with \(\overline{M} + \overline{a} \subseteq \overline{a} + \overline{M}\) for all \(\overline{a} \in \overline{M}\) and it satisfies the conditions of Proposition 5.4, with \(\overline{\phi}({\overline{a}}, {\overline{b}}) = \overline{\lambda\pi(b)}\), for all \(\overline{a}, \overline{b} \in \overline{M}\). In particular, \((\overline{M}, \overline{r})\), where \(\overline{r}(\overline{a}, \overline{b}) = (\overline{\lambda\pi(a)}, \overline{\lambda\pi^{-1}(b)}(c(\overline{a}, \overline{\lambda\pi(b)})))\), for all \(\overline{a}, \overline{b} \in \overline{M}\), is a left non-degenerate set-theoretic solution of the YBE. In particular, \((\overline{X}, \overline{r}|_{\overline{X}})\) is a left non-degenerate solution on the image \(\overline{X}\) of \(X\) in \(\overline{M}\).

We say that a left non-degenerate solution \((X, r)\) of the YBE is injective if the natural map \(X \to M/\eta\) is injective. Obvious such examples are irretractable solutions, and in this case \(r = \overline{r}|_{\overline{X}}\). Note that if \(r\) is also bijective and non-degenerate, then this notion corresponds with the one introduced by Lebed and Vendramin in [24]. In [24] it is also shown that, in this case, several properties of involutive solutions can be generalized to injective ones.

Corollary 5.10. Any left non-degenerate injective set-theoretic solution \((X, r)\) of the YBE is the restriction of the induced left-non-degenerate solution of the YBE determined by a left cancellative semi-truss \((M, +, \circ, \phi)\) with \(M + a \subseteq a + M\) for all \(a \in M\).

However, note that \((\overline{M}, \circ)\) is not necessarily the structure monoid of the solution of \((\overline{X}, \overline{r})\). Indeed, let \(X = \text{Sym}_3\) be the symmetric group of degree 3. Let \((X, r)\) be the bijective non-degenerate solution defined by \(r(a, b) = (aba^{-1}, a)\) for all \(a, b \in X\). Note that the solution \((X, r)\) is non-involutive and irretractable (because the center of \(\text{Sym}_3\) is trivial). So, \(X\) is naturally embedded in \((\overline{M}, \circ) = (M(X, r)/\eta, \circ)\) and \(\overline{r}|_{\overline{X}} = r\). Let us denote the multiplication in the structure monoid \(M(X, r)\) by \(\cdot\). In \((M(X, r), \cdot)\) we have
\[
(1, 2) \cdot (1, 2, 3) \cdot (1, 2, 3) \cdot (1, 2, 3) = (1, 3, 2) \cdot (1, 3, 2) \cdot (1, 3, 2) \cdot (1, 2)
= (1, 3, 2) \cdot (1, 3, 2) \cdot (1, 3, 2) \cdot (1, 3, 2)
= (1, 3, 2) \cdot (1, 3, 2) \cdot (1, 3, 2) \cdot (1, 3, 2)
= (1, 2) \cdot (1, 3, 2) \cdot (1, 3, 2) \cdot (1, 3, 2)
\]
while \((1, 2, 3) \cdot (1, 2, 3) \cdot (1, 2, 3) \cdot (1, 2, 3) \neq (1, 3, 2) \cdot (1, 3, 2) \cdot (1, 3, 2) \cdot (1, 3, 2)\). Hence, \(M(X, r)\) is not left cancellative, while \(\overline{M}\) is left cancellative. Thus \(\overline{M}\) is not the structure monoid of \((X, r)\).

The following problem remains a challenge.
**Question 5.11.** Determine when a left non-degenerate solution \((X, r)\) of the YBE is cancellative injective. If \((X, r)\) is a left non-degenerate solution that is injective, then does there exist a finite left cancellative semi-truss in which \(X\) can be embedded naturally? In case \(r\) also is finite, bijective, and non-degenerate this has been proven by Lebed and Vendramin in [24].

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