

EVALUATION OF POLYNOMIALS OVER FINITE RINGS VIA ADDITIVE COMBINATORICS

GYULA KÁROLYI AND CSABA SZABÓ

Abstract: We give an improved polynomial bound on the complexity of the equation solvability problem, or more generally, of finding the value sets of polynomials over finite nilpotent rings. Our proof depends on a result in additive combinatorics, which may be of independent interest.

2010 Mathematics Subject Classification: Primary: 16Z05; Secondary: 11B75, 13M10, 16P10.

Key words: additive combinatorics, Chevalley’s theorem, dichotomy, nilpotent rings, Olson’s theorem, polynomial method, equation solvability problem.

1. Introduction

Several ‘classical’ algebraic problems are investigated from the computational perspective. The equivalence problem and the equation solvability problem for various algebraic structures have received increasing attention recently. Both problems originate from the theory of rings and fields. The equivalence problem over a finite ring asks whether or not two polynomials define the same function over the given ring. The equation solvability problem is basically looking for the existence of a root or for the solution of an equation. It asks whether or not two polynomials over a ring have at least one substitution, where they attain the same value. Both of these problems are decidable for finite rings: it is enough to substitute every element of the ring into the variables.

Early investigations into the computational complexity of the equivalence problem were carried out by computer scientists at Syracuse University. In the early 1990s it was shown by Hunt and Stearns [11] that the equivalence problem of a finite commutative ring either can be decided in polynomial time or has co-NP-complete complexity. Later Burris and Lawrence proved in [3] that the same holds for rings in general: the problem can be decided in polynomial time if the ring is nilpotent (this part is already found in [11]), and it is co-NP-complete otherwise.

The solvability problem has a somewhat shorter history. Although the borderline is again nilpotency, the proofs are more complicated. Following the idea of Burris and Lawrence it is not hard to verify that the solvability problem for non-nilpotent rings is NP-complete. Horváth ([7]) has proved that for nilpotent rings the problem can be decided in polynomial time. More precisely, let R be a nilpotent ring of size m and nilpotency class ℓ , and let f be a polynomial over R in the (non-commuting) variables x_1, \dots, x_n . In [7] it is shown that $O(\|f\|^t)$ substitutions suffice to decide, whether f has a root or not. Here $\|f\|$ represents the number of operations used to present f . The exponent $t = t(m, \ell)$, obtained in terms of the Ramsey number for a particular coloured hypergraph, is rather enormous, though.

In the present paper, by applying a recent result in additive combinatorics the exponent t is reduced to $m \log m$; see Theorem 4.3 for a more precise formulation. In particular, it is proved that the range of a polynomial can be found using $O(n^{m(\ell-1)})$ many substitutions.

Note that the equivalence and solvability problems for finite groups have proved to be far more challenging. For results and detailed references see for example [9, 10] and the most recent paper of Horváth [8], where most of the existing results for equivalence and equation solvability are brought under a unified theory. For results about these problems on finite monoids and semigroups we refer to [1, 12, 15, 16]; concerning general algebras see e.g. [6].

2. Terminology

Polynomials. Having a diverse readership in mind we introduce our notions in a somewhat informal, albeit precise manner. Let R be an arbitrary ring, and x_1, \dots, x_n symbols we think of as non-commuting variables. By a polynomial in these variables over R we mean an expression that can be constructed in a finite number of steps according to the following rules: (i) every element of $\{x_1, \dots, x_n\} \cup R$ is a polynomial; (ii) if f is a polynomial, then $-f$ is also a polynomial; (iii) if f, g are polynomials, then $f + g$ is also a polynomial; and (iv) if f, g are polynomials, then fg is also a polynomial. The set of all such polynomials we denote by $R[x_1, \dots, x_n]$.

Evaluating a polynomial $f \in R[x_1, \dots, x_n]$ at the point $\mathbf{c} = (c_1, \dots, c_n) \in R^n$, or substituting \mathbf{c} in f , means replacing each occurrence of the variable x_i in f by the corresponding ring element c_i , for every $1 \leq i \leq n$; the result is an element of R denoted by $f(\mathbf{c})$. This way f defines a function from R^n to R ; two polynomials are equivalent if they

define the same function. For example, for $a, b \in R$ the polynomials $(ax)b, a(xb), -(ax)(-b) \in R[x]$ are equivalent, but may not be equivalent to $(ab)x$. One may even identify the first three, but this is irrelevant to our purposes. The polynomials $x_1(x_2x_1)$ and x_1x_2 are also equivalent over $\mathbb{Z}/2\mathbb{Z}$.

Each polynomial is equivalent to a ‘standard’ polynomial, that is, to one that can be written as a sum of monomials. More precisely, a monomial is a polynomial that can be constructed using only the rules (i), (ii), and (iv), where (ii) is only applied in the last step, if at all. Suppressing the brackets, each monomial can be represented in a canonical form as $z_1 \cdots z_t$ or $-z_1 \cdots z_t$, where each $z_i \in \{x_1, \dots, x_n\} \cup R$. If two monomials have the same canonical form, then they are clearly equivalent. A standard polynomial is one obtained from monomials using only the rule (iii). Once again, up to equivalence it does not matter in which order the additions are executed.

The computational model. In our model R denotes a fixed finite ring, presented by its addition and multiplication tables. Thus, its cardinality is $|R| = O(1)$. We may assume that the table of additive inverses and the zero element are also given, for they can be computed from the addition table in $O(1)$ time. In accordance with rules (i)–(iv), the length $\|f\|$ of a polynomial $f \in R[x_1, \dots, x_n]$ is defined as follows: $\|f\| = 0$ if $f \in \{x_1, \dots, x_n\} \cup R$, and if $\|f\|, \|g\|$ are defined, then set $\| - f \| = \|f\| + 1$ and $\|f + g\| = \|fg\| = \|f\| + \|g\| + 1$. Thus, $\|f\|$ is the number of operations used to define f , which may be regarded as the complexity of evaluating f at any single point of R^n . Our goal is to measure the complexity of the problems mentioned in the introduction in terms of $\|f\|$.

Nilpotency. In what follows, $R^{(i)}$ will denote the ideal consisting of all finite sums of terms of the form $c_1 \cdots c_i$, where $c_1, \dots, c_i \in R$. The ring R is nilpotent if there is a positive integer i such that $R^{(i)} = 0$; the smallest such i is called the nilpotency class of R . The standard example for a ring of nilpotency class ℓ is the ring of strictly upper triangular $\ell \times \ell$ matrices over an arbitrary field.

3. A result in additive combinatorics

A standard application of the pigeonhole principle gives that for any sequence g_1, g_2, \dots, g_n of a finite group G there is a subsequence $g_{i_1}, g_{i_2}, \dots, g_{i_t}$ with $t < |G|$ such that $g_{i_1}g_{i_2} \cdots g_{i_t} = g_1g_2 \cdots g_n$. Here we need a more general result that we can establish for finite abelian p -groups.

Let p denote a prime and let G be an arbitrary finite abelian p -group, written additively. Then there exist unique positive integers r and $\alpha_1 \leq \dots \leq \alpha_r$ such that $G \cong \mathbb{Z}/p^{\alpha_1}\mathbb{Z} \oplus \dots \oplus \mathbb{Z}/p^{\alpha_r}\mathbb{Z}$. For a finite set H and a non-negative integer k we denote by $P_k(H)$ the set of all subsets X of H with $|X| \leq k$. Given a function $\varphi: P_k(H) \rightarrow G$, associate with each subset U of H the value

$$\bar{\varphi}(U) = \sum_{X \subseteq U, |X| \leq k} \varphi(X).$$

Thus, $\bar{\varphi}(H)$ is simply the total sum of the values of φ over $P_k(H)$.

Theorem 3.1. *Let H be a finite set, k a non-negative integer. For any function $\varphi: P_k(H) \rightarrow \mathbb{Z}/p^{\alpha_1}\mathbb{Z} \oplus \dots \oplus \mathbb{Z}/p^{\alpha_r}\mathbb{Z}$, there is a set $U \subseteq H$ such that*

$$|U| \leq k \sum_{j=1}^r (p^{\alpha_j} - 1)$$

and $\bar{\varphi}(U) = \bar{\varphi}(H)$.

Remark 3.2. It is easily seen that the case $k = 1$ of the above theorem is equivalent to Olson's classical result [13] on the Davenport constant of finite abelian p -groups.

Remark 3.3. The bound on the cardinality of U cannot be improved upon for any k . Assume that $|H| \geq k \sum_{j=1}^r (p^{\alpha_j} - 1)$. Select pairwise disjoint k -element subsets $H_{j,l}$ of H for $1 \leq j \leq r$, $1 \leq l < p^{\alpha_j}$. Denote by e_j any generating element of the j th direct summand $\mathbb{Z}/p^{\alpha_j}\mathbb{Z}$, and consider the function φ that assigns e_j to each set $H_{j,l}$ and 0 to any other element of $P_k(H)$. Then $\bar{\varphi}(H) = \sum_{j=1}^r (p^{\alpha_j} - 1)e_j$, and it is obvious that $\bar{\varphi}(U) = \bar{\varphi}(H)$ implies $U \supseteq \cup H_{j,l}$.

Our proof depends on the following result of Brink [2], which can be viewed as a generalization of Chevalley's well-known theorem [4] and its somewhat forgotten extension by Schanuel [14].

Theorem 3.4. *Let p denote a prime and let A_1, \dots, A_n be non-empty subsets of \mathbb{Z} such that the natural homomorphism from \mathbb{Z} to $\mathbb{Z}/p\mathbb{Z}$ restricted to A_i is injective for each $1 \leq i \leq n$. Assume that the polynomials $f_1, \dots, f_r \in \mathbb{Z}[x_1, \dots, x_n]$ satisfy*

$$\sum_{i=1}^n (|A_i| - 1) > \sum_{j=1}^r (p^{\alpha_j} - 1) \deg f_j.$$

If the set $\{\mathbf{a} \in A_1 \times \dots \times A_n \mid f_j(\mathbf{a}) \equiv 0 \pmod{p^{\alpha_j}}, 1 \leq j \leq r\}$ is not empty, then it has at least two different elements.

Proof of Theorem 3.1: Putting $|H| = n$, enumerate the elements of H as h_1, \dots, h_n . We may assume that $n \geq k \sum_{j=1}^r (p^{\alpha_j} - 1) + 1$, for the statement is valid with $U = H$ otherwise. For $1 \leq j \leq r$ denote by $\varphi_j: P_k(H) \rightarrow \mathbb{Z}/p^{\alpha_j}\mathbb{Z}$ the j th coordinate function of φ , and consider the polynomial $f_j \in (\mathbb{Z}/p^{\alpha_j}\mathbb{Z})[x_1, \dots, x_n]$ defined by

$$f_j(\mathbf{x}) = \sum_{\nu=0}^k \left(\sum_{1 \leq i_1 < \dots < i_\nu \leq n} \varphi_j(\{h_{i_1}, \dots, h_{i_\nu}\}) \prod_{\alpha=1}^{\nu} x_{i_\alpha} \right).$$

Thus, letting $f = (f_1, \dots, f_r)$ we have $f(\mathbf{0}) = \varphi(\emptyset) = \overline{\varphi}(\emptyset)$, $f(\mathbf{1}) = \overline{\varphi}(H)$, and

$$f(\mathbf{x}) = \overline{\varphi}(U) \quad \text{with} \quad U = U(\mathbf{x}) = \{h_i \mid x_i = 1\}$$

for $\mathbf{x} \in \{0, 1\}^n$ in general.

Define sequences $I_s \subseteq \{1, \dots, n\}$ and $\mathbf{a}(s) \in \{0, 1\}^n$ recursively as follows. Put $I_0 = \{1, \dots, n\}$ and $\mathbf{a}(0) = \mathbf{1}$, then

$$I_0 = \{i \mid \mathbf{a}(0)_i = 1\} \quad \text{and} \quad f(\mathbf{a}(0)) = \overline{\varphi}(H).$$

Assume that I_s and $\mathbf{a}(s)$ are already defined such that

$$I_s = \{i \mid \mathbf{a}(s)_i = 1\} \quad \text{and} \quad f(\mathbf{a}(s)) = \overline{\varphi}(H).$$

If $|I_s| > k \sum_{j=1}^r (p^{\alpha_j} - 1)$, then put $A_i = \{0, 1\}$ for $i \in I_s$ and $A_i = \{0\}$ for $i \in I_0 \setminus I_s$. Note that $\mathbf{a}(s) \in A_1 \times \dots \times A_n$. For each $1 \leq j \leq r$ the polynomial $g_j = f_j - \overline{\varphi}_j(H) \in (\mathbb{Z}/p^{\alpha_j}\mathbb{Z})[x_1, \dots, x_n]$ vanishes at $\mathbf{a}(s)$ and

$$\sum_{i=1}^n (|A_i| - 1) = |I_s| > k \sum_{j=1}^r (p^{\alpha_j} - 1) \geq \sum_{j=1}^r (p^{\alpha_j} - 1) \deg g_j.$$

It follows from Theorem 3.4 that there exists an element

$$\mathbf{a} \in (A_1 \times \dots \times A_n) \setminus \{\mathbf{a}(s)\}$$

such that $g_i(\mathbf{a}) = 0$ in $\mathbb{Z}/p^{\alpha_i}\mathbb{Z}$ for every $i = 1, 2, \dots, r$, and accordingly $f(\mathbf{a}) = \overline{\varphi}(H)$. Let

$$\mathbf{a}(s+1) = \mathbf{a}, \quad I_{s+1} = \{i \mid \mathbf{a}(s+1)_i = 1\};$$

then I_{s+1} is a proper subset of I_s .

This process must terminate at some $t \geq 1$ with $|I_t| \leq k \sum_{j=1}^r (p^{\alpha_j} - 1)$, meaning that the statement of the theorem is valid with

$$U = \{h_i \mid i \in I_t\} = U(\mathbf{a}(t)). \quad \square$$

4. Nilpotent rings

All the results claimed in Section 1 can be easily reduced to the following general statement.

Proposition 4.1. *Let R be an arbitrary finite ring and $g \in R[x_1, \dots, x_n]$ a standard polynomial, written as a sum of monomials. If each such monomial contains at most k different variables, then $O(n^{k|R|})$ evaluations suffice to determine the range of g .*

Proof: Assume first that the cardinality of R is a power of a prime p . That is, the additive group of R is a finite abelian p -group:

$$R^+ \cong \mathbb{Z}/p^{\alpha_1}\mathbb{Z} \oplus \dots \oplus \mathbb{Z}/p^{\alpha_r}\mathbb{Z}.$$

Suppose that g admits the value v : there are elements $c_1, \dots, c_n \in R$ such that $g(c_1, \dots, c_n) = v$. For each subset X of $H = \{1, 2, \dots, n\}$ with $|X| \leq k$, denote by g_X the sum of those monomial terms of g which contain exactly the variables x_i with $i \in X$. Writing $\varphi(X) = g_X(c_1, \dots, c_n)$ we have $v = \sum_{X \in P_k(H)} \varphi(X) = \bar{\varphi}(H)$. Theorem 3.1 guarantees the existence of a subset U of $\{1, 2, \dots, n\}$ with

$$|U| \leq k \sum_{j=1}^r (p^{\alpha_j} - 1) \leq k|R|$$

and $\bar{\varphi}(U) = \bar{\varphi}(H) = v$. Putting $c'_i = c_i$ for $i \in U$ and $c'_i = 0$ for $i \notin U$ we have

$$g_X(c'_1, \dots, c'_n) = \begin{cases} g_X(c_1, \dots, c_n) & \text{if } X \subseteq U, \\ 0 & \text{otherwise.} \end{cases}$$

It follows that

$$\begin{aligned} g(c'_1, \dots, c'_n) &= \sum_{X \in P_k(H)} g_X(c'_1, \dots, c'_n) = \sum_{X \subseteq U} g_X(c_1, \dots, c_n) \\ &= \sum_{X \subseteq U} \varphi(X) = \bar{\varphi}(U) = \bar{\varphi}(H) = v. \end{aligned}$$

Consequently, to determine the range of g it suffices to substitute into g only those n -tuples (c_1, \dots, c_n) in which all but at most $k|R|$ elements are equal to 0. The number of such n -tuples is

$$\sum_{i=0}^{k|R|} \binom{n}{i} |R|^i \leq (k|R| + 1) |R|^{k|R|} n^{k|R|} = O(n^{k|R|}),$$

where the implied constant only depends on k and R .

Turning to the general case, assume that $R^+ = G_1 \oplus \cdots \oplus G_s$, where G_i is an abelian group of order $p_i^{\beta_i}$, the primes p_1, \dots, p_s being pairwise different. Each set defined by

$$R_i = \{r \in R \mid p_i^{\beta_i} r = 0\}$$

is an ideal in R whose additive group must be identical to G_i . It follows that $R = R_1 \oplus \cdots \oplus R_s$, and there are natural homomorphisms $\pi_i: R \rightarrow R_i$ such that $\pi_1 + \cdots + \pi_s$ is the identical map from R to R . Thus, if the n -tuple $\mathbf{c} \in R^n$ is written as $\mathbf{c} = \mathbf{c}_1 + \cdots + \mathbf{c}_s$ with $\mathbf{c}_i \in (R_i)^n$, then $g(\mathbf{c}) = g_1(\mathbf{c}_1) + \cdots + g_s(\mathbf{c}_s)$, where g_i is the polynomial over R_i obtained from g by replacing each coefficient in g by its image under π_i . Accordingly, the value set of g is the Minkowski sum of the value sets of the polynomials g_i , and in view of the first part of the proof it can be found using

$$O\left(\prod_{i=1}^s n^{k|R_i|}\right) = O\left(n^{k \sum_{i=1}^s |R_i|}\right) = O(n^{k|R|})$$

substitutions. □

Remark 4.2. As part of the preprocessing, one can compute in $O(1)$ time the factorization $|R| = p_1^{\beta_1} \cdots p_s^{\beta_s}$, and then the ideals R_i in the direct decomposition of R . To compute the range of g it is then enough to evaluate g on the elements of the small set

$$S = \{\mathbf{c} \in R^n : |\{j : (\mathbf{c}_i)_j \neq 0\}| \leq k p_i^{\beta_i} \text{ for every } 1 \leq i \leq s\}.$$

According to the above proof, $|S| = O(n^{k|R|})$.

Theorem 4.3. *Let R be a nilpotent ring of size m . For an n -variable polynomial $f \in R[x_1, x_2, \dots, x_n]$, it can be decided in $O(\|f\|n^{m \log m})$ time whether or not f has a root in R .*

Proof: Denote by ℓ the nilpotency class of R . Thus, $R^{(\ell)} = 0$. Therefore f is equivalent to a standard polynomial $\bar{f} \in R[x_1, x_2, \dots, x_n]$ in which every monomial term contains at most $\ell - 1$ different variables. By Proposition 4.1, the range of f , which is the same as the range of \bar{f} , can be found by evaluating f on a subset S of R^n , of size $O(n^{(\ell-1)m})$; see Remark 4.2 for an explicit description. In particular, one can check whether f admits the value 0, and it can be also decided if f is identically zero or not. Note that the chain of ideals $R \supset R^{(2)} \supset \cdots \supset R^{(\ell)} = 0$ is strictly decreasing. As $|R^{(i)}/R^{(i+1)}| \geq 2$, it follows that $\ell - 1 \leq \log_2 m$. □

Remark 4.4. During the preparation of this article Földvári ([5]) proved a similar result with an entirely different method. His bound

is $O(\|f\|^{m^2 \log m \log^5 m})$. Note that one may assume that $\|f\| \geq n - 1$, otherwise the set of variables that occur in f can be restricted to a proper subset of $\{x_1, \dots, x_n\}$. If this is the case, then Theorem 4.3 implies the bound $O(\|f\|^{m \log m})$.

Acknowledgements. The first author was supported by the Bolyai Research Fellowship and by National Research, Development, and Innovation Office NKFIH Grant K 120154. The research of the second author was supported by NKFIH Grant K 109185.

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Gyula Károlyi

Alfréd Rényi Institute of Mathematics, 13–15 Reáltanoda utca, 1053 Budapest, Hungary and Institute of Mathematics, Eötvös University, 1/C Pázmány Péter sétány, 1117 Budapest, Hungary

E-mail address: karolyi.gyula@renyi.hu

Csaba Szabó

Institute of Mathematics, Eötvös University, 1/C Pázmány Péter sétány, 1117 Budapest, Hungary

E-mail address: szabo.csaba.mathdid@ttk.elte.hu

Received on April 14, 2020.

Accepted on September 4, 2020.