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A SIMPLE PROOF OF THE OPTIMAL POWER IN LIOUVILLE THEOREMS

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Abstract: Consider the equation $\operatorname{div}(\varphi^2\nabla\sigma)=0$ in $\mathbb{R}^N$, where $\varphi>0$. It is well known [**4, 2**] that if there exists $C>0$ such that $\int_{B_R}(\varphi\sigma)^2\,dx\leq CR^2$ for every $R\geq 1$, then σ is necessarily constant. In this paper we present a simple proof that this result is not true if we replace R^2 with R^k for $k>2$ in any dimension N. This question is related to a conjecture by De Giorgi [**7**].

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1. Introduction and main results

In 1978, E. De Giorgi ([**7**]) stated the following conjecture:

Conjecture. *Let $u\colon\mathbb{R}^N\to(-1,1)$ be a smooth entire solution of the Allen–Cahn equation $-\Delta u=u-u^3$ which is monotone in one direction (for instance $\partial u/\partial_{x_N}>0$ in $\mathbb{R}^N$). Then u depends only on one variable (equivalently, all its level sets are hyperplanes), at least if $N\leq 8$.*

This conjecture was established in 1998 for $N=2$ by Ghoussoub and Gui [**10**], and in 2000 for $N=3$ by Ambrosio and Cabré [**2**]. For dimensions $4\leq N\leq 8$ the conjecture was proved by Savin [**14**] under the additional assumption $\lim_{x_N\to\pm\infty}u(x',x_N)=\pm 1$ for all $x'\in\mathbb{R}^{N-1}$. However, for $4\leq N\leq 8$ the conjecture in its original statement is still open. In 2011, del Pino, Kowalczyk, and Wei ([**8**]) established that the conjecture does not hold for $N\geq 9$, as De Giorgi suggested.

It is easily seen that a monotone solution of the Allen–Cahn equation is stable in the following sense:

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Definition 1.1. Let $G \in C^2(\mathbb{R})$. We say that a solution $u \in C^2(\mathbb{R}^N)$ of $\Delta u = G'(u)$ in $\mathbb{R}^N$ is stable if

$$Q(v) := \int_{\mathbb{R}^N} (|\nabla v|^2 + G''(u)v^2)\, dx \geq 0,$$

for every $v \in C^1(\mathbb{R}^N)$ with compact support in $\mathbb{R}^N$.

Note that the above expression is nothing but the second variation of the energy functional in a bounded domain $\Omega \subset \mathbb{R}^N$:

$$E_\Omega(u) = \int_\Omega \left(\frac{1}{2}|\nabla u|^2 + G(u) \right) dx.$$

On the other hand, note that the Allen–Cahn equation $-\Delta u = u - u^3$ corresponds to the function $G(s) = (1-s^2)^2/4$.

Two crucial ingredients in the proof of the conjecture for $N \leq 3$ are the notion of stability and the following Liouville-type theorem due to Ambrosio and Cabré [**2**], which was motivated by a simpler version in [**4**].

Theorem 1.2 (Ambrosio–Cabré [**2**])**.** *Let $\varphi \in L^\infty_{\mathrm{loc}}(\mathbb{R}^N)$ be a positive function. Assume that $\sigma \in H^1_{\mathrm{loc}}(\mathbb{R}^N)$ satisfies*

$$\sigma \operatorname{div}(\varphi^2 \nabla \sigma) \geq 0 \quad \text{in } \mathbb{R}^N \tag{1.1}$$

in the distributional sense. For every $R > 0$, let $B_R = \{|x| < R\}$ and assume that there exists a constant independent of R such that

$$\int_{B_R} (\varphi\sigma)^2\, dx \leq CR^2 \quad \text{for every } R \geq 1.$$

Then σ is constant.

The idea is the following. If u is a solution in De Giorgi's conjecture, consider the functions $\varphi := \partial u/\partial_{x_N} > 0$ and $\sigma_i := \partial_{x_i} u/\partial_{x_N} u$, for $i = 1, \ldots, N-1$. Since both $\partial_{x_i} u$ and φ solve the same linear equation $-\Delta v = (1-3u^2)v$, an easy computation shows that $\operatorname{div}(\varphi^2 \nabla \sigma_i) = 0$. In dimensions $N \leq 3$ it is proved that $\int_{B_R} |\nabla u|^2\, dx \leq CR^2$, for every $R \geq 1$. Theorem 1.2 gives that σ_i is constant and it follows easily that u is a one-dimensional function. Observe that this reasoning just uses $\operatorname{div}(\varphi^2 \nabla \sigma_i) = 0$, which is a stronger condition than (1.1).

Motivated by the useful applications of Liouville-type theorems to problems of this kind, many authors have posed questions that allow us to understand qualitative properties of the solutions of general nonlinear partial differential equations of the form $\Delta u = G'(u)$ in $\mathbb{R}^N$. Here we show some of them.

Question A (Berestycki–Caffarelli–Nirenberg [**4**]). Let $L = -\Delta - V$ be a Schrödinger operator on $\mathbb{R}^N$ with a smooth and bounded potential V. Suppose that $u \in W^{2,p}_{\mathrm{loc}}$ for some $p > N$ is a bounded and sign-changing solution for $Lu = 0$. Set

$$\lambda_1(V) = \inf \left\{ \frac{\displaystyle\int_{\mathbb{R}^N} |\nabla \psi|^2 - V\psi^2}{\displaystyle\int_{\mathbb{R}^N} \psi^2} ; \psi \in C_0^\infty(\mathbb{R}^N) \right\}.$$

Then is $\lambda_1(V) < 0$?

Question B (Berestycki–Caffarelli–Nirenberg [**4**]). Let $\varphi \in L^\infty_{\mathrm{loc}}(\mathbb{R}^N)$ be a positive function. Assume that $\sigma \in H^1_{\mathrm{loc}}(\mathbb{R}^N)$ is a weak solution of $\mathrm{div}(\varphi^2 \nabla \sigma) = 0$. If $\varphi\sigma$ is bounded, then is σ constant?

In [**4**] a positive answer to Question A is deduced from a positive answer to Question B. By Theorem 1.2, it is easily deduced that the answer to Question B (and therefore also to Question A) is "yes" in dimensions $N = 1, 2$. In [**10**] Ghoussoub and Gui proved that the answer to Question A (and therefore also to Question B) is "no" if $N \geq 7$ by exploiting the idea of differentiating with respect to the first variable x_1 any solution of the semilinear PDE $\Delta v + v^p = 0$, $v > 0$, $v(x) \to 0$ when $|x| \to \infty$ in the whole space $\mathbb{R}^N$, for an appropriate exponent $p > 1$. Shortly after that, Barlow ([**3**]) used probabilistic methods to construct counterexamples giving a negative answer to Questions A and B in any dimension $N \geq 3$.

Another question related to the previous ones, which in fact is the main point of this paper, is raised by Alberti, Ambrosio, and Cabré:

Question C (Alberti–Ambrosio–Cabré [**1**]). If $0 < \varphi \in L^\infty_{\mathrm{loc}}(\mathbb{R}^N)$ and $\sigma \in H^1_{\mathrm{loc}}(\mathbb{R}^N)$ satisfy

$$\mathrm{div}(\varphi^2 \nabla \sigma) = 0 \quad \text{in } \mathbb{R}^N \tag{1.2}$$

in the distributional sense, what is the optimal (maximal) exponent γ_N such that

$$\int_{B_R} (\varphi\sigma)^2 \, dx \leq C R^{\gamma_N} \quad \text{for every } R \geq 1 \Longrightarrow \sigma \text{ constant?}$$

By the result of Barlow [**3**], it is deduced that $\gamma_N < N$ when $N \geq 3$. Also, a sharp choice in the counterexamples of Ghoussoub and Gui [**10**] shows that $\gamma_N < 2 + 2\sqrt{N-1}$ for $N \geq 7$. More recently, Moradifam ([**13**]) used some ideas of the probabilistic methods of Barlow [**3**] to prove that $\gamma_N < 3$ when $N \geq 4$.

On the other hand, assuming only inequality (1.1) (instead of equality (1.2)) Gazzola ([**9**]) proved the sharpness of the exponent 2 in Theorem 1.2 using nondifferentiable counterexamples. In any case, we would like to emphasize that we are more interested in the case of equality (1.2) than in that of inequality (1.1), since in the study of stable solutions it is precisely equality that is obtained.

In this paper we present a simple proof that $\gamma_N = 2$ for every $N \geq 1$. In other words, the exponent 2 in Theorem 1.2 is sharp, when equality holds in (1.1).

In fact, our result could be deduced from a recent paper by the author [**15**], in which more general functions (not only of the type $\Psi(R) = CR^k$) are considered in this kind of Liouville theorems. However, due to the simplicity of the proof of Theorem 1.3 below and the original motivation of this type of questions, we think that it is appropriate to write the proof separately for the case of pure powers.

Theorem 1.3. *Let $k > 2$ be a real number and $N \geq 1$ any dimension. Then there exist $C > 0$, $0 < \varphi \in C^\infty(\mathbb{R}^N)$ such that equation $\operatorname{div}(\varphi^2 \nabla\sigma) = 0$ admits a nonconstant classical solution $\sigma \in C^\infty(\mathbb{R}^N)$ satisfying*

$$\int_{B_R} (\varphi\sigma)^2 \, dx \leq CR^k \quad \textit{for every } R \geq 1.$$

Therefore, to apply the same arguments as in the proof of the conjecture of De Giorgi for dimensions $N \leq 3$, it seems essential to establish a connection between R^{γ_N} and $\int_{B_R} |\nabla u|^2$ (for varying R). Since $|\nabla u|$ is bounded we have the upper bound $\int_{B_R} |\nabla u|^2 \leq KR^N$ ($R > 0$), for some $K > 0$. In fact, for minimizers in any dimension and monotone solutions up to dimension 3 the better upper bound KR^{N-1} can be obtained [**1, 2**]. In the following proposition we obtain a lower bound of this expression. Note that this lower bound cannot be improved, since in the case in which u is one-dimensional we have $\int_{B_R} |\nabla u|^2 \sim CR^{N-1}$ for large R.

Proposition 1.4. *Let $G \in C^2(\mathbb{R})$ be a nonnegative function and $u \in C^2(\mathbb{R}^N)$ be a stable nonconstant bounded entire solution of $\Delta u = G'(u)$. Suppose that G satisfies*

(H) *There exists $K > 0$ such that $-(\sqrt{G})''(s) \geq K$ for every $s \in u(\mathbb{R}^N)$. Then there exist $C, R_0 > 0$ such that*

$$\int_{B_R} |\nabla u|^2 \geq CR^{N-1}, \quad \textit{for all } R \geq R_0. \tag{1.3}$$

Note that in the classical Allen–Cahn equation $-\Delta u = u - u^3$, $|u| < 1$, we have that $G(s) = (1-s^2)^2/4$ satisfies the hypothesis (H), since $-(\sqrt{G})''(s) = 1$ for every $s \in (-1, 1)$.

The lower bound (1.3) follows also from a result of the forthcoming paper by Cabré, Cinti, and Serra [**5**], carried out independently of ours, which shows that Dirichlet energy in a ball controls potential energy in a slightly smaller ball. This bound together with Modica's monotonicity formula (see Theorem 2.2) leads immediately to (1.3).

Finally, to obtain a more precise study on the energy functional E_{B_R}, we provide the following result, which establishes that the Dirichlet and potential energies have the same behavior in B_R, for large R.

Proposition 1.5. *Let $G \in C^2(\mathbb{R})$ be a nonnegative function and $u \in C^2(\mathbb{R}^N)$ be a stable nonconstant bounded entire solution of $\Delta u = G'(u)$. Suppose that G satisfies* (H)*. Then*

$$\lim_{R\to\infty} \frac{\displaystyle\int_{B_R} \frac{1}{2}|\nabla u|^2}{\displaystyle\int_{B_R} G(u)} = 1.$$

2. Relationship between Dirichlet and potential energies

In this section we prove some results concerning the Dirichlet and potential energies. We will use the following two results due to Modica.

Theorem 2.1 (Modica [**11**])**.** *Let $G \in C^2(\mathbb{R})$ be a nonnegative function and $u \in C^2(\mathbb{R}^N)$ be a bounded solution of $\Delta u = G'(u)$ in $\mathbb{R}^N$. Then,*

$$\frac{|\nabla u|^2}{2} \le G(u) \quad \text{in } \mathbb{R}^N.$$

In addition, if u is not constant, then $G(u(x)) > 0$ for all $x \in \mathbb{R}^N$.

In [**11**] this bound was proved under the hypothesis $u \in C^3(\mathbb{R}^N)$. The result as stated above, which applies to all solutions (recall that every solution is $C^{2,\alpha}(\mathbb{R}^N)$ since $G \in C^2(\mathbb{R})$), was established by Caffarelli, Garofalo, and Segala [**6**].

Theorem 2.2 (Modica [**12**])**.** *Let $G \in C^2(\mathbb{R})$ be a nonnegative function and $u \in C^2(\mathbb{R}^N)$ be a bounded solution of $\Delta u = G'(u)$ in $\mathbb{R}^N$. Then,*

$$\Phi(R) := R^{1-N} \int_{B_R} \left(\frac{1}{2}|\nabla u|^2 + G(u) \right) dx$$

is nondecreasing in $R \in (0, \infty)$. In particular, if u is not constant, then there exists a positive constant c_0 such that

$$\int_{B_R} \left(\frac{1}{2}|\nabla u|^2 + G(u) \right) dx \ge c_0 R^{N-1}, \quad \textit{for all } R \ge 1.$$

In the following lemma we give a characterization of stability in terms of an integral inequality involving the potential G.

Lemma 2.3. *Let $G \in C^2(\mathbb{R})$ be a nonnegative function and $u \in C^2(\mathbb{R}^N)$ be a nonconstant bounded entire solution of $\Delta u = G'(u)$. Then u is stable if and only if*

$$\int_{\mathbb{R}^N} G(u)|\nabla\mu|^2\,dx \geq \int_{\mathbb{R}^N} \left(G(u) - \frac{1}{2}|\nabla u|^2\right)(-2(\sqrt{G})''(u))\sqrt{G(u)}\mu^2\,dx,$$

for every $\mu \in C^1(\mathbb{R}^N)$ with compact support in $\mathbb{R}^N$.

Proof: First of all note that, by Theorem 2.1, we have $G(u(x)) > 0$ for every $x \in \mathbb{R}^N$. This means that $G(s) > 0$ for every $s \in I := u(\mathbb{R}^N)$. Then $\sqrt{G} \in C^2(I)$ and the above inequality makes sense.

Let $\mu \in C^1(\mathbb{R}^N)$ have compact support and $\omega \in C^2(\mathbb{R}^N)$. We obtain that

$$\begin{aligned} Q(\omega\mu) &= \int_{\mathbb{R}^N} (\omega^2|\nabla\mu|^2 + \nabla\mu^2\cdot\omega\nabla\omega + \mu^2|\nabla\omega|^2 + G''(u)\omega^2\mu^2) \\ &= \int_{\mathbb{R}^N} (\omega^2|\nabla\mu|^2 - \mu^2\operatorname{div}(\omega\nabla\omega) + \mu^2|\nabla\omega|^2 + G''(u)\omega^2\mu^2) \\ &= \int_{\mathbb{R}^N} (\omega^2|\nabla\mu|^2 + (-\Delta\omega + G''(u)\omega)\omega\mu^2). \end{aligned}$$

Take $\omega := \sqrt{G(u)}$. An easy computation shows that

$$\begin{aligned} -\Delta\omega + G''(u)\omega &= -(\sqrt{G})'(u)\Delta u - (\sqrt{G})''(u)|\nabla u|^2 + G''(u)\sqrt{G(u)} \\ &= -\frac{G'(u)}{2\sqrt{G(u)}}G'(u) - (\sqrt{G})''(u)|\nabla u|^2 + G''(u)\sqrt{G(u)} \\ &= 2\left(G(u) - \frac{1}{2}|\nabla u|^2\right)(\sqrt{G})''(u). \end{aligned}$$

Combining the above two equalities gives

$$Q(\sqrt{G(u)}\mu) = \int_{\mathbb{R}^N} \left(G(u)|\nabla\mu|^2 + 2\left(G(u) - \frac{1}{2}|\nabla u|^2\right)(\sqrt{G})''(u)\sqrt{G(u)}\mu^2\right).$$

Finally, μ is a C^1 function with compact support in $\mathbb{R}^N$ if and only if $\sqrt{G(u)}\mu$ is also so, and the proof is complete. □

The following proposition gives us an upper bound on the difference between the potential and Dirichlet energies.

Proposition 2.4. *Let $G \in C^2(\mathbb{R})$ be a nonnegative function and $u \in C^2(\mathbb{R}^N)$ be a stable nonconstant bounded entire solution of $\Delta u = G'(u)$. Suppose that G satisfies* (H). *Then there exist $C_1, C_2 > 0$ such that*

(i) $$\int_{B_R} \left(G(u) - \frac{1}{2}|\nabla u|^2 \right) \sqrt{G(u)} \le C_1 R^{N-2}, \qquad \textit{for all } R > 0.$$

(ii) $$\int_{B_R} \left(G(u) - \frac{1}{2}|\nabla u|^2 \right) \le C_2 R^{N-4/3}, \quad \textit{for all } R > 0.$$

Proof: By standard regularity arguments, we can take in Lemma 2.3 Lipschitz functions $\mu\colon \mathbb{R}^N \to \mathbb{R}$ with compact support. Fix $R > 0$ and take $\mu\colon \mathbb{R}^N \to \mathbb{R}$ defined by

$$\mu(x) := \begin{cases} 1 & \text{if } \|x\| \le R, \\ 2 - \dfrac{\|x\|}{R} & \text{if } R < \|x\| \le 2R, \\ 0 & \text{if } \|x\| > 2R. \end{cases}$$

Taking into account that $0 < G(u(x)) \le M$ for all $x \in \mathbb{R}^N$ (for some $M > 0$), we see at once that

$$\int_{\mathbb{R}^N} G(u)|\nabla \mu|^2 = \int_{B_{2R}\setminus B_R} G(u) \frac{1}{R^2} \le \int_{B_{2R}\setminus B_R} M \frac{1}{R^2} = M(2^N - 1)|B_1| R^{N-2},$$

$$\begin{aligned} \int_{\mathbb{R}^N} &\left(G(u) - \frac{1}{2}|\nabla u|^2 \right) \left(-2(\sqrt{G})''(u) \right) \sqrt{G(u)} \mu^2 \\ &\ge \int_{B_R} \left(G(u) - \frac{1}{2}|\nabla u|^2 \right) \left(-2(\sqrt{G})''(u) \right) \sqrt{G(u)} \\ &\ge 2K \int_{B_R} \left(G(u) - \frac{1}{2}|\nabla u|^2 \right) \sqrt{G(u)}. \end{aligned}$$

Combining these inequalities with Lemma 2.3 we obtain (i).

To prove (ii), consider the functions:

$$\alpha(x) = \left(\left(G(u(x)) - \frac{1}{2}|\nabla u(x)|^2 \right) \sqrt{G(u(x))} \right)^{2/3}, \quad x \in B_R,$$

$$\beta(x) = \left(\frac{G(u(x)) - \frac{1}{2}|\nabla u(x)|^2}{G(u(x))} \right)^{1/3}, \qquad x \in B_R.$$

By (i) we have

$$\|\alpha\|_{L^{3/2}(B_R)} \le (C_1 R^{N-2})^{2/3}.$$

By Theorem 2.1 we see that $0 \le \beta \le 1$ and consequently

$$\|\beta\|_{L^3(B_R)} \le \|1\|_{L^3(B_R)} = (|B_1| R^N)^{1/3}.$$

Therefore, applying Hölder's inequality to functions α and β we conclude

$$\begin{aligned}\int_{B_R} \left(G(u) - \frac{1}{2}|\nabla u|^2 \right) &= \|\alpha\beta\|_{L^1(B_R)} \le \|\alpha\|_{L^{3/2}(B_R)} \|\beta\|_{L^3(B_R)} \\ &\le (C_1 R^{N-2})^{2/3} (|B_1| R^N)^{1/3} = C_2 R^{N-4/3},\end{aligned}$$

where $C_2 = C_1^{2/3} |B_1|^{1/3}$, and (ii) is proved. □

Proof of Proposition 1.4: Applying part (ii) of Proposition 2.4 and Theorem 2.2 we have

$$\begin{aligned}\int_{B_R} |\nabla u|^2 &= \int_{B_R} \left(\frac{1}{2}|\nabla u|^2 + G(u) \right) - \int_{B_R} \left(G(u) - \frac{1}{2}|\nabla u|^2 \right) \\ &\ge c_0 R^{N-1} - C_2 R^{N-4/3}, \quad \text{for all } R \ge 1.\end{aligned}$$

Choosing $R_0 > 1$ such that $c_0 R^{N-1} - C_2 R^{N-4/3} \ge c_0 R^{N-1}/2$ for every $R \ge R_0$, we complete the proof. □

Proof of Proposition 1.5: Applying Theorems 2.1 and 2.2, we deduce that

$$2 \int_{B_R} G(u) \ge \int_{B_R} \left(G(u) + \frac{1}{2}|\nabla u|^2 \right) \ge c_0 R^{N-1},$$

for every $R \ge 1$, for some $c_0 > 0$. Combining this with part (ii) of Proposition 2.4 we conclude

$$\begin{aligned}1 \ge \frac{\displaystyle\int_{B_R} \frac{1}{2}|\nabla u|^2}{\displaystyle\int_{B_R} G(u)} &= 1 - \frac{\displaystyle\int_{B_R} \left(G(u) - \frac{1}{2}|\nabla u|^2 \right)}{\displaystyle\int_{B_R} G(u)} \\ &\ge 1 - \frac{C_2 R^{N-4/3}}{c_0 R^{N-1}/2} = 1 - \frac{2C_2}{c_0 R^{1/3}},\end{aligned}$$

for every $R \ge 1$, and the proposition follows. □

3. Counterexample

Proof of Theorem 1.3: Let $0 < H \in C^\infty(\mathbb{R}^{N-1})$ satisfying $\int_{\mathbb{R}^{N-1}} H^2 = 1$. Let $k > 2$ and consider an odd function $g \in C^\infty(\mathbb{R})$ satisfying $g(r) = 2 - r^{2-k}$ if $r \geq 1$ and $g'(r) > 0$ for all $r \in \mathbb{R}$.

Define

$$\varphi(x_1, \dots, x_N) := \frac{H(x_1, \dots, x_{N-1})}{\sqrt{g'(x_N)}},$$
$$\sigma(x_1, \dots, x_N) := g(x_N).$$

(If $N = 1$, then define $\varphi(x) = 1/\sqrt{g'(x)}$ and we apply the same reasoning as in the case $N > 1$.)

Clearly $0 < \varphi \in C^\infty(\mathbb{R}^N)$, $\sigma \in C^\infty(\mathbb{R}^N)$ satisfy

$$\nabla\sigma(x_1, \dots, x_N) = (0, \dots, 0, g'(x_N)),$$
$$(\varphi^2\nabla\sigma)(x_1, \dots, x_N) = (0, \dots, 0, H^2(x_1, \dots, x_{N-1})),$$

which implies $\operatorname{div}(\varphi^2\nabla\sigma) = 0$ in $\mathbb{R}^N$.

On the other hand, taking into account that $\sigma^2 < 4$ in $\mathbb{R}^N$, and $B_R \subset \mathbb{R}^{N-1} \times (-R, R)$, we obtain for arbitrary $R \geq 1$:

$$\int_{B_R} (\varphi\sigma)^2\,dx \leq 4 \int_{B_R} \varphi^2\,dx \leq 4 \int_{\mathbb{R}^{N-1}} H^2\,d(x_1, \dots, x_{N-1}) \int_{-R}^{R} \frac{dt}{g'(t)}$$
$$= 8 \int_0^R \frac{dt}{g'(t)} = 8 \int_0^1 \frac{dt}{g'(t)} + 8 \int_1^R \frac{dt}{(k-2)t^{1-k}}$$
$$= 8 \int_0^1 \frac{dt}{g'(t)} + \frac{8(R^k - 1)}{k(k-2)} = C_1 R^k + C_2,$$

for certain $C_1 > 0$, $C_2 \in \mathbb{R}$. Taking $C = C_1 + |C_2|$, the proof is complete. □

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