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LOCALLY COUNTABLE PSEUDOVARIETIES

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Abstract: The purpose of this paper is to contribute to the theory of profinite semigroups by considering the special class consisting of those all of whose finitely generated closed subsemigroups are countable, which are said to be locally countable. We also call locally countable a pseudovariety V (of finite semigroups) for which all pro-V semigroups are locally countable. We investigate operations preserving local countability of pseudovarieties and show that, in contrast with local finiteness, several natural operations do not preserve it. We also investigate the relationship of a finitely generated profinite semigroup being countable with every element being expressible in terms of the generators using multiplication and the idempotent (omega) power. The two properties turn out to be equivalent if there are only countably many group elements, gathered in finitely many regular $\mathcal{J}$-classes. We also show that the pseudovariety generated by all finite ordered monoids satisfying the inequality $1 \leqslant x^n$ is locally countable if and only if $n = 1$.

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1. Introduction

With the advent of modern computers, several mathematical models of computation emerged. One of the simplest and most successful in applications is that of a finite automaton, which computes regular languages. The transition semigroup of the minimal automaton of a regular language, which is also its syntactic semigroup, provides an alternative purely algebraic computation model. This connection with algebra turned out to be rather fruitful and eventually led to a general framework for the translation of combinatorial problems on classes of regular languages to algebraic problems about finite semigroups, which is provided by Eilenberg's correspondence [**19**]. On the algebraic side, one considers so-called pseudovarieties of semigroups, classes of finite semigroups closed under taking homomorphic images, subsemigroups, and finite direct products.

Unlike varieties, pseudovarieties do not in general possess free members. It is possible to overcome this problem by considering more general structures, namely profinite semigroups, which are inverse limits of finite semigroups. Relatively free profinite semigroups are then realized naturally as topological spaces, namely as the Stone duals of Boolean algebras of regular languages [**3**, Theorem 3.6.1] (see also [**36**]). In fact, the algebraic structure of such profinite semigroups is also captured by duality theory [**21, 20, 9**]. Particularly in view of Reiterman's theorem [**37**], which provides a description of pseudovarieties in terms of formal equalities of members of a free profinite semigroup (so-called pseudoidentities), such semigroups thus naturally came to play an important role in the theory of pseudovarieties of semigroups, which remains the core of the theory of finite semigroups [**41**].

A very special type of pseudovarieties is that of locally finite pseudovarieties. They may be characterized by the existence of a finite bound on the cardinality of members on a given number of generators, or by the finiteness of the finitely generated relatively free profinite semigroups. Locally finite pseudovarieties constitute a particularly well-behaved special case, for instance when dealing with the important operation of semidirect product (see [**41**, Section 3.7]).

In this paper, we consider a generalization of locally finite pseudovarieties: locally countable pseudovarieties. These are pseudovarieties whose finitely generated relatively free profinite semigroups are countable. They turn out to have an important property, namely that closed congruences on their finitely generated relatively free profinite semigroups are profinite [**9**], a property that holds for arbitrary profinite groups but that fails in general for profinite semigroups. The authors came across this property in connection with the conjectured completeness of a proof scheme for pseudoidentities [**7**].

It should be noted that an infinite countably generated profinite semigroup is either countable or has the cardinality of the continuum. This follows from the Cantor–Bendixson Theorem (see [**28**, Theorem 6.4]), which holds in every complete metric space with a countable dense subset. In the case of a finitely generated profinite semigroup S, one may say more precisely that, if S is uncountable, then it is the union of a countable set with a closed set homeomorphic to the Cantor set. In particular, from the point of view of cardinality, locally countable pseudovarieties deserve special attention.

There is a duality significance of a Stone space being countable that adds motivation to considering this cardinality condition. Indeed, the Boolean algebras whose dual spaces are countable have been characterized by Day [**17**, Section 4] as the countable superatomic Boolean algebras, where superatomic means that every homomorphic image is atomic.

We proceed to describe briefly the organization and main contributions of this paper. Section 2 provides the preliminary material on semigroups used in the rest of the paper. In Section 3 we start the investigation of locally countable pseudovarieties of semigroups; in particular, we discuss how they behave with respect to the operations of join and semidirect product. The Mal'cev product is considered in Section 4, where a profinite characterization from [**35**] is recalled that plays a role in later sections. A well-known and key result is that the set of locally finite pseudovarieties is closed under Mal'cev product, a fact that comes from Brown's finiteness theorem [**15**]. One of the several proofs in the literature of Brown's theorem is due to Simon [**44**] and uses his factorization forest theorem. This important theorem turns out to have applications also in the study of locally countable pseudovarieties that are based on a theorem presented in Section 5 concerning algebraic generation of a finitely generated profinite semigroup. In Section 6, we discuss profinite semigroups with only finitely many regular $\mathcal{J}$-classes. In particular, we show that such a finitely generated profinite semigroup is countable if and only if it has only countably many group elements, in which case it is algebraically generated by the finite generating set together with the idempotents. The special case where there are only finitely many idempotents is considered in Section 7, where it is shown that the Mal'cev product of a locally countable pseudovariety of semigroups with only one idempotent with a locally finite pseudovariety is again locally countable. Moreover, in such a case, we show that elements of finitely generated relatively free profinite semigroups may be obtained from the generators by using only product and the ω-power, without the need to nest the latter operation; this generalizes a result of the first author [**2**] that has been extensively used. The Mal'cev product of a locally countable pseudovariety with a locally finite pseudovariety in general is considered in Section 8, where it is shown that if the second factor consists of nilpotent semigroups, then the resulting Mal'cev product is locally countable. By examining the atoms in the lattice of pseudovarieties containing non-nilpotent semigroups, we show in Section 9 that the result of Section 8 fails as soon as the second factor contains a non-nilpotent semigroup. Finally, in Section 10 we use aperiodic inverse semigroups to show that certain pseudovarieties of block groups are not locally countable, thus answering some questions raised in our paper [**8**]. In particular, we show that the pseudovariety generated by all finite ordered monoids satisfying the inequality $1 \leqslant x^n$ is not locally countable whenever $n \geqslant 2$.

2. Preliminaries

As the remainder of the paper is concerned with semigroups, we gather in this section the required preliminary material on that subject. For general background, the reader is referred to [**3, 41**].

A fundamental tool in semigroup theory is given by the so-called Green's relations [**22**]. They are binary relations concerning the ideal structure of a semigroup S and are defined as follows. For $s, t \in S$, write $s \leqslant_{\mathcal{R}} t$ if s belongs to the principal right ideal generated by t. Replacing right by left or two-sided, one obtains respectively the relations $\leqslant_{\mathcal{L}}$ and $\leqslant_{\mathcal{J}}$. The intersection of the relations $\leqslant_{\mathcal{R}}$ and $\leqslant_{\mathcal{L}}$ is denoted by $\leqslant_{\mathcal{H}}$. Each of the above relations $\leqslant_{\mathcal{K}}$ is a quasi-order on S and the corresponding equivalence relation $\leqslant_{\mathcal{K}} \cap \geqslant_{\mathcal{K}}$ is denoted by $\mathcal{K}$. The relations $\mathcal{R}$ and $\mathcal{L}$ commute under composition and so $\mathcal{D} = \mathcal{R}\mathcal{L}$ is also an equivalence relation on S. In particular, every $\mathcal{D}$-class D is a union of $\mathcal{R}$-classes and also a union of $\mathcal{L}$-classes, where the intersection of each $\mathcal{R}$-class with each $\mathcal{L}$-class within D is an $\mathcal{H}$-class. It turns out that the relations $\mathcal{D}$ and $\mathcal{J}$ coincide in each compact semigroup (see, for instance, [**41**, Proposition 3.1.10]).

By a *pseudovariety* we mean a class of finite semigroups that is closed under taking homomorphic images, subalgebras, and finite direct products. For a pseudovariety V, a *pro-*V *semigroup* is an inverse limit of semigroups from V. In the case of the pseudovariety S of all finite semigroups, a pro-S semigroup is simply called a *profinite semigroup.* A topological semigroup is *locally countable* if every finitely generated closed subsemigroup is countable. Recall that a pseudovariety V is *locally finite* if every finitely generated pro-V semigroup is finite. We say that a pseudovariety V is *locally countable* if every pro-V semigroup is locally countable. Of course, a locally finite pseudovariety is also locally countable.

The classes of all finite semigroups in which the relations $\mathcal{J}$ and $\mathcal{R}$ are trivial (meaning that they are reduced to the equality relation) are denoted by, respectively, J and R. On the other hand, the class of all $\mathcal{H}$-trivial finite semigroups is denoted by A. All three are pseudovarieties, the reason for the notation A being that the members of A are the finite *aperiodic* semigroups, that is, finite semigroups in which all subgroups are trivial.

An element s of a semigroup S is *regular* if there exists $s' \in S$ such that $s = ss's$. It follows immediately from the definitions that an element is regular if and only if its $\mathcal{R}$-class (respectively its $\mathcal{L}$-class) contains an idempotent. Hence, a $\mathcal{D}$-class consists of regular elements if and only if it contains a regular element, if and only if it contains an idempotent in each $\mathcal{R}$-class and in each $\mathcal{L}$-class. Such $\mathcal{D}$-classes are said to be *regular*. The $\mathcal{H}$-classes of S that contain idempotents are exactly the maximal subgroups of S.

Profinite semigroups with only one $\mathcal{J}$-class are said to be *completely simple*. A profinite semigroup with zero whose non-zero elements form a regular $\mathcal{J}$-class is said to be *completely* 0*-simple*. The finite completely simple semigroups constitute a pseudovariety, denoted by CS. Those whose subgroups lie in a pseudovariety of groups H form a subpseudovariety of CS, denoted by $\mathsf{CS}(\mathsf{H})$.

Given an ideal I of a semigroup S, the union of the equality relation on S with the universal relation on I is a congruence on S. The corresponding quotient semigroup is denoted by S/I and is called the *Rees quotient* of S by I.

For a semigroup S, denote by $E(S)$ the set of its idempotent elements.

Several operators may be defined on pseudovarieties. Let us consider first some unary operators, defined on a pseudovariety V. We denote by DV the class of all finite semigroups whose regular $\mathcal{D}$-classes constitute subsemigroups belonging to V. The class EV consists of all finite semigroups S such that the subsemigroup generated by $E(S)$ belongs to V. For LV, we take the class of all finite semigroups S such that, for every $e \in E(S)$, the subsemigroup $eSe = \{ese : s \in S\}$ belongs to V. The definition of the block operator B is a bit more complicated. Given a regular $\mathcal{D}$-class D of a finite semigroup S, we consider the smallest equivalence relation β on the idempotents of D for which $\mathcal{L}$ or $\mathcal{R}$-equivalent idempotents are equivalent. For each β-class C, we consider the subsemigroup T generated by the union of the $\mathcal{H}$-classes containing members of C and the ideal $I = \{t \in T : t <_{\mathcal{J}} e\}$ of T, where $e \in C$ is arbitrary. The Rees quotient T/I is called a *block* of D. We denote by BV the class of all finite semigroups whose blocks belong to V. It is well-known that BV, DV, EV, and LV are pseudovarieties.

Three binary operators on pseudovarieties play a special role in finite semigroup theory. They correspond to classical algebraic constructions. Given a semigroup S, we denote by S^1 the monoid obtained from S by adding an identity element unless S is already a monoid; we also denote by $\mathrm{End}(S)$ the monoid of all endomorphisms of S. Given another semigroup T and a monoid homomorphism $\varphi\colon T^1 \to \mathrm{End}(S)$, we denote $\varphi(t)(s)$ by ${}^t s$. The Cartesian product $S \times T$ is then a semigroup under the operation $(s,t)(s',t') = (s\,{}^t s', tt')$ which is denoted by $S * T$ and is called the *semidirect product* of S and T determined by φ.

Given a class $\mathcal{C}$ of finite semigroups, there is a smallest pseudovariety containing it, which is called the pseudovariety *generated* by $\mathcal{C}$. Let V

and W be two pseudovarieties. The *join* $\mathsf{V} \vee \mathsf{W}$ is the pseudovariety generated by $\mathsf{V} \cup \mathsf{W}$ and may be thought of as the pseudovariety generated by all direct products $S \times T$ with $S \in \mathsf{V}$ and $T \in \mathsf{W}$. The *semidirect product* $\mathsf{V} * \mathsf{W}$ is the pseudovariety generated by the class of all semidirect products $S * T$ with $S \in \mathsf{V}$ and $T \in \mathsf{W}$. The *Mal'cev* product $\mathsf{V} ⓜ \mathsf{W}$ is the pseudovariety generated by the class of all finite semigroups S such that there exists a homomorphism $\varphi\colon S \to T$ with $T \in \mathsf{W}$ and $\varphi^{-1}(e) \in \mathsf{V}$ for every $e \in E(S)$.

We say that a continuous mapping $\varphi\colon X \to S$ from a topological space X into a topological semigroup S is a *generating mapping* if the subsemigroup of S generated by $\varphi(X)$ is dense; in this case, we also say that S is *X-generated*.

Let V be a pseudovariety and let X be a topological space. We say that a continuous function $\varphi\colon X \to S$ into a pro-V semigroup S defines S as a *free pro-V semigroup over X* if it has the following universal property: for every continuous mapping $\psi\colon X \to T$ into another pro-V semigroup T, there is a unique continuous homomorphism $\hat{\psi}\colon S \to T$ such that $\hat{\psi} \circ \varphi = \psi$. Standard arguments show that such a pro-V semigroup S is unique up to homeomorphic isomorphism respecting the choice of generators and it is denoted by $\overline{\Omega}_X \mathsf{V}$. In the case of X being a finite set, we generally consider X as a discrete space when referring to $\overline{\Omega}_X \mathsf{V}$. Moreover, when we write $\overline{\Omega}_n \mathsf{V}$ for a positive integer n, we mean $\overline{\Omega}_X \mathsf{V}$, where $X = \{1, \ldots, n\}$.

The existence of free pro-V semigroups over an arbitrary topological space X may be established by considering inverse limits of X-generated semigroups from V. In particular, $\overline{\Omega}_X \mathsf{V}$ is X-generated. The generating mapping $\varphi\colon X \to \overline{\Omega}_X \mathsf{V}$ is also called the *natural mapping*. If V is not the trivial pseudovariety I, consisting only of singleton semigroups, and X is a profinite space, then the natural mapping φ into $\overline{\Omega}_X \mathsf{V}$ is injective and we usually identify each $x \in X$ with $\varphi(x)$. Elements of $\overline{\Omega}_X \mathsf{V}$ are sometimes called *pseudowords*. Elements of the subsemigroup of $\overline{\Omega}_X \mathsf{V}$ generated by X are said to be *finite*, the remaining elements being called *infinite*. A word $u = x_1 \cdots x_n$ with each $x_i \in X$ is said to be a *subword of the pseudoword* $v \in \overline{\Omega}_X \mathsf{V}$ if there is a factorization $v = v_0 x_1 v_1 \cdots x_n v_n$ with each v_i in $(\overline{\Omega}_X \mathsf{V})^1$.

Note that the pseudovariety V is locally countable if and only if $\overline{\Omega}_n \mathsf{V}$ is countable for every positive integer n.

For a compact semigroup S and an element $s \in S$, it is well-known that the closed subsemigroup generated by s contains a unique idempotent, which we denote by s^ω. In the case of S being a finite semigroup, s^ω is the unique idempotent power of s.

Given a finite set X and a pseudovariety V, each element $w \in \overline{\Omega}_X\mathsf{V}$ may be viewed as an operation $w_S\colon S^X \to S$ on each pro-V semigroup S as follows: given a function $\psi \in S^X$ and the natural mapping $\varphi\colon X \to \overline{\Omega}_X\mathsf{V}$, we let $w_S(\psi) = \hat{\psi}(w)$, where $\hat{\psi}$ is the unique continuous homomorphism such that the following diagram commutes:

$$\begin{tikzcd} X \arrow[r, "\varphi"] \arrow[dr, "\psi"'] & \overline{\Omega}_X\mathsf{V} \arrow[d, "\hat{\psi}"] \\ & S \end{tikzcd}$$

It may be shown that in this way S becomes a profinite algebra where, for each $n \geqslant 1$, $\overline{\Omega}_n\mathsf{V}$ is the set of n-ary operation symbols. Another convenient signature is that reduced to the ω-power and multiplication, for which every profinite semigroup thus has a natural structure. The elements of the subalgebra of $\overline{\Omega}_X\mathsf{V}$ generated by X in this signature are called ω-words.

A formal equality $u = v$ of elements of $\overline{\Omega}_X\mathsf{V}$ is said to be a V-*pseudoidentity* over X. It is said to be *satisfied* in the pro-V semigroup S if the equality $u_S = v_S$ holds. The class of all semigroups from V satisfying a set Σ of V-pseudoidentities is a subpseudovariety of V denoted by $[\![\Sigma]\!]_{\mathsf{V}}$ and it is said to be *defined* by Σ. By Reiterman's theorem [**37**], every subpseudovariety is defined by some set of V-pseudoidentities. In the case of V being the pseudovariety S of all finite semigroups, then we omit reference to V. The elements of X are often called *variables*. We may sometimes write $u = 1$ as an abbreviation for the pair of pseudoidentities $ux = x = xu$, where x is a new variable. A similar convention applies to $u = 0$, which abbreviates $ux = u = xu$.

Given a residually finite discrete semigroup S, we may define a metric on S by letting $d(s, s) = 0$ and, for distinct $s, s' \in S$, $d(s, s') = 2^{-r(s,s')}$, where $r(s, s')$ is the minimum cardinality of a finite semigroup T for which there is a homomorphism $\varphi\colon S \to T$ such that $\varphi(s) \neq \varphi(s')$. Note that the multiplication of S is uniformly continuous with respect to d. The completion of the metric space (S, d) therefore has a natural topological semigroup structure. In the case of S being finitely generated, this completion is the *profinite completion* of S, which is denoted by $\widehat{S}$. In the particular case where $S = X^+$ is the free semigroup generated by a finite (discrete) set X, $\widehat{X^+}$ may be shown to be a free profinite semigroup over X. In particular, every element of $\overline{\Omega}_X \mathsf{S}$ is the limit of some sequence of words. For an infinite discrete set X, $\widehat{X^+}$ is not a metric space.

Denote by $\widehat{\mathbb{N}}$ the profinite completion of the additive monoid $\mathbb{N}$ of natural numbers. It may be thought of as the monoid $(\overline{\Omega}_1 \mathsf{S})^1$, written additively. If we denote by x the unique generator of $\overline{\Omega}_1 \mathsf{S}$, the identification between $\widehat{\mathbb{N}}$ and $(\overline{\Omega}_1 \mathsf{S})^1$ sends 0 to the identity element 1 and each positive $n \in \mathbb{N}$ to x^n. In general, we write x^α for the element of $(\overline{\Omega}_1 \mathsf{S})^1$ corresponding to $\alpha \in \widehat{\mathbb{N}}$.

Many pseudovarieties play a role in the sequel. For the moment, for the sake of example, we introduce the following:

- $\mathsf{Sl} = [\![x^2 = x, xy = yx]\!]$ is the pseudovariety of all finite *semilattices*;
- $\mathsf{N} = [\![x^\omega = 0]\!] = \bigcup_{n \geqslant 1} [\![x_1 \cdots x_n = 0]\!]$ is the pseudovariety of all finite *nilpotent* semigroups;
- $\mathsf{G} = [\![x^\omega = 1]\!]$ is the pseudovariety of all finite groups;
- $\mathsf{K} = [\![x^\omega y = x^\omega]\!]$ is the pseudovariety of all finite semigroups in which idempotents are left zeros;
- $\mathsf{IE} = [\![x^\omega = y^\omega]\!]$ is the pseudovariety of all finite semigroups with a unique idempotent.

Further pseudovarieties will be introduced as needed.

3. Locally countable pseudovarieties

We proceed with some examples of locally countable pseudovarieties of semigroups.

It was conjectured by I. Simon and proved by the first author that the pseudovariety J is locally countable [**2**, Corollary 3.4]. In particular, we deduce that the pseudovariety J satisfies the strong form of the conjecture of [**7**].

For groups, local countability does not provide a new class of pseudovarieties.

Theorem 3.1. *The following are equivalent for a pseudovariety of groups* H*:*

(i) $\overline{\Omega}_1 \mathsf{H}$ *is countable;*
(ii) H *is locally countable;*
(iii) H *is locally finite;*
(iv) H *satisfies some identity of the form* $x^n = 1$*, where* n *is a positive integer;*
(v) $\overline{\Omega}_1 \mathsf{H}$ *is finite.*

Proof: (iv) $\Leftrightarrow$ (v) and (iii) $\Rightarrow$ (ii) $\Rightarrow$ (i) are obvious.

(iv) ⇒ (iii) follows from Zelmanov's solution of the restricted Burnside problem [**51, 50, 52**].[1]

(i) ⇒ (iv) Suppose that $\overline{\Omega}_1\mathsf{H}$ is countable and assume that (iv) fails. Then, H must contain groups of arbitrarily large exponent. If the number of primes appearing in such exponents is finite, then there is a prime p such that the cyclic group of order p^n belongs to H for every $n \geqslant 1$. But the inverse limit of such cyclic groups is the additive group of the ring $\mathbb{Z}_p$ of p-adic integers, which is uncountable and a homomorphic image of $\overline{\Omega}_1\mathsf{H}$, contradicting the hypothesis (i). Hence, there is an infinite set of primes $\{p_1, p_2, \dots\}$ such that $\mathbb{Z}/p_i\mathbb{Z}$ belongs to H. For each $\alpha \in \widehat{\mathbb{N}}$, consider the pseudovariety $\mathsf{Ab}_\alpha = [\![x^\alpha = 1, xy = yx]\!]$. Let α be an accumulation point of the sequence $(p_1 \cdots p_n)_n$ in $\widehat{\mathbb{N}}$. By the Chinese remainder theorem, it is easy to show that the mapping $\overline{\Omega}_1\mathsf{Ab}_\alpha \to \prod_{i=1}^{\infty} \overline{\Omega}_1\mathsf{Ab}_{p_i}$ induced by the natural projections $\overline{\Omega}_1\mathsf{Ab}_\alpha \to \overline{\Omega}_1\mathsf{Ab}_{p_i}$ is a bijection. This again contradicts the assumption that H satisfies (i) since Ab_α is contained in H. □

Actually, a more general result is true: every profinite countable group is finite. Several proofs of this fact may be given. A direct proof can be found in [**42**, Proposition 2.3.1]. Other proofs are obtained by applying various results. In [**49**, Section 2.3], one can find immediate proofs based on the Baire category theorem or using the Haar measure. Yet another proof is obtained by considering the set of isolated points of a profinite countable group. By symmetry, if there is such a point, then all points are isolated so that, by compactness, the group is finite. Otherwise, by the Cantor–Bendixson theorem, the group is uncountable. In the proof of Theorem 3.1, thanks to Zelmanov's theorem, we only need to deal with the cyclic case.

Example 3.2. Consider the equational pseudovariety $\mathsf{N}_2 = [\![x^2 = 0]\!]$. Note that it is locally countable. For a finite alphabet A, the semigroup $\overline{\Omega}_A\mathsf{N}_2$ is the Rees quotient of A^+ by the ideal consisting of the words containing some square factor. Hence, $\overline{\Omega}_A\mathsf{N}_2$ is infinite whenever $|A| \geqslant 3$ [**48**]. Thus, in contrast with the group case, a locally countable pseudovariety need not be locally finite, even if it is equational.

[1]As was pointed out by a referee, Zelmanov solved the restricted Burnside problem for prime-power exponents. The case of an arbitrary exponent follows from a theorem of Hall and Higman [**23**, Theorem 4.4.3] which reduces it to the case of prime-power exponents and two properties of simple groups: the Schreier conjecture that the outer automorphism group of a finite simple group is solvable; and, up to isomorphism, there are only finitely many simple groups of given exponent. Both of these properties are known to be consequences of the classification of finite simple groups.

The remainder of this section examines operations that preserve local countability and, therefore, provide methods to produce many examples of locally countable pseudovarieties.

Theorem 3.3. *If* V *and* W *are locally countable pseudovarieties, then so is* $\mathsf{V} \vee \mathsf{W}$*.*

Proof: Let A be a finite set. Simply note that the natural projections $\overline{\Omega}_A(\mathsf{V} \vee \mathsf{W})$ onto $\overline{\Omega}_A\mathsf{V}$ and $\overline{\Omega}_A\mathsf{W}$ yield an embedding of $\overline{\Omega}_A(\mathsf{V} \vee \mathsf{W})$ into the product $\overline{\Omega}_A\mathsf{V} \times \overline{\Omega}_A\mathsf{W}$, which is countable since the factors are countable. $\square$

The next result is an immediate application of a representation theorem for free profinite semigroups over a semidirect product of pseudovarieties $\mathsf{V} * \mathsf{W}$ [**12**]. Indeed, as has already been observed in [**7**], since $\overline{\Omega}_A(\mathsf{V} * \mathsf{W})$ embeds in a certain semidirect product $\overline{\Omega}_B\mathsf{V} * \overline{\Omega}_A\mathsf{W}$, where $B = (\overline{\Omega}_A\mathsf{W})^1 \times A$, we have the following theorem.

Theorem 3.4. *If* V *and* W *are pseudovarieties of semigroups such that* V *is locally countable and* W *is locally finite, then* $\mathsf{V} * \mathsf{W}$ *is also locally countable.*

For example, the pseudovarieties $\mathsf{J} * \mathsf{B}$, where $\mathsf{B} = [\![x^2 = x]\!]$ is the pseudovariety of all finite bands, and $\mathsf{J} * [\![x^n = 1]\!]$ are locally countable. Consider the pseudovariety R. It is known that $\mathsf{Sl} * \mathsf{J} = \mathsf{R}$ [**16**] and that R is not locally countable: in fact R contains the pseudovariety K and even $\overline{\Omega}_2\mathsf{K}$ is uncountable as its infinite elements are in natural bijection with the infinite words over a two-letter alphabet (cf. [**3**, Section 3.7]). Hence, the assumptions on V and W may not be exchanged in the hypothesis of Theorem 3.4. Since it is well-known and easy to check that R is closed under semidirect product, we also have the equality $\mathsf{J} * \mathsf{J} = \mathsf{R}$ and so even the semidirect product of a locally countable pseudovariety with itself may not be locally countable.

For the Mal'cev product, the situation is considerably more complicated and is analyzed in the next six sections.

4. The Mal'cev product and profinite semigroups

Recall from Section 2 the definition of the Mal'cev product of pseudovarieties in terms of generators. A comprehensive description of the Mal'cev product involves the notion of relational morphisms.

By a *relational morphism* $\mu\colon S \longrightarrow\!\!\!\!\!\!\circ\;\, T$ between semigroups S and T we mean a subsemigroup μ of the direct product $S \times T$ such that the projection $\mu \to S$ on the first component is onto. In other words, a relational morphism $S \mathrel{-\!\!\circ\!\!\rightarrow} T$ is a relation with domain S and values in T which is closed under multiplication. Some authors prefer to emphasize the transformation character of a relational morphism, which is viewed as a function associating subsets of T to elements of S, calling the set of pairs $(s,t) \in S \times T$ such that s is related to t, the *graph* of the relational morphism (cf. [**41**]). We make no such distinction as we view relations as sets of ordered pairs, as is common in set theory.

It is well-known that the Mal'cev product $\mathsf{V} \textcircled{m} \mathsf{W}$ consists of all finite semigroups S for which there is a relational morphism $\mu\colon S \mathrel{-\!\!\circ\!\!\rightarrow} T$ into some $T \in \mathsf{W}$ such that $\mu^{-1}(e) \in \mathsf{V}$ for every $e \in E(T)$. The following result, which will be instrumental later in the paper, provides a profinite version of this comprehensive characterization of the Mal'cev product.

Theorem 4.1 ([**35**, Proposition 4.4])**.** *Let S be a profinite semigroup and let V and W be pseudovarieties. Then S is pro-$(\mathsf{V} \textcircled{m} \mathsf{W})$ if and only if there exists a closed relational morphism $\mu\colon S \mathrel{-\!\!\circ\!\!\rightarrow} T$ such that T is a pro-W semigroup and $\mu^{-1}(e)$ is a pro-V semigroup for every $e \in E(T)$.*

In the special case of the profinite semigroup $\overline{\Omega}_A(\mathsf{V} \textcircled{m} \mathsf{W})$, we may say a bit more. Since it is a pro-$(\mathsf{V} \textcircled{m} \mathsf{W})$ semigroup, by Theorem 4.1 there is a closed relational morphism $\mu\colon \overline{\Omega}_A(\mathsf{V} \textcircled{m} \mathsf{W}) \mathrel{-\!\!\circ\!\!\rightarrow} T$ into a pro-W semigroup T such that $\mu^{-1}(e)$ is pro-V for every $e \in E(T)$. We may choose for each $a \in A$ an element $t_a \in \mu(a)$. The closed subsemigroup of μ generated by the set $\{(a, t_a) : a \in A\}$ is still a closed relational morphism sharing the above property with μ, and so we may assume that it coincides with μ. Moreover, for the second component projection $\pi_2\colon \mu \to T$, the semigroup $\pi_2^{-1}(e) = \mu^{-1}(e) \times \{e\}$ is pro-V. By Theorem 4.1, μ is a pro-$(\mathsf{V} \textcircled{m} \mathsf{W})$ semigroup. Since it is A-generated via the mapping $a \mapsto (a, t_a)$, it follows that the first component projection $\pi_1\colon \mu \to \overline{\Omega}_A(\mathsf{V} \textcircled{m} \mathsf{W})$ is an isomorphism. The composite $\pi_1^{-1}\pi_2 = \mu$ is, therefore, a continuous homomorphism $\varphi\colon \overline{\Omega}_A(\mathsf{V} \textcircled{m} \mathsf{W}) \to T$. Since T is pro-W, φ factors through the natural projection $p\colon \overline{\Omega}_A(\mathsf{V} \textcircled{m} \mathsf{W}) \to \overline{\Omega}_A\mathsf{W}$, which is identical on generators, as $\varphi = \psi \circ p$:

$$\begin{tikzcd}
 & \mu \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & \\
\overline{\Omega}_A(\mathsf{V} \textcircled{m} \mathsf{W}) \arrow[rr, "\varphi"] \arrow[dr, "p"'] & & T \\
 & \overline{\Omega}_A\mathsf{W} \arrow[ur, "\psi"'] &
\end{tikzcd}$$

Given an idempotent $e \in \overline{\Omega}_A\mathsf{W}$, by the commutativity of the above diagram, we have the inclusion $p^{-1}(e) \subseteq \varphi^{-1}(\psi(e))$, which entails that $p^{-1}(e)$ is a pro-V semigroup. This discussion proves the following consequence of Theorem 4.1, which could easily have been included in [**35**].

Corollary 4.2. *The natural projection* $p\colon \overline{\Omega}_A(\mathsf{V} ⓜ \mathsf{W}) \to \overline{\Omega}_A\mathsf{W}$ *is such that* $p^{-1}(e)$ *is a pro-*V *semigroup for every* $e \in E(\overline{\Omega}_A\mathsf{W})$. □

From Corollary 4.2 and Theorem 4.1, it is immediate to deduce the so-called Pin–Weil basis theorem for the Mal'cev product [**35**] (see also [**41**, Theorem 3.7.13]): if $\{u_i(x_1, \dots, x_{n_i}) = v_i(x_1, \dots, x_{n_i}) : i \in I\}$ is a basis of pseudoidentities for V, then

$$\{u_i(w_1, \dots, w_{n_i}) = v_i(w_1, \dots, w_{n_i}) : i \in I,\, \mathsf{W} \models w_1^2 = w_1 = \cdots = w_{n_i}\}$$

is a basis of pseudoidentities for $\mathsf{V} ⓜ \mathsf{W}$.

5. An application of the factorization forest theorem

Already the preservation of local finiteness by the Mal'cev product of semigroup pseudovarieties is a non-trivial result depending on a finiteness theorem of Brown [**14, 15**] which was first proved using combinatorial methods. Several proofs of Brown's theorem are available in the literature. See [**41**, Theorem 4.2.4 and Notes to Chapter 4] for an algebraic proof and several references.

A particularly elegant proof of Brown's theorem is due to Simon [**44**] using his factorization forest theorem, which we proceed to recall.

For a set X, let $\mathcal{F}(X)$ be the set of all finite sequences $(x_1, \dots, x_n)$ of elements of X. The *length* of the sequence $s = (x_1, \dots, x_n)$ is n and is denoted by $|s|$. By a *factorization forest over* A we mean a pair $F = (X, d)$, where X is a subset of A^+ and $d\colon X \to \mathcal{F}(X)$ such that, for every $x \in X$, $d(x) = (x_1, \dots, x_n)$ implies $x = x_1 \cdots x_n$.

Let $F = (X, d)$ be a factorization forest over A. If $x \in X$, then we say that the *degree* of $x \in X$ is 0 if $|d(x)| = 1$, while it is $|d(x)|$ otherwise. The *external elements* of F are the elements of X of degree 0. The *height* $h(x)$ of x is defined recursively as follows: $h(x) = 0$ if x is external and $h(x) = 1 + \max\{h(x_i) : 1 \leqslant i \leqslant n\}$ if $d(x) = (x_1, \dots, x_n)$ with $n > 1$. The *height* of F is $h(F) = \sup\{h(x) : x \in X\}$.

Let $f\colon A^+ \to S$ be a semigroup homomorphism. A factorization forest F is *Ramseyan modulo* f if, whenever the degree of x is at least 3 and $d(x) = (x_1, \dots, x_n)$, we have $f(x) = f(x_1) = \cdots = f(x_n)$ and $f(x)$ is an idempotent. We say that f *admits* the factorization forest $F = (X, d)$ if $X = A^+$ and the external set is A.

Theorem 5.1 (Factorization forest theorem [**45**]). *Every homomorphism $f\colon A^+ \to S$ into a finite semigroup S admits a Ramseyan factorization forest of height at most $9|S|$.*

Relaxing the linear bound in the above theorem to an exponential bound, Simon ([**46**]) has also produced a simplified proof based on the Krohn–Rhodes decomposition theorem.

Our proof of the following result relies on the methods and ideas of Simon's proof of Brown's theorem [**44**, Theorem 5].

Theorem 5.2. *Let $\varphi\colon S \to T$ be a continuous homomorphism where S is a profinite semigroup generated by a finite set A and T is a finite semigroup. Then the semigroup S is algebraically generated by A together with the elements that map to idempotents in T.*

Proof: Let $\psi\colon \overline{\Omega}_A\mathsf{S} \to S$ be the continuous homomorphism mapping each generator to itself and let $f\colon A^+ \to T$ be obtained by restricting $\varphi \circ \psi$ to A^+. By Theorem 5.1, f admits a Ramseyan factorization forest $F = (A^+, d)$ of finite height H. For each integer $k \in [0, H]$, consider the subset S_k of S consisting of all elements of the form $\psi(w)$ where $w = \lim w_n$ for a sequence of words $w_n \in A^+$ with $h(w_n) \leqslant k$. Note that $S_H = S$. Let $E = E(T)$. By induction on k, we show that every element of S_k is a (finite) product of elements of $A \cup \varphi^{-1}(E)$.

Since the factorization forest F is admitted by f, the set of external elements of F is A. As A is a finite set, it follows that $S_0 = A$. Assume now, inductively, that $k > 0$ and S_{k-1} is contained in the subsemigroup $\langle A \cup \varphi^{-1}(E)\rangle$ generated by $A \cup \varphi^{-1}(E)$. Let $s \in S_k$ and let $(w_n)_n$ be a convergent sequence of words from A^+ of height at most k such that $s = \psi(\lim w_n)$. We claim that $s \in \langle A \cup \varphi^{-1}(E)\rangle$.

In the case of an infinite number of terms w_n of the sequence having height less than k, then $s \in S_{k-1}$ and the induction hypothesis yields the claim. On the other hand, if an infinite number of the w_n have degree 2, say $d(w_n) = (u_n, v_n)$, then $h(u_n)$ and $h(v_n)$ are both less than k. By compactness, there is a strictly increasing sequence of indices n_r such that the sequences $(u_{n_r})_r$ and $(v_{n_r})_r$ converge. Let $s' = \psi(\lim u_{n_r})$ and $s'' = \psi(\lim v_{n_r})$, so that $s = s's''$. Since s' and s'' belong to $\langle A\cup\varphi^{-1}(E)\rangle$ by the induction hypothesis, it follows that so does s.

Hence, we may assume that every w_n has degree at least 3. Since F is Ramseyan modulo f, we conclude that $f(w_n)$ is an idempotent. Since T is finite and f is continuous, it follows that $\varphi(s)$ is also an idempotent, that is, $s \in \varphi^{-1}(E)$. This concludes the induction step and the proof. □

6. Profinite semigroups with only finitely many regular $\mathcal{J}$-classes

In this section, we investigate profinite semigroups with only finitely many regular $\mathcal{J}$-classes and their relationship with pseudovarieties consisting of semigroups with only one regular $\mathcal{J}$-class, that is, subpseudovarieties of LG. We start with a couple of general results about Green's relations in profinite semigroups.

Proposition 6.1. *Let S be a profinite semigroup and let $\mathcal{K}$ be any of the Green's relations $\mathcal{L}$, $\mathcal{R}$, $\mathcal{J}$, $\mathcal{H}$, or the corresponding quasi-orders $\leqslant_{\mathcal{L}}$, $\leqslant_{\mathcal{R}}$, $\leqslant_{\mathcal{J}}$, and $\leqslant_{\mathcal{H}}$. Then two elements of S are $\mathcal{K}$-related if and only if their images under every continuous homomorphism onto a finite semigroup are $\mathcal{K}$-related.*

Proof: The proof is a more or less standard compactness argument which works more generally for a system of equations with given parameters (see [**3**, Section 5.6], where the argument is formulated only for relatively free profinite semigroups). For the sake of completeness, we present the proof in the case of $\mathcal{L}$. The other cases may be proved similarly. Since homomorphisms preserve the relation $\mathcal{L}$, we may consider a pair of elements $u, v \in S$ such that $\varphi(u)$ and $\varphi(v)$ are $\mathcal{L}$-related for every continuous homomorphism $\varphi\colon S \to T$ onto a finite semigroup T. We may consider an inverse system S_i of finite semigroups S_i indexed by a directed set I and onto homomorphisms $S_i \to S_j$ $(i \geqslant j)$ whose inverse limit is isomorphic to S. For each $i \in I$, let $\varphi_i\colon S \to S_i$ be the projection homomorphism. Let X_i be the set of all pairs (x, y) of elements of S^1 such that $\varphi_i(xu) = \varphi_i(v)$ and $\varphi_i(yv) = \varphi_i(u)$. Since φ_i is continuous, X_i is a closed subset of $S^1 \times S^1$. On the other hand, since I is a directed set and $X_i \subseteq X_j$ whenever $i \geqslant j$, every finite subfamily of the family of closed sets $(X_i)_{i\in I}$ has non-empty intersection. By compactness, the set $X = \bigcap_{i\in I} X_i$ is non-empty. Given a pair $(x, y) \in X$, we must have $xu = v$ and $yv = u$ since the two sides of each of these equations have the same image under every homomorphism φ_i. This shows that $u \mathrel{\mathcal{L}} v$. $\square$

The following is a simple application of Proposition 6.1.

Corollary 6.2. *Let S be a profinite semigroup. Then each of the Green's relations $\mathcal{L}$, $\mathcal{R}$, $\mathcal{J}$, and $\mathcal{H}$ is a closed equivalence relation and the corresponding quotient topological space is profinite.*

Proof: Since the argument is similar for all the considered relations, we deal only with $\mathcal{L}$. By Proposition 6.1, the relation $\mathcal{L}_S$ on S is the intersection of the closed relations $(\varphi \times \varphi)^{-1}\mathcal{L}_T$, where φ runs over all continuous homomorphisms $S \to T$ onto finite semigroups T. Hence, $\mathcal{L}_S$ is closed. Moreover, if $u, v \in S$ are not $\mathcal{L}$-related, then, again by Proposition 6.1, there exists a continuous homomorphism $\varphi\colon S \to T$ onto a finite semigroup T such that $\varphi(u)$ and $\varphi(v)$ are not $\mathcal{L}$-related. Since the $\mathcal{L}$ relation is preserved by homomorphisms, we obtain the following commutative diagram of continuous functions, where the vertical arrows are the natural mappings:

$$\begin{tikzcd} S \arrow[r, "\varphi"] \arrow[d] & T \arrow[d] \\ S/\mathcal{L}_S \arrow[r, dashed] & T/\mathcal{L}_T \end{tikzcd}$$

Thus, distinct points in the quotient space $S/\mathcal{L}_S$ may be distinguished by continuous mappings onto finite discrete sets, which shows that $S/\mathcal{L}_S$ is a profinite space. □

Proposition 6.3. *A countably generated pro-*LG *semigroup is countable if and only if it has only countably many idempotents and its subgroups are finite.*

Proof: Let S be a countably generated pro-LG semigroup. If S is countable, then it certainly has only countably many idempotents and all its subgroups are countable, whence finite (cf. Section 3). For the converse, the hypothesis guarantees that the minimum ideal K of S is countable since it is the union of the subgroups contained in it. Furthermore, since S is pro-LG, the Rees quotient S/K is a countably generated pro-N semigroup, whence countable. Thus, S is countable. □

The following result shows how LG may be used to decompose semigroups with only finitely many regular $\mathcal{J}$-classes. It depends on the well-known fact that a profinite semigroup is pro-LG if and only if the only semilattice that embeds in it is the trivial one.

Proposition 6.4. *Let S be a profinite semigroup with only finitely many regular $\mathcal{J}$-classes. Then, there is a continuous homomorphism $\varphi\colon S \to T$ onto a finite semigroup such that $\varphi^{-1}(e)$ is a pro-*LG *semigroup for every idempotent $e \in T$.*

Proof: Choose an element s_J from each regular $\mathcal{J}$-class J of S. By Proposition 6.1, there exists a continuous homomorphism $\varphi\colon S \to T$ onto a finite semigroup T such that the restriction of the $\mathcal{J}$-order of S to the elements s_J is isomorphic under φ to the restriction of the $\mathcal{J}$-order of T to the $\varphi(s_J)$.

We claim that $\varphi^{-1}(e)$ is pro-LG for every idempotent $e \in T$. Indeed, from the choice of φ it follows that there cannot be a pair of idempotents f and g in $\varphi^{-1}(e)$ such that $f <_{\mathcal{J}} g$. This implies that $\varphi^{-1}(e)$ is a pro-LG semigroup. □

We may now give a characterization of finitely generated profinite semigroups with only finitely many regular $\mathcal{J}$-classes in terms of Mal'cev products.

Corollary 6.5. *Let S be a finitely generated profinite semigroup. Then S has only finitely many regular $\mathcal{J}$-classes if and only if S is a pro-$(\mathsf{LG} \circledm \mathsf{V})$ semigroup for some locally finite pseudovariety V.*

Proof: Suppose first that S has only finitely many regular $\mathcal{J}$-classes. Consider a continuous homomorphism $\varphi\colon S \to T$ given by Proposition 6.4 and let V be the pseudovariety generated by the finite semigroup T. Then, by Theorem 4.1, S is a pro-$(\mathsf{LG} \circledm \mathsf{V})$ semigroup.

Conversely, if S is a pro-$(\mathsf{LG} \circledm \mathsf{V})$ semigroup with V locally finite, then Theorem 4.1 provides a closed relational morphism $\mu\colon S \multimap T$ into a pro-V semigroup T. Since S is finitely generated, we may assume that so is T. Since V is locally finite, it follows that T is finite. Given a regular element $s \in S$, there exists $s' \in S$ such that $ss's = s$ and $s'ss' = s'$. Take an idempotent e in the subsemigroup $\mu(ss')$ of T. Then, the idempotent ss' belongs to $\mu^{-1}(e)$. Since there is only one regular $\mathcal{J}$-class J in $\mu^{-1}(e)$, as $\mu^{-1}(e)$ is pro-LG, we conclude that s belongs to the $\mathcal{J}$-class of S containing J. Hence, there are only finitely many regular $\mathcal{J}$-classes in S. □

Pseudovarieties of the form $\mathsf{LG} \circledm \mathsf{V}$ play a special role in the semilocal theory of Rhodes (cf. [**41**, Theorem 4.6.50]).

The following theorem gives factorizations for elements of a finitely generated profinite semigroup with only finitely many regular $\mathcal{J}$-classes.

Theorem 6.6. *If S is a profinite semigroup generated by a finite set A and S has only finitely many regular $\mathcal{J}$-classes, then S is algebraically generated by A together with the group elements of S. Moreover, in the case of S having only countably many group elements, S is algebraically generated by $A \cup E(S)$.*

Proof: By Proposition 6.4, there is a continuous homomorphism $\varphi\colon S \to T$ onto a finite semigroup such that $\varphi^{-1}(e)$ is a pro-LG semigroup for every idempotent $e \in T$. By Theorem 5.2, S is algebraically generated by $A \cup \varphi^{-1}(E(T))$. By induction on the depth of an idempotent $e \in E(T)$ in the poset $E(T)/\mathcal{J}|_{E(T)}$, we show that every element in $\varphi^{-1}(e)$ is a product of elements of A and group elements. Indeed, given an idempotent $e \in T$, since $\varphi^{-1}(e)$ is pro-LG, its elements are either group elements or products of elements of $A \cup \varphi^{-1}(E_e)$, where E_e denotes the set of all idempotents of T strictly $\mathcal{J}$-above e. By the induction hypothesis, in the latter case such elements are themselves products of elements of A and group elements of S.

In the special case where S has only countably many group elements, we may choose a maximal subgroup G_e of $\varphi^{-1}(e)$ for each idempotent $e \in E(T)$. By assumption, G_e is countable, whence finite. There is, therefore, a continuous homomorphism $\psi\colon S \to U$ onto a finite semigroup U that separates the points of G_e for every $e \in E(T)$.

Now, let $g \in \varphi^{-1}(e)$ be a group element. Let ε be the idempotent in G_e and, in the semigroup $\varphi^{-1}(e)$, let f be the idempotent in the $\mathcal{R}$-class of g that is $\mathcal{L}$-equivalent to ε. Since the product $\varepsilon g f$ belongs to G_e and S is topologically generated by A, there exists a product w of elements of A such that $\psi(w) = \psi(\varepsilon g f)$ and $\varphi(w) = e$. As $\psi(G_e)$ is a group and $\psi(\varepsilon)$ is its idempotent, we have $\psi(w) = \psi(\varepsilon w \varepsilon)$. Since $\varphi^{-1}(e)$ is a pro-LG semigroup, $\varepsilon w \varepsilon$ is an element of G_e. As ψ separates the elements of G_e, we deduce from the equalities $\psi(\varepsilon w \varepsilon) = \psi(w) = \psi(\varepsilon g f)$ that $\varepsilon w \varepsilon = \varepsilon g f$. Taking into account that g^ω is an idempotent $\mathcal{R}$-equivalent to f and $g^{\omega+1} = g$, it follows that

$$f \varepsilon w \varepsilon g^\omega = f \varepsilon g f g^\omega = f g f g^\omega = g^{\omega+1} = g.$$

This shows that g is a product of elements of $A \cup E(S)$ and, in view of the first part of the proof, completes the proof of the theorem. $\square$

For the pseudovariety S of all finite semigroups, since the regular $\mathcal{J}$-classes of $\overline{\Omega}_A\mathsf{DS}$ are characterized by their content (cf. [**3**, Theorem 8.1.7]), Theorem 6.6 applies to every finitely generated pro-DS semigroup. Hence, Theorem 6.6 generalizes a result of Azevedo and the first author [**4**] (see also [**3**, Section 8.1]).

Corollary 6.7. *If S is a finitely generated profinite semigroup with only finitely many regular $\mathcal{J}$-classes and countably many group elements, then S is countable.*

Proof: By Theorem 6.6, S is algebraically generated by a countable set and, therefore, it is countable. $\square$

7. Profinite semigroups with only finitely many idempotents

In this section, we consider the special case of the setting of Section 6 when there are only finitely many idempotents.

Recall that the pseudovariety IE consists of all finite semigroups with only one idempotent.

Proposition 7.1. *A subpseudovariety* V *of* IE *is locally countable if and only if it is contained in* $\mathsf{N} \vee \mathsf{H}$ *for some locally finite pseudovariety* H *of groups.*

Proof: Suppose that V is locally countable. Then, so is the intersection $\mathsf{H} = \mathsf{V} \cap \mathsf{G}$. By Theorem 3.1, the pseudovariety H is locally finite. As observed in [**3**, Section 9.1], it follows that $\mathsf{V} \subseteq \mathsf{N} \vee \mathsf{H}$.

The converse follows from Theorem 3.3. □

The following is an elementary but useful observation that holds more generally for compact semigroups.

Lemma 7.2. *Let* S *be a profinite semigroup and suppose that its minimum ideal has a unique idempotent* e*. Then* e *is central in* S*.*

Proof: Let $s \in S$ and let K be the minimum ideal of S. As $(se)^\omega$ and $(es)^\omega$ are both idempotents in K, they are equal to e. Hence, we have

$$se = s(es)^\omega = (se)^\omega s = es. \qquad \square$$

The relationship between the cardinality of the set of idempotents of a profinite semigroup and the pseudovariety IE is considered in the following result.

Theorem 7.3. *Let* S *be a profinite semigroup.*

(i) *The set* $E(S)$ *is a singleton if and only if* S *is pro-*IE*.*

(ii) *If* $E(S)$ *is finite, then* S *is pro-*$(\mathsf{IE} \,ⓜ\, \mathsf{V})$ *for some locally finite pseudovariety* V*.*

(iii) *If* V *is a locally finite pseudovariety and* S *is a finitely generated pro-*$(\mathsf{IE} \,ⓜ\, \mathsf{V})$ *semigroup, then* $E(S)$ *is finite.*

Proof: (i) If S has two distinct idempotents, then they may be separated by a continuous homomorphism onto a finite semigroup T. Since T has at least two idempotents, it follows that T is not in IE. Hence, S is not pro-IE. Conversely, if S is pro-IE, then it embeds in a product of semigroups with only one idempotent and so it has only one idempotent.

(ii) Now, suppose that the profinite semigroup S has only finitely many idempotents. Since S is residually finite, there is an idempotent-separating continuous homomorphism $\varphi\colon S \to T$ onto a finite semigroup T. By (i), since $\varphi^{-1}(e)$ has only one idempotent for each $e \in E(T)$, the profinite semigroup $\varphi^{-1}(e)$ is pro-IE. Hence, by Theorem 4.1, S is pro-$(\mathsf{IE} \,ⓜ\, \mathsf{V}(T))$, where $\mathsf{V}(T)$ is the (locally finite) pseudovariety generated by T.

(iii) Suppose that S is a finitely generated pro-$(\mathsf{IE} \textcircled{m} \mathsf{V})$ semigroup where V is a locally finite pseudovariety. To show that S has finitely many idempotents, it suffices to show that so does $\overline{\Omega}_A(\mathsf{IE} \textcircled{m} \mathsf{V})$ for every finite set A. Consider the natural continuous homomorphism $\varphi\colon \overline{\Omega}_A(\mathsf{IE} \textcircled{m} \mathsf{V}) \to \overline{\Omega}_A\mathsf{V}$. Then, for each idempotent $e \in E(\overline{\Omega}_A\mathsf{V})$, the profinite semigroup $\varphi^{-1}(e)$ is pro-IE by Corollary 4.2, whence it has only one idempotent by (i). Since the image of every idempotent of $\overline{\Omega}_A(\mathsf{IE} \textcircled{m} \mathsf{V})$ is an idempotent of $\overline{\Omega}_A\mathsf{V}$, there are no further idempotents to consider other than those from the $\varphi^{-1}(e)$ with $e \in E(\overline{\Omega}_A\mathsf{V})$. Hence, $\overline{\Omega}_A(\mathsf{IE} \textcircled{m} \mathsf{V})$ has the same number of idempotents as the finite semigroup $\overline{\Omega}_A\mathsf{V}$. □

Note that the hypothesis that S is finitely generated may not be dropped from the statement (iii) of Theorem 7.3. Indeed, if A is the one-point compactification of an infinite discrete set, then the semigroup $\overline{\Omega}_A\mathsf{Sl}$ consists of idempotents and it is uncountable for, as is observed in [**12**, Example, Section 1.2], $\overline{\Omega}_A\mathsf{Sl}$ is isomorphic to the semilattice of closed subsets of A under union.

Theorem 7.4. *Let* V *be a locally countable subpseudovariety of* IE *and let* W *be a locally finite pseudovariety. Then* $\mathsf{V} \textcircled{m} \mathsf{W}$ *is locally countable.*

Proof: By Proposition 7.1, there is a locally finite pseudovariety of groups H such that $\mathsf{V} \subseteq \mathsf{N} \vee \mathsf{H}$. Let A be a finite set and consider the natural continuous homomorphism $\varphi\colon \overline{\Omega}_A(\mathsf{V} \textcircled{m} \mathsf{W}) \to \overline{\Omega}_A\mathsf{W}$, which is idempotent-separating since the preimage of each idempotent belongs to IE. By Theorem 6.6, we know that every element of $\overline{\Omega}_A(\mathsf{V} \textcircled{m} \mathsf{W})$ is a product of elements of A and group pseudowords. Hence, to show that $\overline{\Omega}_A(\mathsf{V} \textcircled{m} \mathsf{W})$ is countable, it suffices to show that it has only countably many group elements. In fact, we show that it has only finitely many group elements, that is, that every subgroup is finite. Since $\overline{\Omega}_A(\mathsf{V} \textcircled{m} \mathsf{W})$ has only finitely many idempotents, namely as many as $\overline{\Omega}_A\mathsf{W}$, that goal is achieved by showing, equivalently, that every regular $\mathcal{J}$-class J of $\overline{\Omega}_A(\mathsf{V} \textcircled{m} \mathsf{W})$ is finite.

Consider the closed subsemigroup S of $\overline{\Omega}_A(\mathsf{V} \textcircled{m} \mathsf{W})$ generated by J. The restriction of φ to S is still an idempotent-separating continuous homomorphism. The fact that the preimage of each idempotent belongs to IE guarantees that $\varphi(J) \cap \varphi(S \setminus J) = \emptyset$: if $u \in J$ and $s \in S$ are such that $\varphi(u) = \varphi(s)$, then there exists $v \in J$ such that uv is an idempotent from J and so $\varphi(uv) = \varphi((sv)^\omega)$; since $\varphi^{-1}(\varphi(uv))$ is pro-IE, it follows that $uv = (sv)^\omega$, which entails that $s \in J$. Hence, the profinite completely 0-simple semigroup given by the Rees quotient $S_0 = S/(S \setminus J)$ admits an idempotent-separating continuous homomorphism ψ onto a finite semigroup T. If s is an element of J but not a group element, then we claim that $\varphi(s)$ cannot be idempotent. Otherwise, we have $\varphi(s) = \varphi(s^\omega) = \varphi(0)$; since φ preserves $\mathcal{J}$-equivalence, all of J must be mapped to $\varphi(0)$, which contradicts the choice of φ. Hence, the preimage of each idempotent is a pro-H group and, therefore, it is locally finite. By Brown's theorem, we conclude that S_0 is locally finite. Observing that taking the Rees quotient by the ideal $S \setminus J$ to obtain S_0 does not affect J, we conclude that, to prove that J is finite, it suffices to show that S_0 is finitely generated.

Returning to $\overline{\Omega}_A(\mathsf{V} ⓜ \mathsf{W})$, given an idempotent $e \in J$, the profinite semigroup $U = \varphi^{-1}(\varphi(e))$ is open. Since the subsemigroup of $\overline{\Omega}_A(\mathsf{V} ⓜ \mathsf{W})$ generated by A is dense, we deduce that its intersection V with U is dense in U. Given $v \in V$, we have $e = v^\omega = (eve)^\omega$ since e is central in U by Lemma 7.2. Hence, eve lies in the maximal subgroup H_e containing e and H_e is generated by the elements of the form eve with $v \in V$. If $v = aw$ with $a \in A$ and $|w| = |v| - 1$, then ea and we belong to J and, by Green's lemma, since $eawe \in H_e$, there is an idempotent $e_1 \in J$ in the intersection of the $\mathcal{L}$-class of ea with the $\mathcal{R}$-class of we and so $eve = eae_1we$. Similarly, if $w = bu$ with $b \in A$ and $|u| = |w| - 1$, then there is an idempotent $e_2 \in J$ such that $e_1we = e_1be_2ue$. Proceeding inductively in this manner, we conclude that eve may be factorized as a product of elements of the form $e'ae''$ where $a \in A$ and e' and e'' are idempotents of J. In conclusion, finitely many elements of the special form $e'ae''$ ($a \in A$, $e', e'' \in E(J)$) generate a profinite subsemigroup T of S_0 that contains all maximal subgroups in J. Since S_0 is locally finite, we deduce that T is finite and, hence, the maximal subgroups in J are finite. Since J has only finitely many idempotents, we also conclude that J is finite. □

We may now improve Theorem 6.6 in the special case of the pseudovarieties of Theorem 7.4 as follows.

Corollary 7.5. *If* $\mathsf{H} \subseteq \mathsf{G}$ *and* $\mathsf{V} \subseteq \mathsf{S}$ *are locally finite pseudovarieties, then every pseudoword over* $(\mathsf{N} \vee \mathsf{H}) ⓜ \mathsf{V})$ *is an* ω*-word of height at most* 1.

Proof: By Theorem 6.6, it suffices to show that every idempotent e of the semigroup $\overline{\Omega}_A((\mathsf{N} \vee \mathsf{H}) ⓜ \mathsf{V})$ is an ω-word of height 1. Let $(u_n)_n$ be a sequence of words converging to e. Then we have $\lim u_n^\omega = e$. But, since $\overline{\Omega}_A((\mathsf{N} \vee \mathsf{H}) ⓜ \mathsf{V})$ has only finitely many idempotents, the convergent sequence $(u_n^\omega)_n$ must eventually stabilize at its limit e. Hence, e is indeed an ω-word of height 1. □

The special case where the pseudovariety V of Corollary 7.5 is Sl leads to quite familiar pseudovarieties. Indeed, it is easy to see that $(\mathsf{N} \vee \mathsf{H}) ⓜ \mathsf{Sl} = \mathsf{DH}$ and, in particular, $\mathsf{N} ⓜ \mathsf{Sl} = \mathsf{J}$. Thus, Corollary 7.5

generalizes the first author's result that every pseudoword over J is an ω-word of height at most 1 [**2**].

In view of the results of this section, it is natural to ask whether a countable finitely generated profinite semigroup is necessarily finitely generated in the signature consisting of multiplication and ω-power. This is left as an open problem.

8. Mal'cev product with locally finite pseudovarieties

The question addressed in this and the next sections is whether the Mal'cev product of a locally countable pseudovariety with a locally finite pseudovariety is also locally countable. First we consider the special case where the locally finite pseudovariety consists of nilpotent semigroups.

Theorem 8.1. *Suppose that S is a profinite semigroup and $\varphi\colon S \to N$ is a continuous homomorphism into a finite nilpotent semigroup N such that $\varphi^{-1}(0)$ is locally countable. If S is finitely generated, then so is $\varphi^{-1}(0)$. In particular, S is locally countable.*

Proof: Suppose that S is generated as a topological semigroup by a finite subset A. Let n be such that N satisfies the identity $x_1 \cdots x_n = 0$. We claim that $\varphi^{-1}(0)$ is generated by the following set, where A^k denotes the set of all words of length k in the alphabet A:

$$B = \varphi^{-1}(0) \cap \bigcup_{k<2n} A^k.$$

Note that the finite set B contains all A^k with $n \leqslant k < 2n$ since products of length at least n are zero in N. Given an arbitrary element s of $\varphi^{-1}(0)$, since $\varphi^{-1}(0)$ is an open set and the subsemigroup T of S generated algebraically by A is dense in S, there is a sequence $(w_r)_r$ of elements of $T \cap \varphi^{-1}(0)$ converging to s. Each w_r admits a factorization in the elements of A, say in k_r factors. If $k_r < n$, then w_r belongs to $A^k \cap \varphi^{-1}(0)$ with $k = k_r$. Otherwise, w_r admits a factorization into elements of $\bigcup_{n \leqslant k<2n} A^k$, simply by taking successive blocks of n factors of A and combining the remainder block with the previous block. Hence, s belongs to the closed subsemigroup generated by B, which shows that $\varphi^{-1}(0)$ is finitely generated as a topological semigroup. Since $\varphi^{-1}(0)$ is locally countable by assumption, it follows that it is countable. To conclude that S is countable it now suffices to apply Theorem 5.2. $\square$

In contrast, if $\varphi\colon G \to H$ is a continuous homomorphism from an infinite finitely generated profinite group to a finite group, then $\varphi^{-1}(1)$ is finitely generated as a topological group. To prove it, it suffices to consider the case where G is a free profinite group. The preimage $\varphi^{-1}(1)$ is then a clopen subgroup in which the intersection I with the free group F on the same finite set of generators is dense. Since I is a subgroup of finite index of F, I is finitely generated by the Nielsen–Schreier theorem. Hence $\varphi^{-1}(1)$ is finitely generated as a topological (semi)group. A more precise statement can be found in [**42**, Section 3.6].

Corollary 8.2. *Let* V *be a locally countable pseudovariety and let* W *be a locally finite pseudovariety of nilpotent semigroups. Then the pseudovariety* $\mathsf{V} \,ⓜ\, \mathsf{W}$ *is locally countable.* □

It is well-known that the atoms in the lattice of all pseudovarieties of semigroups are $[\![xy = 0]\!]$, Sl, the pseudovariety Ab_p of all finite elementary Abelian p-groups, where p is prime, and the pseudovarieties LZ and RZ, respectively of all finite left-zero and right-zero semigroups. Note that the largest pseudovariety not containing all but the first one is precisely N. The next theorem shows that the pseudovariety N is optimal in Theorem 8.1.

Theorem 8.3. *For every non-nilpotent finite semigroup T, there is a non-locally countable profinite semigroup S and a continuous homomorphism $\varphi\colon S \to U$ into a divisor of T such that $\varphi^{-1}(e)$ is locally countable for every $e \in E(U)$.*

Proof: It suffices to show that, if W is any of the atoms Sl, Ab_p, LZ, or RZ, then there is a locally countable pseudovariety V such that $\mathsf{V} \,ⓜ\, \mathsf{W}$ is not locally countable. This is proved in the next section for all but the atom RZ, for which the result follows from the case of LZ by left/right duality. □

Corollary 8.4. *Let* W *be a locally finite pseudovariety. Then* $\mathsf{V} \,ⓜ\, \mathsf{W}$ *is locally countable for every locally countable pseudovariety* V *if and only if* W *is contained in* N. □

9. Mal'cev products with non-nilpotent atoms

In this section we prove the claim stated in the proof of Theorem 8.3 that for every non-nilpotent atom W in the lattice of pseudovarieties of semigroups there is a locally countable pseudovariety V such that $\mathsf{V} \,ⓜ\, \mathsf{W}$ is not locally countable.

9.1. The atom Sl. The following construction seems to have been first used in [**40**] in the so-called synthesis theory and plays a role in several contexts [**38, 39, 18, 6, 10**]. Here, we extend it to profinite semigroups.

Let S and T be profinite semigroups and let $f\colon S \to T$ be a continuous mapping. Consider the Cartesian product $K = S \times T \times S$. We define on $U(S,T,f) = S \uplus K$ a multiplication extending that of S as follows:

- $(s_1, t, s_2)(s_1', t', s_2') = (s_1, tf(s_2 s_1')t', s_2')$;
- $s(s_1, t, s_2) = (ss_1, t, s_2)$;
- $(s_1, t, s_2)s = (s_1, t, s_2 s)$.

It is easy to see that this multiplication on $U(S,T,f)$ is associative (this has been done in the purely algebraic setting, for instance in [**6**, Lemma 3.1]). If we endow $U(S,T,f)$ with the coproduct topology of the space S with the product space K, then $U(S,T,f)$ is a compact zero-dimensional space in which the multiplication is continuous. By a theorem of Numakura [**32**, Theorem 1], it follows that $U(S,T,f)$ is a profinite semigroup.

Theorem 9.1. *For every non-trivial locally finite pseudovariety of groups* H*, the join* $\mathsf{N} \vee \mathsf{CS}(\mathsf{H})$ *is locally countable but the Mal'cev product* $(\mathsf{N} \vee \mathsf{CS}(\mathsf{H})) \circledcirc \mathsf{SI}$ *is not.*

Proof: Since it is well-known that $\mathsf{CS}(\mathsf{H}) = \mathsf{H} * \mathsf{RZ}$, it follows from Theorems 3.4 and 3.3 that the join $\mathsf{N} \vee \mathsf{CS}(\mathsf{H})$ is locally countable. To prove that $(\mathsf{N} \vee \mathsf{CS}(\mathsf{H})) \circledcirc \mathsf{SI}$ is not locally countable, we exhibit an uncountable finitely generated profinite semigroup U and a continuous homomorphism $\varphi\colon U \to F$ onto a finite semilattice F such that, for each $e \in F$, the subsemigroup $\varphi^{-1}(e)$ is pro-$(\mathsf{N} \vee \mathsf{CS}(\mathsf{H}))$. The theorem then follows from Theorem 4.1.

To construct U, consider the monoid $M = \mathbb{N} \cup \{\infty\}$ under addition, where the topology is given by the one-point compactification of the discrete set $\mathbb{N}$ of natural numbers. Let $G = \overline{\Omega}_M \mathsf{H}$ and let $f\colon M \to G$ be the natural generating function. We take $U = U(M,G,f)$ to be the profinite semigroup defined above.

For F, we take the meet-semilattice given by the three-element chain $0 \leqslant 1 \leqslant 2$. The mapping φ sends $0 \in M$ to 2, $M \setminus \{0\}$ to 1, and $K = M \times G \times M$ to 0. Note that φ is a continuous homomorphism. We claim that U is not locally countable.

We first observe that U is generated as a topological monoid by the two elements $a = 1 \in M$ and $b = (0,1,0)$, where the middle 1 is the idempotent of the group G. Indeed, $1 \in M$ generates the topological monoid M, while the elements $ba^ib = (0, f(i), 0)$ generate the topological group $\{0\} \times G \times \{0\}$ and $a^i(0,g,0)a^j = (i,g,j)$. Moreover, U is uncountable since G is isomorphic to each of the maximal subgroups in its minimum ideal and G is an infinite profinite group (as the function f is injective), whence uncountable, as observed in Section 3.

To prove the claim, it remains to show that K is locally finite. This follows from showing that K is pro-$\mathsf{CS}(\mathsf{H})$: the calculation

$$(i,g,j) = (i, gh^{-1}f(k)^{-1}, 0)(k,h,\ell)(0, f(\ell)^{-1}, j)$$

shows that K has only one $\mathcal{J}$-class while the maximal subgroups of K are isomorphic to G.

To conclude the proof, it suffices to observe that $\varphi^{-1}(2) = \{1\}$, $\varphi^{-1}(1) \simeq \overline{\Omega}_1\mathsf{N}$, and $\varphi^{-1}(0) = K$ is pro-$\mathsf{CS}(\mathsf{H})$, as shown above. By Theorem 4.1 it follows that the profinite monoid U is pro-$(\mathsf{N}\vee\mathsf{CS}(\mathsf{H}))ⓜ\mathsf{Sl}$. □

The semigroup U of the preceding proof may also be used to establish the following result, which shows that the hypothesis that W is locally finite may not be dropped in Corollary 8.2.

Theorem 9.2. *There is a locally finite pseudovariety* V *such that* $\mathsf{V}ⓜ\mathsf{N}$ *is not locally countable.*

Proof: Let S be the profinite semigroup that is obtained from U by removing the identity element. Note that, in the notation of the proof of Theorem 9.1, $I = K \cup \{\infty\}$ is an ideal of S and S/I is isomorphic to $\overline{\Omega}_1\mathsf{N}$. Taking into account that K is locally finite, it is easy to see that I is also locally finite. By the proof of Theorem 9.1, S is not locally countable. Moreover, if V is a pseudovariety such that I is pro-V, then S is pro-$(\mathsf{V}ⓜ\mathsf{N})$ by Theorem 4.1. Thus, to complete the proof, it suffices to exhibit a locally finite pseudovariety V for which I is pro-V. We claim that the locally finite pseudovariety $\mathsf{Sl}\vee\mathsf{CS}(\mathsf{H})$ has the required property.

Consider the product $P = K\times\{0,1\}$ of K with the two-element semilattice and the subset $T=(K\times\{0\})\cup\{(e,1)\}$, where $e = (\infty, f(\infty)^{-1}, \infty)$. Since e is an idempotent, T is a closed subsemigroup of P. The profinite semigroup P is pro-$(\mathsf{Sl}\vee\mathsf{CS}(\mathsf{H}))$ since K is pro-$\mathsf{CS}(\mathsf{H})$. To establish the claim, it thus suffices to show that T is isomorphic to I, which is a consequence of the following calculations: for (x,g,y) in K, we have

- $e(x,g,y) = (\infty, f(\infty)^{-1}\cdot f(\infty)\cdot g, y) = (\infty, g, y) = \infty(x,g,y)$;
- dually, $(x,g,y)e = (x,g,y)\infty$. □

9.2. The atom LZ. Our treatment of the atom LZ requires a more casuistic and complicated construction than that of Sl. This subsection provides a somewhat long and technical proof of the following result, while the terminology is explained later.

Theorem 9.3. *There is an aperiodic locally countable pseudovariety* U *of semigroups of dot depth* 1 *such that* $\mathsf{U}ⓜ\mathsf{LZ}$ *is not locally countable.*

Let k be a positive integer. We consider the semigroup S_k with zero given by the following presentation:

$$\langle a,b \mid a^{k+1}=a^k,\ b^{k+1}=b^k,\ a^kb^ka^k=a^k,\ b^ka^kb^k=b^k,\\ a^nb^na=b^na^nb=0\,(n<k)\rangle.$$

The elements of S_k may be represented by words of the form

$$w=a^{\gamma_0}b^{\gamma_1}\cdots a^{\gamma_{2\ell}}b^{\gamma_{2\ell+1}}. \tag{1}$$

To describe the unique representation of each element different from 0, we first denote by $E_k=\{0,1,\ldots,k\}$ the set of potential exponents, since we may assume that $\{\gamma_0,\gamma_1,\ldots,\gamma_{2\ell+1}\}\subseteq E_k$ as higher powers may be reduced by applying the rules $a^{k+1}\mapsto a^k$ and $b^{k+1}\mapsto b^k$. And, of course, we assume that only γ_0 and $\gamma_{2\ell+1}$ may take the value 0. Then we consider the relation $\prec_k$ on E_k given by the formula $i\prec_k j$ if $i<j$ or $i=j=k$. We may assume that for each $0\leqslant i<2\ell-1$ we have $\gamma_i\prec_k\gamma_{i+1}$, since otherwise we have $w=0$ by the defining relations $a^nb^na=b^na^nb=0$. For the same reason, we may assume that $\gamma_{2\ell-1}\prec_k\gamma_{2\ell}$ in the case $\gamma_{2\ell+1}\neq 0$. Furthermore, we may assume that at most two exponents take the value k, since otherwise we could shorten the word by applying the transformations $a^kb^ka^k\mapsto a^k$ or $b^ka^kb^k\mapsto b^k$. This describes canonical forms of words in S_k. The canonical form may be obtained from each word by applying rules which are oriented by the defining relations from longer words to shorter ones. To say it more formally, we may add a new symbol 0 and the rules $a0,0a,b0,0b\mapsto 0$, and mention that the resulting rewriting system is obviously confluent with the canonical forms described above. Finally, notice that the semigroup S_k is finite, since there are only finitely many canonical words.

Now, let V be the pseudovariety generated by the set of all such semigroups S_k with $k\geqslant 1$. Notice that all semigroups S_k are aperiodic and, therefore, we have $\mathsf{V}\subseteq\mathsf{A}$. More precisely, S_k is a semigroup of "dot depth 1". The Straubing–Thérien dot depth hierarchy is a filtration of A as an infinite increasing chain of pseudovarieties, which has been extensively studied. The natural definition of the hierarchy comes from language theory (cf. [**34**]) via Eilenberg's correspondence between pseudovarieties of semigroups and so-called "varieties of regular languages" [**19**]. The second level of the hierarchy (which starts at level 0, given by the trivial pseudovariety I) is known as dot depth 1 and has been shown by Knast [**29**] to be defined by the pseudoidentity

$$(exfye)^\omega xft(ezfte)^\omega=(exfye)^\omega(ezfte)^\omega, \tag{2}$$

where $e=u^\omega$, $f=v^\omega$, and t, u, v, x, y, z are distinct variables.

Lemma 9.4. *The semigroup S_k is of dot depth* 1.

Proof: All non-zero idempotents of S_k lie in the same $\mathcal{D}$-class D, which consists of all elements in canonical form (1) for which the first or second non-zero exponent is k. It follows that, if a product of the form esf is a factor of a non-zero idempotent, where e and f are idempotents, then esf belongs to D. Thus, esf is the only element in the intersection of the $\mathcal{R}$-class of e with the $\mathcal{L}$-class of f. Hence, if fte is also a factor of a non-zero idempotent, then the equality $esfte = e$ follows from Green's lemma (cf. [**26**, Proposition 2.3.7]) and aperiodicity. This shows that S_k satisfies the pseudoidentity (2). □

The next step for the proof of Theorem 9.3 is the following result.

Lemma 9.5. *The pseudovariety* V *is not locally countable.*

Proof: We fix the alphabet $A = \{a, b\}$ and consider the relatively free profinite semigroups $\overline{\Omega}_A\mathsf{A}$ and $\overline{\Omega}_A\mathsf{V}$. We denote by $\eta\colon \overline{\Omega}_A\mathsf{A} \to \overline{\Omega}_A\mathsf{V}$ and $\psi_k\colon \overline{\Omega}_A\mathsf{V} \to S_k$ ($k \geqslant 1$) the natural continuous homomorphisms; all these homomorphisms map the generating set A identically. We describe an uncountable subset X in $\overline{\Omega}_A\mathsf{A}$ such that the restriction of η to X is an injective mapping.

Let $s = (s_i)_{i\in\mathbb{N}}$ be an increasing sequence of natural numbers. For each such s, we consider the following sequence of words: we put $w_1 = a^{s_1}b^{s_2}$ and, for each $i > 1$, we let $w_i = w_{i-1}a^{s_{2i-1}}b^{s_{2i}}$. We claim that the sequence $(w_i)_{i\in\mathbb{N}}$ converges in $\overline{\Omega}_A\mathsf{A}$. Indeed, let $\alpha\colon \overline{\Omega}_A\mathsf{A} \to S$ be a continuous homomorphism into a finite aperiodic semigroup S. Then, for some n, we have $u^n = u^{n+1}$ for every element $u \in S$. In particular, we have $\alpha(a)^m = \alpha(a)^n$ and $\alpha(b)^m = \alpha(b)^n$ for every $m > n$. Thus, for such $m > n$, we have $\alpha(w_m) = \alpha(w_n(a^nb^n)^{m-n})$ and therefore $\alpha(w_m) = \alpha(w_n(a^nb^n)^n)$ for every $m \geqslant 2n$ and we see that the sequence $(\alpha(w_i))_{i\in\mathbb{N}}$ is eventually constant. Let w_s be the limit in $\overline{\Omega}_A\mathsf{A}$ of the converging sequence $(w_i)_{i\in\mathbb{N}}$. Let X be the set of all w_s's obtained in this way.

Let s and t be distinct increasing sequences of natural numbers. Let j be the minimum index such that $s_j \neq t_j$. Without loss of generality, we may assume that $s_j < t_j$. Now we put $k = s_j + 1$ and consider the semigroup S_k. Let us assume first that j is an even index. We see that, for i such that $2i > j$, we have $\psi_k(\eta(w_i)) = a^{s_1}b^{s_2}\cdots a^{s_{j-1}}b^{s_j}a^kb^k$. Let $w'_i = a^{t_1}b^{t_2}\cdots a^{t_{2i-1}}b^{t_{2i}}$ be the words in the given sequence converging to w_t. Since $t_j \geqslant k$, we get that, for every i such that $2i > j$, the equality $\psi_k(\eta(w'_i)) = a^{s_1}b^{s_2}\cdots a^{s_{j-1}}b^k$ holds. Therefore, the element $\psi_k(\eta(w_s))$ of S_k is equal to $a^{s_1}b^{s_2}\cdots a^{s_{j-1}}b^{s_j}a^kb^k$, while $\psi_k(\eta(w_t))$ is equal to $a^{s_1}b^{s_2}\cdots a^{s_{j-1}}b^k$. Thus, $\psi_k(\eta(w_s)) \neq \psi_k(\eta(w_t))$ in the case of even j. The case of odd j may be treated in the same way. Hence, we have $\eta(w_s) \neq \eta(w_t)$ and we conclude that X is uncountable and the restriction of η to the set X is an injective mapping, as claimed. $\square$

For each k, let T_k be the subsemigroup of S_k consisting of 0 and the elements in canonical form (1) with $\gamma_0 \neq 0$. Let U be the pseudovariety generated by all finite semigroups T_k ($k \geqslant 2$). Thus, we have $\mathsf{U} \subseteq \mathsf{V} \subseteq \mathsf{A}$.

Let C be a finite set. It is known that, for arbitrary $u \in \overline{\Omega}_C\mathsf{A}$ and every factor $x \in C$ of u, there are $u', u'' \in (\overline{\Omega}_C\mathsf{A})^1$ such that $u = u'xu''$ and x is not a factor of u', where u' is uniquely determined (see [**11**]).[2] We talk about the *first occurrence* of x in u. We denote by $\flat^x_y(u)$ the *number of occurrences* of an element y of C in u': if it exists, $\flat^x_y(u)$ is the maximum non-negative number n such that y^n is a subword of u'; otherwise, $\flat^x_y(u)$ is the symbol ∞. Thus, $\flat^x_y(u)$ is defined for any pair of distinct letters x, y such that x occurs in u. Dually, if we consider the last occurrence of x in u, then we get $u = v'xv''$, where x does not appear in the uniquely determined v''; we denote by $\sharp^x_y(w)$ the number of occurrences of y in v''. Furthermore, for each pair of not necessarily distinct letters x, y we consider the set $M_u(x,y)$ of all words $w \in C^*$ in which x and y do not appear and such that xwy is a factor of u. By Higman's theorem [**25**], for every set of words the set of its minimal members in the subword ordering is finite. We denote this finite subset of $M_u(x,y)$ by $m_u(x,y)$. Notice that $m_u(x,y)$ is empty if and only if u has no finite factor of the form xwy; also, $m_u(x,y) = \{1\}$ if and only if xy is a factor of u.

The following technical lemma plays a crucial role in our investigation of the pseudovariety U.

Lemma 9.6. *Let $u, v \in \overline{\Omega}_C\mathsf{A}$ be a pair of infinite pseudowords satisfying the following assumptions:*

(i) *Both u and v contain the word x^3 as a subword for every $x \in C$.*
(ii) *For every pair $x, y \in C$ of distinct letters, we have $\flat^x_y(u) = \flat^x_y(v)$ and $\sharp^x_y(u) = \sharp^x_y(v)$.*
(iii) *For each $x, y \in C$ we have $m_u(x,y) = m_v(x,y)$.*

Then $\mathsf{U} \models u = v$.

Proof: Let $\varphi\colon C \to T_k$ be an arbitrary mapping with $k \geqslant 2$; we denote by the same symbol the unique extension to a continuous homomorphism $\varphi\colon \overline{\Omega}_C\mathsf{A} \to T_k$. We want to show that $\varphi(u) = \varphi(v)$. Let us assume for a contradiction that $\varphi(u) \neq \varphi(v)$.

[2] Actually, by "equidivisibility" of the pseudovariety A, u'' is also unique. See [**5**], where equidivisible pseudovarieties are characterized.

If $\varphi(x) = 0$ for some $x \in C$, then $\varphi(u) = \varphi(v) = 0$. Thus, we may assume that, for each $x \in C$, the image $\varphi(x)$ is a canonical word of the form (1) with $\gamma_0 \neq 0$. Assume for a moment that some exponent γ_{2i+1} of b in $\varphi(x)$ is smaller than k. Since x^3 is a subword of u, we may factorize u in the following way: $u = u_0 x u_1 x u_2 x u_3$ for some pseudowords $u_0, \dots, u_3 \in (\overline{\Omega}_C \mathsf{A})^1$. Since the canonical forms of both $\varphi(u_1 x)$ and $\varphi(u_2 x)$ start with a, a representation of $\varphi(x u_1 x u_2 x)$ in the form (1) contains two factors of the form $ab^{\gamma_{2i+1}}a$, which means that $\varphi(u) = 0$. So, we reach a contradiction and from hereon we may assume that $\varphi(x)$ is of the form $a^{\gamma_0} b^k a^{\gamma_2}$ or it is a power of a. Denote by X the set of all $x \in C$ such that $\varphi(x)$ is a power of a. If all letters are mapped to powers of a, then we see that $\varphi(u) = \varphi(v) = a^k$ since both u and v are infinite pseudowords. Hence, we may assume that there is $y \in C \setminus X$ such that there is a factorization $u = u_0 y u'$ with $u_0 \in \overline{\Omega}_X \mathsf{A}$. By (i), there is also a factorization $v = v_0 y v'$ where y is not a factor of v_0. By (ii), the factors of v_0 from C are the same as those in u_0. So, y is also the leftmost factor of v from $C \setminus X$. Taking also into account the left/right dual argument, we conclude that we may assume that there are letters $y, z \in C \setminus X$ and factorizations $u = u_0 y u_1 z u_2$ and $v_0 y v_1 z v_2$ such that $u_0, v_0, u_2, v_2 \in (\overline{\Omega}_X \mathsf{A})^1$ are uniquely determined. By the assumption (ii), we get that $\varphi(u_0) = \varphi(v_0) = a^m$ and $\varphi(u_2) = \varphi(v_2) = a^n$ for some natural numbers m and n. Then there are just two possible values of $\varphi(u)$ and $\varphi(v)$, namely 0 and $a^{m+\gamma_0} b^k a^j$ with $j = k$ or $j = \gamma_2 + n$, where γ_0 is the exponent of the first a in $\varphi(y)$ and γ_2 is the exponent of the last a in $\varphi(z)$.

Now, since the images $\varphi(u)$ and $\varphi(v)$ are distinct, we see that exactly one of them is equal to 0. We may assume without loss of generality that it is $\varphi(u)$. Let $(u_n)_n$ be a sequence of words from C^+ converging to u in $\overline{\Omega}_C \mathsf{A}$. Since φ is continuous, without loss of generality we may assume that $\varphi(u_n) = 0$ for every n. Since every letter in $C \setminus X$ maps to an element of T_k of the form $a^{\gamma_0} b^k a^{\gamma_2}$ and those in X map to powers of a, we deduce that there is a finite factor $y'w'z'$ of u_n such that $\varphi(y') = a^{\alpha_0} b^k a^{\alpha_2}$, $\varphi(z') = a^{\beta_0} b^k a^{\beta_2}$, $\varphi(w') = a^m$ (where we let $a^0 = 1$), and $\alpha_2 + m + \beta_0 < k$. As there are only finitely many possibilities for such factors $y'w'z'$, for $|w'| < k$, there are infinitely many values of n for which we may choose the same such factor $y'w'z'$. Hence, there is also such a factor $y'w'z'$ of u. Using the hypothesis (iii) for the pair of letters y', z', we know that there is a word w'' which is a subword of w' and such that $y'w''z'$ is a factor of v. Clearly, $\varphi(w'') = a^n$ with $n \leqslant m$ and, therefore, we have $\varphi(v) = 0$, which contradicts the assumption at the beginning of the paragraph. □

We are now ready to establish the following result, which is part of the proof of Theorem 9.3.

Proposition 9.7. *The pseudovariety* U *is locally countable.*

Proof: Let C be a finite set and consider $\overline{\Omega}_C\mathsf{U}$ and $\overline{\Omega}_C\mathsf{A}$. We prove inductively with respect to the size of C that $\overline{\Omega}_C\mathsf{U}$ is countable. Since $\mathsf{U} \subseteq \mathsf{A}$, we see that $\overline{\Omega}_C\mathsf{U}$ is countable for every singleton set C. We next assume that $\overline{\Omega}_B\mathsf{U}$ is countable for every proper subset B of C. We show that there is a countable subset W of $\overline{\Omega}_C\mathsf{A}$ such that for each $u \in \overline{\Omega}_C\mathsf{A}$ there is $w \in W$ such that $\mathsf{U} \models u = w$. This proves that the cardinality of $\overline{\Omega}_C\mathsf{U}$ is at most the cardinality of the set W, whence $\overline{\Omega}_C\mathsf{U}$ is countable.

At first, assume that u does not contain all words from C^* as subwords and consider some shortest word v that is not a subword of u. Then, there is a factorization $v = a_1a_2\cdots a_na_{n+1}$, where $a_1a_2\cdots a_n$ is a subword of u. Moreover, there is a factorization of u of the form $u = u_1a_1u_2a_2\cdots a_nu_{n+1}$ such that each letter a_i does not occur in u_i for $1 \leqslant i \leqslant n+1$. Therefore, each u_i belongs to $\overline{\Omega}_B\mathsf{A}$ for some proper subset B of C. By the induction hypothesis, there is a countable set $W_B \subseteq \overline{\Omega}_B\mathsf{A}$ of representatives of elements of $\overline{\Omega}_B\mathsf{U}$. We construct the set $\overline{W}$ of all pseudowords of the form $w_0a_1w_1a_2\cdots a_nw_{n+1}$ with $w_i \in \overline{\Omega}_{B_i}\mathsf{A}$, where $B_i = C \setminus \{a_i\}$ $(i = 1, \ldots, n+1)$. The set $\overline{W}$ is countable and for each pseudoword u which does not contain all words from C^* as subwords there is a pseudoword $w \in \overline{W}$ such that $\mathsf{U} \models u = w$.

Thus, it remains to deal with the case when u contains all words from C^* as subwords. Denote by Z the set of all such pseudowords u. We consider an equivalence relation $\sim$ on Z such that $u \sim v$ if u and v satisfy the conditions (ii) and (iii) of Lemma 9.6; note that the definition of Z entails that its elements are infinite and that condition (i) of Lemma 9.6 is satisfied for every pair of elements of Z. Thus, the relation $\sim$ is in fact an intersection of finitely many equivalence relations on Z each of which has countably many classes. Hence, also $\sim$ has countably many classes and we may choose from each class one representative and collect them into the countable subset W' of $Z \subseteq \overline{\Omega}_C\mathsf{A}$. By Lemma 9.6, we deduce that for each $u \in Z$ there is $w \in W'$ such that $\mathsf{U} \models u = w$. We have proved that there is a countable set $W = \overline{W} \cup W'$ with the required property. □

Proposition 9.8. *The pseudovariety* $\mathsf{U} \circledcirc \mathsf{LZ}$ *is not locally countable.*

Proof: Let S be the two-element set $\{a, b\}$ and consider the operation on S given by $xy = x$. Thus, S is a left zero semigroup. For every $k \geqslant 2$, we denote by R_k the subsemigroup of $S_k \times S$, which is generated by the pair of elements (a,a) and (b,b). Clearly, both projections $\pi_1\colon S_k\times S\to S_k$ and $\pi_2\colon S_k\times S\to S$ to the first and second coordinates are surjective when restricted to R_k. Note that $\pi_2^{-1}(a)=T_k\times\{a\}\in\mathsf{U}$. Also, the subsemigroup $\pi_2^{-1}(b)$ is isomorphic to T_k by applying the automorphism of S_k that exchanges the generators a and b. Hence R_k belongs to $\mathsf{U}ⓜ\mathsf{LZ}$ and since S_k is a homomorphic image of R_k, we get $\mathsf{V}\subseteq\mathsf{U}ⓜ\mathsf{LZ}$. By Lemma 9.5, it follows that $\mathsf{U}ⓜ\mathsf{LZ}$ is not locally countable. □

Proof of Theorem 9.3: The result follows from Propositions 9.7 and 9.8 together with Lemma 9.4. □

9.3. The atom Ab_p. Throughout this subsection, p denotes a fixed prime number. Our aim is to establish the following result.

Theorem 9.9. *There is a locally countable pseudovariety U_p such that the Mal'cev product $\mathsf{U}_pⓜ\mathsf{Ab}_p$ is not locally countable.*

The proof follows the same lines as the proof of Theorem 9.3, based on a similar construction. We include just those details in which the new construction differs from the previous one.

Let k be a positive integer. We consider the semigroup $S_k(p)$ with zero given by the following presentation:

$$\langle a,b\mid a^2=0,\ b^{k+1}=b^k,\ b^k(ab^k)^p=b^k,\ b^nab^na=0\,(n<k)\rangle.$$

Since $a^2=0$, the elements of $S_k(p)$ different from 0 may be represented by words of the form

$$w=b^{\beta_0}ab^{\beta_1}ab^{\beta_2}a\cdots ab^{\beta_\ell}. \tag{3}$$

Here, the integer ℓ denotes the number of occurrences of the letter a in the word, including the case $\ell=0$, where the considered word is b^{β_0} with a positive exponent β_0. To describe canonical forms of words in $S_k(p)$, we use the same notation E_k and $i\prec_k j$ as in Subsection 9.2. So, we assume that $\{\beta_0,\dots,\beta_\ell\}\subseteq E_k$ since $b^{k+1}=b^k$, and only β_0 and β_ℓ may take the value 0, where b^0 is interpreted as the empty word. Furthermore, for $0\leqslant i<\ell-1$ we assume that $\beta_i\prec_k\beta_{i+1}$, and at most p of the exponents β take the value k, since otherwise we could shorten the word by applying the equality $b^k(ab^k)^p=b^k$. In particular, under all these assumptions, for canonical words of the form (3) we have $\ell\leqslant k+p$ and we conclude that there are only finitely many canonical forms representing elements of the semigroup $S_k(p)$, which is consequently finite. Moreover, for w in the canonical form (3) with $\ell\neq0$ and such that $w^3\neq0$, we see that $\beta_1=\cdots=\beta_{\ell-1}=k\leqslant\beta_0+\beta_\ell$. Then, for each $m\geqslant3$, we have $w^m=b^{\beta_0}(ab^k)^{m\ell-1}ab^{\beta_\ell}$. Thus $w^{p+3}=w^3$, an equality that holds also in the case where $w^3=0$. If w is the canonical word (3) with

$\ell = 0$, then $w = b^{\beta_0}$ and we have $w^{k+1} = w^k$. Hence $S_k(p)$ satisfies the identity $x^{\max\{k,3\}+p} = x^{\max\{k,3\}}$.

Let V_p be the pseudovariety generated by the semigroups $S_k(p)$ with $k \geqslant 2$. By the arguments at the end of the previous paragraph we obtain the inclusion $\mathsf{V}_p \subseteq [\![x^{\omega+p} = x^\omega]\!]$. We denote by W_p the latter pseudovariety which consists of all finite semigroups containing as non-trivial subgroups only groups of exponent p. Notice that $\mathsf{A} \subseteq \mathsf{W}_p$ for an arbitrary prime p.

Lemma 9.10. *For each prime p, the pseudovariety V_p is not locally countable.*

Proof: Although the proof is a quite straightforward modification of the proof of Lemma 9.5, we include the details since the canonical forms of elements in $S_k(p)$ are slightly different from those in S_k.

Let $\eta\colon \overline{\Omega}_A\mathsf{W}_p \to \overline{\Omega}_A\mathsf{V}_p$ and $\psi_k\colon \overline{\Omega}_A\mathsf{V}_p \to S_k(p)$ $(k \geqslant 2)$ be the natural continuous homomorphisms which map the generating set $A = \{a, b\}$ identically. Our aim is to show that $\overline{\Omega}_A\mathsf{V}_p$ is not countable.

For an increasing sequence of natural numbers $s = (s_i)_{i\geqslant 1}$, we consider the following sequence of words: $w_1 = ab^{ps_1}ab^{ps_2}\cdots ab^{ps_p}$ and for each $i > 1$ we put

$$w_i = w_{i-1}\, ab^{ps_{p(i-1)+1}}\, ab^{ps_{p(i-1)+2}} \cdots ab^{ps_{pi}}.$$

We show that the sequence $(w_i)_{i\geqslant 1}$ converges in $\overline{\Omega}_A\mathsf{W}_p$. Let S be an arbitrary finite semigroup from W_p and let $\alpha\colon \overline{\Omega}_A\mathsf{W}_p \to S$ be a continuous homomorphism. There is an integer n such that $u^n = u^{n+p}$ for every element $u \in S$. Clearly, if we fix such n (depending just on the semigroup S), we also have $u^m = u^{m+p}$ for every $m > n$. In particular, for every $m > n$, $u^{pm} = u^{pn}$ is a unique idempotent which is a power of $u \in S$. To simplify the notation, we denote $a_S = \alpha(a)$ and $b_S = \alpha(b)$. Then we have $\alpha(ab^{ps_m}) = a_S b_S^{ps_m} = a_S b_S^{pn}$ for every $m > n$ because $s_m \geqslant m$. Now, for each $m \geqslant 2n$, we deduce that $\alpha(w_m) = \alpha(w_n)(a_S b_S^{pn})^{p(m-n)} = \alpha(w_n)(a_S b_S^{pn})^{pn}$. Hence, the sequence $(\alpha(w_i))_{i\geqslant 1}$ is eventually constant and $(w_i)_{i\geqslant 1}$ converges. Let w_s be the limit of the converging sequence $(w_i)_{i\geqslant 1}$.

We show that $\eta(w_s) \neq \eta(w_t)$ for every pair of distinct increasing sequences of natural numbers $s \neq t$. This proves that $\overline{\Omega}_A\mathsf{V}_p$ is uncountable as there are uncountably many increasing sequences of natural numbers.

Assume that $(w_i)_{i\geqslant 1}$ and $(w'_i)_{i\geqslant 1}$ are the constructed converging sequences for s and t respectively. Let j be the minimum index such that $s_j \neq t_j$ and assume that $s_j < t_j$ as the opposite case can be treated in the same way. We put $k = p(s_j + 1)$ and consider the semigroup $S_k(p)$.

Denote by j' the unique integer such that $p(j'-1) < j \leqslant pj'$. For each $i \geqslant j'$, we have

$$\psi_k(\eta(w_i)) = ab^{ps_1}ab^{ps_2}\cdots ab^{ps_{j-1}}ab^{ps_j}(ab^k)^{pj'-j}(ab^k)^{p(i-j')}.$$

Thus we deduce that $\psi_k(\eta(w_s))$ is equal to $ab^{ps_1}\cdots ab^{ps_{j-1}}ab^{ps_j}(ab^k)^\ell$, for a certain $\ell \in \{1,\ldots,p\}$. On the other hand, the word

$$w_i' = ab^{pt_1}ab^{pt_2}\cdots ab^{pt_{pi}}$$

is such that, for every $i \geqslant j'$, the equality

$$\psi_k(\eta(w_i')) = ab^{ps_1}ab^{ps_2}\cdots ab^{ps_{j-1}}ab^k(ab^k)^{pj'-j}(ab^k)^{p(i-j')}$$

holds. Thus, $\psi_k(\eta(w_t))$ is equal to $ab^{ps_1}\cdots ab^{ps_{j-1}}(ab^k)^{\ell+1}$ with the same ℓ as above. We conclude that $\psi_k(\eta(w_s)) \neq \psi_k(\eta(w_t))$ in $S_k(p)$, which implies that $\eta(w_s) \neq \eta(w_t)$. $\square$

For each k, we denote by $T_k(p)$ the subsemigroup of $S_k(p)$ consisting of the element 0 and all elements in canonical form (3), where the number of occurring a's, that is, ℓ, is divisible by p. The pseudovariety generated by all semigroups $T_k(p)$ is denoted by U_p. Notice that $\mathsf{U}_p \subseteq \mathsf{V}_p \subseteq \mathsf{W}_p$.

Our aim is the same as in the previous subsection, namely to prove inductively with respect to the size of a finite set C that $\overline{\Omega}_C\mathsf{U}_p$ is countable. First of all, for $u \in \overline{\Omega}_C\mathsf{W}_p$, we may define all technical notions, that is, $\flat_y^x(u)$, $\sharp_y^x(w)$, $M_u(x,y)$, and $m_u(x,y)$, in the same way, because W_p satisfies the same conditions as A which enable us to apply the results from [**11**] when we define the first occurrence of the letter in the pseudoword u.

We formulate a slight modification of Lemma 9.6.

Lemma 9.11. *Let $u,v \in \overline{\Omega}_C\mathsf{W}_p$ be a pair of infinite pseudowords satisfying the following assumptions:*

(i) *For every $x \in C$, both u and v contain x^2 as a subword.*
(ii) *For every pair $x,y \in C$ of distinct letters, we have $\flat_y^x(u) = \flat_y^x(v)$ and $\sharp_y^x(u) = \sharp_y^x(v)$.*
(iii) *For each $x,y \in C$, we have $m_u(x,y) = m_v(x,y)$.*

Then $\mathsf{U}_p \models u = v$.

Proof: We proceed in the same way as in the proof of Lemma 9.6. Let $\varphi\colon \overline{\Omega}_C\mathsf{W}_p \to T_k(p)$ be a continuous homomorphism and assume for a contradiction that $\varphi(u) \neq \varphi(v)$.

The case when $\varphi(x) = 0$ for some $x \in C$ is clear. So, assume that, for each $x \in C$, the element $\varphi(x) \in T_k(p)$ is represented by the canonical word $\varphi(x) = b^{\beta_0}ab^{\beta_1}\cdots ab^{\beta_\ell}$ with ℓ divisible by p. If there is

$1 \leqslant i < \ell$ such that $\beta_i < k$, then $\varphi(u) = \varphi(v) = 0$ since x^2 is a subword of both u and v. Thus we may also assume that every $\varphi(x)$ is of the form $b^{\beta_0}(ab^k)^{p-1}ab^{\beta_p}$ or b^{β_0}. Now we denote by X the set of all $x \in C$ such that $\varphi(x)$ is a power of b and the rest of the proof is essentially the same as in the case of Lemma 9.6 and is omitted. $\square$

Proposition 9.12. *For each prime p, the pseudovariety U_p is locally countable.*

Proof: This result can be proved in the same way as Proposition 9.7. The pseudovariety A is replaced by W_p and the pseudovariety U by U_p. At the induction basis, we again have that $\overline{\Omega}_C \mathsf{W}_p$ is countable whenever C is a singleton set. $\square$

Proposition 9.13. *For every prime p, the pseudovariety $\mathsf{U}_p \circledcirc{m} \mathsf{Ab}_p$ is not locally countable.*

Proof: Let $G = \{1, a, a^2, \dots, a^{p-1}\}$ be a cyclic group of order p. For every $k \geqslant 2$, we consider the semigroup $S_k(p) \times G$ and its subsemigroup R_k generated by (a, a) and $(b, 1)$. Let $\pi_2 \colon R_k \to G$ be the restriction of the projection from $S_k(p) \times G$ onto the second coordinate. Then $\pi_2^{-1}(1) = T_k(p) \in \mathsf{U}_p$, where 1 is a unique idempotent in the group G. Thus R_k belongs to $\mathsf{U}_p \circledcirc{m} \mathsf{Ab}_p$. Considering the restriction of the projection from $S_k(p) \times G$ onto the first coordinate, we see that $S_k(p)$ is a homomorphic image of R_k. Thus, we have $\mathsf{V}_p \subseteq \mathsf{U}_p \circledcirc{m} \mathsf{Ab}_p$. We deduce that $\mathsf{U}_p \circledcirc{m} \mathsf{Ab}_p$ is not locally countable by Lemma 9.10. $\square$

Proof of Theorem 9.9: It suffices to invoke Propositions 9.12 and 9.13. $\square$

Unlike in the case of the atom LZ, we do not have a familiar upper bound for the pseudovariety V_p. Yet, the same argument as in the proof of Lemma 9.4 shows that $S_k(p)$ satisfies the pseudoidentity obtained from (2) by raising both sides to the pth power.

10. Pseudovarieties of aperiodic inverse semigroups

Recall that a semigroup is said to be an *inverse semigroup* if, for every element s, there is a unique element t such that $sts = s$ and $tst = t$; an element t satisfying these equalities is called an *inverse* of s. The existence of an inverse characterizes regular elements. An inverse semigroup may also be characterized as a semigroup in which every element is regular and in which idempotents commute. Another important property of inverse semigroups is that they arise precisely as semigroups of partial bijections, that is, of bijections between two subsets of a fixed set; the operation is composition and the set should be closed under taking function inverses.

Denote by Inv the class of all finite inverse semigroups. The pseudovariety of semigroups generated by Inv turns out to be ESl [**13**], which may be decomposed both as $\mathsf{Sl} * \mathsf{G}$ and as $\mathsf{Sl} ⓜ \mathsf{G}$. It is natural to ask whether a pseudovariety bound on the groups in a class of finite inverse semigroups leads to a smaller bound on a pseudovariety containing the class. The negative answer to this question may be found in [**24**, Theorem 5.3]: in particular, if $\mathsf{Inv} \cap \mathsf{A} \subseteq \mathsf{DA} ⓜ \mathsf{H}$ for a pseudovariety of groups H, then $\mathsf{H} = \mathsf{G}$.

The main purpose of this section is to show that the pseudovariety $\mathsf{ESl} \cap \mathsf{A}$ is not locally countable. The proof of this result requires some combinatorial tools which we proceed to introduce.

Given a word $w = a_1 a_2 \cdots a_n$ over an alphabet A, we consider the following linear A-labeled digraph Λ_w:

$$q_1 \xrightarrow{a_1} q_2 \xrightarrow{a_2} q_3 \longrightarrow \cdots \xrightarrow{a_n} q_{n+1}.$$

The vertices q_1 and q_{n+1} are called respectively the *beginning* and *end* vertices. For words $w_1, \dots, w_r$, the *flower digraph* $\Phi(w_1, \dots, w_r)$ is obtained by taking the disjoint union of the digraphs Λ_{w_i} $(i = 1, \dots, r)$ and identifying all beginning and end vertices to a single vertex which is denoted by q_0.

Consider the Prouhet–Thue–Morse endomorphism μ of the free monoid $\{a, b\}^*$, which is defined by $\mu(a) = ab$ and $\mu(b) = ba$. Noting that $\mu^n(a)$ is a prefix of $\mu^{n+1}(a)$, we see that the sequence of words $(\mu^n(a))_n$ determines an infinite word, denoted by $\mathbf{t}$, which is also known as the *Prouhet–Thue–Morse word*: the word $\mu^n(a)$ is the prefix of $\mathbf{t}$ of length 2^n. See [**1**] for the name attribution and many relevant properties. In particular, Thue ([**48**]) proved that $\mathbf{t}$ is *cube-free* and even *overlap-free* in the sense that it contains respectively no factor of the forms w^3 and $uvuvu$ with $u, w \in \{a, b\}^+$ and $v \in \{a, b\}^*$. The latter corresponds to a minimal situation where we find overlapping occurrences of the same word. By direct inspection, one verifies that the only words in $\{a, b\}^+$ of length at most 3 that are not factors of $\mathbf{t}$ are the cubes a^3 and b^3. Note also that if the finite word w is a factor of $\mathbf{t}$, then so is $\mu(w)$.

For $n \geqslant 0$, we let $\Gamma_n = \Phi(\mu^n(a), \mu^n(b))$, with vertex set V_n, where μ^0 means the identity mapping. The vertex q_0 of the graph Γ_n is denoted by 0_n. The first three $\{a, b\}$-labeled digraphs Γ_n are depicted in Figure 1.

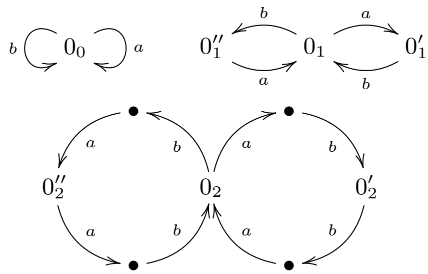

Figure 1. The flower digraphs Γ_0, Γ_1, and Γ_2.

For an A-labeled digraph Δ, one may view each letter $a \in A$ as representing a binary relation on the set of vertices, namely the relation $\Delta(a)$ consisting of the pairs of vertices (p, q) such that there is an edge $p \xrightarrow{a} q$ in the digraph. The monoid of binary relations generated by all $\Delta(a)$ with $a \in A$ is called the *transition* monoid of Δ and is denoted by $T(\Delta)$. Note that, for a word $w = a_1a_2 \cdots a_n$ $(a_i \in A)$, a pair of vertices (p, q) belongs to the product $\Delta(a_1)\Delta(a_2)\cdots\Delta(a_n)$ if and only if there is a path in Δ beginning at p and ending at q for which w is the concatenation of the labels of the successive edges; in this case, we also write $p \xrightarrow{w} q$. If the relations $\Delta(a)$ $(a \in A)$ are partial functions, then every word w in A^* induces a right action on the vertices of Δ by $p \cdot w = q$ if $p \xrightarrow{w} q$. For simplicity, we then also say that words of A^* *act* on Δ.

In the case where distinct edges either starting or ending at the same vertex have distinct labels, the relations $\Delta(a)$ are partial bijections. Since the words $\mu^n(a)$ and $\mu^n(b)$ start with different letters and end with different letters, that property holds for the flower digraphs Γ_n. We denote by M_n the transition monoid of the disjoint union $\tilde{\Gamma}_n = \biguplus_{i=0}^{n} \Gamma_i$. Since it is generated by partial bijections of a finite set, M_n embeds in the inverse monoid of all partial bijections of the vertex set of $\tilde{\Gamma}_n$ and so M_n belongs to the pseudovariety ESl. Note that the restriction of the mappings from M_{n+1} to the graph $\tilde{\Gamma}_n$ defines an onto homomorphism $\varphi_n \colon M_{n+1} \to M_n$. Also note that, if we let $T_n = T(\Gamma_n)$, then M_n is a subdirect product of the monoids $T_0, \ldots, T_n$. The fact that the identity mapping on generators does not extend to a homomorphism $T_{n+1} \to T_n$ explains the need to work with M_n to obtain an inverse system. For instance, $\mu^n(a^3)$ acts as a non-empty transformation on V_n while, as the arguments in the proof of Lemma 10.5 show, it acts as the empty transformation on V_{n+1}.

Recall that a language $L \subseteq A^*$ over a finite alphabet A is said to be *star-free* if it may be expressed in terms of the languages $\emptyset$, $\{1\}$, and $\{a\}$ ($a \in A$) using only the operations of finite union, finite product, and complement (in A^*) [**34**].

There is a nice alternative description of the semigroup T_n which we proceed to give. Consider Γ_n as a (deterministic, incomplete) automaton $\mathcal{A}_n$ with 0_n as the only initial and terminal state. The language it recognizes is simply the image $\operatorname{Im}\mu^n$ of the homomorphism μ^n. Since no quotient of the automaton recognizes the same language, T_n is the syntactic monoid of $\operatorname{Im}\mu^n$. We claim that the language $\operatorname{Im}\mu^n$ is star-free. We prove this below by induction on $n \geqslant 0$. The induction step depends on the following result.

Lemma 10.1. *Let η be an injective endomorphism of the free monoid A^* such that $\operatorname{Im}\eta$ is star-free. If $L \subseteq A^*$ is a star-free language, then so is $\eta(L)$.*

Proof: By assumption, L is obtained from the languages $\emptyset$, $\{1\}$, and $\{a\}$ ($a \in A$) by applying a finite number of times the operations of finite union, finite product, and complement in A^*. Proceeding by induction on the number of times the operations are applied, $\eta(L)$ may be obtained from the languages $\emptyset$, $\{1\}$, and $\{\eta(a)\}$ ($a \in A$) by applying the operations of finite union, finite product, and $\eta(K) \mapsto \eta(A^* \setminus K)$. Since finite languages are star-free and homomorphisms preserve union and multiplication, all we need to show is that, in the induction process, the image under η of the complement also produces star-free languages. Indeed, since η is injective, we obtain the following equalities:

$$\eta(A^* \setminus K) = \{\eta(w) : w \in A^* \setminus K\} = (\operatorname{Im}\eta) \cap (A^* \setminus \eta(K)).$$

By the induction hypothesis, we may assume that $\eta(K)$ is star-free. Hence, so is $\eta(A^* \setminus K)$ as $\operatorname{Im}\eta$ is assumed to be star-free. □

It is well-known and easy to check that syntactic monoids of star-free languages are aperiodic. The converse is a key result of Schützenberger [**43**].

We may now easily prove our claim.

Proposition 10.2. *The language $\operatorname{Im}\mu^n$ is star-free.*

Proof: We first note that an elementary hand (or computer) calculation shows that T_1 is a 15-element aperiodic inverse monoid. Hence, by Schützenberger's theorem, the language $\operatorname{Im}\mu = \{ab, ba\}^*$ is star-free.[3]

[3] The calculation is carried out in detail in Example 2.1 of Chapter 4 of [**33**], where a star-free expression for the language $\{ab, ba\}^*$ is also derived.

Furthermore, the language $\operatorname{Im}\mu^0 = \{a,b\}^*$ is star-free, being the complement of the empty language. Also note that μ is injective. Finally, as $\operatorname{Im}\mu^{n+1} = \mu(\operatorname{Im}\mu^n)$, assuming inductively that $\operatorname{Im}\mu^n$ is star-free, Lemma 10.1 yields that so is $\operatorname{Im}\mu^{n+1}$. □

As an immediate consequence of Proposition 10.2, we obtain the following result.

Corollary 10.3. *The monoid T_n is aperiodic.* □

The following is a little technical observation about the action of certain words on V_n.

Lemma 10.4. *The two words $\mu^{n+1}(a)$ and $\mu^{n+1}(b)$ act in the same way on V_n, namely as the identity at the single vertex 0_n.*

Proof: To prove the lemma, note that, by construction of the graph Γ_n, the words $\mu^n(a)$ and $\mu^n(b)$ label circuits at the vertex 0_n. Since $\mu^{n+1}(a) = \mu^n(a)\mu^n(b)$, this word also labels a circuit at the vertex 0_n. We need to show that it does not label a path starting at any other vertex of Γ_n. Indeed, otherwise, there would be some word $u = u_1u_2u_3$ ($u_i \in \{a,b\}$) and a factorization

$$\mu^n(u) = x\mu^{n+1}(a)y = x\mu^n(a)\mu^n(b)y$$

with $1 \leqslant |x| < 2^n$. In particular, $\mu^n(a)$ is a factor of $\mu^n(u_1u_2)$ which overlaps with $\mu^n(u_2)$ and $\mu^n(b)$ is a factor of $\mu^n(u_2u_3)$ which also overlaps with $\mu^n(u_2)$. Since both $\mu^n(u_1u_2)$ and $\mu^n(u_2u_3)$ are factors of $\mathbf{t}$ and no two consecutive occurrences of a factor of $\mathbf{t}$ may overlap, by considering respectively the factor $\mu^n(a)$ of $\mu^n(u_1u_2)$ and the factor $\mu^n(b)$ of $\mu^n(u_2u_3)$ we conclude that u_2 may be neither a nor b, which is absurd. Interchanging the roles of a and b, we conclude that also $\mu^{n+1}(b)$ acts on V_n precisely as the identity on the vertex 0_n. □

There is a natural graph homomorphism respecting labels $\gamma_n : \Gamma_{n+1} \to \Gamma_n$: it maps 0_{n+1} to 0_n and each vertex $0_{n+1} \cdot w$ to $0_n \cdot w$; this mapping is well defined since the two simple cycles at 0_{n+1} in Γ_{n+1} are labeled by the words $\mu^{n+1}(a)$ and $\mu^{n+1}(b)$, which fix 0_n. The preimage of 0_n in Γ_{n+1} consists of three vertices, namely 0_{n+1}, $0'_{n+1} = 0_{n+1} \cdot \mu^n(a)$ and $0''_{n+1} = 0_{n+1} \cdot \mu^n(b)$. The two vertices $0'_{n+1}$ and $0''_{n+1}$ are distinguished in particular by the fact that from the first only leaves an arrow labeled b (which is the first letter of $\mu^n(b)$) while from the second only leaves an arrow labeled a (the first letter of $\mu^n(a)$).

The following result was discovered by computer calculation for small values of n. The proof that it holds for all $n \geqslant 0$ is somewhat technical.

Lemma 10.5. *The monoid M_n satisfies the identity $x^4 = x^3$.*

Proof: It suffices to show that each monoid T_n $(n \geqslant 1)$ satisfies the identity $x^4 = x^3$.

We claim that if $w \in \{a,b\}^+$, then, on V_n, either w^3 acts as the empty transformation or w acts as a local identity. Indeed, having a path from p to q labeled w^3 implies that w^r is a factor of $\mu^n(u)$ for some word u that we may assume to be of minimum length. More precisely, there is a factorization $\mu^n(u) = xw^3y$ with $0 \leqslant \max\{|x|,|y|\} < 2^n$ and $0_n \cdot x = p$ (cf. Figure 2, where $u = u_1u_2\cdots u_m$, with the $u_i \in \{a,b\}$).

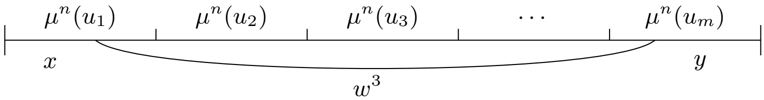

Figure 2. A factorization involving w^r.

Since $\mathbf{t}$ is cube-free, $\mu^n(u)$ cannot be a factor of $\mathbf{t}$ and, therefore, neither can u. Hence, we must have $|u| \geqslant 3$ and at least one of the words $\mu^n(a)$ or $\mu^n(b)$ must be a factor of w^3. Since the two situations are symmetric, we assume that $\mu^n(a)$ is a factor of w^3. By Lemma 10.4, $\mu^n(a)$ acts as the identity at the single vertex 0_{n-1} of Γ_{n-1}. We deduce that the domain of the action of $\mu^n(a)$ on V_n consists of the vertices 0_n and $0''_n$. Moreover, by definition of the graph Γ_n, $\mu^n(a)$ fixes both 0_n and $0''_n$.

If either $\mu^n(a)$ or $\mu^n(b)$ is actually a factor of w, then the domain of the (partial injective) action of w on V_n has at most two vertices. Since $p \cdot w^3$ is defined, either $p \cdot w = p$, and we are done, or $p \cdot w \neq p$ and $p \cdot w^2 = p$. The latter case is excluded because T_n is aperiodic by Corollary 10.3. Hence, we may assume that neither $\mu^n(a)$ nor $\mu^n(b)$ is a factor of w.

Suppose now that w is not a factor of either $\mu^n(a)$ or $\mu^n(b)$. In this case, the beginning of the factorization of $\mu^n(u)$ must be as in Figure 3.

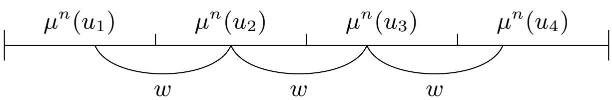

Figure 3. A factorization involving w^3.

Again, since $\mathbf{t}$ is cube-free, the word $u_1u_2u_3u_4$ cannot be a factor of $\mathbf{t}$ and so either the first three or the last three letters of $u_1u_2u_3u_4$ must be equal. In either case, there is a letter c such that w^2 appears as a factor of $\mu^n(c^3)$ as in Figure 4.

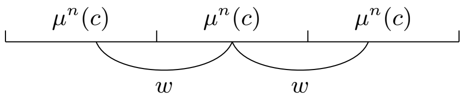

Figure 4. A factorization involving w^2.

If $|w| \neq 2^n$, as in Figure 4, then comparing the two factorizations of $\mu^n(c^2)$, we find an overlap of w within $\mathbf{t}$ (see Figure 5), which is impossible.

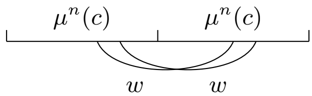

Figure 5. A factorization involving $\mu^n(c^2)$.

Hence, we must have $|w| = 2^n$ and w and $\mu^n(c)$ are conjugate words, that is, there is a factorization $\mu^n(c) = \alpha\beta$ such that $w = \beta\alpha$. It follows that $w^3 = \beta(\alpha\beta)^2\alpha = w^2$ in T_n as $\mu^n(c)$ acts as a local identity on V_n, and this fulfills our claim since w acts as a partial bijection on V_n.

It remains to consider the case where, in the factorization of Figure 2, there is at least one of the factors w in w^3 that falls completely and properly within one of the factors $\mu^n(u_i)$. Looking at such a factor $\mu^n(u_i)$, within which appears a full factor w in the factorization of Figure 2, we find a word $z = z_1z_2z_3$ (with $z_i \in \{a, b\}$) such that w^3 is a factor of $\mu^n(z)$. Since the word w^3 is not a factor of $\mathbf{t}$, the infinite word $\mathbf{t}$ cannot have z as a factor. Hence, we must find w^3 as a factor of $\mu^n(c^3)$ for some letter c, but not as a factor of $\mu^n(c^2)$. Depending on where each factor w starts, the picture may look different, but one possible configuration is represented in Figure 6.

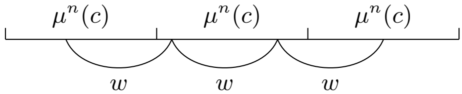

Figure 6. Another factorization involving w^3.

In any case, comparing again the two factorizations of $\mu^n(c^2)$, since $|w| < 2^n$, we find again an overlap within $\mu^n(c^2)$ unless $|w| = 2^{n-1}$ (see

Figure 7), which is impossible. In the exceptional case, looking at the middle factor $\mu^n(c)$, we see that there is a factorization $\mu^n(c) = rws$, where r is a suffix of w and s is a prefix of w; since $|\mu^n(c)| = 2|w|$, it follows that $w = sr$ and so $(rs)^2 = \mu^n(c) = \mu^{n-1}(c)\mu^{n-1}(d)$, where $\{c, d\} = \{a, b\}$, which is impossible since the factors $\mu^{n-1}(c)$ and $\mu^{n-1}(d)$ have the same length and start with distinct letters.

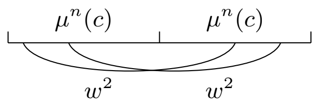

FIGURE 7. Another factorization involving $\mu^n(c^2)$.

This completes the proof of the lemma. □

Since M_n is defined as a monoid of binary relations, it is ordered by the relation $s \leqslant t$ if $s \supseteq t$ in the sense that $\leqslant$ is a partial order that is stable under multiplication.

The pseudovariety of ordered monoids $[\![1 \leqslant x^n]\!]$ is the main theme of the authors' paper [**8**]. The main problem considered in that paper is to determine the pseudovariety of monoids (or semigroups) $\langle[\![1 \leqslant x^n]\!]\rangle$ generated by the ordered monoids in $[\![1 \leqslant x^n]\!]$ once the order is forgotten. The conjectured result is the subpseudovariety of BG defined by the pseudoidentities $(xy^n)^\omega = (y^n x)^\omega$ and $x^{\omega+n} = x^\omega$, which is denoted by $(\mathsf{BG})_n$.

Lemma 10.6. *The ordered monoid M_n satisfies the inequality $1 \leqslant x^k$ for every $k \geqslant 3$.*

Proof: Note that, since idempotents of M_n are partial bijections, they are identity mappings of some subset of the vertex set of $\tilde{\Gamma}_n$. Thus, M_n satisfies the inequality $1 \leqslant x^\omega$. By Lemma 10.5, M_n satisfies the identity $x^\omega = x^k$ for every $k \geqslant 3$ and whence also the inequality $1 \leqslant x^k$. □

To simplify the proof and as it is sufficient for our purposes, the following lemma states only a special case of a much more general phenomenon.

Lemma 10.7. *Given any transformation $t \in T(\Gamma_n)$ of domain 0_n, there are words u and v whose action on V_n coincides with t and whose actions on V_{n+1} are distinct transformations of domain 0_{n+1}.*

Proof: Suppose that t is given by $t\colon 0_n \to p$. Let w be the label of the shortest path from 0_n to p. Since w is a prefix of $\mu^n(c)$ for some $c \in \{a, b\}$ (there is a choice for c only in the case $w = 1$), we consider such a letter c. Let $u = \mu^{n+2}(a)w$ and $v = \mu^{n+2}(a)\mu^n(d)w$, where d is the letter such that $\{c, d\} = \{a, b\}$. By Lemma 10.4, the domain of the actions of u and v on V_{n+1} is reduced to 0_{n+1} and they both act like t on V_n. $\square$

The homomorphisms $\varphi_n\colon M_{n+1} \to M_n$ constitute a chain and determine an inverse limit $\varprojlim M_n$. Our key result of this section is the following.

Theorem 10.8. *The inverse limit $\varprojlim M_n$ is uncountable.*

Proof: We may build a tree by taking as vertices at level n the elements of M_n and letting the sons of a vertex $s \in M_n$ be the elements of $\varphi_n^{-1}(s)$. The elements of $\varprojlim M_n$ may then be identified with the infinite simple paths from the root. The result will follow from showing that the complete infinite binary tree embeds in our tree. The existence of such an embedding follows from Lemma 10.7, which provides two alternatives for lifting elements of M_n whose action on V_n is a transformation with domain 0_n to transformations whose action on V_{n+1} has domain 0_{n+1}. $\square$

We conclude this section with consequences of Theorem 10.8.

Corollary 10.9. *None of the pseudovarieties $\mathsf{A} \cap \mathsf{ESl}$, BV (for every pseudovariety V), $\langle [\![1 \leqslant x^n]\!] \rangle$, and $(\mathsf{BG})_n$ ($n \geqslant 3$) is locally countable.*

Proof: By Lemma 10.5, the monoid M_n is aperiodic and its idempotents commute. Hence, the semigroup $S = \varprojlim M_n \setminus \{1\}$ is a pro-($\mathsf{A} \cap \mathsf{ESl}$) semigroup on two generators. Since S is uncountable by Theorem 10.8, so is $\overline{\Omega}_2(\mathsf{A} \cap \mathsf{ESl})$. For BV, the result follows from the inclusions $\mathsf{A} \cap \mathsf{ESl} \subseteq \mathsf{Bl} \subseteq \mathsf{BV}$. For the remaining pseudovarieties, the result follows from the inclusion $\langle [\![1 \leqslant x^n]\!] \rangle \subseteq (\mathsf{BG})_n$ [**8**, Proposition 4.2], Lemma 10.6, and Theorem 10.8. $\square$

The questions as to whether $(\mathsf{BG})_n$ or $\mathsf{B}[\![x^n = 1]\!]$ are locally countable were raised in our paper [**8**, Section 8]. With the restriction $n \geqslant 3$, Corollary 10.9 does not handle the case of $(\mathsf{BG})_2$ and $\langle [\![1 \leqslant x^2]\!] \rangle$, while $(\mathsf{BG})_1 = \langle [\![1 \leqslant x]\!] \rangle = \mathsf{J}$ is locally countable. We will not go into details, but the case of $n = 2$ may be treated similarly by working with a suitable substitution generating an infinite square-free word instead of the Prouhet–Thue–Morse substitution. One such substitution over a three-letter alphabet is given by $\varphi(a) = abc$, $\varphi(b) = ac$, $\varphi(c) = b$ [**30**, Proposition 2.3.2]. The idea is to start with the flower digraph $\Phi(\varphi^n(a), \varphi^n(b), \varphi^n(c))$ and apply the Stallings folding procedure (see [**47, 31, 27**]) to reduce it to a labeled digraph Λ_n in which each letter acts as a partial bijection. The graphs Λ_n for $n = 1, \dots, 5$ are drawn in Figure 8. The transition semigroup of Λ_n plays the role of the semigroup T_n in the above argument. Then, one may prove analogs of the previous results in this section, which yield an extension of Corollary 10.9 to cover the case $n = 2$. We leave open the question as to whether the profinite semigroups $\overline{\Omega}_2\langle[\![1 \leqslant x^2]\!]\rangle$ and $\overline{\Omega}_2(\mathsf{BG})_2$ are countable.

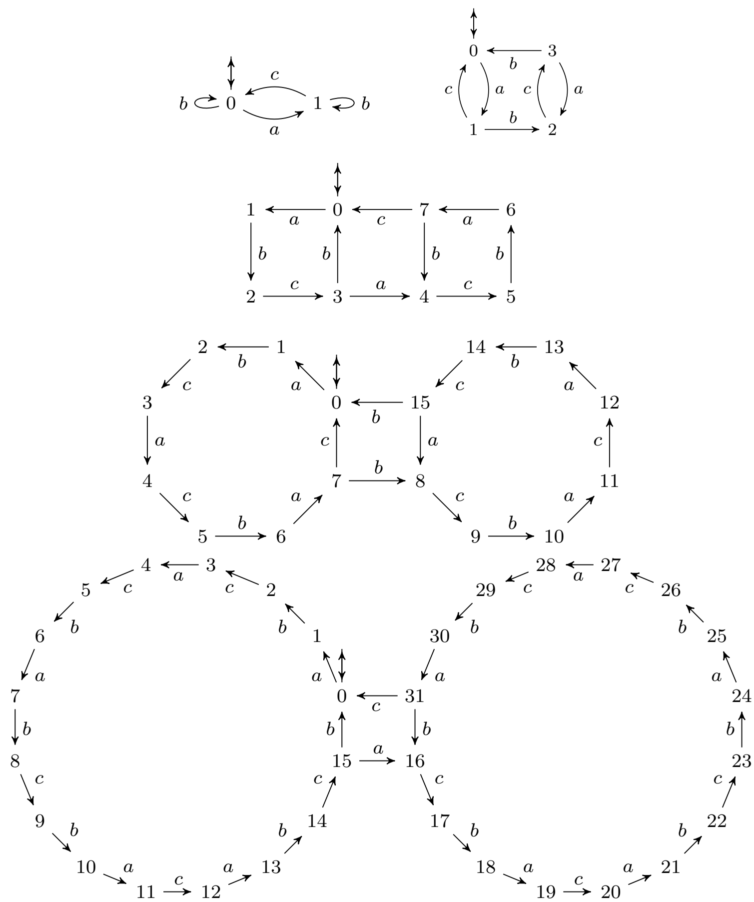

FIGURE 8. The labeled digraphs Λ_n as automata for $n = 1, \dots, 5$.

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