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PROBLEMS ON CHARACTERS: SOLVABLE GROUPS

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Dedicated to the memory of Carlo Casolo

Abstract: I review some problems on characters of finite solvable groups, while introducing new ones.

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1. Introduction

The world of representation theory of finite groups is often too much focused on simple groups. This is not a complaint: simple groups are fascinating objects, with an enthralling history, that have deep connections with important parts of mathematics, and that attract the attention of profound mathematicians. Furthermore, many problems on finite groups reduce to questions on simple groups, and most of the time these questions are solved (using the classification of finite simple groups, CFSG from now on), hence settling the problems. These are often quite difficult questions that require Deligne–Lusztig theory (in the case of groups of Lie type), sophisticated combinatorics (in the case of the alternating groups), or incredibly clever ad hoc arguments (in the sporadic groups). Sometimes these questions even require investigations of the covering groups of the simple groups (quasi-simple groups), or of their automorphism groups (almost simple groups). The modular (characteristic p) representation theory of finite simple groups is far from understood: there is a vast and exciting territory on simple groups that remains to be discovered. If there has to be a complaint, it is that once a problem is solved using the CFSG it is rarely revisited to explore a CFSG-free proof; and if there has to be another complaint, it is that we are never certain that a proof that uses the CFSG is *fully understood*. But perhaps

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this is a feature of finite group theory: certain theorems are true, almost surely, because simple groups satisfy certain properties.

Where do solvable groups fit into this picture? Certainly, the history of solvable groups, from E. Galois to P. Hall, from Burnside's $p^a q^b$ theorem to the Feit–Thompson odd order theorem, is no less extraordinary. These days the representation theory of finite groups is overwhelmed with a myriad of extraordinarily deep problems (from Brauer's to the Alperin–Broué–Dade–McKay–Robinson conjectures, and many more), and it is true (particularly in the above mentioned cases) that they have already been solved for solvable groups. (As a matter of fact, some of them have already been solved for simple groups too.) A consequence of this is that one often has the impression that there is little left to do for solvable groups, or that solvable groups are *easy*. But this is not the whole story.

Aside from the beauty of some of the character theory of solvable groups (for instance, the so called Isaacs π-theory, Dolfi's theorem on graphs, Berger's characterization of primitive characters, Dade–Isaacs theorems on monomial groups, etc.), for me solvable groups is also the territory to test and develop theories and conjectures for all finite groups. Even to outline possible proofs for some of the most important conjectures. For instance, our approach to the McKay conjecture in [**29**] is based on the proof of the p-solvable case and the Glauberman correspondence.

Unfortunately, it is sometimes forgotten how difficult solvable groups can be. Many times, I measure the depth of a conjecture by checking how hard it is to prove it for solvable groups. Richard Brauer's $k(B)$-conjecture—completed by D. Gluck, K. Magaard, U. Riese, and P. Schmid for p-solvable groups, after work of R. Gow, R. Knörr, G. R. Robinson, J. G. Thompson, and others—or one direction of Brauer's height zero conjecture—solved by D. Gluck and T. R. Wolf for p-solvable groups—are examples of incredibly hard mathematics. (As a matter of fact, these two conjectures remain open for general finite groups: if a problem is difficult for solvable groups, hold no doubt that it is going to be very difficult to prove it for general finite groups or to reduce it to a question on simple groups.)

My purpose here is modest. For some years, I have been collecting what I think are interesting problems on characters of solvable (mostly p-solvable) groups. Some of them are false outside the realm of these groups. Some others are convenient reformulations of less established conjectures that are still open for solvable groups. If we can solve them for these groups, there is a good chance that they will be true in general.

(I always remember J. L. Alperin saying: "a conjecture is *safe* if it is true for solvable groups, symmetric groups, the general linear groups, and the sporadic groups.") Hence, a conjecture on finite groups that has not been proven for solvable groups is either in danger or, perhaps, it is a promise for exciting mathematics ahead.

I avoid here the well-known open topics on character degrees of solvable groups, such as the Isaacs–Seitz conjecture on the derived length and the number of character degrees; Huppert's ρ-σ conjecture; Gluck's largest character degree conjecture; and some others already mentioned in [**48**]. Let me add now that in the ten years since its publication, there has been spectacular progress on the problems in [**48**]; in fact, many have been solved: McKay's conjecture for $p = 2$ in [**41**] (Problem 2.1 in [**48**]); one implication of Brauer's height zero conjecture in [**35**] (Problem 4.1); Conjecture 4.2 on the generalized Gluck–Wolf theorem in [**51**]; Conjecture 7.1 on the divisibility of the Glauberman correspondents in [**15**]; Problem 8.4 on character degrees in [**49**]; and a counterexample to Conjecture 14.3 on rational groups and Sylow 2-subgroups appears in [**30**]. In contrast, on solvable groups, there has been little or no progress.

Some of the problems below appear here for the first time. Of course, these new problems (let me avoid the overused term "conjecture") are here because I haven't been able to solve them. (This indicates either that they are not completely trivial or perhaps that they are completely false. In either case, an answer would be more than welcome.) I, together with some colleagues, have put forward some conjectures—sometimes questions—without having a proof even for solvable groups. As I said above, this is a risky business. (An anecdote: the main conjecture of the recent [**14**], proved for many classes of simple groups, is false: `SmallGroup`(192,955) is a counterexample for $p = 2$. This is a small counterexample, but as is well known, solvable counterexamples might be quite big and remain out of reach of the power of computers, like Hertweck's counterexample on the isomorphism problem [**22**].) Perhaps only a Richard Brauer can propose the $k(B)$-conjecture or the height zero conjecture, without having a proof for solvable groups, and still be (most probably) right.

In summary, let me write down a list of these problems not only as a tribute to solvable groups, the area where I grew up, but with the hope that they can interest someone, or reactivate interest in them, perhaps enough to try to solve them or to provide counterexamples. Surely there are other interesting open problems on characters of solvable groups that do not appear here, either because I am overlooking them, or because they lie a bit outside of my main interests right now. I apologize for that.

2. Problems on blocks: inequalities

We fix a prime p. In most of the problems in this paper we consider finite p-solvable groups, which are the natural extension of solvable groups from the perspective of the prime p. Recall that a finite group is *p-solvable* if the order of each of its composition factors is either p or not divisible by p. (Hence a finite group is solvable if and only if it is p-solvable for all primes p.) In any case, as a first approach, the reader might ignore p-solvability and focus on solvable groups throughout the paper. My notation for finite groups is [**26**], for ordinary characters [**25**], and for blocks and Brauer (modular) characters [**46**]. Thus $\mathrm{Irr}(G)$ is the set of complex irreducible characters of G, and $\mathrm{IBr}(G)$ is a set of irreducible Brauer characters of G. If G is p-solvable, then $\mathrm{IBr}(G)$ is uniquely defined (see Corollary 10.4 of [**46**]).

Brauer's p-blocks are the subject of some famous conjectures, some arising as early as the 1950's. By results of P. Fong and I. M. Isaacs, most of the block theory for p-solvable groups can be entirely rephrased within ordinary character theory. Recall that if F is an algebraic closure of the field of p elements, then the *blocks* of a finite group G are the two-sided indecomposable ideals into which the group algebra FG decomposes. Each irreducible complex or modular character of G *belongs* exactly to a p-block B, and B uniquely determines a conjugacy class of p-subgroups, the *defect groups*, which in some sense measure the complexity of B. If $\chi \in \mathrm{Irr}(G)$ belongs to B, and B has defect group D of order p^d, then it is a fact that

$$\chi(1)_p = \frac{|G|_p}{p^d} p^h,$$

where $h \geq 0$ is the *height* of χ. (In this paper, n_p is the largest power of p dividing the integer n, and $n_{p'} = n/n_p$.) If B is a p-block, then $k(B)$ is the number of irreducible ordinary characters in B, and $l(B)$ is the number of irreducible Brauer characters in B. We have that $l(B) \leq k(B)$ (see, for instance, the paragraphs after Theorem 3.3 in [**46**]). The *principal* block is the block containing the trivial character.

Brauer's two most famous conjectures in blocks are the $k(B)$-conjecture, which asserts that the number of complex irreducible characters in B is less than or equal to p^d, the order of any defect group D; and Brauer's height zero conjecture, which asserts that D is abelian if and only if the heights of the complex irreducible characters in B are all zero.

The first block problems in this section are, after standard Clifford theoretical reductions, translations to p-solvable groups of some questions raised by G. Malle and me in [**37**] (which generalize Brauer's $k(B)$-conjecture). Affirmative answers to these have been given for classical families of simple groups, most recently in [**36**], or for certain types of defect groups in [**57**]. There is even a generalization of these problems to fusion systems ([**34**, Conjecture 2.10]). Therefore it is natural to think that p-solvable groups is the next territory to check the inequalities proposed. As we wrote in [**37**], "for p-solvable groups, however, it seems possible that these inequalities might be too optimistic." In any case, it is frustrating that no progress (on p-solvable groups) has been made on this since 2006.

Blocks B of arbitrary finite groups with exactly one irreducible Brauer character (we usually write $l(B) = 1$) are extraordinarily important: these are the blocks such that $B/\mathbf{J}(B)$ is a matrix algebra, where $\mathbf{J}(B)$ is the Jacobson radical of the algebra B. R. Kessar and M. Linckelmann have conjectured that if $l(B) = 1$, then B is Morita equivalent to a block of a p-solvable group. If this is true, but even if not, Problem 2.1 below (which deals with this type of blocks) has interest.

First, some notation. If G is a finite group, $N \triangleleft G$, and $\theta \in \mathrm{Irr}(N)$, then $\mathrm{Irr}(G|\theta)$ is the set of irreducible constituents of the induced character θ^G. By Frobenius reciprocity, these are also the irreducible characters χ of G that have θ as an irreducible constituent of the restriction character χ_N. In general, if $N \triangleleft G$ has order not divisible by p, B is a p-block of G, and $\mathrm{Irr}(B)$ is the set of complex irreducible characters in B, then it is easy to check that there exists some $\theta \in \mathrm{Irr}(N)$ such that $\mathrm{Irr}(B) \subseteq \mathrm{Irr}(G|\theta)$ (using the theory of covering blocks; see Theorem 9.2 of [**46**]). Furthermore, if $N = \mathbf{O}_{p'}(G)$ is the largest normal subgroup of G of order not divisible by p, θ is G-invariant and G is p-solvable, then

$$\mathrm{Irr}(B) = \mathrm{Irr}(G|\theta),$$

by a theorem of Fong (Theorem 10.20 of [**46**]). Using the theory of central extensions, it is no loss to assume in many problems that N is even contained in the center of G.

If G is a finite group, we write $k(G)$ for the number of conjugacy classes of G, and $l(G)$ for the number of conjugacy classes of G consisting of elements of order not divisible by p. (These are the *p-regular* elements of G.) A p-solvable group has a unique conjugacy class of *p-complements*, also called Hall p'-subgroups (after P. Hall), which are the subgroups H of G such that $|G : H|$ is a power of p and p does not divide $|H|$.

Problem 2.1. *Let G be a finite p-solvable group, $P \in \mathrm{Syl}_p(G)$, and let H be a Hall p-complement of G. Suppose that $Z = \mathbf{O}_{p'}(G)$ is central in G, and let $\lambda \in \mathrm{Irr}(Z)$. Assume that $\lambda^H = e\alpha$, for some $\alpha \in \mathrm{Irr}(H)$ and $e \geq 1$. Is it true that $|\mathrm{Irr}(G|\lambda)| \leq k(P)$?*

Problem 2.1 is a particular case of a *projective version* of a conjecture of Pyber–Sangróniz which I mentioned in [**48**]: if N is a central normal subgroup of G, $\theta \in \mathrm{Irr}(N)$, and $G = AB$ for some subgroups A, B containing N with $(|A/N|, |B/N|) = 1$, then

$$|\mathrm{Irr}(G|\theta)| \leq |\mathrm{Irr}(A|\theta)||\mathrm{Irr}(B|\theta)|.$$

Notice that Problem 2.1 is really about solvable groups since, by the Howlett–Isaacs theorem (Theorem 8.13 of [**50**]), we have that H/Z is solvable, and thus G is solvable.

If N is a normal p'-subgroup of G (that is, of order not divisible by p), and $\lambda \in \mathrm{Irr}(N)$, then $\lambda \in \mathrm{IBr}(N)$, and we write $\mathrm{IBr}(G|\lambda)$ for the set of irreducible Brauer characters of G which are constituents of the induced Brauer character λ^G. If G is p-solvable and H is a p-complement of G, it is not difficult to show that $|\mathrm{IBr}(G|\lambda)| = 1$ if and only if $\lambda^H = e\alpha$, for some $\alpha \in \mathrm{Irr}(H)$. We see then that Problem 2.1 is also a particular case of a more general question in [**37**], which for p-solvable groups reduces to the following:

Problem 2.2. *Let G be a finite p-solvable group, and $P \in \mathrm{Syl}_p(G)$. Suppose that $Z = \mathbf{O}_{p'}(G)$ is central in G, and let $\lambda \in \mathrm{Irr}(Z)$. Let $\mathrm{IBr}(G|\lambda)$ be the set of irreducible Brauer characters of G over λ. Is it true that*

$$|\mathrm{Irr}(G|\lambda)| \leq |\mathrm{IBr}(G|\lambda)|k(P)?$$

In the important case where $\mathbf{O}_{p'}(G) = 1$, Problem 2.2 proposes an appealing group theoretical question.

Problem 2.3. *If G is p-solvable and $\mathbf{O}_{p'}(G) = 1$, is it true that $k(G) \leq l(G)k(P)$?*

In fact, at the time of this writing, I cannot even find an example of a finite p-solvable group G, with $\mathbf{O}_{p'}(G) = 1$, in which $k(G) > l(G)k_p(G)$, where $k_p(G)$ is the number of conjugacy classes of p-elements of G.

After reading a draft of this paper, B. Sambale has suggested that the inequality in Problem 2.1 might hold "height to height." That is, if $k_i(G|\lambda)$ denotes the number of irreducible characters of G over λ whose degree has p-part p^i, is it true in Problem 2.1 that we have that $k_i(G|\lambda) \leq k_i(P)$? Notice that this "height to height" version certainly does not hold in Problems 2.2 or 2.3 (as shown by $\mathrm{GL}_2(3)$ and $p = 2$).

Yet another block inequality was studied in [**37**]. For p-solvable groups in the principal block case (using that the McKay conjecture is true for p-solvable groups) it reduces to the following group theoretical problem.

Problem 2.4. *Let G be a finite p-solvable group, $P \in \mathrm{Syl}_p\, G$, and assume that $\mathbf{O}_{p'}(G) = 1$. Is it true that*

$$k(G) \leq k(\mathbf{N}_G(P)/P')k(P')?$$

Using the $k(GV)$-theorem (that is, the solution to Brauer's $k(B)$-conjecture for p-solvable groups) and that $\mathbf{O}_{p'}(\mathbf{N}_G(P))$ is contained in $\mathbf{O}_{p'}(G)=1$ (by standard but non-trivial finite group theory), notice that we have that

$$k(\mathbf{N}_G(P)/P') \leq |P/P'|,$$

if $\mathbf{O}_{p'}(G) = 1$. Since $k(P') \leq |P'|$, the right hand side quantity in Problem 2.4 is usually much smaller than $|P|$ (the quantity with which Brauer bounds $k(G)$ in the principal block case of the $k(B)$-conjecture).

We end this section with a lower bound for $k(B)$ proposed in [**23**]: $k(B) \geq 2\sqrt{p-1}$, whenever $k(B) > 1$. Using the usual standard reductions, and assuming the Alperin–McKay conjecture, this is equivalent to the following innocent-looking problem.

Problem 2.5. *Let V be a finite-dimensional $\mathbb{F}_p H$-module, where H is a p'-group and $\mathbb{F}_p$ is the field of p elements. Let $G = VH$ be the semidirect product, and set $Z = \mathbf{O}_{p'}(G)$. Let $\lambda \in \mathrm{Irr}(Z)$ and assume that $Z \subseteq \mathbf{Z}(G)$. Is it true that $|\mathrm{Irr}(G|\lambda)| \geq 2\sqrt{p-1}$?*

A. Maróti proved in [**43**] that if G is a finite group and p divides $|G|$, then $k(G) \geq 2\sqrt{p-1}$. This takes care of the case $Z = 1$, in Problem 2.5.

3. Problems on Brauer characters: inequalities

A totally different type of inequality was proposed in [**39**]. The new subject now is the number of modular irreducible constituents that a complex character has, after reducing it modulo p. It is completely open for p-solvable groups, even in the case where the Sylow p-subgroup is normal! If $\chi \in \mathrm{Irr}(G)$, then we denote by $\mathrm{IBr}(\chi^0)$ the set of the irreducible p-Brauer characters that appear in the decomposition of χ^0, the restriction of χ to the p-regular elements of G. In general, recall that

$$\chi^0 = \sum_{\varphi \in \mathrm{IBr}(G)} d_{\chi\varphi}\varphi,$$

where the non-negative integers $d_{\chi\varphi}$ are the *decomposition numbers.* Therefore $|\mathrm{IBr}(\chi^0)| = |\{\varphi \in \mathrm{IBr}(G) \mid d_{\chi\varphi} \neq 0\}|$.

We recall now that p-blocks can be characterized by *linking* via decomposition numbers. If we link $\chi, \psi \in \mathrm{Irr}(G)$ if there exists $\varphi \in \mathrm{IBr}(G)$ such that $d_{\chi\varphi} \neq 0 \neq d_{\psi\varphi}$, then the irreducible characters in blocks are the connected components of the associated graph, called the *Brauer* *graph* (Theorem 3.9 of [**46**]). Likewise, if we link $\varphi, \mu \in \mathrm{IBr}(G)$ if there exists $\chi \in \mathrm{Irr}(G)$ such that $d_{\chi\varphi} \neq 0 \neq d_{\chi\mu}$, then, again, the irreducible Brauer characters in blocks are the connected components of this graph (Problem 3.4 of [**46**]). We shall use these facts in what follows.

Problem 3.1. *Let G be a finite p-solvable group and let $\chi \in \mathrm{Irr}(G)$. Is it true that*

$$|\mathrm{IBr}(\chi^0)| \leq |G|_p/\chi(1)_p?$$

There is some evidence for an affirmative answer to this question. First, we claim that it follows from Brauer's $k(B)$-conjecture if χ has height zero (for any finite group). Indeed, if $\chi \in \mathrm{Irr}(B)$ and $d_{\chi\varphi} \neq 0$, then $\varphi \in \mathrm{IBr}(B)$. Now, we know that $l(B) \leq k(B)$, in general. If we assume Brauer's conjecture, then $k(B) \leq p^d = |G|_p/\chi(1)_p$, whenever χ has height zero, hence proving the claim. If $|G|_p/\chi(1)_p = 1$, then we know that $\chi^0 \in \mathrm{IBr}(G)$, by the celebrated Brauer–Nesbitt theorem. Thus $|\mathrm{IBr}(\chi^0)| = 1$. We also know that Problem 3.1 has an affirmative solution if $|G|_p/\chi(1)_p = p$ (using Brauer's defect one theory); or if $|G|_p/\chi(1)_p \leq p^2$ and G is p-solvable ([**39**]). Also, we know that it holds for symmetric groups and groups of Lie type in the defining characteristic (as shown in [**39**]).

It is somewhat curious that the *dual* question of Problem 3.1 is true (and easy). If Ψ is a character of G, then we write $\mathrm{Irr}(\Psi)$ for the set of the irreducible constituents of Ψ. If $\varphi \in \mathrm{IBr}(G)$, then Φ_φ is the *projective indecomposable character* associated with φ; that is,

$$\Phi_\varphi = \sum_{\chi \in \mathrm{Irr}(G)} d_{\chi\varphi}\chi.$$

Lemma 3.2. *Suppose that G is p-solvable and let $\varphi \in \mathrm{IBr}(G)$. Then*

$$|\mathrm{Irr}(\Phi_\varphi)| \leq |G|_p/\varphi(1)_p.$$

We also have that equality holds if and only if the block of φ is nilpotent with an abelian defect group.

Proof: By Fong's dimensional formula (Corollary 10.14 of [**46**]), we have that $\Phi_\varphi(1) = |G|_p\varphi(1)_{p'}$. If $\chi \in \mathrm{Irr}(G)$ and $d_{\chi\varphi} \neq 0$, notice that $\chi(1) \geq \varphi(1)$. Therefore

$$\Phi_\varphi(1) \geq |\mathrm{Irr}(\Phi_\varphi)|\varphi(1),$$

and the first part follows.

Notice that the equality above holds if and only if whenever $d_{\chi\varphi} \neq 0$, then $\chi^0 = \varphi$. Let B be the block of φ, and let D be a defect group of B. We prove now that this condition is equivalent to B being nilpotent and D abelian.

If B is nilpotent and D is abelian, then this follows from the main result of [**2**]. For the converse, first we claim that $l(B) = 1$. Indeed, suppose that $\mu \in \mathrm{IBr}(B)$ is linked to φ in the sense of Problem 3.4 of [**46**]. Then there exists $\psi \in \mathrm{Irr}(G)$ such that $d_{\psi\varphi} \neq 0 \neq d_{\psi\mu}$. Then $\psi^0 = \varphi$. Since μ is also an irreducible constituent of ψ^0, it follows that $\varphi = \mu$. The claim follows.

Once we have that $l(B) = 1$, then it follows by our hypothesis that all $\mathrm{Irr}(B)$ have the same degree, and then we apply Proposition 1 and Theorem 3 of [**54**] to deduce that D is abelian and B has inertial index 1. Now, blocks with abelian defect group and with inertial index 1 are nilpotent by Example (1.ex.3) of [**2**]. □

In general, $|\mathrm{Irr}(\Phi_\varphi)|$ can be bigger than $|G|_p/\varphi(1)_p$, as shown by $G = 2.\mathsf{A}_6$, and $p = 2$. If Q is a *vertex* of $\varphi \in \mathrm{IBr}(G)$ and G is p-solvable, then it is known that $|Q| = |G|_p/\varphi(1)_p$. Outside p-solvable groups, we only have that $|G|_p/\varphi(1)_p \leq |Q|$. Perhaps, $|\mathrm{Irr}(\Phi_\varphi)| \leq |Q|$, in general.

In 1984, M. Murai ([**45**]) proposed an interesting inequality conjecture involving a somewhat mysterious quantity. Again, this remains open for p-solvable groups. Recall that if G^0 is the set of p-regular elements of G, and if $|G|_{p'}$ is the largest positive integer not divisible by p dividing $|G|$, then $|G^0|/|G|_{p'}$ is an integer. (See Problem 2.3 of [**46**].) This is a particular case of a well-known theorem of Frobenius. Furthermore, Frobenius conjectured that if $|G^0|/|G|_{p'} = 1$, then G has a normal p-complement. (This is now known to be a consequence of the CFSG.)

Conjecture 3.3 (Murai)**.** *Let G be a finite group, and let B be the principal block of G. Then $l(B) \leq |G^0|/|G|_{p'}$.*

As we said, if $|G^0| = |G|_{p'}$, then G has a normal p-complement, and it is trivial to see that in this case $l(B) = 1$. Murai proved that in order to prove his conjecture in the p-solvable case, one can assume that $G = VH$, where V is a normal elementary abelian p-group and H has order not divisible by p. (Be aware that Murai also proved that Conjecture 3.3 implies the $k(GV)$-conjecture.)

B. Sambale and I have wondered if the following natural generalization of Murai's conjecture holds true. If G is a finite group and ψ, χ are class functions defined over G^0, we denote

$$[\psi, \chi]^0 = \frac{1}{|G|} \sum_{x \in G^0} \psi(x)\chi(x^{-1}).$$

Problem 3.4. *Let B be a p-block with defect group of order p^d. If $\chi \in \mathrm{Irr}(B)$, is it true that*

$$l(B) \leq p^d [\chi, \chi]^0?$$

A positive solution to Conjecture 3.3 and Problem 3.4, if there is one, should be quite hard to obtain, because of the following observation (by B. Sambale):

Theorem 3.5. *Let B be a block with defect group of order p^d. If Problem 3.4 has a positive solution for B, then $k(B) \leq p^d$.*

Proof: We have that

$$\begin{aligned} k(B)l(B) &\leq p^d \sum_{\chi\in\mathrm{Irr}(B)} [\chi,\chi]^0 \\ &= p^d \sum_{\chi\in\mathrm{Irr}(B)} \left[\sum_{\varphi\in\mathrm{IBr}(G)} d_{\chi\varphi}\varphi, \sum_{\mu\in\mathrm{IBr}(G)} d_{\chi\mu}\mu \right]^0 \\ &= p^d \sum_{\varphi,\mu\in\mathrm{IBr}(B)} \left(\sum_{\chi\in\mathrm{Irr}(B)} d_{\chi\varphi}d_{\chi\mu} \right) [\varphi,\mu]^0 \\ &= p^d \sum_{\varphi,\mu\in\mathrm{IBr}(B)} c_{\varphi\mu}[\varphi,\mu]^0 \\ &= p^d\,\mathrm{Trace}(I_{l(B)}) = p^d l(B), \end{aligned}$$

as desired. (In the next to last equality, we have used Theorem 2.13 of [**46**], which asserts that the inverse of the Cartan matrix is precisely $([\varphi,\mu]^0)$.) □

We end this section with an inequality proposed in [**7**] by J. P. Cossey.

Problem 3.6. *Suppose that G is p-solvable and $\varphi \in \mathrm{IBr}(G)$ has vertex Q. Is it true that the number of irreducible lifts of φ is less than or equal to $|Q/Q'|$?*

Notice that Problem 3.6 can be stated without the p-solvability hypothesis, and, in principle, it might even hold in full generality.

In general, I have mixed feelings about inequalities: there is always the possibility that we are trying to relate quantities that are not meant to be related.

4. Problems on blocks of p-solvable groups: equalities

Broué–Puig nilpotent blocks are ubiquitous in the block theory of finite groups. (We already saw them in Lemma 3.2.) Their definition [**2**], however, requires knowing a good deal of the local structure of the group.

In [**38**], we proposed two easy-to-state global characterizations of nilpotent blocks. After the usual arguments, they reduce to the following problems (for p-solvable groups).

Problem 4.1. *Let G be a finite p-solvable group, $P \in \mathrm{Syl}_p(G)$, and assume that $Z = \mathbf{O}_{p'}(G)$ is central in G. Let $\lambda \in \mathrm{Irr}(Z)$. Suppose that all the irreducible characters of G of degree not divisible by p that lie over λ have the same degree. Is $G = Z \times P$?*

Problem 4.2. *Let G be a finite p-solvable group, $P \in \mathrm{Syl}_p(G)$, and assume that $Z = \mathbf{O}_{p'}(G)$ is central in G. Let $\lambda \in \mathrm{Irr}(Z)$. Suppose that each irreducible character of G of degree not divisible by p that lies over λ lifts an irreducible Brauer character of G. Is $G = Z \times P$?*

If P is abelian, then these questions have affirmative solutions by using the main result of [**54**]. As we explained in [**38**], a natural orbit problem would solve Problems 4.1 and 4.2 for p-solvable groups.

Problem 4.3. *Let $G > 1$ be a finite group with a normal p-complement K and let $P \in \mathrm{Syl}_p(G)$. Suppose that V is a finite faithful completely reducible G-module over a field of characteristic p. Does there exist $v \in \mathbf{C}_V(P)$ such that $|K : \mathbf{C}_K(v)|^2 > |K|$?*

D. Gluck has proved this is the case in [**19**] for groups of odd order. The case where K is nilpotent was solved in [**9**]. The case where $K = G$ is a celebrated theorem of S. Dolfi [**8**] and Halasi–Podoski [**20**].

To end this short section, let us remind the reader of the Eaton–Moretó conjecture [**11**], which proposes an exciting block equality. After standard reductions, the following is the p-solvable principal block case.

Conjecture 4.4 (Eaton–Moretó)**.** *Let G be a finite p-solvable group with $\mathbf{O}_{p'}(G) = 1$. Suppose that $P \in \mathrm{Syl}_p(G)$ is not abelian. If p^a is the smallest non-linear character degree of P, then the smallest p-part of the degrees of the irreducible characters of G of degree divisible by p is exactly p^a.*

The Eaton–Moretó conjecture was checked for many families of quasi-simple groups in [**3**], and for symmetric groups, general linear groups in characteristic p and sporadic groups in [**11**]. Hence, if we share Alperin's opinion, we have to check p-solvable groups for this conjecture to be *safe*.

5. Problems on zeros of characters

Zeros of characters is a classical topic in character theory. It probably starts with Burnside's theorem stating that if G is a finite group, and $\chi \in \mathrm{Irr}(G)$ is non-linear, then there exists $g \in G$ such that $\chi(g) = 0$. A famous theorem of Brauer and Nesbitt asserts that if $\chi(1)_p = |G|_p$, then $\chi(g) = 0$ whenever p divides the order of g.

If $x \in G$, we say that x is *non-vanishing* (in G) if $\chi(x) \neq 0$ for all $\chi \in \mathrm{Irr}(G)$. If $x \in \mathbf{Z}(G)$ is a central element of G, then x is non-vanishing in G, because if $\mathcal{X}$ is any representation affording χ, then $\mathcal{X}(x)$ is a scalar (non-zero) matrix. The converse is certainly not true. Generically, however, one expects x to lie in a nilpotent normal subgroup of G ([**10**]). For solvable groups, this was conjectured in [**32**], and there is a great deal of literature following it. In this section, again we fix a prime p and consider the elements of finite p-solvable groups G on which the irreducible characters of G of degree not divisible by p, which we denote by $\mathrm{Irr}_{p'}(G)$, do not vanish. It would be interesting to know which elements these are. For instance:

Problem 5.1. *Let G be a finite p-solvable group with $\mathbf{O}_{p'}(G) = 1$, where p is a prime, and let $x \in G$. Is it true that $\chi(x) \neq 0$ for all $\chi \in \mathrm{Irr}_{p'}(G)$ if and only if x is a p-element?*

If $\chi \in \mathrm{Irr}_{p'}(G)$ and $x \in G$ is a p-element, it is known that $\chi(x) \neq 0$, using some elementary facts on p-power roots of unity (see for instance Corollary 4.20 of [**50**]). Hence, Problem 5.1 discusses conditions for the converse to hold. To have an affirmative answer to Problem 5.1 some p-solvability hypotheses are needed, as shown, for instance, by the elements of order 3 in S_5, with $p = 2$. Problem 5.1, if we do not require that $\mathbf{O}_{p'}(G) = 1$, cannot have an affirmative answer. Indeed, notice that the intersection of the kernels of the p'-degree characters of G is not necessarily a p-group (but a group with a normal p-complement); hence some hypothesis regarding normal p'-subgroups is necessary. Let us record (without proof) the following essentially well-known result.

Theorem 5.2. *Let G be a finite group, p a prime, and $P \in \mathrm{Syl}_p(G)$. Let*

$$K = \bigcap_{\chi \in \mathrm{Irr}_{p'}(G)} \ker(\chi).$$

Then $K = \mathrm{core}_G(NP')$, where N is the largest normal subgroup of G such that $\mathbf{C}_N(P) = 1$.

Notice that an affirmative answer to Problem 5.1 would easily solve the main problem in [**32**]. We denote by $\mathbf{F}(G)$ the Fitting subgroup of G. Also, recall that if $x \in G$, then x can be uniquely written as $x = x_p x_{p'}$, where x_p, $x_{p'}$ are powers of x, having orders a power of p and not divisible by p, respectively.

Theorem 5.3. *Let G be a finite solvable group and let $x \in G$ be non-vanishing in G. If Problem 5.1 has an affirmative answer, then $x \in \mathbf{F}(G)$.*

Proof: If p is any prime, then $x_{p'} \in \mathbf{O}_{p'}(G)$, by Problem 5.1 applied in $G/\mathbf{O}_{p'}(G)$. Thus, if $q \neq p$ is a prime, then $x_q \in \mathbf{O}_{p'}(G)$. Hence, holding q fixed and varying p, we have that

$$x_q \in \bigcap_{p \neq q} \mathbf{O}_{p'}G = \mathbf{O}_q(G),$$

as desired. □

We should mention that the conclusion of Theorem 5.3 is still not known if $o(x)$ is even, even in the case where $\mathbf{F}(G)$ is a Sylow subgroup of G. (This could be territory for a negative answer in Problem 5.1. Perhaps the possible groups involved are too big to be detected by computers.)

Next we provide some evidence for a possible affirmative answer to Problem 5.1, using a non-standard variation of a well-known regular orbit theorem, that might have some interest on its own. The proof that we present here is a simplification by I. M. Isaacs of our original one.

Theorem 5.4. *Suppose that G is a finite group that has an abelian normal p-complement A, and let $P \in \mathrm{Syl}_p(G)$. Let V be a finite-dimensional G-module in characteristic p. Then there exists $v \in \mathbf{C}_V(P)$ such that $\mathbf{C}_A(v) = \mathbf{C}_A(V)$.*

Proof: We can assume that $V \neq 0$, and we proceed by induction on the dimension of V. First, $\mathbf{C}_A(V)$ is normal in G, so we can replace G by $G/\mathbf{C}_A(V)$, and assume that A acts faithfully on V. Suppose that $V = X \oplus Y$, where X and Y are non-trivial G-submodules. By the inductive hypothesis, there exists $x \in \mathbf{C}_X(P)$ and $y \in \mathbf{C}_Y(P)$ such that $\mathbf{C}_A(x) = \mathbf{C}_A(X)$ and $\mathbf{C}_A(y) = \mathbf{C}_A(Y)$. Writing $v = x + y$, we have that $v \in \mathbf{C}_V(P)$, and $\mathbf{C}_A(v) = \mathbf{C}_A(x) \cap \mathbf{C}_A(y) = \mathbf{C}_A(X) \cap \mathbf{C}_A(Y) = \mathbf{C}_A(V) = 1$. We can thus assume that V is indecomposable as a G-module.

Now suppose that $1 < B \leq A$, with $B \triangleleft G$. By Fitting's lemma,

$$V = \mathbf{C}_V(B) \oplus [V, B]$$

and both summands are G-invariant. It follows from the previous paragraph that one of these summands must be trivial. But A acts faithfully on V and B is non-trivial, so $[V, B] > 0$, and we deduce that $\mathbf{C}_V = 0$.

Now let U be a simple G-submodule of V. If we let $B = \mathbf{C}_A(U)$, then B is normal in G, and since $0 < U \subseteq \mathbf{C}_V(B)$, it follows from the previous paragraph that $B = 1$ and thus A acts faithfully on U. We can thus replace V by U, so we can assume that V is a simple G-module.

If V is homogeneous as an A-module, it follows from the fact that A is abelian and acts faithfully on V that the action of A on V is Frobenius. In this case, we can choose v to be an arbitrary non-zero element of $\mathbf{C}_V(P)$, and we have $\mathbf{C}_A(v) = 1$, as wanted.

Finally, we can assume that V is not homogeneous as an A-module, and thus there exists $H \triangleleft G$ of index p and H-submodules $U_i \subseteq V$ for $0 \le i < p$ such that

$$V = U_0 \oplus U_1 \oplus \cdots \oplus U_{p-1}$$

and the U_i's are transitively permuted by G/H.

Let $Q = H \cap P$, and apply the inductive hypothesis to the action of H on U_0 to choose $u_0 \in \mathbf{C}_{U_0}(Q)$ such that $\mathbf{C}_A(u_0) = \mathbf{C}_A(U_0)$. Let $x \in P - Q$, and note that $x^p \in Q$. We can assume that the subspaces U_i are numbered so that $(U_0)x^i = U_i$ for $0 \le i < p$. Now write $u_i = u_0 x^i$ for $0 < i < p$, and let $v = \sum_i u_i$. Then $u_i \in U_i$ for $0 \le i < p$, and since x normalizes Q, we see that $u_i \in \mathbf{C}_V(Q)$ for all i, and thus Q centralizes v. Since $x^p \in Q$ and Q centralizes u_0, it follows that $(u_{p-1})x = u_0$, and thus x centralizes v, so $v \in \mathbf{C}_V(P)$. Also, since $\mathbf{C}_A(u_0) = \mathbf{C}_A(U_0)$, it follows by conjugation by powers of x that $\mathbf{C}_A(u_i) = \mathbf{C}_A(U_i)$ for all i. Since the spaces U_i are A-invariant, we have

$$\mathbf{C}_A(v) = \bigcap_i \mathbf{C}_A(u_i) = \bigcap_i \mathbf{C}_A(U_i) = \mathbf{C}_A(V) = 1,$$

and the proof is complete. □

Notice that the condition of A being abelian cannot be relaxed to A being nilpotent, since there are examples of nilpotent p'-groups that do not have regular orbits on simple modules in characteristic p.

We now give an affirmative answer to Problem 5.1 in a special case.

Theorem 5.5. *Let G be a finite solvable group with $\mathbf{O}_{p'}(G) = 1$, such that all the p'-factors in the ascending p-p'-series of G are abelian. Suppose that $x \in G$ is such that $\chi(x) \neq 0$ for all $\chi \in \mathrm{Irr}_{p'}(G)$. Then x is a p-element.*

Proof: We can assume that G is not a p-group, and we proceed by induction on $|G|$. Write $V = \mathbf{O}_p(G)$, and let $\bar{G} = G/\Phi(V)$, where $\Phi(V)$ is the Frattini subgroup of V. Then $\mathbf{O}_{p'}(\bar{G}) = 1$ using standard group theory, and $\bar{G}$ satisfies the hypotheses with respect to the element $\bar{x}$. If $|\bar{G}| < |G|$, then the inductive hypotheses guarantee that $\bar{x}$ is a p-element, and thus x is a p-element, and there is nothing further to prove. We can assume, therefore, that $\Phi(V) = 1$, so V is elementary abelian. Now let $M/V = \mathbf{O}_{p'}(G/V)$, and note that M/V is abelian by hypothesis. Also, since $V < G$, we have $M > V$, and thus M is not a p-group. Let $N = \mathbf{O}^p(M)$, and note that $N > 1$. If $K/N = \mathbf{O}_{p'}(G/N)$, then $K \cap M = N$, and we will have that KM/M is a p'-group. Therefore $K = N$. It is straightforward to check that G/N satisfies the hypothesis of the theorem. Therefore, we deduce that Nx is a p-element.

Write $W = N \cap V$ and observe that the p'-group N/W acts faithfully on W. Indeed, if $U/W \subseteq N/W$ acts trivially on W, then $[V, U/W, U/W] = 1$ and by coprime action, $[V, U] = 1$. However, $\mathbf{C}_G(V) = V$, since G is p-solvable.

Let $X = N\langle x \rangle$, and note that X/N is a cyclic p-group. Now choose $P \in \mathrm{Syl}_p(G)$ so that $X \subseteq NP$, and note that $P \cap N = W$. Let $I = \mathrm{Irr}(W)$, and observe that I can be viewed as an NP/W-module on which N/W acts faithfully. By Theorem 5.4, there exists a P-invariant linear character $\lambda \in I$ such that the stabilizer of λ in N is W, and thus $\theta = \lambda^N$ is irreducible and P-invariant. Also, $\theta(1) = |N/W|$ is a p'-number. Now $\mathbf{O}^p(N) = N$, so θ has p'-determinantal order as well as p'-degree, and thus θ extends to $\psi \in \mathrm{Irr}(NP)$ (by Corollary 6.2 of [**50**]). Since NP has p'-index in G, it follows that ψ^G has an irreducible constituent $\chi \in \mathrm{Irr}_{p'}(G)$. By hypothesis, $\chi(x) \neq 0$, so there exists some irreducible constituent ξ of the restriction χ_X such that $\xi(x) \neq 0$. Now χ lies over $\theta \in \mathrm{Irr}(N)$, so ξ_N is a sum of G-conjugates of θ, and all of these have p'-degree.

First, suppose that p divides $\xi(1)$, so ξ_N is reducible. Since X/N is cyclic, ξ_N cannot be homogeneous, so ξ is induced from some character of a subgroup T with $N \subseteq T < X$. Since $X = N\langle x \rangle$, no X-conjugate of x lies in T, and thus $\xi(x) = 0$, which is a contradiction. We can now assume that ξ has p'-degree, and we write $Y = X \cap P$. Then $|X : Y| = |N : W| = \theta(1) \leq \xi(1)$. Since Y is a p-group and ξ has p'-degree, we deduce that ξ_Y has some linear constituent β. Then ξ is a constituent of β^X, which has degree $|X : Y| \leq \xi(1)$, and it follows that $\xi = \beta^X$. Since $\xi(x) \neq 0$, we see that some conjugate of x lies in the p-group Y, and in particular, x is a p-element. This completes the proof. □

The following easy block-theoretical result gives a criterion for certain elements to have non-zero values. We let $\mathbf{R}$ be the ring of algebraic integers in $\mathbb{C}$.

Theorem 5.6. *Let G be a finite group, $P \in \mathrm{Syl}_p(G)$. Let $x \in G$ be such that $x_{p'} \in \mathbf{O}_{p'}(\mathbf{N}_G(P))$. Then $\chi(x) \not\equiv 0$ mod $p\mathbf{R}$ for every $\chi \in \mathrm{Irr}_{p'}(G)$ in the principal p-block of G.*

Proof: Suppose that $\chi \in \mathrm{Irr}_{p'}(G)$ in the principal p-block of G is such that $\chi(x) \equiv 0$ mod $p\mathbf{R}$. Let M be a maximal ideal of $\mathbf{R}$ containing p. Write $y = x_{p'}$. By Lemma 4.19(b) of [**50**], we have that $\chi(y) \equiv \chi(x) \equiv 0$ mod M. Let $L = y^G$, the conjugacy class of y. Since $[P, y] = 1$, we have that $|L| \not\equiv 0$ mod p. However,

$$\chi(1)|L| \equiv \chi(1)\left(\frac{|L|\chi(y)}{\chi(1)}\right) = |L|\chi(y) \equiv 0 \bmod M.$$

Therefore p divides $\chi(1)|L|$, and this is not possible. □

The conclusion in Theorem 5.6 is really about characters in the principal block. For instance, `SmallGroup`(160,235) has an irreducible character χ of degree 5 and value 3 in an element x of order 2. Therefore if $p=3$, then $\{\chi\}$ is a 3-block of maximal defect and of course $x \in \mathbf{O}_{p'}(\mathbf{N}_G(P))$, in this case (because 3 does not divide the order of the group). The converse of Theorem 5.6 is not true, even assuming that $\chi(x) \not\equiv 0 \bmod p\mathbf{R}$ for all χ in the principal block of G (as shown by S_3, $p=2$).

6. Problems on zeros of characters: natural correspondences

We start this section by speculating that there might exist a "row version" of Problem 5.1 that is related to some recent results. Suppose that $X(G)$ is the character table of the finite group G. In Problem 5.1, we wish to infer a property of a fixed element of G (a column in $X(G)$) provided that the p'-degree characters of G do not vanish on this element. Once characters of p'-degree are in the picture, due to the McKay conjecture, p-Sylow normalizers show up naturally. (Recall that the McKay conjecture asserts that $|\mathrm{Irr}_{p'}(G)| = |\mathrm{Irr}_{p'}(\mathbf{N}_G(P))|$ if $P \in \mathrm{Syl}_p(G)$.) It would be interesting to know the answer to the following dual question.

Problem 6.1. *Let G be a p-solvable group. Let $\chi \in \mathrm{Irr}(G)$ be such that $\chi(x) \neq 0$ for all $x \in \mathbf{N}_G(P)$, where $P \in \mathrm{Syl}_p(G)$. Is the degree of χ not divisible by p?*

Some solvability hypotheses are again needed in order to have an affirmative answer to Problem 6.1, as shown by $\mathrm{PSL}_2(11)$ for $p=2$, or M_{22} for $p=3$. (Certainly, Problem 6.1 does not have an affirmative answer in p-solvable groups if $\mathbf{N}_G(P)$ is replaced by P, as shown by a 2-Sylow normalizer of Suz(8), unless, perhaps, the hypothesis that $\mathbf{O}^{p'}(G)=G$ is added.) On the other hand, for many groups, such as the alternating or symmetric groups, the answer to Problem 6.1 seems to be affirmative, even only assuming non-zero values on p-elements. (The case of the symmetric and alternating groups is proved in [**17**].)

An affirmative answer to Problem 6.1 implies that if G is p-solvable, $P \in \mathrm{Syl}_p(G)$, and $\mathbf{N}_G(P)=P$, then the number of irreducible characters of G which are never zero on P is $|P/P'|$. (This is a particular case of a conjecture of this author on Carter subgroups of solvable groups that is mentioned in [**27**]. See Problem 6.7 below.) Let us now prove the following.

Theorem 6.2. *Let G be a p-solvable group. Let $\chi \in \mathrm{Irr}(G)$ be such that $\chi(x) \neq 0$ for all $x \in \mathbf{N}_G(P)$, where $P \in \mathrm{Syl}_p(G)$. If P is maximal in G or P is abelian, then χ has degree not divisible by p.*

Proof: The Hall–Higman lemma 1.2.3 asserts that $\mathbf{C}_G(\mathbf{O}_{p'p}(G)/\mathbf{O}_{p'}(G))$ is contained in $\mathbf{O}_{p'p}(G)/\mathbf{O}_{p'}(G)$. Hence, if P is abelian, then we have that $G/\mathbf{O}_{p'}(G)$ has a normal Sylow p-subgroup, and then $\mathbf{O}_{p'pp'}(G) = G$. Let us prove that χ has p'-degree under the more general assumption that G has an abelian normal p-subgroup K such that $\mathbf{O}_{p'pp'}(G/K) = G/K$. Indeed, suppose that $K \subseteq L \subseteq M$ are normal subgroups of G such that L/K is p', M/L is a p-group, and G/M is p'. Now, G/L has a normal Sylow p-subgroup, and by elementary group theory, we have that $G = L\mathbf{N}_G(P)$. By Corollary C of [**47**], we have that $\chi_L = \theta$ is irreducible. Since L has an abelian normal Sylow p-subgroup K, we have that $\theta(1) = \chi(1)$ divides $|L : K|$, and therefore $\chi(1)$ is not divisible by p.

Suppose now that P is maximal in G. In order to prove this case, we heavily use the results of [**27**] and the McKay conjecture for p-solvable groups. If P is normal in G, then necessarily χ is linear by hypothesis and Burnside's theorem on zeros of characters. Hence, we may assume that $P = \mathbf{N}_G(P)$ is a Carter subgroup of G. Therefore $|\mathrm{Irr}_{p'}(G)| = |P/P'|$ by Theorem 9.4 of [**50**], for instance. Notice that if $\chi \in \mathrm{Irr}_{p'}(G)$, then χ_P is never zero, for instance by Corollary 4.20 of [**50**]. Now P is a Carter subgroup of G, and by Theorem A of [**27**], $|P/P'|$ is also the number of the *head characters* of G. By hypothesis and Theorem 6.1 of [**27**], we have that the number of *head characters* of G is the number of irreducible characters which do not vanish on P. The claim follows easily now. □

If G is p-solvable and $P = \mathbf{N}_G(P)$, we remind the reader that there is a natural bijection $^*\colon \mathrm{Irr}_{p'}(G) \to \mathrm{Irr}_{p'}(P)$ (see Theorem 9.4 of [**50**]). This leads us to the main topic of this section: zeros of characters and natural correspondences.

It is my belief, as a somewhat vague general principle, that if H is a *distinguished* (up to G-conjugacy) subgroup of a finite group G, $\mathcal{A}$ and $\mathcal{B}$ are canonical subsets of $\mathrm{Irr}(G)$ and $\mathrm{Irr}(H)$, respectively, and $^*\colon \mathcal{A} \to \mathcal{B}$ is a natural bijection, then the image χ^* of $\chi \in \mathcal{A}$ is linear if and only if χ_H is never zero. (I shall not even attempt to make a general definition of when a bijection is natural, when a subgroup H is *distinguished* or when a subset is canonical—although in all the cases, I certainly mean *choice-free*.) All this is better explained with an example.

Problem 6.3. *Suppose that A acts coprimely on G, let $\mathrm{Irr}_A(G)$ be the set of A-invariant irreducible characters of G, let $C = \mathbf{C}_G(A)$, and let $^*\colon \mathrm{Irr}_A(G) \to \mathrm{Irr}(C)$ be the Glauberman–Isaacs correspondence. Let $\chi \in \mathrm{Irr}_A(G)$. Is it true that $\chi^*(1) = 1$ if and only if χ_C is never zero?*

I recognize that the evidence supporting an affirmative answer in Problem 6.3 is quite scarce. I do not even know how to prove the case where G is nilpotent, and this should be territory for counterexamples.

On the other hand, an argument involving roots of unity (which we shall use in Theorem 6.6 below) shows that if A is a p-group and $\chi^*(1) = 1$, then χ_C is never zero.

Besides the case where $P = \mathbf{N}_G(P)$ and G is p-solvable, there are some other (few but) important cases where we have *McKay natural correspondences of characters*

$$^*\colon \operatorname{Irr}_{p'}(G) \to \operatorname{Irr}_{p'}(\mathbf{N}_G(P)).$$

For instance, if $p = 2$ and G is solvable ([**24**]); if $p = 2$ and $G = \mathsf{S}_n$, $G = \mathrm{GL}_n(q)$, or $G = \mathrm{GU}_n(q)$, where q is odd ([**16**]); if p is any prime and G has a normal p-complement (Theorem 9.2 of [**50**]); or if p is any odd prime and $P = \mathbf{N}_G(P)$ ([**53**]). In these cases, we also believe that there might be a connection between non-vanishing and linear characters.

The following has been proved by E. Giannelli (for symmetric groups) and P. H. Tiep in private communication.

Theorem 6.4. *Suppose that $p = 2$ and $G = \mathsf{S}_n$, $G = \mathrm{GL}_n(q)$, or $G = \mathrm{GU}_n(q)$, where q is odd. Let $P \in \mathrm{Syl}_2(G)$, $N = \mathbf{N}_G(P)$, and let*

$$^*\colon \operatorname{Irr}_{p'}(G) \to \operatorname{Irr}_{p'}(N)$$

be the McKay natural correspondence. Let $\chi \in \mathrm{Irr}(G)$. Then χ_N is never zero if and only if χ has p'-degree and $\chi^*(1) = 1$. In particular, in all these cases, the number of irreducible characters of G which are never zero on N is $|N/N'|$.

In the remaining cases where we have a McKay natural correspondence, it seems natural to guess that the same is going to happen, but I am unable to prove it. The main obstruction is that we do not know much about the relationship between Clifford correspondents and zeros of characters.

Problem 6.5. *Let G be a finite group, let p be a prime, let $P \in \mathrm{Syl}_p(G)$, and $N = \mathbf{N}_G(P)$. Assume one of the following hypotheses:*

(a) *$p = 2$ and G is solvable;*
(b) *G has a normal p-complement;*
(c) *p is odd and $N = P$.*

Let $\chi \in \mathrm{Irr}(G)$. Is it true that χ_N is never zero if and only if χ has p'-degree and $\chi^*(1) = 1$?

Notice that, in particular, this would imply that in all the cases enumerated above, the number of irreducible characters of G which are never zero on N is $|N/N'|$.

There is a connection between Problem 6.5(b) and Problem 6.3. This follows by using properties of the Glauberman correspondence.

Theorem 6.6. *If Problem 6.3 has an affirmative answer whenever A is a p-group, then Problem 6.5(b) has an affirmative answer.*

Proof: Suppose that G has a normal p-complement K, and a Sylow p-subgroup P. Then $N = \mathbf{N}_G(P) = C \times P$, where $C = \mathbf{C}_K(P)$. Let $\chi \in \mathrm{Irr}(G)$.

Suppose that χ_N is never zero. Then χ_P is never zero, and therefore $\chi_K = \theta \in \mathrm{Irr}(K)$, for instance by Corollary C of [**47**]. Therefore χ has p'-degree. If $\hat{\theta} \in \mathrm{Irr}(G)$ is the canonical extension of θ, then $\chi = \lambda\hat{\theta}$ for some linear $\lambda \in \mathrm{Irr}(G/K)$, and $\chi^* = \lambda_P \times \theta^*$, where $\theta^* \in \mathrm{Irr}(C)$ is the P-Glauberman correspondent of θ. Therefore, we only need to prove that $\theta^*(1) = 1$. Since χ_C is never zero by hypothesis, this follows from Problem 6.3.

Conversely, assume that $\chi^*(1) = 1$ and χ has p'-degree. Write $\chi_K = \theta \in \mathrm{Irr}(K)$ and $\chi = \lambda\hat{\theta}$ for some linear $\lambda \in \mathrm{Irr}(G/K)$, where $\hat{\theta} \in \mathrm{Irr}(G)$ is the canonical extension. Again, $\chi^* = \lambda_P \times \theta^*$, and we have that $\theta^*(1) = 1$. We need to prove that $\hat{\theta}(cx) \neq 0$ for all $x \in P$ and all $c \in C$. By the Glauberman correspondence, we can write

$$\theta_C = e\theta^* + p\Delta,$$

where $e \geq 1$ is an integer not divisible by p, and Δ is a character of C or zero. By Theorem 13.6 and Theorem 13.14 of [**25**], we have that

$$\hat{\theta}(cx) = \epsilon\beta(c),$$

where $\beta \in \mathrm{Irr}(\mathbf{C}_G(x))$ is the $\langle x\rangle$-Glauberman correspondent of θ. By the Glauberman correspondence, we can write

$$\theta_{\mathbf{C}_G(x)} = d\beta + p\Xi,$$

where $d \geq 1$ is an integer not divisible by p and Ξ is a character of $\mathbf{C}_G(x)$ or zero. Hence

$$\chi_C = e\theta^* + p\Delta = d\beta_C + p\Xi_C.$$

From this equation, we deduce that

$$\beta_C = f\theta^* + p\rho,$$

where $f \geq 1$ is an integer not divisible by p, and ρ is a character of C or zero. Now, if $\beta(c) = 0$, then $(-f)\theta^*(c) = p\rho(c)$. Now choose a maximal ideal M of the ring of algebraic integers $\mathbf{R}$ in $\mathbb{C}$ containing p. Then

$$(-f)\theta^*(c) \equiv 0 \bmod M.$$

Therefore either $f \equiv 0 \bmod M$ or $\theta^*(c) \equiv 0 \bmod M$. Since p does not divide f, we have that $f \not\equiv 0 \bmod M$. Since $\theta^*(c)$ is a root of unity, $\theta^*(c) \not\equiv 0 \bmod M$, and this is a contradiction. □

I. M. Isaacs brought to my attention the following example. Suppose that G is a finite group, N is a normal subgroup of G, and that $\theta \in \mathrm{Irr}(N)$ has a canonical (*choice-free*) extension ψ to G. Assume now that N has a complement H in G. Hence, we have that the map $^*\colon \mathrm{Irr}(H) \to \mathrm{Irr}(G)$ given by $\alpha \mapsto \alpha\psi$ is a canonical injection. Hence, according to our "principle" before Theorem 5.3, we *should* have that ψ_H is never zero. This does happen if $(|N|, |H|) = 1$, using the Glauberman correspondence. (Thus our *principle* is right if H happens to be a *distinguished* subgroup of G, in this case, a Hall subgroup of G.) Now, if we take G the wreath product of A_4 with A_5, and $N \triangleleft G$ is the base group, then N has a G-invariant irreducible character θ of degree 3^5, which extends to G, by well-known properties of the wreath products. Since G/N is perfect, by the Gallagher correspondence we have that θ has a unique (and therefore) canonical extension $\psi \in \mathrm{Irr}(G)$. Now, G has five conjugacy classes of subgroups H isomorphic to A_5, and all of them are complements to N in G. It turns out that ψ_H is never zero exactly in one of these conjugacy classes of subgroups: those that are conjugate to the natural complement of N in G. Perhaps this is serendipity, or perhaps not.

Without natural correspondences, it is definitely not true that the number of (p'-degree) irreducible characters of a finite group G which are never zero on $N = \mathbf{N}_G(P)$ equals $|N/N'|$ (even if G is p-solvable and $\mathbf{O}_{p'}(G) = 1$, as shown by $(C_3 \times C_3) : \mathrm{GL}_2(3)$ for $p = 3$). In fact these numbers can be bigger (A_5 with $p = 2$) or smaller (A_5 with $p = 5$).

In [**27**], Isaacs recently constructed injections from the linear characters of a Carter subgroup of a solvable group G into $\mathrm{Irr}(G)$. Although the injections are not totally canonical, their image is. As is written in [**27**], I have wondered if this image is the set of the irreducible characters of G which are never zero on C.

Problem 6.7. *Let G be a solvable group, and let C be a Carter subgroup of G. Is it true that the number of irreducible characters of G which do not vanish on C is $|C/C'|$?*

Of course, if G has a self-normalizing Sylow p-subgroup P, then P is a Carter subgroup of G, and this case of Problem 6.7 is included in Problem 6.5.

7. Problems on fields of values

Recall that if χ is a character of a finite group G, then

$$\mathbb{Q}(\chi) = \mathbb{Q}(\chi(g) \mid g \in G),$$

the *field of values* of χ, is the smallest subfield of $\mathbb{C}$ containing the values of χ. Also, the *conductor* $c(\chi)$ of χ is the smallest positive integer n such that $\mathbb{Q}(\chi)$ is contained in the n-th cyclotomic field $\mathbb{Q}_n$.

A recent conjecture, proposed in [**52**], would explain the abelian extensions $\mathbb{Q}(\chi)/\mathbb{Q}$ of the p'-degree irreducible characters of finite groups.

Conjecture 7.1. *Suppose that G is a finite group, and let p be a prime. Let $\chi \in \mathrm{Irr}_{p'}(G)$, and let $P \in \mathrm{Syl}_p(G)$. Write $c(\chi) = p^a m$, where m is not divisible by p, and $a \geq 0$ is an integer. Is it true that the extension $\mathbb{Q}_{p^a}/\mathbb{Q}(\chi_P)$ has degree not divisible by p?*

This would imply, for instance, that if $\chi \in \mathrm{Irr}(G)$ has odd degree and $P \in \mathrm{Syl}_2(G)$, then $\mathbb{Q}(\chi_P)$ is always a full cyclotomic field. (For instance, if λ is a linear character of order 8, then $1 + \lambda + \lambda^3$ cannot be the restriction of an odd-degree irreducible character to a Sylow 2-subgroup.) Also, Conjecture 7.1 implies that $\chi \in \mathrm{Irr}(G)$ of odd degree is 2-*rational* (i.e., $c(\chi)$ is odd) if and only if χ_P is a rational-valued character. There are many indications (see [**52**]) that Conjecture 7.1 has an affirmative answer for every finite group. Again, solvable groups seem to be blocking a reduction of it to a problem on simple groups. (*Added in proof:* Conjecture 7.1 is now known to be true for p-solvable groups by unpublished work by I. M. Isaacs and myself [**31**].)

Brauer's Problem 7 in [**1**] proposes "to study the irreducible characters of p-groups." I would add the following.

Problem 7.2. *If P is a p-group, study the characters χ_P, where P is a Sylow p-subgroup of some finite group G, and $\chi \in \mathrm{Irr}(G)$.*

A small piece of information was discussed in [**18**]: if p divides $\chi(1)$ and χ_P has a linear constituent, then it appears that at least it should have p. (Now, this has been extended in [**56**].)

While checking Conjecture 7.1, we came across the following problem. Perhaps there is an easy explanation or a counterexample.

Problem 7.3. *Let G be a finite group, and let $\chi \in \mathrm{Irr}(G)$. If $\chi(1)$ is odd or χ is 2-rational, is it true that there is $g \in G$ such that $\mathbb{Q}(\chi) = \mathbb{Q}(\chi(g))$?*

Suppose that $\mathbb{Q}(G)$ is the smallest field containing $\mathbb{Q}(\chi)$ for all $\chi \in \mathrm{Irr}(G)$. In 1972, B. Fein and B. Gordon ([**13**]) proposed to study the abelian extensions $\mathbb{Q}(G)/\mathbb{Q}$. As far as we know, the following remains an open problem.

Problem 7.4. *For which finite abelian extensions $F/\mathbb{Q}$ does there exist a finite group G such that $F = \mathbb{Q}(G)$?*

It is true, by work in [**12**], that there is no solvable group G such that $\mathbb{Q}(G) = \mathbb{Q}(i\sqrt{11})$, for instance. However, $\mathbb{Q}(M_{12}) = \mathbb{Q}(\sqrt{11}i)$, where

M_{12} is the Mathieu group. I would not be surprised if "for all" is the answer to Problem 7.4, but at this time, I cannot find a finite group G such that $\mathbb{Q}(G) = \mathbb{Q}(\sqrt{11})$.

In Problem 7.4, one can also restrict the families of groups under consideration. For instance, I cannot find any solvable group G such that $\mathbb{Q}(G) = \mathbb{Q}(\sqrt{5}i)$.

Finally, it is a well-known theorem of J. G. Thompson that the values of any irreducible character are zeros or roots of unity for more than a third of the elements of the group. (See Problem 3.15 of [**25**].) Recently A. R. Miller conjectured in [**44**] that the following also holds.

Conjecture 7.5 (Miller)**.** *If G is a finite group and $\chi \in \mathrm{Irr}(G)$, then the values of χ are zeros or roots of unity for at least half of the elements of G.*

Conjecture 7.5 is open for solvable groups.

8. Problems on character degrees and conjugacy classes

The analogies between characters and conjugacy classes—between rows and columns of the character table—is a source of inspiration for many results in character theory. In this last section, let me remind the reader of a few (perhaps) not so well-known interesting open problems.

I have mentioned in the Introduction Dolfi's spectacular result on graphs of solvable groups: if p and q are different primes dividing an irreducible character degree $\chi(1)$, then there exists a conjugacy class K of G whose size is divisible by p and q. Using the CFSG, this result was extended to every finite group by Casolo and Dolfi many years later ([**4**]). Inspired by Dolfi's result, two new ideas were discussed in [**28**]. It is well known that conjugacy class sizes are, in general, much bigger than irreducible character degrees, if only by looking at the equations

$$\sum_{K \in \mathrm{Cl}(G)} |K| = |G| = \sum_{\chi \in \mathrm{Irr}(G)} \chi(1)^2.$$

(Here $\mathrm{Cl}(G)$ denotes the set of conjugacy classes of G.) But could it be that for each irreducible character degree $\chi(1)$ there is a conjugacy class size $|K|$ such that $\chi(1)$ divides $|K|$? Unfortunately, there are a few examples where this does not happen. It seems, however, that in these cases χ is always induced from a proper subgroup (that is, χ is not *primitive*). This is the content of Conjecture C in [**28**], which is open even for solvable groups. (Recall that we are using g^G to denote the conjugacy class of g in G.)

Conjecture 8.1. *Let χ be a primitive irreducible character of a finite group G. Then $\chi(1)$ divides $|g^G|$ for some $g \in G$.*

Conjecture 8.1 is true for simple groups and symmetric groups by work in [**5**]. Given the failure to prove this conjecture for solvable groups, some attempts have been made to try to prove a weak local version (one prime at a time). This was done recently in [**42**] for groups of odd order, and in [**33**] for solvable groups.

The following related problem (also from [**28**]) remains open. If n is an integer, we let $\pi(n)$ be the set of primes dividing n.

Problem 8.2. *Let χ be an irreducible character of a finite group G. Is there a conjugacy class K of G such that $\pi(\chi(1)) \subseteq \pi(|K|)$?*

These problems somewhat resemble an intriguing conjecture of G. Qian ([**55**]).

Problem 8.3. *Let G be a finite group, and let $g \in G$. Is there some $\chi \in \mathrm{Irr}(G)$ such that $o(g)$ divides $|G : \ker(\chi)|/\chi(1)$?*

As is explained in [**40**], this would be a consequence of the following.

Conjecture 8.4. *Let G be a finite group, and let H be a cyclic subgroup of G. If $\theta \in \mathrm{Irr}(H)$, then there exists an irreducible constituent χ of θ^G such that $\chi(1)$ divides $|G : H|$.*

In [**40**], we have a strong version of Conjecture 8.4 which we have reduced to decorated simple groups—in particular giving a stronger solution to Problem 8.3 for solvable groups. Also, as discussed in [**40**], in Conjecture 8.4, cyclic might be replaced by abelian (or even nilpotent), at least in solvable groups. In [**6**], the modular version of this conjecture is studied.

Finally, K. Harada proposed the following in [**21**].

Conjecture 8.5. *Let G be a finite group. Then*

$$\prod_{\chi \in \mathrm{Irr}(G)} \chi(1) \quad \textit{divides} \quad \prod_{K \in \mathrm{Cl}(G)} |K|.$$

We are not aware that this has been solved, even for solvable groups.

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