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LOGARITHMIC HARDY–LITTLEWOOD–SOBOLEV INEQUALITY ON PSEUDO-EINSTEIN 3-MANIFOLDS AND THE LOGARITHMIC ROBIN MASS

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Abstract: Given a three-dimensional pseudo-Einstein CR manifold $(M, T^{1,0}M, \theta)$, we study the existence of a contact structure conformal to θ for which the logarithmic Hardy–Littlewood–Sobolev (LHLS) inequality holds. Our approach closely follows [**30**] in the Riemannian setting, yet the differential operators that we are dealing with are of very different nature. For this reason, we introduce the notion of Robin mass as the constant term appearing in the expansion of the Green's function of the P'-operator. We show that the LHLS inequality appears when we study the variation of the total mass under conformal change. This can be tied to the value of the regularized Zeta function of the operator at 1 and hence we prove a CR version of the results in [**27**]. We also exhibit an Aubin-type result guaranteeing the existence of a minimizer for the total mass which yields the classical LHLS inequality.

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1. Introduction and statement of the results

The logarithmic Hardy–Littlewood–Sobolev (LHLS) inequality is an important inequality in analysis since it appears as the borderline case of the classical Hardy–Littlewood–Sobolev inequality, which in turn represents the dual form of the classical Sobolev embeddings. We refer the reader for instance to [**6, 25**] and the references therein. We recall that in the standard sphere (S^n, g_0) this inequality reads as:

$$\frac{2}{n!}\int_{S^n} F\ln(F)\,dv_{g_0} - \int_{S^n} FA_n^{-1}F\,dv_{g_0} \geq 0, \tag{1}$$

for all $F\colon S^n \to \mathbb{R}_+$ such that $\int_{S^n} F\,dv = 1$ with $\int_{S^n} F\ln(F)\,dv < \infty$. Here A_n is the Paneitz operator defined by its action on the spherical harmonics Y_k by

$$A_nY_k = k(k+1)\cdots(k+n-1)Y_k.$$

The dual of (1) is the classical Beckner–Onofri inequality [**2, 6, 13**], which states that for $u \in H^{\frac{n}{2}}(S^n)$

$$\frac{1}{2n!}\fint_{S^n} uA_n u\, dv_{g_0} + \fint_{S^n} u\, dv_{g_0} - \ln\left(\fint_{S^n} e^u\, dv_{g_0}\right) \geq 0.$$

From a spectral point of view, the LHLS inequality appears in estimating the regularized spectral Zeta function of the operator A_n as proved in [**27**]: if $\int_{S^n} F\, dv_{g_0} = 1$, then

$$\tilde{Z}_{\tilde{g}}(1) - \tilde{Z}_{g_0}(1) = \frac{2}{n!}\int_{S^n} F\ln(F)\, dv_{g_0} - \int_{S^n} FA_n^{-1}F\, dv_{g_0}, \tag{2}$$

where $\tilde{g} = F^{\frac{2}{n}}g_0$ and $\tilde{Z}_{g_0}$ is the regularized Zeta function of the operator A_n. We recall here that the spectral Zeta function of the operator A_n is defined by

$$\zeta_{A_n}(s) = \operatorname{trace}(A_n^{-s}) = \sum_{j\geq 1}\frac{1}{\lambda_j^s},$$

where $\lambda_0 < \lambda_1 \leq \cdots \leq \lambda_j \leq \cdots$ is the sequence of eigenvalues of A_n and $\operatorname{Re}(s) > 1$. ζ_{A_n} can be extended to a meromorphic function having a simple pole at $s = 1$. The regularized Zeta function $\tilde{Z}_{g_0}$ is then defined by

$$\tilde{Z}_{g_0}(s) = \zeta_{A_n}(s) - \frac{\operatorname{Res}_{s=1}(\zeta_{A_n})}{s-1}.$$

This notion of a regularized Zeta function was originally introduced in [**27**]. Notice that from (4), $\tilde{Z}_{g_0}(1)$ can be seen as a regularized version of $\operatorname{trace}(A_n^{-1})$ as in [**34, 35, 30**], since the latter quantity is not well defined and hence we shall understand it in the regularized sense. This spectral property was then investigated in [**30**], in the case of general Riemannian manifolds:

Theorem 1.1 ([**30**]). *Let Γ_V be a conformal class of metrics on M^n with a fixed volume V. Then*

$$\inf_{g\in\Gamma_V} \operatorname{trace} A_n^{-1}(M,V) \leq \operatorname{trace} A_n^{-1}(S^n,V),$$

where $A_n(M,V)$ is the critical GJMS operator on the manifold M with volume V ([**19**]*). Moreover, if the inequality is strict, then the infimum is attained by a smooth metric in Γ_V with constant logarithmic mass.*

This result was proved by introducing the notion of mass for the Green's function of the critical GJMS operator (see [**30, 34, 35**]). Indeed, as in the case of the mass for the Yamabe-type problems, the Robin mass is the constant term appearing after the logarithmic singularity in the expansion of the Green's function.

In this work we will focus on the three-dimensional CR setting. With this setting, there are fundamental differences compared to the Riemannian setting. In fact, one does not have a Moser–Trudinger inequality with the same conformal invariance properties as in the Euclidean space, unless the study is restricted to pluriharmonic functions $\mathcal{P}$. For more information about the GJMS operators and their properties we refer the reader to [**11, 16, 19**]. Hence, the right substitute for the critical GJMS operator in this case is the P'-Paneitz-type operator. This operator was first introduced on S^{2n+1} in [**4**] and defined as

$$P' \sum_j (Y_{0,j} + Y_{j,0}) = \sum_j \lambda_j (Y_{0,j} + Y_{j,0}),$$

where $\lambda_j = j(j+1)\cdots(j+n)$ and $Y_{0,j}$, $Y_{j,0}$ form an L^2-orthonormal basis of the L^2 pluriharmonic functions on S^{2n+1} denoted by $\hat{\mathcal{P}}$. This operator was denoted by A'_Q in [**4**] and referred to as a "conditional intertwinor" because the operator intertwines with the conformal automorphisms modulo functions orthogonal to $\mathcal{P}$ (see Proposition 2.6 below). Moreover, as shown in [**4**], one has the following Moser–Trudinger inequality:

$$\frac{1}{2(n+1)!} \fint_{S^{2n+1}} F P' F \, dv + \fint_{S^{2n+1}} F \, dv - \ln \fint_{S^{2n+1}} e^F \, dv \geq 0,$$

for $F \in \hat{\mathcal{P}} \cap W^{2,2}(M)$. Its dual, also derived in [**4**], can be stated as follows: for any $G\colon S^{2n+1} \to \mathbb{R}$ with $G \geq 0$, $G \in L \log L$, and $\fint_{S^{2n+1}} G \, dv = 1$, we have

$$\frac{(n+1)!}{2} \fint_{S^{2n+1}} (G-1) P'^{-1} \tau (G-1) \, dv \leq \fint_{S^{2n+1}} G \ln(G) \, dv.$$

The conditional intertwinor $P' = A'_Q$ introduced in [**4**] is defined on $\hat{\mathcal{P}}$ and valued in $\hat{\mathcal{P}}$. In [**10**], the authors extended the construction of the operator P' from the standard sphere to a general three-dimensional pseudo-Einstein CR manifold. In fact, the authors construct an explicit differential operator P'_θ (see Section 2 below) that coincides with A'_Q if projected on $\hat{\mathcal{P}}$. In particular, we have $A'_Q = P' = \tau P'_\theta =: \overline{P}'$, where τ is the L^2-projection on $\hat{\mathcal{P}}$, the completion of $\mathcal{P}$ under the L^2-norm. This last remark also shows another fundamental difference from the Riemannian setting especially from a spectral point of view since the operator $\overline{P}'$

is not elliptic or sub-elliptic and does not have an invertible principal symbol. Instead, it can be seen as a Toeplitz operator.

In this paper we propose to study the notion of Robin mass in the three-dimensional CR setting and relate it to the LHLS inequality. Indeed, given an embeddable pseudo-Einstein manifold $(M, T^{1,0}, \theta)$, then the $\overline{P}'$ operator is well defined, and its Green's function G_θ takes the form
$$G_\theta(x,y) = -\gamma_3 \ln(d_\theta(x,y)) + O(1),$$
where $\gamma_3 = \frac{1}{4\pi^2}$ and $O(1)$ is a bounded quantity when $y \to x$.

Definition 1.2. Given a compact embeddable pseudo-Einstein manifold $(M, T^{1,0}M, \theta)$, the CR-Robin mass is defined by
$$m_\theta(x) = \lim_{y\to x}(G_\theta(x,y) + \gamma_3 \ln(d_\theta(x,y))),$$
where d_θ is the horizontal quasi-distance induced by the Levi form L_θ and defined in Subsection 2.2.

The total mass of $(M, T^{1,0}M, \theta)$ is then defined by
$$\mathcal{M}_\theta(M) := \int_M m_\theta \, dv_\theta.$$

For example, an easy computation in the case of the standard sphere $(S^3, T^{1,0}S^3, \theta_0)$ yields
$$m_{\theta_0} = \frac{1}{8\pi^2}\ln(2).$$
We define the space $\mathcal{L}(M)$ by
$$\mathcal{L}(M) := L^1\log(L^1)^+(M) = \left\{F\colon M \to \mathbb{R}^+;\ \int_M F\ln(F)\,dv_\theta < \infty\right\}.$$
On the Heisenberg group $\mathbb{H}$, we let
$$\mathcal{L}_c(\mathbb{H}) := \Big\{f\colon \mathbb{H} \to \mathbb{R}^+;\ f \text{ is measurable and compactly supported} \\ \text{and } \int_{\mathbb{H}} f\ln(f)\,dx < \infty\Big\}.$$
We now fix a pseudo-Einstein structure $(M, T^{1,0}M, \theta)$, such that P'_θ is non-negative and $\ker P'_\theta = \mathbb{R}$ and without loss of generality we can assume that
$$V := \int_M \theta\wedge d\theta = \int_{S^3} \theta_0 \wedge d\theta_0.$$
We also let $\theta_F = F^{\frac{1}{2}}\theta$. Notice that θ_F induces a pseudo-Einstein structure if and only if $\ln(F) \in \mathcal{P}$. We set τ_F to be the orthogonal projection

on $\hat{\mathcal{P}}$ with respect to the L^2-inner product induced by θ_F. We can then define the operators

$$A_\theta := \tau P'_\theta \quad \text{and} \quad A_{\theta_F} = \tau_F(F^{-1}A_\theta).$$

Here the operator A_θ is defined on $\mathcal{P}$. In fact, using the expression (5) below, we can take $H^4(M) \cap \hat{\mathcal{P}}$ as its domain. We will also write V_F for the volume of M with respect to dv_{θ_F}, that is,

$$V_F = \int_M \theta_F \wedge d\theta_F = \int_M F\theta \wedge d\theta.$$

Conventions.

Under the assumption that P'_θ is non-negative and $\ker P'_\theta = \mathbb{R}$, here we will make a very important convention that will be carried throughout the paper:

Consider the operator A_θ. Then we will let the operator A_θ^{-1} act on all functions in $\hat{\mathcal{P}}$ with the convention that

$$A_\theta \circ A_\theta^{-1}\tau u = \tau u - \frac{1}{V}\int_M u\, dv_\theta \text{ and } A_\theta^{-1}1 = 0.$$

We will state below our main results, and when there is no confusion we will drop the dependence of the total mass on the manifold M. We have then the following result, which is the CR version of (2) proved in [**27**].

Theorem 1.3. *Consider an embeddable pseudo-Einstein compact 3-manifold $(M, T^{1,0}M, \theta)$ such that A_θ is non-negative and $\ker A_\theta = \mathbb{R}$.*

If m_θ is constant and under the constraint of $V_F = V$, one has:

$$\mathcal{M}_{\theta_F} - \mathcal{M}_\theta = \frac{\gamma_3}{4}\int_M F\ln(F)\, dv_\theta - \frac{1}{V}\int_M FA_\theta^{-1}\tau F\, dv_\theta, \tag{3}$$

for all $F \in C^\infty(M)$ with $F > 0$. In particular, on the standard sphere $(S^3, T^{1,0}S^3, \theta_0)$ one has

$$\mathcal{M}_{\theta_F}(S^3) - \mathcal{M}_{\theta_0}(S^3) = \frac{\gamma_3}{4}\int_{S^3} F\ln(F)\, dv_\theta - \frac{1}{V}\int_{S^3} FA_\theta^{-1}\tau F\, dv_\theta \geq 0,$$

with equality if and only if $F = |J_k|$ with $k \in \mathrm{Aut}(S^3)$, normalized to have volume V.

Notice that the right-hand side of (3) is well defined for all $F \in \mathcal{L}(M)$. Hence, even when F is not smooth, one can make sense of $\mathcal{M}_{\theta_F}$ without going through the Green's function G_{θ_F} by simply writing

$$\mathcal{M}_{\theta_F} = \mathcal{M}_\theta + \frac{\gamma_3}{4}\int_M F\ln(F)\, dv_\theta - \frac{1}{V}\int_M FA_\theta^{-1}\tau F\, dv_\theta.$$

This is a similar situation to the classical Yamabe energy, which can be seen as the normalized total scalar curvature but for metrics with conformal factor possibly in just $H^1(M)$. In the same spirit, $\mathcal{M}_{\theta_F}$ should be seen as a functional rather than a geometric quantity, in the non-smooth case.

As we will show in Subsection 3.1, if $\tilde{Z}_\theta(1)$ is the regularized Zeta function of A_θ at 1, then there exists a constant c such that

$$\mathcal{M}_\theta = \tilde{Z}_\theta(1) + c.$$

Therefore, the previous theorem can be reformulated as follows:

Corollary 1.4. *Assume that $(M, T^{1,0}M, \theta)$ is an embeddable pseudo-Einstein manifold such that A_θ is non-negative and $\ker A_\theta = \mathbb{R}$. If m_θ is constant, then under the constraint $V_F = V$, we have*

$$\tilde{Z}_{\theta_F}(1) - \tilde{Z}_\theta(1) = \frac{\gamma_3}{4}\int_M F\ln(F)\,dv_\theta - \frac{1}{V}\int_M FA_\theta^{-1}\tau F\,dv_\theta,$$

for all $F \in \mathcal{L}(M)$. In particular, on the standard sphere $(S^3, T^{1,0}S^3, \theta_0)$ one has

$$\tilde{Z}_{\theta_F, S^3}(1) - \tilde{Z}_{\theta_0, S^3}(1) = \frac{\gamma_3}{4}\int_{S^3} F\ln(F)\,dv_\theta - \frac{1}{V}\int_{S^3} FA_\theta^{-1}\tau F\,dv_\theta \geq 0, \tag{4}$$

with equality if and only if $F = |J_k|$ with $k \in \mathrm{Aut}(S^3)$, normalized to have volume V.

A truncated version of (4) was proved in [**4**, Proposition 3.5]. Indeed, the authors prove the following Hersch-type result: if $\lambda_k(\theta)$ denotes the k^{th} eigenvalue of A_θ, then

$$\sum_{k=1}^{4} \frac{1}{\lambda_k(\theta_F)} \geq \sum_{k=1}^{4} \frac{1}{\lambda_k(\theta_0)}.$$

It is important to point out that inequality (4) is a characterization of the extremals of the regularized Zeta function on the conformal class of the standard CR sphere. To the best of our knowledge, this is the first result of this kind in the CR setting with a clear link between the spectral properties of the operator A_θ and the LHLS inequality.

Next, we will deduce a result that can be seen as an Aubin-type result as in [**1**] for the Yamabe problem and [**23**] for the CR-Yamabe problem. This result is the CR analogue of Theorem $1'$ in [**30**].

Theorem 1.5. *We define*

$$\mathcal{M}([\theta], M) := \inf_{F \in \mathcal{L}(M);\, V_F = V} \mathcal{M}_{\theta_F}(M).$$

Then under the assumptions of Theorem 1.3 we have

(i) $\mathcal{M}([\theta], M) \leq \mathcal{M}([\theta_0], S^3)$.

(ii) *If* $\mathcal{M}([\theta], M) < \mathcal{M}([\theta_0], S^3)$, *then the infimum is achieved.*

Moreover, if $\mathcal{M}([\theta], M)$ *is achieved by a function* F_0, *then the contact form* θ_{F_0} *has constant mass and the LHLS inequality holds, i.e., for all* $F \in \mathcal{L}(M)$, *such that* $V = V_F$, *we have*

$$\frac{\gamma_3}{4}\int_M F\ln(F)\,dv_{\theta_{F_0}} - \frac{1}{V}\int_M F A^{-1}_{\theta_{F_0}}\tau F\,dv_{\theta_{F_0}} \geq 0.$$

If in addition $m_\theta \in \mathcal{P}$, *then* θ_{F_0} *is pseudo-Einstein.*

Based on the work in [**14**] and [**10**], the assumptions that M is embeddable, P'_θ is non-negative, and $\ker P'_\theta = \mathbb{R}$ can be replaced by the non-negativity of the Paneitz operator P_θ and that the conformal class $[\theta]$ carries a pseudo-Einstein structure with non-negative Webster curvature but non-identically zero.

One is also hoping to have a positive mass type theorem as in [**15**], stating that if $\mathcal{M}([\theta], M) = \mathcal{M}([\theta_0], S^3)$, then (M, θ) is CR-equivalent to the standard sphere (S^3, θ_0), but for now, this type of result is beyond the work done in this paper and it needs a more refined blow-up analysis of the functional $J(\cdot, M)$ defined below.

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2. Preliminaries and setting

In this section we survey the main quantities and properties that we will be using during our investigation.

2.1. Pseudo-Hermitian geometry. We will closely follow the notations in [**10**]. Let M be a smooth, oriented three-dimensional manifold. A CR structure on M is a one-dimensional complex sub-bundle $T^{1,0} \subset T_{\mathbb{C}}M := TM \otimes \mathbb{C}$ such that $T^{1,0} \cap T^{0,1} = \{0\}$ for $T^{0,1} := \overline{T^{1,0}}$. Let $H = \operatorname{Re} T^{1,0}$ and let $J\colon H \to H$ be the almost complex structure defined by $J(Z + \overline{Z}) = i(Z - \overline{Z})$, for all $Z \in T^{1,0}$. The condition that $T^{1,0} \cap T^{0,1} = \{0\}$ is equivalent to the existence of a contact form θ such that $\ker\theta = H$. We recall that a 1-form θ is said to be a contact form if $\theta \wedge d\theta$ is a volume form on M. Since M is oriented, a contact form always exists, and is determined up to multiplication by a positive real-valued smooth function. We say that $(M, T^{1,0}M)$ is strictly pseudo-convex if the

Levi form $d\theta(\cdot, J\cdot)$ on $H \otimes H$ is positive definite for some, and hence any, choice of contact form θ. We shall always assume that our CR manifolds are strictly pseudo-convex.

Notice that in a CR manifold, there is no canonical choice of the contact form θ. A pseudo-Hermitian manifold is a triple $(M, T^{1,0}M, \theta)$ consisting of a CR manifold and a contact form. The Reeb vector field T is the vector field such that $\theta(T) = 1$ and $d\theta(T, \cdot) = 0$. The choice of θ induces a natural L^2-dot product $\langle \cdot, \cdot \rangle$, defined by

$$\langle f, g \rangle = \int_M f(x) g(x)\, \theta \wedge d\theta.$$

A $(1,0)$-form is a section of $T^*_{\mathbb{C}}M$ which annihilates $T^{0,1}$. An admissible coframe is a non-vanishing $(1,0)$-form θ^1 in an open set $U \subset M$ such that $\theta^1(T) = 0$. Let $\theta^{\bar{1}} := \overline{\theta^1}$ be its conjugate. Then $d\theta = i h_{1\bar{1}} \theta^1 \wedge \theta^{\bar{1}}$ for some positive function $h_{1\bar{1}}$. The function $h_{1\bar{1}}$ is equivalent to the Levi form. We set $\{Z_1, Z_{\bar{1}}, T\}$ to the dual of $(\theta^1, \theta^{\bar{1}}, \theta)$. The geometric structure of a CR manifold is determined by the connection form ${\omega_1}^1$ and the torsion form $\tau_1 = A_{11}\theta^1$ defined in an admissible coframe θ^1 and is uniquely determined by

$$\begin{cases} d\theta^1 = \theta^1 \wedge {\omega_1}^1 + \theta \wedge \tau^1, \\ \omega_{1\bar{1}} + \omega_{\bar{1}1} = dh_{1\bar{1}}, \end{cases}$$

where we use $h_{1\bar{1}}$ to raise and lower indices. The connection forms determine the pseudo-Hermitian connection ∇, also called the Tanaka–Webster connection, by

$$\nabla Z_1 := {\omega_1}^1 \otimes Z_1.$$

The scalar curvature R of θ, also called the Webster curvature, is given by the expression

$$d{\omega_1}^1 = R\theta^1 \wedge \theta^{\bar{1}} \mod \theta.$$

Definition 2.1. A real-valued function $w \in C^\infty(M)$ is CR pluriharmonic if locally $w = \operatorname{Re} f$ for some complex-valued function $f \in C^\infty(M, \mathbb{C})$ satisfying $Z_{\bar{1}} f = 0$.

Equivalently ([**24**]), w is a CR pluriharmonic function if

$$P_3 w := \nabla_1 \nabla_1 \nabla^1 w + i A_{11} \nabla^1 w = 0,$$

for $\nabla_1 := \nabla_{Z_1}$. We denote by $\mathcal{P}$ the space of all CR pluriharmonic functions and let $\tau \colon L^2(M) \to \hat{\mathcal{P}}$ be the orthogonal projection on the space of L^2 pluriharmonic functions, the completion of $\mathcal{P}$ under the L^2-norm. If $S \colon L^2(M) \to \ker \overline{\partial}_b$ denotes the Szegő kernel, then

$$\tau = S + \overline{S} + \mathcal{F},$$

where $\mathcal{F}$ is a smoothing kernel as shown in [**22**]. In particular, one has that τ is a bounded operator from $W^{k,p}(M) \to W^{k,p}(M)$ for $1 < p < \infty$ and $k \in \mathbb{N}$ (see [**31**]). In fact, this last property can be directly deduced from the work [**22**], since the author provides an expansion of the kernel of τ that we will still denote by τ:

Theorem 2.2 ([**22**]). *Assume that $(M, T^{1.0}M)$ is a compact embeddable strongly pseudo-convex CR manifold, then there exist $F_1, G_1 \in C^\infty(M \times M)$ such that*

$$\tau(x,y) = 2\,\mathrm{Re}(F_1(-i\varphi(x,y))^{-2} + G_1 \ln(-i\varphi(x,y))),$$

with

$$F_1 = a_0(x,y) + a_1(x,y)(-i\varphi(x,y)) + f_1(x,y)(-i\varphi(x,y))^2,$$

where $\tau(x,y)$ is the distribution kernel of the operator τ, $f_1 \in C^\infty(M \times M)$, and φ is such that $\mathrm{Im}(\varphi) \geq 0$ and has the following expansion in local coordinates near $x_0 \in M$ with $x = x_3 + iz$, $y = y_3 + iw$, and $\frac{\partial}{\partial x_3}$ is transversal to the CR structure,

$$\begin{aligned}\varphi(x,y) = -x_3 + y_3 + i|z-w|^2 &\\ &+ (i(\overline{z}w - z\overline{w}) + c(-zx_3 + wy_3) + \overline{c}(-\overline{z}x_3 + \overline{w}y_3)) \\ &+ |x_3 - y_3| f(x,y) + O(|(x,y)|^3),\end{aligned}$$

and f is a smooth real function such that $f(0,0) = 0$.

In particular, one can check that the first term of the expansion of τ coincides with the real part of the Szegő projection in $\mathbb{H}$.

The Paneitz operator P_θ is the differential operator

$$P_\theta(w) := 4\,\mathrm{div}(P_3 w) = \Delta_b^2 w + T^2 - 4\,\mathrm{Im}\,\nabla^1(A_{11}\nabla^1 f),$$

for $\Delta_b := \nabla^1\nabla_1 + \nabla^{\bar{1}}\nabla_{\bar{1}}$ the sub-Laplacian. In particular, $\mathcal{P} \subset \ker P_\theta$. Hence, $\ker P_\theta$ is infinite-dimensional. For a thorough study of the analytical properties of P_θ and its kernel, we refer the reader to [**22, 7, 9**]. The main property of the Paneitz operator P_θ is that it is CR covariant [**20**]. That is, if $\hat{\theta} = e^w\theta$, then $e^{2w}P_{\hat{\theta}} = P_\theta$. In [**10**], the authors actually write a general formula for the conformally covariant operator $P_{\theta,n}$ on manifolds of dimension $2n+1$ such that $P_{\theta,1} = P_\theta$. The P'-operator is then obtained by a limiting process from $P_{\theta,n}$ as $n \to 1$ as follows:

$$P'_\theta := \lim_{n\to 1} \frac{1}{n-1} P_{\theta,n|\mathcal{P}}.$$

An explicit formula is then provided:

Proposition 2.3 ([**10**]). *Let $(M^3, T^{1,0}M, \theta)$ be a pseudo-Hermitian manifold. The Paneitz-type operator $P'_\theta\colon \mathcal{P} \to C^\infty(M)$ has the following expression:*

$$\begin{aligned} P'_\theta f &= 4\Delta_b^2 f - 8\operatorname{Im}(\nabla^\alpha(A_{\alpha\beta}\nabla^\beta f)) - 4\operatorname{Re}(\nabla^\alpha(R\nabla_\alpha f)) \\ &\quad + \frac{8}{3}\operatorname{Re}(\nabla_\alpha R - i\nabla^\beta A_{\alpha\beta})\nabla^\alpha f - \frac{4}{3} f\nabla^\alpha(\nabla_\alpha R - i\nabla^\beta A_{\alpha\beta}) \end{aligned} \tag{5}$$

for $f \in \mathcal{P}$.

The main property of the operator P'_θ is its "almost" conformal covariance as shown in [**5, 10**]. That is, if $(M, T^{1,0}M, \theta)$ is a pseudo-Hermitian manifold, $w \in C^\infty(M)$, and we set $\hat{\theta} = e^w\theta$, then

$$e^{2w}P'_{\hat{\theta}}(u) = P'_\theta(u) + P_\theta(uw)$$

for all $u \in \mathcal{P}$. In particular, since P_θ is self-adjoint and $\mathcal{P} \subset \ker P_\theta$, we have that the operator P' is conformally covariant, mod $\mathcal{P}^\perp$.

Definition 2.4. A pseudo-Hermitian manifold $(M, T^{1,0}M, \theta)$ is pseudo-Einstein if

$$\nabla_\alpha R - i\nabla^\beta A_{\alpha\beta} = 0.$$

Moreover, if θ induces a pseudo-Einstein structure, then $e^u\theta$ is pseudo-Einstein if and only if $u \in \mathcal{P}$. The definition above was stated in [**10**], but it was implicitly mentioned in [**20**]. In particular, if $(M^3, T^{1,0}M, \theta)$ is pseudo-Einstein, then P'_θ takes a simpler form:

$$P'_\theta f = 4\Delta_b^2 f - 8\operatorname{Im}(\nabla^1(A_{11}\nabla^1 f)) - 4\operatorname{Re}(\nabla^1(R\nabla_1 f)).$$

To finish this part, we state the following result related to the Green's function G_θ of the operator $A_\theta = \tau P'_\theta$:

Proposition 2.5. *Assume that $(M^3, T^{1,0}M, \theta)$ is a compact embeddable pseudo-Einstein CR manifold. Then the Green's function G_θ of A_θ has the following expansion for y close to x:*

$$G_\theta(x, y) = -\gamma_3 \ln(d_\theta(x, y)) + \mathcal{K}(x, y), \tag{6}$$

where $\mathcal{K}(x, y)$ is bounded and d_θ is defined at the end of Subsection 2.2.

Proof: Using the notations in [**8**] (more precisely, equations (27) and (29)), we define $\tilde{G} = \ln(G_L)$, where G_L is the Green's function of the conformal sub-Laplacian. Then we have

$$P'_\theta(\tilde{G})(x, y) = 8\pi^2\tau(x, y) + A(x, y),$$

where A is a bounded function, smooth away from the diagonal. On the other hand, we have

$$P_\theta(\tilde{G}) = B(x,y),$$

where $B(x,y)$ is also a bounded function smooth away from the diagonal. Now, we can write $\tilde{G} = \tau(\tilde{G}) + \tilde{G}^\perp$. In particular,

$$P_\theta(\tilde{G}^\perp) = B(x,y).$$

Using the sub-ellipticity of P_θ on $\mathcal{P}^\perp$ ([**12**, Corollary 4.1]), we have that $\tilde{G}^\perp$ is bounded and smooth away from the diagonal. One then notices that

$$A_\theta\left(G_\theta - \frac{1}{8\pi^2}\tau(\tilde{G})\right) = C(x,y).$$

Here, $C(\cdot,y)$ is a function in $L^2(M)$. Using the ellipticity of A_θ on $\mathcal{P}$ as proved in [**9**], we have

$$G_\theta - \frac{1}{8\pi^2}\tau(\tilde{G}) \in H^2(M).$$

Combining this last statement and the boundedness of $\tilde{G}^\perp$ yields the expansion (6). $\square$

For the rest of the paper, $(M, T^{1,0}M, \theta)$ will always be assumed to be embeddable with P'_θ non-negative and $\ker P'_\theta = \mathbb{R}$.

2.2. The Heisenberg group. We identify the Heisenberg group $\mathbb{H}$ with $\mathbb{C}\times\mathbb{R} \simeq \mathbb{R}^3$ with elements $w = (z,t) = (x+iy,t) \simeq (x,y,t) \in \mathbb{R}\times\mathbb{R}\times\mathbb{R}$ and group law

$$w\cdot w' = (z,t)\cdot(z',t') = (z+z', t+t'+2\operatorname{Im}(z\overline{z'})) \quad \forall\, w,w' \in \mathbb{H},$$

where Im denotes the imaginary part of a complex number and $z\overline{z'}$ is the standard Hermitian inner product in $\mathbb{C}$. The dilations in $\mathbb{H}$ are

$$\delta_\lambda \colon \mathbb{H} \to \mathbb{H}, \quad \delta_\lambda(z,t) = (\lambda z, \lambda^2 t) \quad \forall\, \lambda > 0.$$

The natural distance that we will adopt in our setting is the Korányi distance, given by

$$d_{\mathbb{H}}((z,t),(z',t')) = (|z-z'|^4 + (t-t'-2\operatorname{Im}(z\overline{z'}))^2)^{\frac{1}{4}}.$$

We denote by

$$\Theta = \mathrm{d}t + 2\sum_{j=1}^{N}(x_i\mathrm{d}y_i - y_i\mathrm{d}x_i)$$

the standard contact form on $\mathbb{H}$ and by $dv_{\mathbb{H}}$ the volume form associated to Θ. The Heisenberg group can be identified with the unit sphere in $\mathbb{C}^2$

minus a point through the Cayley transform $\mathcal{C}\colon \mathbb{H} \to S^3 \setminus \{(0,0,0,-1)\}$ defined as follows:

$$\mathcal{C}(z,t) = \left(\frac{2z}{1+|z|^2+it}, \frac{1-|z|^2-it}{1+|z|^2+it} \right).$$

On the unit sphere $S^3 = \{\zeta \in \mathbb{C}^2 : |\zeta| = 1\}$ we consider the distance

$$d_{S^3}(\zeta,\eta)^2 = 2|1-\zeta\overline{\eta}|, \quad \zeta,\eta \in \mathbb{C}^2.$$

With this definition of d_{S^3}, the relation between the distance of two points $w=(z,t)$, $w'=(z',t')$ in $\mathbb{H}$ and the distance of their images $\mathcal{C}(w)$, $\mathcal{C}(w')$ in S^3 is given by

$$d_{S^3}(\mathcal{C}(w),\mathcal{C}(w')) = d_{\mathbb{H}}(w,w') \left(\frac{4}{(1+|z|^2)^2+t^2} \right)^{\frac{1}{4}} \left(\frac{4}{(1+|z'|^2)^2+t'^2} \right)^{\frac{1}{4}}.$$

On S^3, we consider the standard contact form

$$\theta_0 = i \sum_{j=1}^{N+1} (\zeta_j \mathrm{d}\overline{\zeta}_j - \overline{\zeta}_j \mathrm{d}\zeta_j),$$

and we denote by dv_0 the volume form associated to θ_0. With this notation we have that

$$(\mathcal{C}^{-1})^* \theta_0 = |J_{\mathcal{C}}|^{\frac{1}{2}} \Theta,$$

where $|J_{\mathcal{C}}| = \frac{8}{[(1+|z|^2)^2+t^2]^2}$ is the Jacobian of $\mathcal{C}$. For $h \in \mathrm{Aut}(\mathbb{H})$, we can parametrize their Jacobian $|J_h|$ as follows:

$$|J_h| = \frac{C}{||z|^2+it+2zw+\lambda|^4},$$

where $C>0$, $\lambda, w \in \mathbb{C}$, and $\mathrm{Re}(\lambda) > |w|^2$. We also recall that

$$\mathrm{Aut}(S^3) = \{k;\, k = \mathcal{C} \circ h \circ \mathcal{C}^{-1},\, h \in \mathrm{Aut}(\mathbb{H})\}.$$

Hence, the Jacobian of J_k can be parametrized as follows:

$$|J_k| := \frac{C}{|1-w\cdot\zeta|^4},$$

where $C>0$, $w \in \mathbb{C}^2$, $|w|<1$, and $\zeta \in S^3$. We can now state an important property which is satisfied by the operator P'_{θ_0} and is the reason for the name "conditional intertwinor" introduced in [**4**]:

Proposition 2.6 ([**4**])**.** *Let $u \in C^\infty(S^3) \cap \hat{\mathcal{P}}$ and $k \in \mathrm{Aut}(S^3)$. Then we have*

$$|J_k|(P'_{\theta_0}u)\circ k = P'_{\theta_0}(u\circ k) + \frac{1}{2} P_{\theta_0}(\ln(|J_k|)(u\circ k)).$$

We finish this section by this theorem regarding the pseudo-Hermitian normal coordinates:

Theorem 2.7 ([**17**])**.** *Let (M,θ) be a pseudo-Hermitian manifold. Given $p\in M$, there exist neighborhoods U_p of p in M and V of the origin of $\mathbb{H}$ and a diffeomorphism $\Psi_p\colon U_p\to V$ such that*

(i) $(\Psi^{-1})^*\theta=(1+O_1)\Theta$,

(ii) $(\Psi^{-1})^*(\theta\wedge d\theta)=(1+O_1)\Theta\wedge d\Theta$,

where O_1 is a function satisfying $|O_1(x)|\le C|x|$. Moreover, if we let $\Psi(p,q)=\Psi_p(q)$, then we have

$$\Psi(p,q)=-\Psi(q,p)=\Psi(p,q)^{-1}.$$

Using the theorem above one can then properly set $d_\theta(p,q)=d_{\mathbb{H}}(0,\Psi(p,q))=|\Psi(p,q)|$, where $|\cdot|$ is the Heisenberg norm. One of the main properties of d_θ is the quasi-triangle inequality. That is, if $d_\theta(p,q)\le 1$ and $d_\theta(q,\ell)\le 1$, then there exists a constant $C\ge 1$ such that

$$d_\theta(p,r)\le C(d_\theta(p,q)+d_\theta(q,r)).$$

3. Properties of the mass and proof of Theorem 1.3

First, we start by determining the change of the mass m_θ under a conformal change of the contact form $\theta\mapsto\theta_F=F^{\frac12}\theta$.

Proposition 3.1. *If $\theta_F=F^{\frac12}\theta$, then*

$$m_{\theta_F}=m_\theta(x)+\frac{\gamma_3}{4}\ln(F(x))-\frac{2}{V_F}A_\theta^{-1}\tau F(x)+\frac{1}{V_F^2}\int_M FA_\theta^{-1}\tau F\,dv_\theta.$$

Proof: The proof of this proposition is similar to the one in the Riemannian setting in [**30**, Lemma 2.1]. Nonetheless, we state it here for the convenience of the reader. We recall that, based on our convention, the Green's function of the operator A_θ has the following properties:

$$\begin{cases} A_{\theta,x}G_\theta(x,y)=-\dfrac{1}{V}, \text{ for } x\neq y,\\ G_\theta(x,y)+\gamma_3\ln(d_\theta(x,y))\in L^\infty(M),\\ \displaystyle\int_M G_\theta(x,y)\,dv_\theta(y)=0.\end{cases}$$

We then introduce the function

$$H_\theta(x,y,z):=G_\theta(x,y)-G_\theta(z,y).$$

Then one has:

$$A_{\theta,y}H_\theta=\delta_x-\delta_z.$$

Now, notice that by definition of the Green's function

$$A_{\theta,y}(H_{\theta_F} - H_\theta) = 0.$$

Moreover, $H_{\theta_F} - H_\theta$ is bounded. Hence, $H_{\theta_F} - H_\theta = \text{constant}$. Thus

$$G_{\theta_F}(x,y) - G_{\theta_F}(z,y) = G_\theta(x,y) - G_\theta(z,y) + C.$$

Integrating with respect to $dv_{\theta_F}(y)$ yields

$$A_\theta^{-1}\tau F(x) - A_\theta^{-1}\tau F(z) + CV_F = 0.$$

Hence, $C = \frac{1}{V_F}A_\theta^{-1}\tau F(z) - A_\theta^{-1}\tau F(x))$. In particular,

$$G_{\theta_F}(x,y) - G_{\tilde{\theta}}(z,y) = G_\theta(x,y) - G_\theta(z,y) + \frac{1}{V_F}A_\theta^{-1}\tau F(z) - A_\theta^{-1}\tau F(x)).$$

We now integrate with respect to $dv_{\theta_F}(z)$ to get

$$\begin{aligned} G_{\theta_F}(x,y) = G_\theta(x,y) &- \frac{1}{V_F}A_\theta^{-1}\tau F(y) \\ &+ \frac{1}{V_F^2}\int_M F A_\theta^{-1}\tau F\, dv_\theta - \frac{1}{V_F}A_\theta^{-1}\tau F(x). \end{aligned}$$

This last formula first appeared in [**28**], in the context of compact Riemannian manifolds. It then follows that

$$m_{\theta_F}(x) = m_\theta(x) + \frac{\gamma_3}{4}\ln(F(x)) - \frac{2}{V_F}A_\theta^{-1}\tau F(x) + \frac{1}{V_F^2}\int_M F A_\theta^{-1}\tau F\, dv_\theta. \quad \square$$

We also point out that a different proof of this result can be deduced from Lemma A.1 in the Appendix. A direct consequence of the previous proposition is

$$\mathcal{M}_{\theta_F} = \int_M m_\theta F\, dv_\theta + \frac{\gamma_3}{4}\int_M F\ln(F)\, dv_\theta - \frac{1}{V_F}\int_M F A_\theta^{-1}\tau F\, dv_\theta.$$

In particular, if m_θ is constant, then under the constraint $V_F = V$ one has

$$\mathcal{M}_{\theta_F} - \mathcal{M}_\theta = \frac{\gamma_3}{4}\int_M F\ln(F)\, dv_\theta - \frac{1}{V}\int_M F A_\theta^{-1}\tau F\, dv_\theta.$$

On the standard sphere (S^3, θ_0), one has

$$\mathcal{M}_{\theta_F} - \mathcal{M}_{\theta_0} = \frac{\gamma_3}{4}\int_{S^3} F\ln(F)\, dv_\theta - \frac{1}{V}\int_{S^3} F A_\theta^{-1}\tau F\, dv_\theta \geq 0,$$

with equality if and only if $F = |J_k|$ with $k \in \text{Aut}(S^3)$ normalized to have volume V. This follows from the LHLS inequality proved in [**4**] and this finishes the proof of Theorem 1.3. In fact, this is also the CR version of the spectral inequality in [**27**], as we will detail in the next section.

3.1. The regularized Zeta function and the mass. In this section we will establish a link between the total mass and the regularized Zeta function leading to Corollary 1.4. We want to point out that in the Riemannian case this link was established in [**29**, Section 5] for a general pseudo-differential operator having a leading term of $\Delta_g^{\frac{d}{2}}$, where d is the dimension of the manifold without any mention of the concept of mass. In [**30**], the author introduced the mass. Both of these proofs rely on the heat kernel estimates and an explicit expansion of the Green's function of the fractional power of the operator. In our case, we avoid the use of the heat kernel expansion since the operator A_θ does not have an invertible symbol as an operator in $\Psi_H(M)$ (but A_θ is a generalized Toeplitz operator which is invertible in the Toeplitz algebra [**3**]). Therefore one cannot use the Volterra calculus and the heat kernel expansion developed in [**33**]. Our proof relies on the non-commutative residue introduced in [**32**].

We will be using the same notations as [**33, 32**]. We consider the operator $P_0 \in \Psi_H(M)$ with Schwartz kernel $K_0 \in \mathcal{K}^0(M \times M)$ such that in a coordinate patch around $x \in M$ we have $K_0(x,y) = -\gamma_3 \ln(d_\theta(x,y))$ (here we disregard the factor related to the Jacobian of the change of coordinates for the sake of notation). Then we have

$$\mathcal{M}_\theta = \int_M \lim_{x \to y} (G_\theta(x,y) - K_0(x,y))\, dv_\theta(x).$$

Therefore, since $A_\theta^{-1}\tau - P_0$ is a trace class operator,

$$\mathcal{M}_\theta = \mathrm{TR}(A_\theta^{-1}\tau - P_0).$$

We now consider the holomorphic family $s \mapsto A_\theta^{-s}$ defined in a neighborhood of zero, where for $\Re(s) > 0$ we have

$$A_\theta^{-s} := q(s) \int_0^\infty t^{-s}(A_\theta + r + t)^{-1}\, dt,$$

where $q(s) = \frac{1}{\int_0^\infty t^{-s}(1+t)^{-1}\,dt}$ and $r\colon \mathcal{P} \to \ker A_\theta$ is the L^2-orthogonal projection. Notice that A_θ^{-s} is defined on $\mathcal{P}$ and can be extended by 0 to $\mathcal{P}^\perp$. We also have $\mathrm{ord}(A_\theta^{-s}) = -4s$.

Notice that with the previous notation $\lim_{s\to 0^+} A_\theta^{-s}u = u - r(u)$ for all $u \in \mathcal{P}$. So we let $T_s = A_\theta^{-s} \oplus \tau^\perp \oplus r$. We will be using this family as a gauge for $A_\theta^{-1}\tau$ since, for $\Re(s) > 0$ and small, $T_s A_\theta^{-1}\tau$ is a trace class operator.

Recall that, [**32**, Proposition 3.17], $\mathrm{TR}(T_s A_\theta^{-1}\tau)$ has a simple pole at $s=0$ and the residue at this pole is

$$\mathrm{Res}_{s=0}(\mathrm{TR}(T_s A_\theta^{-1}\tau)) = -\operatorname{Res}(A_\theta^{-1}\tau) = -\gamma_3 V.$$

Similarly $\mathrm{TR}(T_s P_0)$ has the same residue at the pole $s=0$. Hence,

$$\begin{aligned}
&\lim_{s\to 0} \mathrm{TR}(T_s A_\theta^{-1}\tau) - \mathrm{TR}(T_s P_0)\\
&\quad = \lim_{s\to 0} \mathrm{TR}(T_s A_\theta^{-1}\tau) - \frac{(-\gamma_3 V)}{s} + \frac{(-\gamma_3 V)}{s} - \mathrm{TR}(T_s P_0)\\
&\quad = \tilde{Z}_\theta(1) + c,
\end{aligned} \tag{7}$$

where

$$c = \lim_{s\to 0} \frac{(-\gamma_3 V)}{s} - \mathrm{TR}(T_s P_0)$$

is a constant that might depend on V. The last equality in (7) follows from the fact that $T_s A_\theta^{-1}\tau = A_\theta^{-s} A_\theta^{-1}\tau$. But

$$\lim_{s\to 0} \mathrm{TR}(T_s A_\theta^{-1}\tau - T_s P_0) = \mathrm{TR}(A_\theta^{-1}\tau - P_0)) = \mathcal{M}_\theta.$$

Therefore,

$$\tilde{Z}_\theta(1) = \mathcal{M}_\theta - c.$$

4. Proof of Theorem 1.5(i)

The proof of Theorem 1.5 that we will be presenting here is an adaptation of the proof of Theorem 1′ in [**30**] to the CR setting. Hence, we will only show the parts which present some specific differences due to the different geometry.

We define the functional $J(\cdot, M)\colon \mathcal{L}(M) \to \mathbb{R}$ by

$$J(F,M) := \int_M m_\theta F\, dv_\theta + \frac{\gamma_3}{4}\int_M F\ln(F)\, dv_\theta - \frac{1}{V_F}\int_M F A_\theta^{-1}\tau F\, dv_\theta.$$

In a similar way, for the Heisenberg group, we define the functional $J(\cdot, \mathbb{H})\colon \mathcal{L}_c(\mathbb{H}) \to \mathbb{R}$ by

$$J(f,\mathbb{H}) := \frac{\gamma_3}{4}\left(\int_{\mathbb{H}} f\ln(f)\, dx - \frac{4}{V_f}\int_{\mathbb{H}}\int_{\mathbb{H}} f(x)\ln\Big(\frac{1}{|xy^{-1}|}\Big) f(y)\, dx\, dy\right),$$

and we let

$$\mathcal{M}(\mathbb{H}, V) = \inf_{f\in\mathcal{L}_c(\mathbb{H});\, V_f = V} J(f,\mathbb{H}).$$

We claim that

$$\mathcal{M}(\mathbb{H}, V) = \mathcal{M}([\theta_0], S^3). \tag{8}$$

Indeed, from Theorem 1.3, we have that

$$\mathcal{M}([\theta_0], S^3) = \mathcal{M}_{\theta_0}(S^3).$$

The equality in (8) then follows from an easy computation starting from the LHLS inequality in $\mathbb{H}$, proved in [**4, 18**] and stated in the theorem below.

Theorem 4.1. *For any measurable function $g\colon \mathbb{H} \to \mathbb{R}$ such that $g \geq 0$, $\int_{\mathbb{H}} g(x)\,dx = \omega_3 := 2\pi^2$, and $\int_{\mathbb{H}} g \ln(1+|x|^2)\,dx < \infty$, we have*

$$\frac{2}{\omega_3^2}\int_{\mathbb{H}\times\mathbb{H}} \ln\left(\frac{2}{|xy^{-1}|}\right) g(x)g(y)\,dx\,dy \leq \frac{1}{\omega_3}\int_{\mathbb{H}} g\ln(g)\,dx + \ln(2),$$

with equality if and only if $g = (|J_{\mathcal{C}}| \circ h)|J_h|$ with $h \in \mathrm{Aut}(\mathbb{H})$.

Next, we claim that

$$\mathcal{M}([\theta], M) \leq \mathcal{M}(\mathbb{H}, V).$$

But, in order to show this, we need an intermediate localization lemma.

Lemma 4.2. *Given $\varepsilon > 0$, and $p \in M$, there exists $\delta > 0$ and a new coordinate system in $B_\delta(p)$ defined by a diffeomorphism ϕ, such that*

$$(\phi^{-1})^*(\theta \wedge d\theta) = \Theta \wedge d\Theta,$$

and

$$e^{-\varepsilon} \leq \frac{d_\theta(p,q)}{d_{\mathbb{H}}(\phi(p),\phi(q))} \leq e^{\varepsilon},$$

for all $q \in B_\delta(p)$.

Proof: First notice that, using Theorem 2.7, we have that for every $p \in M$ there exists $\delta > 0$ and a diffeomorphism $\Psi\colon B_\delta(p) \to V$, where V is a neighborhood of the origin in $\mathbb{H}$, such that

$$(\Psi^{-1})^*\theta = (1 + O(\delta))\Theta,$$

and

$$(\Psi^{-1})^*(\theta \wedge d\theta) = (1 + O(\delta))\Theta \wedge d\Theta.$$

Now using Gray's theorem, we can find new coordinate systems in $\mathbb{H}$, defined by a diffeomorphism Φ such that

$$(\Phi^{-1})^*(1 + O(\delta))(\Theta \wedge d\Theta) = \Theta \wedge d\Theta.$$

Since $(1 + O(\delta))\Theta$ is close to Θ for δ small enough, and $\phi = \Phi \circ \Psi$, then given $\varepsilon > 0$ there exists $\delta > 0$ such that

$$e^{-\varepsilon} \leq \frac{d_\theta(p,q)}{d_{\mathbb{H}}(\phi(p),\phi(q))} \leq e^{\varepsilon},$$

for all $q \in B_\delta(p)$. $\square$

Lemma 4.3. *Given $\varepsilon > 0$, there exists $\delta > 0$ such that, for $F \in \mathcal{L}(M)$ supported in $B_\delta(p)$, there exists $f \in \mathcal{L}_c(\mathbb{H})$ compactly supported, such that*

$$|J(F, M) - J(f, \mathbb{H})| \leq \varepsilon V_F.$$

Similarly, for any $f \in \mathcal{L}_c(\mathbb{H})$, there exists $F \in \mathcal{L}(M)$ such that F is supported in $B_\delta(p)$ and

$$|J(F, M) - J(f, H)| \leq \varepsilon V_f.$$

Proof: Using the compactness of M and a covering argument, we can always find $\delta > 0$ such that for every $p \in M$, $B_\delta(p)$ is in a coordinate chart as described in Lemma 4.2. So we fix $\varepsilon > 0$. Taking $\delta > 0$ even smaller if necessary, we can assume that

$$|G_\theta(p, q) + \gamma_3 \ln(|\phi(p)\phi(q)^{-1}|) - m_\theta(p)| < \varepsilon.$$

Hence if F is supported in $B_\delta(p)$, taking $U = \phi(B_\delta(p))$, $x = \phi(p)$, $y = \phi(q)$, and $f(x) = F(p)$, we have

$$\begin{aligned} J(F, M) = & \frac{\gamma_3}{4} \int_U f \ln(f)\, dv_\Theta - \frac{\gamma_3}{V_F} \int_{U \times U} f(x) \ln(|xy^{-1}|) f(y)\, dv_\Theta(y)\, dv_\Theta(y) \\ & + \frac{1}{V_F} \int_{M \times M} F(p)\eta(p, q)F(q)\, dv_\theta(p)\, dv_\theta(q) \\ = & J(f, \mathbb{H}) + \frac{1}{V_F} \int_{M \times M} F(p)\eta(p, q)F(q)\, dv_\theta(p)\, dv_\theta(q). \end{aligned}$$

Hence,

$$|J(F, M) - J(f, \mathbb{H})| \leq \varepsilon V_F.$$

In a similar way, the second assertion follows easily from the invariance of the functional $J(\cdot, \mathbb{H})$ by the scaling

$$f \mapsto \frac{1}{\lambda^4} f(\delta_{\frac{1}{\lambda}} \cdot), \tag{9}$$

where δ_λ is the dilation in the Heisenberg group. So one can shrink the support and then lift it to a function on M via the diffeomorphism ϕ. □

Corollary 4.4. *Let*

$$\mathcal{M}_\delta([\theta], M) := \inf_{F \in \mathcal{L}(M);\, V_F = V \text{ and } \operatorname{supp}(F) \subset B_\delta(p)} J(F, M).$$

Then, one has

$$\lim_{\delta \to 0} \mathcal{M}_\delta([\theta], M) = \mathcal{M}(\mathbb{H}, V).$$

In particular,

$$\mathcal{M}([\theta], M) \leq \mathcal{M}(\mathbb{H}, V).$$

5. Proof of Theorem 1.5(ii)

5.1. Concentration and improved LHLS inequality. We state here some useful results that will be brought into play in the main proof of (ii). These results follow mainly from the expansion of the Green's function (6) and the convexity of the function $t \mapsto t\ln(t)$. The proofs then follow the same algebraic manipulations as in [**30**] and hence will be omitted.

Proposition 5.1. *There exists $C > 0$, depending on M and V, such that if $V_F = V$, one has*

$$\frac{1}{V}\int_M F A_\theta^{-1}\tau F\, dv_\theta \leq \frac{\gamma_3}{4}\int_M F\ln(F)\, dv_\theta + C.$$

Moreover, for any $\varepsilon > 0$, there exists $C_\varepsilon > 0$ such that for all $F \in \mathcal{L}(M)$

$$\|A_\theta^{-1}\tau F\|_\infty \leq (1+\varepsilon)\frac{\gamma_3}{4}\int_M F\ln(F)\, dv_\theta + C_\varepsilon\left(\int_M F\, dv_\theta + 1\right).$$

As was noted in the proof of Lemma 4.3, the functional $J(\cdot, \mathbb{H})$ is invariant under the scaling (9), which leaves the volume or the L^1-norm invariant. This hints at a concentration phenomena that can happen locally for the functional $J(\cdot, M)$. So we start by investigating the effect of concentration on the functional $J(\cdot, M)$.

Definition 5.2. We say that a sequence $(F_j)_{j\in\mathbb{N}} \in L^1(M)$ is a concentrating sequence if there exists a sequence of points $p_j \in M$ and numbers $\delta_j \to 0$ such that

$$\int_{B_{\delta_j}(p_j)} F_j\, dv_\theta \geq (1-\delta_j)V_{F_j}.$$

We then have this lower bound on the energy of concentrating sequences:

Proposition 5.3. *Let $(F_j)_{j\in\mathbb{N}}$ be a concentrating sequence in $\mathcal{L}(M)$ with constant volume V. Then*

$$\liminf_{j\to\infty} J(F_j, M) \geq \mathcal{M}(\mathbb{H}, V).$$

Since concentration tends to localize the problem in such a way that it becomes similar to the Heisenberg case, one expects to obtain an improved logarithmic HLS inequality in the case of absence of concentration and this can be quantified by the following:

Lemma 5.4. *Fix $0<\delta<1$, then there exists $C(\delta,M,\theta)>0$ such that for any $F\in\mathcal{L}(M)$ satisfying $V_F=V$ and*

$$\int_{B_\delta(x)} F\,dv_\theta<(1-\delta)V,$$

for all $x\in M$, we have

$$(1-\delta)\frac{\gamma_3}{4}\int_M F\ln(F)\,dv_\theta+C\geq\frac{1}{V}\int_M FA_\theta^{-1}\tau F\,dv_\theta.$$

5.2. Sub-critical approximation. The main idea of the proof of the rest of Theorem 1.5 is to construct an adequate minimizing sequence and show its convergence by discarding its concentration. Again, the steps of the proof follow [**30**] but there are some technical difficulties specific to the CR setting and the operator A_θ that one needs to be mindful of. In the Riemannian setting, the Fourier transform and manipulations of the principal symbol of the operators used play a key role in the proof of the convergence of the minimizing sequence. In the CR setting, this becomes a bit challenging and we take a different approach more adapted to the operator A_θ. This approach relies mainly on the ellipticity of A_θ when restricted to $\mathcal{P}$, as proved in [**9**].

We start by considering a sub-critical approximation of our original functional:

$$J_\varepsilon(F,M)=\int_M m_\theta F\,dv_\theta+\frac{\gamma_3}{4}\int_M F\ln(F)\,dv_\theta-\frac{(1-\varepsilon)\lambda_1^\varepsilon}{V_F}\int_M FA_\theta^{-1-\varepsilon}\tau F\,dv_\theta,$$

where λ_1 is the first non-zero eigenvalue of A_θ and $\varepsilon>0$.

Lemma 5.5. *There exists $F_\varepsilon\in C^\infty(M)$ that minimizes the functional J_ε. That is,*

$$\inf_{f\in\mathcal{L}(M);\,V_F=V} J_\varepsilon(F,M)=J_\varepsilon(F_\varepsilon,M).$$

Proof: First notice that

$$J_\varepsilon(F,M)\geq\int_M m_\theta F\,dv_\theta+\frac{\gamma_3}{4}\int_M F\ln(F)\,dv_\theta-\frac{(1-\varepsilon)}{V_F}\int_M FA_\theta^{-1}\tau F\,dv_\theta.$$

Now using Proposition 5.1, we have

$$\begin{aligned}J_\varepsilon(F,M)&\geq\frac{\varepsilon\gamma_3}{4}\int_M F\ln(F)\,dv_\theta-C(1-\varepsilon)+\inf_M(m_\theta)V\\&\geq\frac{\varepsilon\gamma_3}{4}\int_M F\ln(F)\,dv_\theta-C_1.\end{aligned}$$

Therefore, if $(F_k)_{k\in\mathbb{N}}$ is a minimizing sequence for J_ε, then $\int_M F_k \ln(F_k)\, dv_\theta$ is bounded above, independently of k. To finish our argument, we use the weak convergence result in [**30**, Lemma 2.11] with $G(t) = t\ln(t)$ and the sequence $(F_k)_{k\in\mathbb{N}}$, we have the existence of $F_\varepsilon \in \mathcal{L}(M)$ such that $F_k \to F_\varepsilon$ weakly in $L^1(M)$ and

$$\int_M F_\varepsilon \ln(F_\varepsilon)\, dv_\theta \leq \liminf_{k\to\infty} \int_M F_k \ln(F_k)\, dv_\theta.$$

Since F_k is bounded in $\mathcal{L}(M)$, we have from Proposition 5.1 that $A_\theta^{-1}\tau F_k$ is uniformly bounded in $L^p(M)\cap\hat{\mathcal{P}}$ for all $1 \leq p \leq \infty$. Now by ellipticity of A_θ on $\hat{\mathcal{P}}$, we have that $A_\theta^{-\varepsilon}\tau$ is a pseudo-differential operator of order -2ε. Hence, $A_\theta^{-1-\varepsilon}\tau F_k$ is uniformly bounded in $W^{2\varepsilon,p}(M)$. Taking $p > \frac{3}{2\varepsilon}$, we see that $(A_\theta^{-1-\varepsilon}\tau F_k)_{k\in\mathbb{N}}$ is compact in $C(M)$. Therefore, we can extract a convergent subsequence, that we still denote by $(F_k)_{k\in\mathbb{N}}$ such that $A_\theta^{-1-\varepsilon}\tau F_k \to A_\theta^{-1-\varepsilon}\tau F_\varepsilon$, since $F_k \to F_\varepsilon$ weakly in $L^1(M)$. Hence,

$$\int_M F_k A_\theta^{-1-\varepsilon}\tau F_k\, dv_\theta \to \int_M F_\varepsilon A_\theta^{-1-\varepsilon}\tau F_\varepsilon\, dv_\theta.$$

Therefore,

$$\lim_{k\to\infty} J_\varepsilon(F_k, M) \geq J_\varepsilon(F_\varepsilon, M).$$

Showing that F_ε is bounded below by a positive constant follows exactly the same proof as in [**30**], hence we will omit it.

Now the Euler–Lagrange equation for the constraint minimization of J_ε yields the equation

$$m_\theta + \frac{\gamma_3}{4}(\ln(F_\varepsilon) + 1) - \frac{2(1-\varepsilon)\lambda_1^\varepsilon}{V} A_\theta^{-1-\varepsilon}\tau F_\varepsilon = \lambda_\varepsilon,$$

where λ_ε is the constant coming from the Lagrange multiplier. Therefore, by ellipticity of A_θ restricted to $\mathcal{P}$ and smoothness of m_θ, we get the smoothness of F_ε. □

At this stage, we have the required ingredients to finish the proof of Theorem 1.5. The idea is to extract a convergent subsequence of F_ε when $\varepsilon \to 0$. Notice that we have

$$\lim_{\varepsilon\to 0} J(F_\varepsilon, M) = \mathcal{M}([\theta], M).$$

So $(F_\varepsilon)_{\varepsilon>0}$ is a minimizing sequence the convergence of which we need to show. But since $\mathcal{M}([\theta], M) < \mathcal{M}([\theta_0], S^3)$, it follows from Proposition 5.3 that $(F_\varepsilon)_{\varepsilon>0}$ does not concentrate. We then combine Lemma 5.4 and the boundedness of $J(F_\varepsilon, M)$ to get

$$\int_M F_\varepsilon \ln(F_\varepsilon)\, dv_\theta \leq C,$$

with C being independent of ε. This, combined with

$$J_\varepsilon(1,M) \geq J_\varepsilon(F_\varepsilon, M) \geq \mathcal{M}([\theta], M),$$

yields the uniform boundedness of $\int_M F_\varepsilon A_\theta^{-1-\varepsilon}\tau F_\varepsilon\, dv_\theta$.

We now recall that F_ε satisfies the equation

$$m_\theta + \frac{\gamma_3}{4}(\ln(F_\varepsilon)+1) - \frac{2(1-\varepsilon)\lambda_1^\varepsilon}{V} A_\theta^{-1-\varepsilon}\tau F_\varepsilon = \lambda_\varepsilon. \tag{10}$$

The Lagrange multiplier λ_ε can be obtained by multiplying (10) by F_ε and then integrating:

$$\begin{aligned}\lambda_\varepsilon V = &\int_M m_\theta F_\varepsilon\, dv_\theta + \frac{\gamma_3}{4}\int_M F_\varepsilon \ln(F_\varepsilon)\, dv_\theta \\ &- \frac{2(1-\varepsilon)\lambda_1^\varepsilon}{V}\int_M F_\varepsilon A_\theta^{-1-\varepsilon}\tau F_\varepsilon\, dv_\theta + \frac{\gamma_3}{4}V.\end{aligned}$$

Hence, λ_ε is uniformly bounded. Using Proposition 5.1, we have that $(A_\theta^{-1}F_\varepsilon)_\varepsilon$ is uniformly bounded in $C(M)\cap\hat{\mathcal{P}}$. A bootstrap argument for equation (10) provides us with the smoothness of F_ε. So if we set $u_\varepsilon = A_\theta^{-1}\tau F_\varepsilon$, then one has

$$\int_M u_\varepsilon A_\theta u_\varepsilon\, dv_\theta \leq C.$$

Therefore, from the Moser–Trudinger inequality in [**9**], we have that u_ε is uniformly bounded in $W^{2,2}(M)$ and e^{u_ε} is uniformly bounded in $L^p(M)$ for all $1 \leq p < \infty$. Since the family $(A_\theta^{-\varepsilon})_\varepsilon$ is uniformly bounded in $W^{2,2}(M)\cap\hat{\mathcal{P}}$, we have the uniform boundedness in $W^{2,2}(M)\cap\hat{\mathcal{P}}$ of $v_\varepsilon := A_\theta^{-\varepsilon}u_\varepsilon$. Again, using the Moser–Trudinger inequality, we get that e^{v_ε} is uniformly bounded in $L^p(M)$. But since

$$F_\varepsilon = e^{R_\varepsilon} e^{\frac{8(1-\varepsilon)\lambda_1^\varepsilon}{\gamma_3 V}v_\varepsilon},$$

and R_ε is uniformly bounded in $L^\infty(M)$, we have the uniform boundedness of F_ε in $L^2(M)$. So, using the regularizing effect of A_θ^{-1}, we see that $(A_\theta^{-1-\varepsilon}\tau F_\varepsilon)_\varepsilon$ is compact in $C(M)\cap\hat{\mathcal{P}}$. Therefore, we can extract a convergent subsequence of $(F_\varepsilon)_\varepsilon$ that we denote by $(F_{\varepsilon_k})_{k\in\mathbb{N}}$ such that $F_{\varepsilon_k}\to F_0$ in $C(M)$, and via a diagonal process we get that

$$J(F_0, M) = \inf_{F\in\mathcal{L}(M);\, V_F = V} J(F, M). \qquad \square$$

It is then easy to see that if $\mathcal{M}([\theta], M)$ is achieved by a function F_0, then it satisfies the equation

$$m_\theta + \frac{\gamma_3}{4}(\ln(F_0) + 1) - \frac{2}{V} A_\theta^{-1} \tau F_0 = \lambda_0.$$

In particular, a bootstrapping argument yields the regularity of F_0 and if $m_\theta \in \mathcal{P}$, then so is $\ln(F_0)$. Using Proposition 3.1, one sees that m_{θ_F} is constant. In fact, we have

$$m_{\theta_{F_0}} = \frac{\mathcal{M}([\theta], M)}{V}.$$

Appendix A. Geometric CR mass

In this section we will add a geometric correction to the mass that makes it independent of the point on the sphere. We start with the following:

Lemma A.1.

$$A_{\tilde{\theta}}^{-1} \tau_F f = A_\theta^{-1} \tau(Ff) - \frac{\int_M Ff \, dv_\theta}{V_F} A_\theta^{-1} \tau(F) - a_1 + a_2,$$

where $a_1 = \frac{\int_M F A_\theta^{-1} \tau(Ff) \, dv_\theta}{V_F}$, $a_2 = \frac{\int_M Ff \, dv_\theta \int_M F A_\theta^{-1} \tau(F) \, dv_\theta}{V_F^2}$.

Proof: Recall that from our convention we have that

$$A_{\theta_F} A_{\theta_F}^{-1} \tau_F f = \tau f - \frac{\int_M f \, dv_{\theta_F}}{V_F}.$$

But $A_{\theta_F} = \tau_F(F^{-1} A_\theta)$. Hence, one has

$$A_\theta A_{\theta_F}^{-1} \tau_F f = \tau(Ff) - \frac{\tau(F) \int_M f \, dv_{\theta_F}}{V_F}.$$

Therefore,

$$A_{\theta_F}^{-1} \tau_F f = A_\theta^{-1} \tau(Ff) - \frac{\int_M f \, dv_{\theta_F}}{V_F} A_\theta^{-1} \tau(F) - \frac{1}{V_F} \int_M A_\theta^{-1} \tau(Ff) \, dv_{\theta_F}$$
$$+ \frac{\int_M f \, dv_{\theta_F}}{V_F^2} \int_M A_\theta^{-1} \tau F \, dv_{\theta_F}. \qquad \square$$

We also recall here the scalar invariant related to the operator P'_θ, namely, the Q'-curvature. Indeed, we set

$$Q'_\theta := 2\Delta_b R - 4|A|^2 + R^2.$$

Then for $w \in \mathcal{P}$ and $\hat{\theta} = e^w \theta$ we have

$$e^{2w} Q'_{\hat{\theta}} = Q'_\theta + P'_\theta(w) + \frac{1}{2} P_\theta(w^2).$$

In our case, we are more interested in the quantity $\overline{Q}'_\theta = \tau Q'_\theta$. For more information about the Q'-curvature we refer the reader to [**5, 10**] and for problems related to prescribing the $\overline{Q}'_\theta$ we refer the reader to [**9, 21, 26**].

Lemma A.2. *Assume that* $\ln(F) \in \mathcal{P}$*, then we have*

$$\tau_F Q'_{\theta_F} = \tau_F(F^{-1}Q'_\theta) + \frac{1}{2}A_{\theta_F}\ln(F).$$

Proof: Recall that under the conformal change $\theta \to \theta_F$ the Q'-curvature changes as follows:

$$\frac{1}{2}P'_\theta \ln(F) + Q'_\theta = Q'_{\theta_F}F + \frac{1}{8}P_\theta((\ln(F))^2).$$

Thus,

$$Q'_{\theta_F} = \frac{F}{2}P'_\theta \ln(F) + FQ'_\theta - \frac{F}{8}P_\theta((\ln(F))^2).$$

Hence,

$$\tau_F Q'_{\theta_F} = \tau_F(F^{-1}Q'_\theta) + \frac{1}{2}A_{\theta_F}\ln(F). \qquad \square$$

Now define the geometric mass as in [**34, 35**], by

$$\mathcal{N}_\theta(x) := m_\theta(x) - \frac{\gamma_3}{2}A_\theta^{-1}\tau Q'_\theta(x).$$

A direct substitution then shows that if $\ln(F) \in \mathcal{P}$, then

$$\mathcal{N}_{\theta_F} - \mathcal{N}_\theta = \frac{\frac{\gamma_3}{2}\int_M Q'_\theta\, dv_\theta - 2}{V_F}A_\theta^{-1}\tau F + \frac{1 - \frac{\gamma_3}{2}\int_M Q'_\theta\, dv_\theta}{V_F^2}\int_M FA_\theta^{-1}\tau F\, dv_\theta$$
$$+ \frac{\gamma_3}{4V_F}\int_M F\ln(F)\, dv_\theta - \frac{\gamma_3}{2V_F}\int_M FA_\theta^{-1}\tau Q'_\theta\, dv_\theta.$$

In particular, on the sphere S^3, since we have $\int_{S^3} Q'_\theta\, dv_\theta = 16\pi^2$, we have

Proposition A.3.

$$\mathcal{N}_{\theta_F}(S^3)(x) - \mathcal{N}_{\theta_0}(S^3)(x) = \frac{\gamma_3}{4V_F}\int_M F\ln(F)\, dv_\theta - \frac{1}{V_F^2}\int_M FA_\theta^{-1}\tau F\, dv_\theta \geq 0.$$

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