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AN EXPLICIT FORMULA FOR THE SECOND MOMENT OF MAASS FORM SYMMETRIC SQUARE L-FUNCTIONS

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Abstract: We prove an explicit formula for the second moment of symmetric square L-functions associated to Maass forms for the full modular group. In particular, we show how to express the considered second moment in terms of dual second moments of symmetric square L-functions associated to Maass cusp forms of levels 4, 16, and 64.

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1. Introduction

Let us denote by $\{u_j(z)\}_{j=1}^{\infty}$ an orthonormal basis of the space of Maass cusp forms for the full modular group. This basis can be chosen such that $u_j(z)$ for $j = 1, 2, \dots$ are eigenfunctions not only of the hyperbolic Laplacian, but also of all Hecke operators. Let $\lambda_j(n)$ be the eigenvalue of the n-th Hecke operator corresponding to the eigenfunction $u_j(z)$ and let $\kappa_j = 1/4 + t_j^2$ be the eigenvalue of the Laplace operator corresponding to $u_j(z)$. The following Fourier expansion takes place:

$$u_j(x+iy) = \sqrt{y} \sum_{n \neq 0} \rho_j(n) K_{it_j}(2\pi|n|y) e(nx),$$

where $K_\alpha(x)$ is the K-Bessel function and $\rho_j(n) = \rho_j(1)\lambda_j(n)$. The symmetric square L-function associated to $u_j(z)$ can be defined by the Dirichlet series

$$L(\operatorname{sym}^2 u_j, s) := \zeta(2s) \sum_{n=1}^{\infty} \frac{\lambda_j(n^2)}{n^s}, \quad \Re s > 1. \tag{1.1}$$

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This L-function admits an analytic continuation to the whole complex plane and satisfies a functional equation.

In order to study moments of L-functions we introduce the normalizing coefficient

$$\alpha_j := \frac{|\rho_j(1)|^2}{\cosh \pi t_j}.$$

Now we can describe our main result, which is an explicit formula for the second moment of symmetric square L-functions. Let

$$\mathcal{M}_2(h;u,v) := \sum_j h(t_j)\alpha_j L(\mathrm{sym}^2 u_j, s_-)L(\mathrm{sym}^2 u_j, s_+), \tag{1.2}$$

where

$$s_+ = 1/2+u+v, \quad s_- = 1/2+u-v, \tag{1.3}$$

and $h(t)$ is a function satisfying the conditions:

(C1) $h(t)$ is even;
(C2) $h(t)$ is holomorphic in the strip $|\Im(t)| < \Delta$ for some $\Delta > 1/2$;
(C3) $h(t)$ is of rapid decay, that is, $h(t) \ll (1+|t|)^{-2-\epsilon}$ in the strip $|\Im(t)| < \Delta$;
(C4) $h(\pm(m+1/2)i) = 0$ for $m = 0,1,\dots,M-1$, where $0 < M < \Delta - 1/2$.

We denote by $\mathcal{M}_4^{(c)}(h;u,v)$ the fourth moment of the Riemann zeta function

$$\mathcal{M}_4^{(c)}(h;u,v) = \frac{\zeta(s_+)\zeta(s_-)}{\pi} \times \int_{-\infty}^{\infty} \frac{h(r)\zeta(s_+ + 2ir)\zeta(s_+ - 2ir)\zeta(s_- + 2ir)\zeta(s_- - 2ir)}{|\zeta(1+2ir)|^2}\, dr. \tag{1.4}$$

Let $H_k^*(M,\chi_{-4})$ be an orthonormal basis of holomorphic Hecke newforms of an odd weight $k > 0$, level M, and nebentypus χ_{-4}. Let $B_1^*(M,\chi_{-4})$ be an orthonormal basis of Hecke–Maass newforms of weight 1, level $M \equiv 0 \pmod 4$, and nebentypus χ_{-4}. Let $1/4 + t_f^2$ be a Laplace eigenvalue for the function $f \in B_1^*(M,\chi_{-4})$.

For cusps $\mathfrak{b} = \infty$ or $\mathfrak{b} = 0$ of $\Gamma_0(N)$ with $N \equiv 0 \pmod 4$, let

$$\mathfrak{M}_{\infty,\mathfrak{b}}(N,u,v) = \mathfrak{M}^{\mathrm{hol}}_{\infty,\mathfrak{b}}(N,u,v) + \mathfrak{M}^{\mathrm{disc}}_{\infty,\mathfrak{b}}(N,u,v) \tag{1.5}$$

with holomorphic and discrete parts given by

$$\mathfrak{M}^{\mathrm{hol}}_{\infty,\infty}(N,u,v) := \sum_{\substack{k>1\\ k \text{ odd}}} \psi_H(k)\Gamma(k) \sum_{\substack{LM=N\\ M\equiv 0 \pmod 4}} \sum_{f\in H_k^*(M,\chi_{-4})} |\rho_{f\infty}(1)|^2 \times L(\overline{\mathrm{sym}^2 f}, s_-)L(\mathrm{sym}^2 f, s_+)P_{\infty,\infty},$$

$$\mathfrak{M}^{\mathrm{hol}}_{\infty,0}(N,u,v) := \sum_{\substack{k>1\\ k \text{ odd}}} \psi_H(k)\Gamma(k) \sum_{\substack{LM=N\\ M\equiv 0 \pmod 4}} \sum_{f\in H_k^*(M,\chi_{-4})} \varepsilon_0\overline{\rho_{f\infty}(1)}^2 \times L(\overline{\mathrm{sym}^2 f}, s_-)L(\overline{\mathrm{sym}^2 f}, s_+)P_{\infty,0},$$

$$\mathfrak{M}^{\mathrm{disc}}_{\infty,\infty}(N,u,v) := \sum_{\substack{LM=N\\ M\equiv 0 \pmod 4}} \sum_{f\in B_1^*(M,\chi_{-4})} |\rho_{f\infty}(1)|^2 \frac{\psi_D(t_f)}{\cosh(\pi t_f)} \times L(\overline{\mathrm{sym}^2 f}, s_-)L(\mathrm{sym}^2 f, s_+)P_{\infty,\infty},$$

$$\mathfrak{M}^{\mathrm{disc}}_{\infty,0}(N,u,v) := \sum_{\substack{LM=N\\ M\equiv 0 \pmod 4}} \sum_{f\in B_1^*(M,\chi_{-4})} \varepsilon_0\overline{\rho_{f\infty}(1)}^2 \frac{\psi_D(t_f)}{\cosh(\pi t_f)} \times L(\overline{\mathrm{sym}^2 f}, s_-)L(\mathrm{sym}^2 f, s_+)P_{\infty,0}.$$

Here the symmetric square L-functions are defined by (2.35), the coefficient ε_0 relates Fourier coefficients at cusps ∞ and 0 (see (2.44)), and $P_{\infty,\infty}$, $P_{\infty,0}$ are some complicated expressions given by (2.57) and (2.58).

The functions $\psi_D(t)$ and $\psi_H(k)$ are the integral transforms of $h(r)$ defined by (4.4) and (4.9) subject to (2.23), (2.19).

Finally, we define the continuous part:

$$\mathfrak{M}^{\mathrm{con}}(F;u,v) = \frac{1}{4\pi i}\int_{-\infty}^{\infty} \frac{\psi_D(t)\sinh(\pi t)}{t\cosh(\pi t)} \times \frac{\zeta(s_+ + 2it)\zeta(s_+ - 2it)\zeta(s_- + 2it)\zeta(s_- - 2it)}{L(\chi_{-4}, 1+2it)L(\chi_{-4}, 1-2it)} F(t)\,dt. \tag{1.6}$$

Theorem 1.1. *For $0 \leq \Re(u) < 1/2$, $\Re(v) = 0$, the following identity holds:*

$$\begin{aligned} \mathcal{M}_2(h;u,v) + \mathcal{M}_4^{(c)}(h;u,v) &= \mathcal{FMT}(u,v) \\ &+ \frac{L(\chi_{-4}, s_+)L(\chi_{-4}, s_-)}{(2\pi)^{1-2u}\pi^{2u}(1-2^{-2s_-})(1-2^{-2s_+})}\mathfrak{M}^{\text{con}}(F,u,v) \\ &+ \frac{(1-2^{-2s_+})^{-1}(1-2^{-2s_-})^{-1}}{(2\pi)^{1-2u}\pi^{2u}} \\ &\times\Big(-4i\mathfrak{M}_{\infty,\infty}(4,u,v) + 2^{1-2u}\mathfrak{M}_{\infty,0}(4,u,v) \\ &+(8-2^{3-s_-}-2^{3-s_+})\mathfrak{M}_{\infty,0}(16,u,v) + 16\mathfrak{M}_{\infty,0}(64,u,v)\Big), \end{aligned} \tag{1.7}$$

where $\mathcal{FMT}(u,v)$ is defined by (6.1)*, s_- and s_+ are given by* (1.3)*, and $F = F(t)$ can be found in* (5.46)*.*

Theorem 1.1 is an analogue of the explicit formula for the fourth moment of Maass form L-functions obtained by Kuznetsov [**29**] and Motohashi [**34**], [**38**]. Furthermore, a class of spectral identities of this kind was proved by Biró [**8**].

Formula (1.7) can also be viewed as a new addition to the collection of various reciprocity formulas expressing one moment of some L-functions in terms of other moments. One of the most famous formulas of this type, proved by Motohashi [**36**] (see also [**37**, Theorem 4.2]), represents the fourth moment of the Riemann zeta function in terms of cubic moments of $GL(2)$-automorphic L-functions. It is quite natural that these cubic moments can also be expressed in terms of the fourth moment of the Riemann zeta function; see [**18**]. These results have recently been generalized in various directions by Petrow [**43**], Blomer, Humphries, Khan, and Milinovich [**9**], Nelson [**40**], Wu [**49**], Balkanova, Frolenkov, and Wu [**6**], and Kaneko [**24**] (see also the references therein).

We remark that reciprocity formulas are also known for other families of L-functions. In particular, during recent years enormous numbers of such results have been discovered.

For example, the papers of Bettin [**7**], Conrey [**15**], and Young [**50**] provide reciprocity formulas for the second moment of Dirichlet L-functions.

For the family of automorphic L-functions, reciprocity formulas can be classified into two types. The first type, called spectral reciprocity, is concerned with L-functions associated to Maass forms of large spectral parameter. The second type, called level reciprocity, is concerned with twisted moments of L-functions associated to forms of large level.

A recent example of the first type of reciprocity formulas was discovered by Blomer, Li, and Miller [**12**] for the case of $GL(4) \times GL(2)$ Rankin–Selberg L-functions. Another example, given by Humphries and Khan [**21**], is concerned with a spectral reciprocity formula for some mixed moments, which has applications in the study of arithmetic quantum chaos for dihedral Maass forms. Furthermore, Kwan ([**32**]) obtained a spectral reciprocity formula for $GL(3) \times GL(2)$ Rankin–Selberg L-functions.

Now let us list some recent examples of reciprocity formulas of the second type. Andersen and Kıral ([**1**]) proved level reciprocity for the twisted second moment of $GL(2) \times GL(2)$ Rankin–Selberg L-functions. Their result was generalized by Zacharias [**51**] to an arbitrary number field. Blomer and Khan (see [**10**] and [**11**]) proved level reciprocity formulas for different twisted fourth moments of $GL(2)$ automorphic L-functions (in fact, in [**10**] a mixed moment of $GL(3) \times GL(2)$ and $GL(2)$ L-functions was considered). The results of [**10**] were generalized by Nunes [**41**] to the case of a number field with the use of a quite different adelic approach. Recently the result of [**10**] was also generalized to the higher rank case of the product of $GL(n+1) \times GL(n)$ and $GL(n) \times GL(n-1)$ Rankin–Selberg L-functions independently in the works of Miao [**33**] and Jana and Nunes [**23**].

Standard applications of reciprocity formulas include non-vanishing results (see [**12**], [**23**]) and subconvexity estimates (see [**10**], [**9**], [**11**]). Furthermore, in some cases reciprocity formulas allow one to deduce estimates on moments in short intervals. Unfortunately, Theorem 1.1 does not produce any direct corollaries, as explained in Section 6. With a different method, Khan and Young ([**25**]) proved the best known estimate on the second moment of Maass form symmetric square L-functions in short intervals as well as a new subconvexity result for the given family of L-functions in some range of the critical line.

The structure of the proof of Theorem 1.1 can be described as follows. We start by considering the left-hand side of (1.7) assuming that $\Re(u) > 1$, $\Re(v) = 0$ (see Section 3). As long as these conditions are satisfied, we can write the L-functions in (1.2) in terms of the Dirichlet series (1.1). After that we can apply the Kuznetsov trace formula to the sums of Fourier coefficients of Maass forms. As a result, we have to work with sums of Kloosterman sums weighted by some special function. However, similarly to [**34**] and [**38**], this function turns out not to have sufficiently good behavior for our purposes. For this reason, it is required to modify the Kuznetsov trace formula via the process of regularization described in Subsection 2.3. Consequently, we have to deal with

$$\frac{1}{2\pi i}\int_{(\alpha)}\hat{\Phi}_{\mathcal{N}}(s)\sum_{q=1}^{\infty}\sum_{m,n=1}^{\infty}\frac{(m/n)^{v}}{(mn)^{1/2+u+s}}\frac{S(m^2,n^2;q)}{q^{1-s}}\frac{ds}{(4\pi)^s}.$$

Splitting the sums over m and n into arithmetic progressions modulo q, we obtain a product of two Lerch zeta functions (required information about these functions is given in Subsection 2.1). For each of the Lerch zeta functions we apply the functional equation. Subsequently, after some transformations, we reduce our problem to the study of the sums

$$K(n,l;q) := \sum_{c,d \pmod q} S(c^2,d^2,q)e\left(\frac{nc+ld}{q}\right).$$

It turns out that this double sum can be written in terms of Gauss sums, and consequently can be evaluated explicitly (see [**2**, Section 3]). As a result, we have an expression for $K(n,l;q)$ in terms of generalized Kloosterman sums; see Subsection 2.2. Therefore, for $\Re(u)>1$, $\Re(v)=0$, the left-hand side of (1.7) can be written as a sum of generalized Kloosterman sums (see Section 3). As the next step, for each of these sums we apply a suitable version of the Kuznetsov trace formula formulated in Subsection 2.5. To this end, it is required to verify that the resulting weight functions satisfy conditions of applicability of the Kuznetsov trace formula. This is done in the first lemma of Section 4. Furthermore, we need to make sure that the summands obtained after applying the Kuznetsov trace formula on the right-hand side of (1.7) are absolutely convergent. To this end, we prove some estimates on $\psi_D(t)$ and $\psi_H(k)$ in Section 4. The final step is to prove an analytic continuation of (1.7) from the region $\Re(u)>1$, $\Re(v)=0$ to the region $0\le\Re(u)<1/2$, $\Re(v)=0$, which is done in Section 5. Combining all these results, we complete the proof of Theorem 1.1 in Section 6.

2. Preliminaries

Let $\hat{f}(s)$ denote the Mellin transform of $f(x)$:

$$\hat{f}(s)=\int_0^{\infty}f(x)x^{s-1}\,dx.$$

For a complex number α, we define the divisor function

$$\tau_\alpha(n) := \sum_{n_1n_2=n}\left(\frac{n_1}{n_2}\right)^{\alpha} = n^{-v}\sigma_{2\alpha}(n),$$

where

$$\sigma_\alpha(n) := \sum_{d|n} d^{\alpha}.$$

The following identity holds:

$$\sum_{n=1}^{\infty} \frac{\tau_{ir}(n^2)}{n^s} = \frac{\zeta(s)\zeta(s+2ir)\zeta(s-2ir)}{\zeta(2s)}. \tag{2.1}$$

Let us also define the divisor function twisted by a character:

$$\sigma_s(\chi; n) := \sum_{d|n} \chi(d) d^s.$$

For even n the following formula holds:

$$\sigma_s\left(\chi_{-4}; \left(\frac{n}{2}\right)^2\right) = \sigma_s(\chi_{-4}; n^2). \tag{2.2}$$

According to [**2**, Lemma 2.2], the Dirichlet series

$$Z(z,s) := \sum_{n=1}^{\infty} \frac{\sigma_s(\chi_{-4}; n^2)}{n^z}$$

satisfies

$$Z(z,s) = \frac{1-2^{2s-z}}{1-2^{2s-2z}} \frac{L(\chi_{-4}, z-s)\zeta(z)\zeta(z-2s)}{\zeta(2z-2s)}. \tag{2.3}$$

2.1. Lerch zeta function and Zagier L-series. The Lerch zeta function

$$\zeta(\alpha, \beta; s) = \sum_{n+\alpha>0} \frac{e(n\beta)}{(n+\alpha)^s}, \quad e(x) = \exp(2\pi i x) \tag{2.4}$$

satisfies the functional equation (see [**42**, (25.13.3)]):

$$\zeta(\alpha, 0; s) = \frac{\Gamma(1-s)}{(2\pi)^{1-s}} (-ie(s/4)\zeta(0, \alpha; 1-s) + ie(-s/4)\zeta(0, -\alpha; 1-s)). \tag{2.5}$$

Similarly to the Riemann zeta function, the function $\zeta(\alpha, 0; s)$ (this particular case is called the Hurwitz zeta function) has a simple pole at $s = 1$ with residue equal to 1.

The Zagier L-series can be defined for $\Re s > 1$ as

$$\mathfrak{L}_n(s) = \frac{\zeta(2s)}{\zeta(s)} \sum_{q=1}^{\infty} \frac{b_q(n)}{q^s},$$

where

$$b_q(n) := \#\{x \pmod{2q} : x^2 \equiv n \pmod{4q}\}.$$

This L-series is closely related to moments of symmetric square L-functions (see [**4**], [**52**]), as well as to the prime geodesic theorem (see [**48**], [**14**]).

If $n = Dl^2$, where D is a fundamental discriminant, then (see [**48**, Lemma 2.1]) the following decomposition takes place:

$$\mathfrak{L}_n(s) = L(\chi_D, s) \sum_{fr=l} f^{1-2s} \sum_{k|r} \frac{\mu(k)\chi_D(k)}{k^s}, \tag{2.6}$$

where $L(\chi_D, s)$ is the Dirichlet L-function for a primitive Dirichlet character χ_D.

Values of Zagier L-series of index $(n^2 - 4l^2)$ can be expressed (see [**4**, (4-9)]) in terms of sums of Kloosterman sums as follows:

$$\mathfrak{L}_{n^2-4l^2}(s) = \zeta(2s) \sum_{q=1}^{\infty} \frac{1}{q^{1+s}} \sum_{c \pmod q} S(l^2, c^2; q) e\left(\frac{nc}{q}\right). \tag{2.7}$$

Lemma 2.1. *For $u, v \neq 0$ and $\Re(u-v) > 0$ the following identity holds:*

$$\sum_{q=1}^{\infty} \frac{1}{q^{5/2+u+v}} \sum_{c,d \pmod q} S(c^2, d^2; q) \zeta\left(\frac{d}{q}, 0; 1+2v\right) = \frac{\zeta(1+2u)\zeta(1+2v)}{\zeta(1+2u-2v)} \frac{L(\chi_{-4}, 1/2+u-v)}{L(\chi_{-4}, 3/2+u+v)}. \tag{2.8}$$

Proof: Let $\Re v > 0$ and $\Re(u-v) > 1$. These conditions guarantee absolute convergence of all series that arise in the proof. Applying (2.4) and (2.7), we have

$$\begin{aligned} &\sum_{q=1}^{\infty} \frac{1}{q^{5/2+u+v}} \sum_{c,d \pmod q} S(c^2, d^2; q) \zeta\left(\frac{d}{q}, 0; 1+2v\right) \\ &= \sum_{q=1}^{\infty} \frac{1}{q^{3/2+u-v}} \sum_{l=1}^{\infty} \frac{1}{l^{1+2v}} \sum_{c \pmod q} S(l^2, c^2; q) \\ &= \frac{1}{\zeta(1+2u-2v)} \sum_{l=1}^{\infty} \frac{\mathfrak{L}_{-4l^2}(1/2+u-v)}{l^{1+2v}}. \end{aligned} \tag{2.9}$$

Computing the sum over l by applying (2.6) with $D = -4$, we find that

$$\begin{aligned} &\sum_{l=1}^{\infty} \frac{\mathfrak{L}_{-4l^2}(1/2+u-v)}{l^{1+2v}} \\ &= L(\chi_{-4}, 1/2+u-v) \sum_{l=1}^{\infty} \frac{1}{l^{1+2v}} \sum_{f|l} f^{-2u+2v} \sum_{k|(l/f)} \frac{\mu(k)\chi_{-4}(k)}{k^{1/2+u-v}} \\ &= L(\chi_{-4}, 1/2+u-v)\zeta(1+2u) \sum_{l=1}^{\infty} \frac{1}{l^{1+2v}} \sum_{k|l} \frac{\mu(k)\chi_{-4}(k)}{k^{1/2+u-v}} \\ &= \zeta(1+2u)\zeta(1+2v) \frac{L(\chi_{-4}, 1/2+u-v)}{L(\chi_{-4}, 3/2+u+v)}. \end{aligned} \tag{2.10}$$

Substituting (2.10) into (2.9), we prove (2.8) for $\Re v > 0$ and $\Re(u-v) > 1$. Now since the right-hand side of (2.8) is analytic in the region $u, v \neq 0$ and $\Re(u-v) > 0$, the identity holds in this region due to the process of analytic continuation. □

2.2. Kloosterman sums. The classical Kloosterman sum

$$S(n,m;c) := \sum_{\substack{a \pmod c \\ (a,c)=1}} e\left(\frac{an+\overline{a}_c m}{c}\right), \quad a\overline{a}_c \equiv 1 \pmod c$$

satisfies the Weil bound

$$|S(m,n;c)| \leq \tau_0(c)\sqrt{(m,n,c)}\sqrt{c}. \tag{2.11}$$

Let us also define the generalized Kloosterman sums. To this end, we need to introduce some more notation. We denote by $\Gamma = \Gamma_0(N)$ the Hecke congruence subgroup of level N. The stabilizer of a cusp $\mathfrak{a}$ is given by

$$\Gamma_{\mathfrak{a}} := \{\gamma \in \Gamma : \gamma\mathfrak{a} = \mathfrak{a}\}.$$

In addition, for each cusp $\mathfrak{a}$, we define a scaling matrix $\sigma_{\mathfrak{a}} \in \mathbf{SL}_2(\mathbf{R})$, which is determined by the following conditions:

$$\sigma_{\mathfrak{a}}\infty = \mathfrak{a}, \quad \sigma_{\mathfrak{a}}^{-1}\Gamma_{\mathfrak{a}}\sigma_{\mathfrak{a}} = \left\{\pm\begin{pmatrix}1 & n\\ 0 & 1\end{pmatrix} : n \in \mathbf{Z}\right\} := \Gamma_\infty.$$

Furthermore, we define one more matrix $\lambda_{\mathfrak{a}}$ as follows: $\sigma_{\mathfrak{a}}^{-1}\lambda_{\mathfrak{a}}\sigma_{\mathfrak{a}} = \left(\begin{smallmatrix}1 & 1\\ 0 & 1\end{smallmatrix}\right)$.

Any Dirichlet character χ modulo N can be considered as a character on Γ:

$$\chi(\gamma) = \chi(d), \quad \gamma = \begin{pmatrix}a & b\\ cN & d\end{pmatrix} \in \Gamma.$$

A cusp $\mathfrak{a}$ is called singular with respect to χ if $\chi(\lambda_{\mathfrak{a}}) = 1$.

We denote by κ the sign of χ so that $\chi(-1) = (-1)^\kappa$. If cusps $\mathfrak{a}$ and $\mathfrak{b}$ are singular with respect to χ, then the generalized Kloosterman sum is defined by (see [**26**, Equation 2.3])

$$S_{\mathfrak{ab}}(m,n;c;\chi) := \sum_{\gamma = \left(\begin{smallmatrix}a & b\\ c & d\end{smallmatrix}\right) \in \Gamma_\infty\backslash\sigma_{\mathfrak{a}}^{-1}\Gamma\sigma_{\mathfrak{b}}/\Gamma_\infty} \chi(\operatorname{sgn}(c))\overline{\chi(\sigma_{\mathfrak{a}}\gamma\sigma_{\mathfrak{b}}^{-1})}e\left(\frac{am+dn}{c}\right).$$

Note that the sum is non-empty only if $|c|$ belongs to the following set:

$$C_{\mathfrak{a},\mathfrak{b}}(N) = \{\gamma > 0 \text{ such that } \left(\begin{smallmatrix}* & *\\ \gamma & *\end{smallmatrix}\right) \in \sigma_{\mathfrak{a}}^{-1}\Gamma\sigma_{\mathfrak{b}}\}. \tag{2.12}$$

Our computations involve the generalized Kloosterman sums only for two cusps: ∞ and 0. It turns out that in these two cases the generalized

Kloosterman sums can be expressed in terms of the classical Kloosterman sums (see [**26**, (2.20), (2.23)]) as follows:

$$S_{\infty,0}(m,n;c\sqrt{N};\chi)=\overline{\chi}(c)S(\overline{N}m,n;c),\quad (c,N)=1, \tag{2.13}$$

$$S_{\infty,\infty}(m,n;c;\chi)=\sum_{ab\equiv 1 \pmod c} e\Big(\frac{am+bn}{c}\Big)\overline{\chi}(b),\quad c\equiv 0 \pmod N. \tag{2.14}$$

According to (2.12) and [**26**, (2.15)] we have

$$C_{\infty,\infty}(4)=\{\gamma=4q>0\}, \tag{2.15}$$

$$\begin{aligned} C_{\infty,0}(4)&=\{\gamma=2q>0,\ (q,4)=1\},\\ C_{\infty,0}(16)&=\{\gamma=4q>0,\ (q,2)=1\},\\ C_{\infty,0}(64)&=\{\gamma=8q>0,\ (q,2)=1\}.\end{aligned} \tag{2.16}$$

Let us define the double sum

$$K(n,l;q):=\sum_{c,d \pmod q} S(c^2,d^2,q)e\Big(\frac{nc+ld}{q}\Big). \tag{2.17}$$

This sum appears naturally in the evaluation of the second moment of symmetric square L-functions and can be expressed in terms of the Gauss sums. Consequently, $K(n,l;q)$ can be computed explicitly, as has been shown in [**2**, Section 3].

Lemma 2.2 ([**2**, Lemma 3.2]). *For odd q the following equality holds:*

$$K(n,l;q)=q\chi_{-4}(q)\sum_{\substack{a,b \pmod q\\ ab\equiv 1 \pmod q}} e\Big(-\frac{\overline{4}_q bn^2+\overline{4}_q al^2}{q}\Big).$$

Lemma 2.3 ([**2**, Lemma 3.3]). *Assume that q is even and $n+l$ is odd. Then*

$$K(n,l;q)=0.$$

Lemma 2.4 ([**2**, Lemma 3.4]). *Assume that q, n, and l are even. Then $K(n,l;q)=0$ if $q\equiv 2 \pmod 4$, and*

$$K(n,l;q)=2iq\sum_{\substack{a,b \pmod q\\ ab\equiv 1 \pmod q}} \chi_{-4}(a)e\Big(-\frac{a(l/2)^2+b(n/2)^2}{q}\Big),$$

if $q\equiv 0 \pmod 4$.

Lemma 2.5 ([**2**, Lemma 3.5]). *Assume that q is even and n and l are odd. Then $K(n,l;q)=0$ if $q\equiv 0 \pmod 4$. If $q\equiv 2 \pmod 4$, then $r:=q/2$ is odd and*

$$K(n,l;q)=2q\chi_{-4}(r)S(\overline{(8)}_r n^2,\overline{(8)}_r l^2;r).$$

2.3. Kuznetsov trace formula and regularization. At the end of the 1970s Kuznetsov ([**28**]) discovered a formula that relates sums of Fourier coefficients of Maass forms and sums of Kloosterman sums. Assume that $m, n > 0$. Then for any function $h(t)$ satisfying the conditions (C1)–(C3) we have

$$\sum_j h(t_j)\alpha_j\lambda_j(m)\lambda_j(n) + \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{\sigma_{2ir}(m)\sigma_{2ir}(n)}{(mn)^{ir}}\frac{h(r)}{|\zeta(1+2ir)|^2}\,dr \tag{2.18}$$
$$= \frac{\delta_{m,n}}{\pi^2}\int_{-\infty}^{\infty} rh(r)\tanh(\pi r)\,dr + \sum_{q=1}^{\infty}\frac{S(m,n;q)}{q}\phi\left(\frac{4\pi\sqrt{mn}}{q}\right),$$

where $\delta_{m,n} = 1$ if $m = n$ and $\delta_{m,n} = 0$ if $m \neq n$, and $\phi(x)$ is defined as an integral transform of $h(r)$:

$$\phi(x) = \frac{2i}{\pi}\int_{-\infty}^{\infty} J_{2ir}(x)\frac{rh(r)\,dr}{\cosh(\pi r)}. \tag{2.19}$$

Around the same time, a similar result was proved independently by Bruggeman [**13**]. The difference is that Bruggeman obtained a trace formula only of type (2.18), while Kuznetsov proved also the inversion of formula (2.18), showing how to evaluate sums of Kloosterman sums with a given weight function.

It can be shown (see the proof of [**37**, Theorem 2.2]) that the Mellin transform of the function $\phi(x)$ is equal to

$$\hat{\phi}(s) = \frac{2^s i}{\pi}\int_{-\infty}^{\infty}\frac{rh(r)}{\cosh(\pi r)}\frac{\Gamma(s/2+ir)}{\Gamma(1-s/2+ir)}\,dr.$$

If the function $h(t)$ satisfies not only the conditions (C1)–(C3), but also the condition (C4), then for $-1-2M < \Re s < 3/2$ the following estimate holds (see [**3**, Lemma 4]):

$$\hat{\phi}(s) \ll (1+|\Im s|)^{\Re s - 1}. \tag{2.20}$$

An application of the Kuznetsov trace formula (2.18) is the first step towards explicit formulas for moments of L-functions attached to Maass forms. After that it is required to sum the right-hand side of (2.18) over m and n. To this end, it is helpful to separate the arithmetic part, namely $S(m,n;q)$, from the analytic part given by $\phi\left(\frac{4\pi\sqrt{mn}}{q}\right)$. This is achieved by means of the inverse Mellin transform of $\phi(x)$. As a result, we have to work with multiple sums and integrals, containing the Mellin inversion of $\hat{\phi}(s)$. It turns out that the estimate (2.20) is not sufficiently good to secure the absolute convergence of the resulting expression. This fact was first observed by Motohashi in the paper [**34**] (see also [**38**]) based on the work of Kuznetsov [**29**], where the absolute convergence issue was omitted. In order to avoid this problem, it is possible to regularize the Kuznetsov formula by subtracting from it several Petersson trace formulas

$$\sum_{f\in H_{2l}(1)} \alpha_f\lambda_f(m)\lambda_f(n) = \delta_{m,n} + 2\pi i^{2l}\sum_{c=1}^{\infty}\frac{S(m,n;c)}{c}J_{2l-1}\left(\frac{4\pi\sqrt{mn}}{c}\right), \tag{2.21}$$

$$\alpha_f := \frac{12\zeta(2)}{(2l-1)L(\mathrm{sym}^2 f,1)}$$

for the spaces of holomorphic cusp forms of level 1 and weight $2l$, where l belongs to a set $\mathcal{L}$, multiplied by suitably chosen constants $c(l)$. The role of these constants is to make the Mellin transform of the weight function (which appears in the sums of Kloosterman sums after subtracting Petersson trace formulas)

$$\Phi_{\mathcal{N}}(x) = \phi(x) - \sum_{l\in\mathcal{L}} c(l)J_{2l-1}(x)$$

decay faster than (2.20). This process is called regularization of the Kuznetsov trace formula. The underlying idea of regularization was first formulated by Murty [**39**] (see the concluding remark on p. 18 of [**38**] about the Kuznetsov and Motohashi variants of regularization). Here we describe the most general form of the regularized Kuznetsov trace formula, proved by Kuznetsov in [**31**, Lemma 3.14].

Lemma 2.6. *Let $\mathcal{L} = \{l_1,\dots,l_{\mathcal{N}}\}$ be a finite set of numbers such that $1 \le l_1 < l_2 < \cdots < l_{\mathcal{N}}$. Assume that $h(\cdot)$ is a function satisfying the conditions (C1)–(C4). We define $\mathcal{N}$ coefficients $c(l)$ as solutions of the following system of linear equations:*

$$\sum_{l\in\mathcal{L}}(l-1/2)^{2m}(-1)^l c(l) = \frac{2(-1)^m}{\pi}\int_{-\infty}^{\infty} r^{2m+1}h(r)\tanh(\pi r)\,dr, \tag{2.22}$$

where $0 \le m \le \mathcal{N}-1$. Let

$$\Phi_{\mathcal{N}}(x) := \phi(x) - \sum_{l\in\mathcal{L}} c(l)J_{2l-1}(x). \tag{2.23}$$

Then for any $m,n \ge 1$ the following identity holds:

$$\sum_j h(t_j)\alpha_j\lambda_j(m)\lambda_j(n) + \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{\sigma_{2ir}(m)\sigma_{2ir}(n)}{(mn)^{ir}}\frac{h(r)}{|\zeta(1+2ir)|^2}\,dr \tag{2.24}$$

$$= \sum_{q=1}^{\infty}\frac{S(m,n;q)}{q}\Phi_{\mathcal{N}}\left(\frac{4\pi\sqrt{mn}}{q}\right) + \frac{1}{2\pi}\sum_{l\in\mathcal{L}}(-1)^l c(l)\sum_{f\in H_{2l}(1)}\alpha_f\lambda_f(m)\lambda_f(n).$$

Furthermore, for $\max(-1-2M, 1-2l_1) < \Re s < 3/2$ *we have*

$$\hat{\Phi}_{\mathcal{N}}(s) \ll (1+|\Im s|)^{\Re s-1-\mathcal{N}}. \tag{2.25}$$

If $0 < \Re s < 3/2$, *then*

$$\begin{aligned} \hat{\Phi}_{\mathcal{N}}(s) &= \hat{\phi}(s) - 2^{s-1}\sum_{l\in\mathcal{L}} c(l)\frac{\Gamma((s-1)/2+l)}{\Gamma(l+(1-s)/2)} \\ &= \hat{\phi}(s) - \frac{2^s\cos(\pi s/2)}{2\pi}\sum_{l\in\mathcal{L}}(-1)^l c(l)\Gamma\left(\frac{s-1}{2}+l\right)\Gamma\left(\frac{s+1}{2}-l\right), \end{aligned} \tag{2.26}$$

where

$$\hat{\phi}(s) = \frac{2^s\cos(\pi s/2)}{\pi^2}\int_{-\infty}^{\infty} r\tanh(\pi r)h(r)\Gamma(s/2-ir)\Gamma(s/2+ir)\,dr. \tag{2.27}$$

The set of weights $\mathcal{L}$ can be chosen as a subset of $\{2,3,4,5,7\}$. It is known that for such values of l the space of holomorphic cusp forms is empty, and consequently on the right-hand side of (2.24) there will be no sum over holomorphic cusp forms.

Note that the right-hand side of (2.24) does not contain any diagonal term with $\delta_{m,n}$. This is because after combining (2.18) and sums of Petersson trace formulas (2.21), the summands with $\delta_{m,n}$ canceled each other out due to our choice of $c(l)$, as in (2.22).

In addition, Kuznetsov ([**31**, (3.130), (3.131)]) proved an explicit formula for coefficients $c(l)$ defined by the system (2.22). More precisely, he showed that

$$(-1)^l c(l) = \frac{2}{\pi}\int_{-\infty}^{\infty} m_l(r) r h(r)\tanh(\pi r)\,dr, \tag{2.28}$$

where

$$m_l(r) = \prod_{\substack{k\in\mathcal{L}\\ k\neq l}}\frac{r^2+(k-1/2)^2}{(k-1/2)^2-(l-1/2)^2} = \prod_{\substack{k\in\mathcal{L}\\ k\neq l}}\frac{r^2+(k-1/2)^2}{(k-l)(k+l-1)}. \tag{2.29}$$

2.4. Holomorphic cusp forms, Maass forms, and Eisenstein series. Let χ be a primitive character modulo q_χ, where $q_\chi|N$. We denote by $S_k(N,\chi)$ the space of holomorphic cusp forms of weight $k>0$, level N, and nebentypus χ. The space $S_k(N,\chi)$ is non-empty only if $k\equiv\kappa \pmod 2$, where κ is the sign of χ so that $\chi(-1)=(-1)^\kappa$.

Let $H_k(N,\chi)$ be an orthonormal basis of holomorphic cusp forms of weight $k>0$, level N, and nebentypus χ. Let $H_k^*(N,\chi)$ be an orthonormal basis of Hecke newforms of weight $k>0$, level N, and nebentypus χ.

Any $f \in H_k^*(N,\chi)$ is an eigenfunction of every Hecke operator T_n, with an eigenvalue $\lambda_f(n)$. One has

$$\lambda_f(m)\lambda_f(n) = \sum_{d|(m,n)} \chi(d)\lambda\left(\frac{mn}{d^2}\right). \tag{2.30}$$

According to Atkin–Lehner theory

$$S_k(N,\chi) = \bigoplus_{\substack{yLM=N\\ M\equiv 0 \pmod{q_\chi}}} \bigoplus_{f\in H_k^*(M,\chi)} S_k^{\text{old}}(L,\chi),$$

where

$$S_k^{\text{old}}(L,\chi) = \text{span}\{l^{k/2}f(lz) : l|L\}.$$

For the space $S_k^{\text{old}}(L,\chi)$ we choose an orthonormal basis constructed in [**46**, Theorem 9], which provides us with the following decomposition:

$$H_k(N,\chi) = \coprod_{\substack{LM=N\\ M\equiv 0 \pmod{q_\chi}}} \coprod_{f\in H_k^*(M,\chi)} \coprod_{g|L} \left\{ f^{(g)}(z) = \sum_{d|g} \xi_g(d) d^{k/2} f(dz)\right\}, \tag{2.31}$$

where the coefficients $\xi_g(d)$ are defined explicitly on p. 2490 of [**44**] and $f \in H_k^*(M,\chi)$ is $L^2(\Gamma_0(N)\backslash H)$ normalized.

Consider a singular cusp $\mathfrak{a}$ and let $\sigma_{\mathfrak{a}}$ denote a scaling matrix for $\mathfrak{a}$. Then the form $f \in H_k(N,\chi)$ admits the following Fourier expansion:

$$f(\sigma_{\mathfrak{a}}z)i(\sigma_{\mathfrak{a}},z)^{-k} = \sum_{m\geq 1} \frac{\rho_{f_{\mathfrak{a}}}(m)}{\sqrt{m}}(4\pi m)^{k/2}e(mz),$$

where $i(\sigma_{\mathfrak{a}},z) := cz+d$ for $\sigma_{\mathfrak{a}} = \left(\begin{smallmatrix} * & * \\ c & d\end{smallmatrix}\right)$. If f is a newform, then $\rho_{f_\infty}(n) = \rho_{f_\infty}(1)\lambda_f(n)$. Using the definition of $f^{(g)}$ in (2.31) one has

$$\rho_{f_\infty^{(g)}}(n) = \sum_{d|(g,n)} \xi_g(d)\sqrt{d}\lambda_f(n/d)\rho_{f_\infty}(1). \tag{2.32}$$

We denote by $A_\kappa(N,\chi)$ the space of Maass cusp forms of level N, nebentypus χ, and weight $\kappa \in \{0,1\}$ such that $\chi(-1) = (-1)^\kappa$ and by $B_\kappa(N,\chi)$ an orthonormal basis of this space. Let $B_\kappa^*(N,\chi)$ be an orthonormal basis of Hecke newforms of weight $\kappa \in \{0,1\}$, level N, and nebentypus χ. Any $f \in B_\kappa^*(N,\chi)$ is an eigenfunction of every Hecke operator T_n, with an eigenvalue $\lambda_f(n)$. According to Atkin–Lehner theory

$$A_\kappa(N,\chi) = \bigoplus_{\substack{LM=N\\ M\equiv 0 \pmod{q_\chi}}} \bigoplus_{f\in B_\kappa^*(M,\chi)} A_\kappa^{\text{old}}(L,\chi),$$

where

$$A_\kappa^{\text{old}}(L,\chi) = \text{span}\{f(dz) : d|L\}.$$

For the space $A^{\rm old}_{\kappa}(L,\chi)$ we choose the basis constructed in [**20**, Lemma 3.15], which provides us with the following decomposition:

$$B_\kappa(N,\chi)=\coprod_{\substack{LM=N\\ M\equiv 0 \pmod{q_\chi}}}\ \coprod_{f\in B^*_\kappa(M,\chi)}\ \coprod_{d|L}\{f^{(d)}=\sum_{l|d}\xi_f(l,d)f(lz)\},\tag{2.33}$$

where the coefficients $\xi_f(l,d)$ are defined explicitly in [**20**, Lemma 3.15] and $f\in B^*_\kappa(M,\chi)$ is $L^2(\Gamma_0(N)\backslash H)$ normalized. Note that coefficients $\xi_f(l,d)$ in (2.33) coincide with $\xi_g(d)$ in (2.31), that is, $\xi_f(l,d)=\xi_d(l)$.

Let $1/4+t_f^2$ be a Laplace eigenvalue for the function $f\in B_\kappa(N,\chi)$. Any $f\in B_\kappa(N,\chi)$ has a Fourier–Whittaker expansion of the form

$$f(\sigma_{\mathfrak a}z)e^{-i\kappa\arg i(\sigma_{\mathfrak a},z)}=\sum_{m\neq 0}\frac{\rho_{f_{\mathfrak a}}(m)}{\sqrt{m}}W_{\frac{|m|}{m}\frac{\kappa}{2},it_f}(4\pi|m|y)e(mx).$$

Here $z=x+iy$ and $W_{\lambda,\mu}(z)$ is a Whittaker function (see [**19**, Section 9.22]). If f is a newform, then $\rho_{f_\infty}(n)=\rho_{f_\infty}(1)\lambda_f(n)$. Using the definition of $f^{(d)}$ in (2.33) one has

$$\rho_{f^{(d)}_\infty}(n)=\sum_{l|(d,n)}\xi_f(l,d)\sqrt{l}\lambda_f(n/l)\rho_{f_\infty}(1).\tag{2.34}$$

For $f\in H^*_k(N,\chi)$ or $f\in B^*_\kappa(N,\chi)$, a real character $\chi \pmod N$, and $\Re s>1$, the associated symmetric square L-functions can be defined ([**47**]) by the series

$$L(\mathrm{sym}^2 f,s)=\zeta^{(N)}(2s)\sum_{l=1}^{\infty}\frac{\lambda_f(l^2)}{l^s},\quad L(\overline{\mathrm{sym}^2 f},s)=\zeta^{(N)}(2s)\sum_{l=1}^{\infty}\frac{\overline{\lambda_f(l^2)}}{l^s},\tag{2.35}$$

where

$$\zeta^{(N)}(z)=\prod_{p\nmid N}(1-p^{-z})^{-1}.$$

For the Hecke congruence subgroup $\Gamma=\Gamma_0(N)$ and nebentypus χ with $\kappa=1$, we define the Eisenstein series associated to a singular cusp $\mathfrak c$ as follows:

$$E_{\mathfrak c}(z,s):=\sum_{\gamma\in\Gamma_{\mathfrak c}\backslash\Gamma}\overline{\chi}(\gamma)j_{\sigma_{\mathfrak c}^{-1}\gamma}(z)^{-1}(\Im(\sigma_{\mathfrak c}^{-1}\gamma z))^s,$$

where $\sigma_{\mathfrak c}$ is a scaling matrix for $\mathfrak c$ and

$$j_\gamma(z):=\frac{cz+d}{|cz+d|}=e^{i\arg(cz+d)},\quad \gamma=\begin{pmatrix}a&b\\c&d\end{pmatrix}.$$

The Fourier–Whittaker expansion for the given Eisenstein series is obtained in [**2**, Theorem 2.3]. Here we just state the final result.

Theorem 2.7. *Assume that* $\mathfrak{c}$ *is a singular cusp with respect to* χ *and* $\mathfrak{a}$ *is an Atkin–Lehner cusp. Then the following Fourier–Whittaker expansion holds:*

$$E_{\mathfrak{c}}(\sigma_{\mathfrak{a}}z,s)j_{\sigma_{\mathfrak{a}}}(z)^{-1}=\delta_{\mathfrak{ac}}y^s+\rho_{\mathfrak{a},\mathfrak{c}}(0,s)y^{1-s}+\sum_{m\neq 0}\rho_{\mathfrak{a},\mathfrak{c}}(m,s)e(mx)W_{\frac{|m|}{2m},s-1/2}(4\pi|m|y),$$

where

$$\rho_{\mathfrak{a},\mathfrak{c}}(0,s)=-\frac{\sqrt{\pi}i\Gamma(s)}{\Gamma(s+1/2)}\phi_{\mathfrak{a},\mathfrak{c}}(0,s,\chi),$$

$$\rho_{\mathfrak{a},\mathfrak{c}}(m,s)=-\frac{\pi^s i|m|^{s-1}}{\Gamma(s+1/2)}\phi_{\mathfrak{a},\mathfrak{c}}(m,s,\chi),$$

$$\phi_{\mathfrak{a},\mathfrak{c}}(m,s,\chi)=\sum_{c\in C_{\mathfrak{c},\mathfrak{a}}(N)}\frac{S_{\mathfrak{ca}}(0,m;c;\chi)}{c^{2s}}.$$

2.5. Kuznetsov trace formula for generalized Kloosterman sums. The Kuznetsov trace formula for generalized Kloosterman sums (defined in Subsection 2.2) can be found, for example, in [**27**, Section 3.3], [**16**, Section 4.1.3], and [**2**, Section 2.5]. Here we state only the case of odd characters (i.e. $\kappa=1$) required for our computations.

Assume that the function $\psi\in C^{\infty}$ satisfies the following conditions:

$$\psi(0)=\psi'(0)=0,\quad \psi^{(j)}(x)\ll(1+x)^{-2-\eta},\quad j=0,1,2,3, \tag{2.36}$$

for some $\eta>0$.

Let us define the integral transforms appearing in the Kuznetsov trace formula:

$$\psi_H(k):=4i^k\int_0^{\infty}J_{k-1}(x)\psi(x)\frac{dx}{x}, \tag{2.37}$$

$$\psi_D(t):=\frac{2\pi it}{\sinh(\pi t)}\int_0^{\infty}(J_{2it}(x)+J_{-2it}(x))\psi(x)\frac{dx}{x}, \tag{2.38}$$

where $J_{\nu}(x)$ denotes the J-Bessel function of order ν. For $m,n>0$ let

$$K_{\mathfrak{a},\mathfrak{b}}(m,n,N;\psi):=\sum_{c\in C_{\mathfrak{a},\mathfrak{b}}(N)}\frac{S_{\mathfrak{ab}}(m,n;c;\chi)}{c}\psi\left(\frac{4\pi\sqrt{mn}}{c}\right). \tag{2.39}$$

Then the Kuznetsov trace formula for generalized Kloosterman sums takes the following form:

$$K_{\mathfrak{a},\mathfrak{b}}(m,n,N;\psi)=H_{\mathfrak{a},\mathfrak{b}}(m,n,N;\psi)+D_{\mathfrak{a},\mathfrak{b}}(m,n,N;\psi)+C_{\mathfrak{a},\mathfrak{b}}(m,n,N;\psi), \tag{2.40}$$

where

$$H_{\mathfrak{a},\mathfrak{b}}(m,n,N;\psi):=\sum_{\substack{k>1\\ k\equiv 1 \pmod 2}}\sum_{f\in H_k(N,\chi)}\psi_H(k)\Gamma(k)\overline{\rho_{f_{\mathfrak{a}}}(m)}\rho_{f_{\mathfrak{b}}}(n), \tag{2.41}$$

$$D_{\mathfrak{a},\mathfrak{b}}(m,n,N;\psi):=\sum_{f\in B_\kappa(N,\chi)}\frac{\psi_D(t_f)}{\cosh(\pi t_f)}\overline{\rho_{f_{\mathfrak{a}}}(m)}\rho_{f_{\mathfrak{b}}}(n), \tag{2.42}$$

$$\begin{aligned}C_{\mathfrak{a},\mathfrak{b}}(m,n,N;\psi):=\sum_{\mathfrak{c}\text{ sing.}}\frac{1}{4\pi}\int_{-\infty}^{\infty}\frac{\psi_D(t)\sinh(\pi t)}{t\cosh(\pi t)}m^{-it}n^{it}\\ \times\overline{\phi_{\mathfrak{a},\mathfrak{c}}(m,1/2+it,\chi)}\phi_{\mathfrak{b},\mathfrak{c}}(n,1/2+it,\chi)\,dt.\end{aligned} \tag{2.43}$$

The summation in (2.43) is taken over all cusps $\mathfrak{c}$ of $\Gamma=\Gamma_0(N)$ that are singular with respect to χ.

Later we will apply (2.40) only for $\mathfrak{a}=\infty$ and $\mathfrak{b}=\infty$ or 0. In these cases using the special choice of bases (2.31) and (2.33) one can rewrite (2.41) and (2.42) in terms of Hecke eigenvalues $\lambda_f(n)$. Applying (2.32), (2.34), and the relation [**45**, (6.9)] (see also [**22**, (13.43)])

$$\rho_{f_0}(n)=\varepsilon_0\overline{\rho_{f_\infty}(n)},\quad\text{with}\quad|\varepsilon_0|=1 \tag{2.44}$$

we obtain

$$\begin{aligned}H_{\infty,\infty}(m,n,N;\psi)=\sum_{\substack{k>1\\ k\equiv 1 \pmod 2}}\sum_{\substack{LM=N\\ M\equiv 0 \pmod{q_\chi}}}\sum_{f\in H_k^*(M,\chi)}\sum_{g|L}\psi_H(k)\\ \times\Gamma(k)\overline{\rho_{f_\infty^{(g)}}(m)}\rho_{f_\infty^{(g)}}(n),\end{aligned} \tag{2.45}$$

$$\begin{aligned}H_{\infty,0}(m,n,N;\psi)=\sum_{\substack{k>1\\ k\equiv 1 \pmod 2}}\sum_{\substack{LM=N\\ M\equiv 0 \pmod{q_\chi}}}\sum_{f\in H_k^*(M,\chi)}\sum_{g|L}\psi_H(k)\\ \times\Gamma(k)\varepsilon_0\overline{\rho_{f_\infty^{(g)}}(m)}\,\overline{\rho_{f_\infty^{(g)}}(n)},\end{aligned} \tag{2.46}$$

where $\rho_{f_\infty^{(g)}}(n)$ are given by (2.32), and

$$D_{\infty,\infty}(m,n,N;\psi)=\sum_{\substack{LM=N\\ M\equiv 0 \pmod{q_\chi}}}\sum_{f\in B_\kappa^*(M,\chi)}\sum_{d|L}\frac{\psi_D(t_f)}{\cosh(\pi t_f)}\overline{\rho_{f_\infty^{(d)}}(m)}\rho_{f_\infty^{(d)}}(n), \tag{2.47}$$

$$D_{\infty,0}(m,n,N;\psi)=\sum_{\substack{LM=N\\ M\equiv 0 \pmod{q_\chi}}}\sum_{f\in B_\kappa^*(M,\chi)}\sum_{d|L}\frac{\psi_D(t_f)}{\cosh(\pi t_f)}\varepsilon_0\overline{\rho_{f_\infty^{(d)}}(m)\rho_{f_\infty^{(d)}}(n)}, \tag{2.48}$$

where $\rho_{f_\infty^{(d)}}(n)$ are given by (2.34). Note that (2.46) and (2.48) match exactly with [**45**, (6.27), (6.25)].

If d is an even power of 2, then we define $\delta_f(d,s) := 1$. If d is an odd power of 2, we let

$$\delta_f(d,s) := \frac{\lambda_f(2)}{2^{s/2}}, \quad \delta_{\overline{f}}(d,s) := \frac{\overline{\lambda_f(2)}}{2^{s/2}}. \tag{2.49}$$

Finally, let

$$\begin{aligned} P_{\infty,\infty} &= P_{\infty,\infty}(s_-, s_+, L; f) \\ &:= \sum_{g|L} \sum_{d_1|g} \sum_{d_2|g} \overline{\xi_g(d_1)} \xi_g(d_2) d_1^{\frac{1-s_-}{2}} d_2^{\frac{1-s_+}{2}} \delta_{\overline{f}}(d_1, s_-) \delta_f(d_2, s_+), \end{aligned} \tag{2.50}$$

and

$$\begin{aligned} P_{\infty,0} &= P_{\infty,0}(s_-, s_+, L; f) \\ &:= \sum_{g|L} \sum_{d_1|g} \sum_{d_2|g} \overline{\xi_g(d_1)}\, \overline{\xi_g(d_2)} d_1^{\frac{1-s_-}{2}} d_2^{\frac{1-s_+}{2}} \delta_{\overline{f}}(d_1, s_-) \delta_{\overline{f}}(d_2, s_+). \end{aligned}$$

Lemma 2.8. *For $N = ML = 2^a$, $g|L$, and $f \in H_k^*(M, \chi_{-4})$ with $4|M$ one has*

$$\begin{aligned} \zeta^N(2s_-)\zeta^N(2s_+) \sum_{m,n=1}^{\infty} \frac{\overline{\rho_{f_\infty^{(g)}}(m^2)} \rho_{f_\infty^{(g)}}(n^2)}{m^{s_-} n^{s_+}} &= |\rho_{f_\infty}(1)|^2 P_{\infty,\infty} L(\overline{\mathrm{sym}^2 f}, s_-) L(\mathrm{sym}^2 f, s_+), \\ \zeta^N(2s_-)\zeta^N(2s_+) \sum_{m,n=1}^{\infty} \frac{\overline{\rho_{f_\infty^{(g)}}(m^2) \rho_{f_\infty^{(g)}}(n^2)}}{m^{s_-} n^{s_+}} &= \overline{\rho_{f_\infty}(1)}^2 P_{\infty,0} L(\overline{\mathrm{sym}^2 f}, s_-) L(\overline{\mathrm{sym}^2 f}, s_+). \end{aligned} \tag{2.51}$$

Proof: Substituting (2.32) instead of $\rho_{f_\infty^{(g)}}(n^2)$ and $\rho_{f_\infty^{(g)}}(m^2)$ we reduce the problem of evaluating (2.51) to

$$\begin{aligned} &\zeta^N(2s_-)\zeta^N(2s_+) |\rho_f(1)|^2 \sum_{d_1|g} \sum_{d_2|g} \overline{\xi_g(d_1)} \xi_g(d_2) \sqrt{d_1 d_2} \\ &\times \sum_{\substack{m=1 \\ m^2 \equiv 0 \pmod{d_1}}}^{\infty} \sum_{\substack{n=1 \\ n^2 \equiv 0 \pmod{d_2}}}^{\infty} \frac{\overline{\lambda_f(m^2/d_1)} \lambda_f(n^2/d_2)}{m^{s_-} n^{s_+}}. \end{aligned} \tag{2.52}$$

Since $g|L|N$ and N is a power of 2, d_1 and d_2 are either 2^{2j+1} or 2^{2j}. Suppose that $d_1 = 2^{2j}$. In this case the condition $m^2 \equiv 0 \pmod{d_1}$ means that $m \equiv 0 \pmod{\sqrt{d_1}}$. Therefore,

$$(2.53)\qquad \begin{aligned}\zeta^N(2s_-)\sum_{\substack{m=1\\ m^2\equiv 0 \ (\mathrm{mod}\ d_1)}}^{\infty}\frac{\overline{\lambda_f(m^2/d_1)}}{m^{s_-}} &= \frac{\zeta^N(2s_-)}{\sqrt{d_1^{s_-}}}\sum_{m=1}^{\infty}\frac{\overline{\lambda_f(m^2/d_1)}}{m^{s_-}}\\ &= \frac{L(\overline{\mathrm{sym}^2 f}, s_-)}{d_1^{s_-/2}}.\end{aligned}$$

If $d_1 = 2^{2j+1}$, then $m \equiv 0 \pmod{\sqrt{2d_1}}$ and since it follows from (2.30) that $\lambda_f(2m) = \lambda_f(2)\lambda_f(m)$ we obtain

$$(2.54)\qquad \begin{aligned}\zeta^N(2s_-)\sum_{\substack{m=1\\ m^2\equiv 0 \ (\mathrm{mod}\ d_1)}}^{\infty}\frac{\overline{\lambda_f(m^2/d_1)}}{m^{s_-}} &= \frac{\zeta^N(2s_-)}{\sqrt{2d_1^{s_-/2}}}\sum_{m=1}^{\infty}\frac{\overline{\lambda_f(2m^2)}}{m^{s_-}}\\ &= \frac{L(\overline{\mathrm{sym}^2 f}, s_-)\overline{\lambda_f(2)}}{\sqrt{2d_1^{s_-/2}}}.\end{aligned}$$

Combining (2.53) and (2.54) one has

$$(2.55)\qquad \zeta^N(2s_-)\sum_{\substack{m=1\\ m^2\equiv 0 \ (\mathrm{mod}\ d_1)}}^{\infty}\frac{\overline{\lambda_f(m^2/d_1)}}{m^{s_-}} = L(\overline{\mathrm{sym}^2 f}, s_-)\delta_{\overline{f}}(d_1, s_-),$$

where $\delta_{\overline{f}}(d_1, s_1)$ is defined by (2.49). In the same way

$$(2.56)\qquad \zeta^N(2s_+)\sum_{\substack{n=1\\ n^2\equiv 0 \ (\mathrm{mod}\ d_2)}}^{\infty}\frac{\lambda_f(n^2/d_2)}{n^{s_+}} = L(\mathrm{sym}^2 f, s_-)\delta_f(d_2, s_+).$$

Substituting (2.55) and (2.56) to (2.52) we prove (2.51). The second relation can be proved in the same way. □

The same formulas also hold in the case of Maass forms, since the expressions (2.32) and (2.34) coincide due to $\xi_f(l,d) = \xi_d(l)$.

When L is a power of 2 and $4|M$ we can evaluate $P_{\infty,\infty}$ and $P_{\infty,0}$ explicitly in terms of $\lambda_f(2)$ since in this case we need only the values of $\xi_g(d)$ for g and d being a power of 2. These values are given on p. 2490 of [**44**] and for $\nu \geq 1$ are equal to

$$\xi_1(1) = 1,$$
$$\xi_{2^\nu}(2^\nu) = \left(1 - \frac{|\lambda_f(2)|^2}{2}\right)^{-1/2},$$
$$\xi_{2^\nu}(2^{\nu-1}) = \frac{-\overline{\lambda_f(2)}}{\sqrt{2}}\left(1 - \frac{|\lambda_f(2)|^2}{2}\right)^{-1/2},$$

and zero in all other cases. Therefore, the summands in (2.50) are non-zero only for d_1 and d_2 being equal to either g or $g/2$. Hence using (2.49) we obtain for g being an even power of 2

$$\sum_{d_1|g} \overline{\xi_g(d_1)} d_1^{\frac{1-s_-}{2}} \delta_{\overline{f}}(d_1, s_-) = g^{\frac{1-s_-}{2}} \left(1 - \frac{|\lambda_f(2)|^2}{2}\right)^{1/2},$$

and for g being an odd power of 2

$$\sum_{d_1|g} \overline{\xi_g(d_1))} d_1^{\frac{1-s_-}{2}} \delta_{\overline{f}}(d_1, s_-) = g^{\frac{1-s_-}{2}} \frac{\frac{\overline{\lambda_f(2)}}{2^{s_-/2}} - \frac{\lambda_f(2)}{2^{-s_-/2}}}{\left(1 - \frac{|\lambda_f(2)|^2}{2}\right)^{1/2}}.$$

Therefore,

$$\begin{aligned} P_{\infty,\infty} &= 1 + \left(1 - \frac{|\lambda_f(2)|^2}{2}\right)\sigma^{\text{ev}}_{1-\frac{s_-+s_+}{2}}(L) \\ &\quad + \frac{\left(\frac{\overline{\lambda_f(2)}}{2^{s_-/2}} - \frac{\lambda_f(2)}{2^{-s_-/2}}\right)\left(\frac{\lambda_f(2)}{2^{s_+/2}} - \frac{\overline{\lambda_f(2)}}{2^{-s_+/2}}\right)}{\left(1 - \frac{|\lambda_f(2)|^2}{2}\right)}\sigma^{\text{od}}_{1-\frac{s_-+s_+}{2}}(L), \end{aligned} \tag{2.57}$$

where

$$\sigma^{\text{ev}}_s(L) = \sum_{\substack{g|L \\ \log_2 g \text{ is even}}} g^s, \quad \sigma^{\text{od}}_s(L) = \sum_{\substack{g|L \\ \log_2 g \text{ is odd}}} g^s.$$

Similarly we find that

$$\begin{aligned} P_{\infty,0} &= 1 + \left(1 - \frac{|\lambda_f(2)|^2}{2}\right)\sigma^{\text{ev}}_{1-\frac{s_-+s_+}{2}}(L) \\ &\quad + \frac{\left(\frac{\overline{\lambda_f(2)}}{2^{s_-/2}} - \frac{\lambda_f(2)}{2^{-s_-/2}}\right)\left(\frac{\lambda_f(2)}{2^{s_+/2}} - \frac{\overline{\lambda_f(2)}}{2^{-s_+/2}}\right)}{\left(1 - \frac{|\lambda_f(2)|^2}{2}\right)}\sigma^{\text{od}}_{1-\frac{s_-+s_+}{2}}(L). \end{aligned} \tag{2.58}$$

2.6. Eisenstein series for $\Gamma_0(4)$, $\Gamma_0(16)$, and $\Gamma_0(64)$. For our purposes, it is required to consider only the case when level is equal to 4, 16, 64, cusps are ∞, 0, and nebentypus is χ_{-4}. More precisely, the following Kloosterman sums arise in our case:

$$K_{\infty,\infty}(m^2,n^2,4;\psi),\quad K_{\infty,0}(m^2,n^2,4;\psi),$$
$$K_{\infty,0}(m^2,n^2,16;\psi),\quad K_{\infty,0}(m^2,n^2,64;\psi).$$

After the application of the Kuznetsov trace formula, we obtain (2.43). Consequently, we need to compute Fourier coefficients of Eisenstein series for all singular cusps. This was done in [**2**] and here we just state the final formulas.

Lemma 2.9 ([**2**, Lemma 5.1])**.** *The following cusps of* $\Gamma_0(4)$ *are singular with respect to* χ_{-4}*:*

$$0,\ \infty.$$

The following cusps of $\Gamma_0(16)$ *are singular with respect to* χ_{-4}*:*

$$0,\ 1/2,\ 1/4,\ 1/8,\ 1/12,\ \infty.$$

The following cusps of $\Gamma_0(64)$ *are singular with respect to* χ_{-4}*:*

$$0,\ 1/2,\ 1/4,\ 1/8,\ 1/12,\ 1/16,\ 1/24,\ 1/32,\ 1/40,\ 1/48,\ 1/56,\ \infty.$$

For convenience, we introduce some notation:

$$\delta_n(m):=\begin{cases}1 & \text{if } n|m\\ 0 & \text{otherwise}\end{cases},$$

$$s(m):=\frac{\sigma_{1-2s}(\chi_{-4};m)}{L(\chi_{-4},2s)}, \tag{2.59}$$

$$t(m):=\frac{\tau(\chi_{-4})\sigma_{2s-1}(\chi_{-4};m)m^{1-2s}}{L(\chi_{-4},2s)}, \tag{2.60}$$

where $\tau(\chi_{-4})$ is the Gauss sum which is equal to $2i$.

Lemma 2.10 ([**2**, Lemma 5.6])**.** *Let $N=64$ and $\mathfrak{a}=\infty$. Then*

$$\phi_{\infty,0}(m,s,\chi_{-4})=\chi_{-4}(-1)\frac{s(m)}{8^{2s}}, \tag{2.61}$$

$$\phi_{\infty,1/2}(m,s,\chi_{-4})=\chi_{-4}(-1)e\left(\frac{m}{2}\right)\frac{s(m)}{8^{2s}},$$

$$\phi_{\infty,\frac{1}{4\Upsilon}}(m,s,\chi_{-4})=\chi_{-4}(-\Upsilon)e\left(-\frac{m\Upsilon}{4}\right)\frac{s(m)}{8^{2s}},\quad \Upsilon=1,3, \tag{2.62}$$

$$\phi_{\infty,\frac{1}{8\Upsilon}}(m,s,\chi_{-4})=\chi_{-4}(-\Upsilon)e\left(-\frac{m\Upsilon}{8}\right)\frac{s(m)}{8^{2s}},\quad \Upsilon=1,3,5,7, \tag{2.63}$$

$$\phi_{\infty,\frac{1}{16\Upsilon}}(m,s,\chi_{-4})=\chi_{-4}(-\Upsilon)4\delta_4(m)e\left(-\frac{m\Upsilon}{16}\right)\frac{s(m)}{16^{2s}},\quad \Upsilon=1,3, \tag{2.64}$$

$$\phi_{\infty,\frac{1}{32}}(m,s,\chi_{-4})=\frac{8}{(32)^{2s}}\delta_8(m)t\left(\frac{m}{8}\right)-\frac{16}{(64)^{2s}}\delta_{16}(m)t\left(\frac{m}{16}\right),$$

$$\phi_{\infty,\infty}(m,s,\chi_{-4})=\frac{16}{(64)^{2s}}\delta_{16}(m)t\left(\frac{m}{16}\right). \tag{2.65}$$

Lemma 2.11 ([**2**, Lemma 5.7])**.** *Let $N=64$ and $\mathfrak{a}=0$. Then*

$$\phi_{0,0}(m,s,\chi_{-4})=\frac{16}{(64)^{2s}}\delta_{16}(m)t\left(\frac{m}{16}\right), \tag{2.66}$$

$$\phi_{0,1/2}(m,s,\chi_{-4})=\frac{8}{(32)^{2s}}\delta_8(m)t\left(\frac{m}{8}\right)-\frac{16}{(64)^{2s}}\delta_{16}(m)t\left(\frac{m}{16}\right),$$

$$\phi_{0,\frac{1}{4\Upsilon}}(m,s,\chi_{-4})=4\delta_4(m)e\left(\frac{m\Upsilon}{16}\right)\frac{s(m)}{16^{2s}},\quad \Upsilon=1,3, \tag{2.67}$$

$$\phi_{0,\frac{1}{8\Upsilon}}(m,s,\chi_{-4})=e\left(\frac{m\Upsilon}{8}\right)\frac{s(m)}{8^{2s}},\quad \Upsilon=1,3,5,7, \tag{2.68}$$

$$\phi_{0,\frac{1}{16\Upsilon}}(m,s,\chi_{-4})=e\left(\frac{m\Upsilon}{4}\right)\frac{s(m)}{8^{2s}},\quad \Upsilon=1,3, \tag{2.69}$$

$$\phi_{0,\frac{1}{32}}(m,s,\chi_{-4})=e\left(\frac{m}{2}\right)\frac{s(m)}{8^{2s}},$$

$$\phi_{0,\infty}(m,s,\chi_{-4})=\frac{s(m)}{8^{2s}}. \tag{2.70}$$

Lemma 2.12 ([**2**, Lemma 5.8])**.** *Let $N = 16$ and $\mathfrak{a} = 0$. Then*

$$\phi_{0,0}(m,s,\chi_{-4}) = \frac{4}{(16)^{2s}}\delta_4(m)t\left(\frac{m}{4}\right), \tag{2.71}$$

$$\phi_{0,1/2}(m,s,\chi_{-4}) = \frac{2}{8^{2s}}\delta_2(m)t\left(\frac{m}{2}\right) - \frac{4}{(16)^{2s}}\delta_4(m)t\left(\frac{m}{4}\right),$$

$$\phi_{0,\frac{1}{4\Upsilon}}(m,s,\chi_{-4}) = e\left(\frac{m\Upsilon}{4}\right)\frac{s(m)}{4^{2s}}, \quad \Upsilon = 1,3, \tag{2.72}$$

$$\phi_{0,\frac{1}{8}}(m,s,\chi_{-4}) = e\left(\frac{m}{2}\right)\frac{s(m)}{4^{2s}},$$

$$\phi_{0,\infty}(m,s,\chi_{-4}) = \frac{s(m)}{4^{2s}}. \tag{2.73}$$

Lemma 2.13 ([**2**, Lemma 5.9])**.** *Let $N = 16$ and $\mathfrak{a} = \infty$. Then*

$$\phi_{\infty,0}(m,s,\chi_{-4}) = \chi_{-4}(-1)\phi_{0,\infty}(m,s,\chi_{-4}) = \chi_{-4}(-1)\frac{s(m)}{4^{2s}}, \tag{2.74}$$

$$\phi_{\infty,1/2}(m,s,\chi_{-4}) = \chi_4(-1)\phi_{0,\frac{1}{8}}(m,s,\chi_{-4}) = \chi_{-4}(-1)e\left(\frac{m}{2}\right)\frac{s(m)}{4^{2s}},$$

$$\begin{aligned}\phi_{\infty,\frac{1}{4\Upsilon}}(m,s,\chi_{-4}) &= \chi_{-4}(-\Upsilon)\phi_{0,\frac{1}{4(-\Upsilon)}}(m,s,\chi_{-4})\\ &= \chi_{-4}(-\Upsilon)e\left(-\frac{m\Upsilon}{4}\right)\frac{s(m)}{4^{2s}}, \quad \Upsilon = 1,3,\end{aligned} \tag{2.75}$$

$$\begin{aligned}\phi_{\infty,\frac{1}{8}}(m,s,\chi_{-4}) &= \phi_{0,1/2}(m,s,\chi_{-4}\\ &= \frac{2}{8^{2s}}\delta_2(m)t\left(\frac{m}{2}\right) - \frac{4}{(16)^{2s}}\delta_4(m)t\left(\frac{m}{4}\right),\end{aligned}$$

$$\phi_{\infty,\infty}(m,s,\chi_{-4}) = \phi_{0,0}(m,s,\chi_{-4}) = \frac{4}{(16)^{2s}}\delta_4(m)t\left(\frac{m}{4}\right). \tag{2.76}$$

Lemma 2.14 ([**2**, Lemma 5.10])**.** *Let $N = 4$. Then*

$$\phi_{0,0}(m,s,\chi_{-4}) = \frac{t(m)}{4^{2s}}, \quad \phi_{0,\infty}(m,s,\chi_{-4}) = \frac{s(m)}{2^{2s}}, \tag{2.77}$$

$$\phi_{\infty,0}(m,s,\chi_{-4}) = \chi_{-4}(-1)\phi_{0,\infty}(m,s,\chi_{-4}), \tag{2.78}$$

$$\phi_{\infty,\infty}(m,s,\chi_{-4}) = \phi_{0,0}(m,s,\chi_{-4}). \tag{2.79}$$

Corollary 2.15 ([**2**, Corollary 5.11])**.** *For $N=64$ we have*

$$\phi_{\infty,\frac{1}{32}}(m^2,s,\chi_{-4})=\phi_{0,\frac{1}{2}}(m^2,s,\chi_{-4})=0. \tag{2.80}$$

For $N=16$ we have

$$\phi_{\infty,\frac{1}{8}}(m^2,s,\chi_{-4})=\phi_{0,\frac{1}{2}}(m^2,s,\chi_{-4})=0. \tag{2.81}$$

3. Explicit formula for large values of u

Consider the second moment of symmetric square L-functions

$$\mathcal{M}_2(h;u,v):=\sum_j h(t_j)\alpha_j L(\operatorname{sym}^2 u_j,1/2+u-v)L(\operatorname{sym}^2 u_j,1/2+u+v). \tag{3.1}$$

We introduce the notation

$$R(u,v):=\frac{\hat{\Phi}_{\mathcal{N}}(1/2-u+v)}{(4\pi)^{1/2-u+v}}\zeta(1+2u)\zeta(1+2v)\zeta(1+2u+2v)\times\frac{L(\chi_{-4},1/2+u-v)}{L(\chi_{-4},3/2+u+v)}, \tag{3.2}$$

$$I(x):=\frac{1}{2\pi i}\int_{(\alpha_1)}\hat{\Phi}_{\mathcal{N}}(s)(\cos(\pi v)+\sin(\pi(u+s)))\times\Gamma(1/2-u+v-s)\Gamma(1/2-u-v-s)x^s\,ds, \tag{3.3}$$

where $\max(-1-2M,1-2l_1)<\alpha_1<1/2-\Re(u)$. Furthermore, let

$$\psi(x):=x^{2u}I(x). \tag{3.4}$$

Theorem 3.1. *Assume that $1<\Re(u)<1/2+\min(1+2M,2l_1-1),\Re(v)=0$. Then*

$$\begin{aligned}\mathcal{M}_2(h;u,v)&=-\mathcal{M}_4^{(c)}(h;u,v)+R(u,v)+R(u,-v)\\&+\frac{\zeta(2s_+)\zeta(2s_-)}{(2\pi)^{1-2u}\pi^{2u}}\sum_{m,n=1}^{\infty}\frac{(m/n)^v}{(mn)^{1/2+u}}\\&\times\Big(-4iK_{\infty,\infty}(m^2,n^2,4;\psi)+2^{1-2u}K_{\infty,0}(m^2,n^2,4;\psi)\\&+(8-2^{3-s_-}-2^{3-s_+})K_{\infty,0}(m^2,n^2,16;\psi)+16K_{\infty,0}(m^2,n^2,64;\psi)\Big),\end{aligned} \tag{3.5}$$

where the definitions of s_+, s_- are given by formulas (1.3), *$\mathcal{M}_4^{(c)}(h;u,v)$ by* (1.4), *and $K_{\mathfrak{a},\mathfrak{b}}(m,n,N;\psi)$ by* (2.39).

Proof: In order to evaluate (3.1) we use the series representation (1.1) for L-functions and apply the regularized Kuznetsov trace formula (2.24) with $\mathcal{L} \subseteq \{2, 3, 4, 5, 7\}$ (which corresponds to weights $\{4, 6, 8, 10, 14\}$). As a result,

$$\mathcal{M}_2(h; u, v) = -\mathcal{M}_4^{(c)}(h; u, v) + \zeta(2s_-)\zeta(2s_+)TS(u, v), \tag{3.6}$$

$$TS(u, v) = \sum_{m,n=1}^{\infty} \frac{(m/n)^v}{(mn)^{1/2+u}} \sum_{q=1}^{\infty} \frac{S(m^2, n^2; q)}{q} \Phi_{\mathcal{N}}\left(\frac{4\pi mn}{q}\right). \tag{3.7}$$

Note that to obtain $\mathcal{M}_4^{(c)}(h; u, v)$ we applied (2.1).

The next step is to show that the assumptions of the theorem guarantee absolute convergence of the triple sum in (3.7). Indeed, estimating the absolute value of (2.19) we show that $\phi(x) \ll x^{-1/2}$. Moving the contour of integration in (2.19) to the line $\Im(r) = \delta = -\Delta + \epsilon$ and estimating the absolute value of the resulting integral, we have $\phi(x) \ll x^{2\delta}$. Combining these two estimates for $\phi(x)$ and using the standard bounds for the Bessel function $J_{2l-1}(x) \ll \min(x^{2l-1}, x^{-1/2})$, we infer that for $c > 1$ (see (2.23))

$$\Phi_{\mathcal{N}}(x) \ll \min(x^c, x^{-1/2}). \tag{3.8}$$

Applying (2.11) and (3.8), we conclude that the triple sum on the right-hand side of (3.6) is absolutely convergent for $\Re(u) > 1$. Using the inverse Mellin transform for the function $\Phi_N(x)$, we conclude that

$$TS(u, v) = \frac{1}{2\pi i} \int_{(\alpha)} \hat{\Phi}_{\mathcal{N}}(s) \sum_{m,n=1}^{\infty} \frac{(m/n)^v}{(mn)^{1/2+u+s}} \sum_{q=1}^{\infty} \frac{S(m^2, n^2; q)}{q^{1-s}} \frac{ds}{(4\pi)^s}, \tag{3.9}$$

where $-1/2 - \Re(u) < \alpha < -1/2$. These limitations on α are sufficient for absolute convergence of the sums and the integral in (3.9). Consequently, we can change the order of summation in (3.9) such that the outer sum becomes the sum over q. After that we split the sums over m and n into arithmetic progressions modulo q. This allows us to apply (2.4) in order to obtain the Lerch zeta functions

$$\sum_{m \equiv c \pmod q} \sum_{n \equiv d \pmod q} \frac{(m/n)^v}{(mn)^{1/2+u+s}}$$
$$= \frac{1}{q^{1+2u+2s}} \zeta\left(\frac{c}{q}, 0; 1/2 + u - v + s\right) \zeta\left(\frac{d}{q}, 0; 1/2 + u + v + s\right).$$

Therefore, (3.9) can be written as

$$TS(u, v) = \frac{1}{2\pi i} \int_{(\alpha)} \hat{\Phi}_{\mathcal{N}}(s) \sum_{q=1}^{\infty} \frac{1}{q^{2+2u+s}} \sum_{c,d \pmod q} S(c^2, d^2; q)$$
$$\times \zeta\left(\frac{c}{q}, 0; 1/2 + u - v + s\right) \zeta\left(\frac{d}{q}, 0; 1/2 + u + v + s\right) \frac{ds}{(4\pi)^s}.$$

Moving the contour of integration to the left, specifically to the line $\Re(s) = \alpha_1$ with $\max(-1-2M, 1-2l_1) < \alpha_1 < 1/2 - \Re(u)$, we cross the poles of the Lerch zeta function at the points $s_1 = 1/2 - u + v$ and $s_2 = 1/2 - u - v$. Let us denote the contribution of these poles as $R_1(u,v)$ and $R_1(u,-v)$, respectively. Then

$$TS(u,v) = R_1(u,v) + R_1(u,-v)\frac{1}{2\pi i}\int_{(\alpha_1)}\hat{\Phi}_{\mathcal{N}}(s)\sum_{q=1}^{\infty}\frac{1}{q^{2+2u+s}} \tag{3.10}$$

$$\times \sum_{c,d \pmod q} S(c^2,d^2;q)\zeta\left(\frac{c}{q},0;1/2+u-v+s\right)\zeta\left(\frac{d}{q},0;1/2+u+v+s\right)\frac{ds}{(4\pi)^s}.$$

Computing the residue at $s_1 = 1/2 - u + v$, we find

$$R_1(u,v) = \frac{\hat{\Phi}_{\mathcal{N}}(1/2-u+v)}{(4\pi)^{1/2-u+v}}\sum_{q=1}^{\infty}\frac{1}{q^{5/2+u+v}}\sum_{c,d \pmod q} S(c^2,d^2;q)\zeta\left(\frac{d}{q},0;1+2v\right).$$

Applying (2.8), we show that

$$R_1(u,v) = \frac{\hat{\Phi}_{\mathcal{N}}(1/2-u+v)}{(4\pi)^{1/2-u+v}}\frac{\zeta(1+2u)\zeta(1+2v)}{\zeta(1+2u-2v)}\frac{L(\chi_{-4},1/2+u-v)}{L(\chi_{-4},3/2+u+v)}.$$

Consider the integral in (3.10). For each of the two Lerch zeta functions, we apply the functional equation (2.5). Consequently, we obtain four combinations with different signs of the products

$$\zeta(0,\pm c/q;\cdot)\zeta(0,\pm d/q;\cdot).$$

Changing the variables $-c$ into c and $-d$ into d yields a unified expression of the form

$$\zeta(0,c/q;\cdot)\zeta(0,d/q;\cdot).$$

More precisely, we show that

$$\sum_{c,d \pmod q} S(c^2,d^2;q)\zeta\left(\frac{c}{q},0;1/2+u-v+s\right)\zeta\left(\frac{d}{q},0;1/2+u+v+s\right)$$

$$= 2(\cos(\pi v)+\sin(\pi(u+s)))\frac{\Gamma(1/2-u+v-s)\Gamma(1/2-u-v-s)}{(2\pi)^{1-2u-2s}}$$

$$\times \sum_{c,d \pmod q} S(c^2,d^2;q)\zeta\left(0,\frac{c}{q};1/2-u+v-s\right)\zeta\left(0,\frac{d}{q};1/2-u-v-s\right).$$

Writing the definition (2.4) for the Lerch zeta functions we have

$$\begin{aligned}&\sum_{c,d \pmod q} S(c^2,d^2;q)\zeta\Big(\frac{c}{q},0;1/2+u-v+s\Big)\zeta\Big(\frac{d}{q},0;1/2+u+v+s\Big)\\ &= 2(\cos(\pi v)+\sin(\pi(u+s)))\frac{\Gamma(1/2-u+v-s)\Gamma(1/2-u-v-s)}{(2\pi)^{1-2u-2s}}\\ &\times\sum_{m,n=1}^{\infty}\frac{(mn)^s(m/n)^v}{(mn)^{1/2-u}}\sum_{c,d \pmod q} S(c^2,d^2;q)e\Big(\frac{md+nc}{q}\Big).\end{aligned} \tag{3.11}$$

Substituting (3.11) into (3.10) and using (2.17), we prove that

$$\begin{aligned}TS(u,v) &= R_1(u,v)+R_1(u,-v)\\ &+2(2\pi)^{2u-1}\sum_{m,n=1}^{\infty}\frac{(m/n)^v}{(mn)^{1/2-u}}\sum_{q=1}^{\infty}\frac{1}{q^{2+2u}}K(n,m;q)I\Big(\frac{\pi mn}{q}\Big),\end{aligned}$$

where $I(x)$ is defined by (3.3). In order to evaluate $K(n,m;q)$ we apply Lemmas 2.2–2.5, showing that

$$\begin{aligned}&TS(u,v) = R_1(u,v)+R_1(u,-v)\\ &+2(2\pi)^{2u-1}\sum_{m,n\equiv 0(2)}\frac{(m/n)^v}{(mn)^{1/2-u}}\sum_{q\equiv 0(4)}\frac{2i}{q^{1+2u}}I\Big(\frac{\pi mn}{q}\Big)\\ &\qquad\times\sum_{\substack{a,b \pmod q\\ ab\equiv 1 \pmod q}}\chi_{-4}(a)e\Big(-\frac{a(m/2)^2+b(n/2)^2}{q}\Big)\\ &+2(2\pi)^{2u-1}\sum_{m,n=1}^{\infty}\frac{(m/n)^v}{(mn)^{1/2-u}}\sum_{q=1}^{\infty}\frac{\chi_{-4}(q)}{q^{1+2u}}I\Big(\frac{\pi mn}{q}\Big)\sum_{\substack{a,b \pmod q\\ ab\equiv 1 \pmod q}}e\Big(-\frac{\overline{4}_q bn^2+\overline{4}_q am^2}{q}\Big)\\ &+2(2\pi)^{2u-1}\sum_{m,n\equiv 1(2)}\frac{(m/n)^v}{(mn)^{1/2-u}}\sum_{\substack{q=2r\\(r,2)=1}}\frac{2\chi_{-4}(r)}{q^{1+2u}}I\Big(\frac{\pi mn}{q}\Big)S(\overline{(8)}_r m^2,\overline{(8)}_r n^2;r).\end{aligned} \tag{3.12}$$

Changing the variables a and b into $-a$ and $-b$ and using (2.14) and (2.15) for the first sum in (3.12), (2.13), and (2.16) for the second sum in (3.12), we have

(3.13)
$$TS(u,v) = R_1(u,v) + R_1(u,-v)$$
$$-\frac{4i(2\pi)^{2u-1}}{\pi^{2u}} \sum_{m,n\equiv 0(2)} \frac{(m/n)^v}{(mn)^{1/2+u}} \sum_{\gamma\in C_{\infty,\infty}(4)} \frac{1}{\gamma}\psi\left(\frac{\pi mn}{\gamma}\right) S_{\infty,\infty}(m^2/4, n^2/4; \gamma; \chi_{-4})$$
$$+\frac{8(2\pi)^{2u-1}}{\pi^{2u}} \sum_{m,n=1}^{\infty} \frac{(m/n)^v}{(mn)^{1/2+u}} \sum_{\gamma\in C_{\infty,0}(16)} \frac{1}{\gamma}\psi\left(\frac{\pi mn}{\gamma}\right) S_{\infty,0}(m^2, n^2; \gamma; \chi_{-4})$$
$$+2(2\pi)^{2u-1} \sum_{m,n\equiv 1(2)} \frac{(m/n)^v}{(mn)^{1/2-u}} \sum_{\substack{q=2r\\(r,2)=1}} \frac{2\chi_{-4}(r)}{q^{1+2u}} I\left(\frac{\pi mn}{q}\right) S(\overline{(8)}_r m^2, \overline{(8)}_r n^2; r),$$

where $\psi(x) = x^{2u}I(x)$. In order to write the third sum in (3.13) similarly to the first and the second sums, we split the summation over m, n into four different cases using the following relation:

$$\sum_{m,n\equiv 1(2)} = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} + \sum_{m,n\equiv 0(2)} - \sum_{m\equiv 0(2)}\sum_{n=1}^{\infty} - \sum_{m=1}^{\infty}\sum_{n\equiv 0(2)}.$$

In the sums over even m we change the variable m into $2m$ (and we do the same with sums over even n). Then

(3.14)
$$2(2\pi)^{2u-1} \sum_{m,n\equiv 1(2)} \frac{(m/n)^v}{(mn)^{1/2-u}} \sum_{\substack{q=2r\\(r,2)=1}} \frac{2\chi_{-4}(r)}{q^{1+2u}} I\left(\frac{\pi mn}{q}\right) S(\overline{(8)}_r n^2, \overline{(8)}_r m^2; r)$$
$$= \frac{16(2\pi)^{2u-1}}{\pi^{2u}} \sum_{m,n=1}^{\infty} \frac{(m/n)^v}{(mn)^{1/2+u}} \sum_{\gamma\in C_{\infty,0}(64)} \frac{1}{\gamma}\psi\left(\frac{\pi mn}{\gamma}\right) S_{\infty,0}(m^2, n^2; \gamma; \chi_{-4})$$
$$+\frac{2^{1-2u}(2\pi)^{2u-1}}{\pi^{2u}} \sum_{m,n=1}^{\infty} \frac{(m/n)^v}{(mn)^{1/2+u}} \sum_{\gamma\in C_{\infty,0}(4)} \frac{1}{\gamma}\psi\left(\frac{\pi mn}{\gamma}\right) \times S_{\infty,0}(m^2, n^2; \gamma; \chi_{-4})$$
$$-\frac{2^{5/2-u+v}(2\pi)^{2u-1}}{\pi^{2u}} \sum_{m,n=1}^{\infty} \frac{(m/n)^v}{(mn)^{1/2+u}} \sum_{\gamma\in C_{\infty,0}(16)} \frac{1}{\gamma}\psi\left(\frac{\pi mn}{\gamma}\right) \times S_{\infty,0}(m^2, n^2; \gamma; \chi_{-4})$$
$$-\frac{2^{5/2-u-v}(2\pi)^{2u-1}}{\pi^{2u}} \sum_{m,n=1}^{\infty} \frac{(m/n)^v}{(mn)^{1/2+u}} \sum_{\gamma\in C_{\infty,0}(16)} \frac{1}{\gamma}\psi\left(\frac{\pi mn}{\gamma}\right) \times S_{\infty,0}(m^2, n^2; \gamma; \chi_{-4}).$$

Substituting (3.14) into (3.13), and using (2.39) and (3.6), we finally prove (3.5). □

4. Special functions arising after applying the Kuznetsov trace formula

The next step in the proof of Theorem 1.1 is the application of the Kuznetsov trace formula (2.40) to the summands on the right-hand side of (3.5).

In this section, we show that the function (3.4) satisfies the conditions of applicability of the Kuznetsov formula. Furthermore, we show that its integral transforms (2.37) and (2.38) decay fast enough for the right-hand side of (2.40) to be absolutely convergent. For that exact reason, we introduced the regularized version of the Kuznetsov trace formula in Lemma 2.6.

Lemma 4.1. *For* $\max(2, 2l_1-1-\mathcal{N}) < 2\Re(u) < \min(2l_1-3, 2l_1-\mathcal{N})$ *and* $\Re(v) = 0$*, the function* $\psi(x)$ *from* (3.4) *as* $x \to 0$ *satisfies the following estimate:*

$$\psi(x) \ll x^{1/2+\Re(u)-\epsilon}, \tag{4.1}$$

and for $x \to \infty$ *we have*

$$\psi(x) \ll \frac{1}{x^{2l_1-1-2\Re(u)-\epsilon}}, \quad \psi^{(j)}(x) \ll \frac{1}{x^{N-\epsilon}}, \tag{4.2}$$

where $j = 1, 2, 3$.

Proof: Applying (2.25) and the Stirling formula

$$\Gamma(\sigma + it) \ll |t|^{\sigma-1/2} \exp(-\pi|t|/2), \quad |t| \to \infty \tag{4.3}$$

for estimating the Gamma functions, we prove that the function under the integral in (3.3) is bounded by $(1 + |\Im s|)^{-1-2\Re(u)-\Re(s)-\mathcal{N}}$. Accordingly, the integral (3.3) is absolutely convergent for $\Re(s) > -2\Re(u) - \mathcal{N}$. Therefore, in (3.3) the contour of integration $\Re s = \alpha$ can be chosen such that (we assume that the parameter M appearing in the condition (C4) is sufficiently large)

$$\max(1 - 2l_1, -2\Re(u) - \mathcal{N}) < \alpha < 1/2 - \Re(u).$$

Note that the conditions of the lemma imply that $1 - 2l_1 > -2\Re(u) - \mathcal{N}$.

In order to prove (4.1), it is sufficient to move the contour of integration in (3.3) to the right on the line $\Re(s) = 1/2 - \Re(u) - \epsilon$ and estimate the absolute value of the integral.

In order to prove the first estimate in (4.2), we move the contour of integration to the left on $\Re(s) = 1 - 2l_1 + \epsilon$ and once again estimate the absolute value of the resulting integral.

In order to prove the second estimate in (4.2), we first differentiate (3.3) with respect to x. The resulting integral converges absolutely for $\Re(s) > j - 2\Re(u) - \mathcal{N}$. Note that the conditions of the lemma imply that

$$1 - 2l_1 < j - 2\Re(u) - \mathcal{N} < 1/2 - \Re u.$$

Moving the contour of integration to $\Re(s) = j - 2\Re(u) - \mathcal{N} + \epsilon$ and estimating the absolute value of the resulting integral, we prove the second estimate in (4.2). □

Let us choose the parameters N and l_1 in Lemma 2.6 in the following way:

$$\mathcal{N} = 3, \quad l_1 = 3.$$

Then Lemma 4.1 implies that for $1 < \Re(u) < 3/2 - \delta$ and $\Re(v) = 0$ the function $\psi(x)$ from (3.4) satisfies the conditions (2.36) of applicability of the Kuznetsov trace formula.

Next, we estimate the transforms (2.37) and (2.38) of the function $\psi(x)$ from (3.4).

Lemma 4.2. *For $0 \leq \Re(u) < 3/2-\delta$ and $\Re(v) = 0$ the following identity holds:*

$$\begin{aligned}\psi_D(t) = \frac{t\cosh(\pi t)}{\pi\sinh(\pi t)} \int_{(a)} & \hat{\Phi}_{\mathcal{N}}(s)\Gamma(1/2 - u + v - s) \\ & \times\Gamma(1/2-u-v-s)\Gamma\left(\frac{s+2u}{2} + it\right)\Gamma\left(\frac{s+2u}{2} - it\right) \\ & \times(\cos(\pi v)+\sin(\pi(u+s)))\sin\left(\frac{\pi(s+2u)}{2}\right)2^{s+2u}\,ds,\end{aligned} \tag{4.4}$$

where $-2\Re(u) < a < 1/2 - \Re(u)$. Furthermore, for $|t| \to \infty$ we have

$$\psi_D(t) \ll \frac{1}{(1+|t|)^{\mathcal{N}}}. \tag{4.5}$$

Proof: Substituting (3.4), (3.3) into (2.38) and using the formula (see [**17**, (1), p. 326]) for $0 < \Re s < 3/2$

$$\int_0^\infty (J_{2it}(x) + J_{-2it}(x))x^{s-1}\,dx = \frac{2^s}{\pi}\Gamma(s/2+it)\Gamma(s/2-it)\cosh(\pi t)\sin\left(\frac{\pi s}{2}\right),$$

we prove (4.4). Estimating the absolute value of the expression under the integral in (4.4) by using (2.25) and Stirling's formula (4.3), we conclude that

$$\begin{aligned}\psi_D(t) \ll (1+|t|)\int_{-\infty}^{\infty} & \frac{\exp(-\pi(|y+t| + |y-t| - 2|y|)/2)}{(1+|y|)^{1+2\Re u+N+a}} \\ & \times(1+|y+t|)^{\Re u+(a-1)/2}(1+|y-t|)^{\Re u+(a-1)/2}\,dy.\end{aligned}$$

Estimating the last integral, we show that (4.5) holds. □

For the sake of convenience, we introduce the notation

$$\begin{aligned} I_0(u,v) := \frac{1}{\pi}\int_{(0)} & \hat{\Phi}_{\mathcal{N}}(s)\Gamma(1/2-u+v-s) \\ & \times\Gamma(1/2-u-v-s)\Gamma\left(\frac{s+u+v+1/2}{2}\right)\Gamma\left(\frac{s+3u-v-1/2}{2}\right) \\ & \times(\cos(\pi v)+\sin(\pi(u+s)))\sin\left(\frac{\pi(s+2u)}{2}\right)2^{s+2u}\,ds. \end{aligned} \tag{4.6}$$

Lemma 4.3. *For* $0 \le \Re(u) < 1/6$ *and* $\Re(v) = 0$ *we have*

$$\begin{aligned} \left.\frac{\psi_D(t)\sinh(\pi t)}{t\cosh(\pi t)}\right|_{2it=1-s_-} &= i2^{5/2-u+v}\hat{\Phi}_{\mathcal{N}}(1/2-3u+v)\Gamma(2u)\Gamma(2u-2v) \\ & \times\Gamma(1/2-u+v)(\cos(\pi v)+\sin(\pi(1/2-2u+v))) \\ & \times\sin\left(\frac{\pi(1/2-u+v)}{2}\right)+I_0(u,v). \end{aligned} \tag{4.7}$$

Proof: It follows from (4.4) that

$$\begin{aligned} \left.\frac{\psi_D(t)\sinh(\pi t)}{t\cosh(\pi t)}\right|_{2it=1-s_-} = \frac{1}{\pi}\int_{(a)} & \hat{\Phi}_{\mathcal{N}}(s)\Gamma(1/2-u+v-s)\Gamma(1/2-u-v-s) \\ & \times\Gamma\left(\frac{s+u+v+1/2}{2}\right)\Gamma\left(\frac{s+3u-v-1/2}{2}\right) \\ & \times(\cos(\pi v)+\sin(\pi(u+s)))\sin\left(\frac{\pi(s+2u)}{2}\right)2^{s+2u}\,ds, \end{aligned}$$

where $1/2 - 3\Re(u) < a < 1/2 - \Re(u)$. Next, we move the contour of integration to the line $\Re s = 0$, crossing a pole at $s = 1/2 - 3u + v$. Computing the residue at this point, we prove (4.7). □

In a similar way we prove the following statement.

Lemma 4.4. *For* $0 \le \Re(u) < 1/6$ *and* $\Re(v) = 0$ *we have*

$$\begin{aligned} \left.\frac{\psi_D(t)\sinh(\pi t)}{t\cosh(\pi t)}\right|_{2it=1-s_+} &= i2^{5/2-u-v}\hat{\Phi}_{\mathcal{N}}(1/2-3u-v)\Gamma(2u)\Gamma(2u+2v) \\ & \times\Gamma(1/2-u-v)(\cos(\pi v)+\sin(\pi(1/2-2u-v))) \\ & \times\sin\left(\frac{\pi(1/2-u-v)}{2}\right)+I_0(u,-v). \end{aligned} \tag{4.8}$$

Lemma 4.5. *For* $0 \leq \Re(u) < 3/2 - \delta$ *and* $\Re(v) = 0$ *we have*

$$\psi_H(k) = \frac{2^{2u+1} i^k}{2\pi i} \int_{(a)} \hat{\Phi}_{\mathcal{N}}(s) \Gamma(1/2 - u + v - s) \Gamma(1/2 - u - v - s) \times \frac{\Gamma((k-1+2u+s)/2)}{\Gamma((k+1-2u-s)/2)} (\cos(\pi v) + \sin(\pi(u+s))) 2^s \, ds, \tag{4.9}$$

where $\max(1 - 2l_1, 1 - k - 2\Re(u)) < a < 1/2 - \Re(u)$. *Moreover, for* $k \to \infty$ *the following estimate holds:*

$$\psi_H(k) \ll \frac{1}{k^{N+1}} + \frac{1}{k^{2l_1}}. \tag{4.10}$$

Proof: Substituting (3.4), (3.3) into (2.37) and using the formula (see [**17**, (1), p. 326])

$$\int_0^\infty J_{k-1}(x) x^{s-1} \, dx = \frac{2^{s-1} \Gamma((k-1+s)/2)}{\Gamma((k+1-s)/2)},$$

we prove (4.9). Estimating the absolute value of the expression under the integral in (4.9) by applying (2.25) and Stirling's formula (4.3), we infer

$$\psi_H(k) \ll \int_{-\infty}^{\infty} \frac{dy}{(1+|y|)^{1+a+\mathcal{N}+2\Re u} (k+|y|)^{1-a-2\Re u}}. \tag{4.11}$$

Since we are interested in the case $k \to \infty$, it is possible to choose $a = 1 - 2l_1 + \epsilon$. Estimating the integral (4.11) for this a, we complete the proof of (4.10). □

Estimates (4.4) and (4.10) guarantee that the series (1.5), arising after the application of the Kuznetsov trace formula, are absolutely convergent.

5. Contribution of the continuous spectrum

Recall that (see (1.3))

$$s_+ = 1/2 + u + v, \quad s_- = 1/2 + u - v.$$

Let us define

$$P_{L,\zeta}(u, v, t) = \frac{L(\chi_{-4}, s_+) L(\chi_{-4}, s_-)}{\zeta(2s_+) \zeta(2s_-)} \times \frac{\zeta(s_+ + 2it) \zeta(s_+ - 2it) \zeta(s_- + 2it) \zeta(s_- - 2it)}{L(\chi_{-4}, 1+2it) L(\chi_{-4}, 1-2it)}. \tag{5.1}$$

Applying the Kuznetsov trace formula (2.40) to the right-hand side of (3.5), we obtain the following sum as a part of the continuous spectrum:

$$\mathcal{C}(u,v) = \frac{\zeta(2s_+)\zeta(2s_-)}{(2\pi)^{1-2u}\pi^{2u}} \sum_{m,n=1}^{\infty} \frac{1}{m^{s_-}n^{s_+}} \tag{5.2}$$
$$\times\Big(-4iC_{\infty,\infty}(m^2,n^2,4;\psi) + 2^{1-2u}C_{\infty,0}(m^2,n^2,4;\psi)$$
$$+(8-2^{3-s_-}-2^{3-s_+})C_{\infty,0}(m^2,n^2,16;\psi)+16C_{\infty,0}(m^2,n^2,64;\psi)\Big).$$

For the sake of convenience, we also define

$$\mathcal{D}_{\mathfrak{a},\mathfrak{b}}(\mathfrak{c},N;t) = \sum_{m,n=1}^{\infty} \frac{\overline{\phi_{\mathfrak{a},\mathfrak{c}}(m^2,1/2+it,\chi_{-4})}\phi_{\mathfrak{b},\mathfrak{c}}(n^2,1/2+it,\chi_{-4})}{m^{s_-+2it}n^{s_+-2it}}.$$

Then it follows from (2.43) that for $C_{\mathfrak{a},\mathfrak{b}}(m^2,n^2,N;\psi)$ we have

$$\sum_{m,n=1}^{\infty} \frac{1}{m^{s_-}n^{s_+}} C_{\mathfrak{a},\mathfrak{b}}(m^2,n^2,N;\psi) \tag{5.3}$$
$$= \frac{1}{4\pi}\int_{-\infty}^{\infty} \frac{\psi_D(t)\sinh(\pi t)}{t\cosh(\pi t)} \sum_{\mathfrak{c}\text{ sing.}} \mathcal{D}_{\mathfrak{a},\mathfrak{b}}(\mathfrak{c},N;t)\,dt.$$

Lemma 5.1. *For $\Re(u) > 1/2$, $\Re(v) = 0$, the following identity holds:*

$$\sum_{m,n=1}^{\infty} \frac{1}{m^{s_-}n^{s_+}} C_{\infty,\infty}(m^2,n^2,4;\psi) \tag{5.4}$$
$$= \frac{1}{4\pi}\int_{-\infty}^{\infty} \frac{\psi_D(t)\sinh(\pi t)}{t\cosh(\pi t)} \frac{P_{L,\zeta}(u,v,t)F_1(t)}{4(1-2^{-2s_-})(1-2^{-2s_+})}\,dt,$$

where

$$F_1(t) = (1-2^{-2it-s_-})(1-2^{2it-s_+}) + (1-2^{2it-s_-})(1-2^{-2it-s_+}). \tag{5.5}$$

Proof: By Lemma 2.9 the following cusps are singular: 0, ∞. Applying (2.79), (2.77), (2.60), and (2.3), we infer

$$\mathcal{D}_{\infty,\infty}(\infty,4;t) = P_{L,\zeta}(u,v,t)\frac{(1-2^{-2it-s_-})(1-2^{2it-s_+})}{4(1-2^{-2s_-})(1-2^{-2s_+})}. \tag{5.6}$$

Using (2.78), (2.77), (2.59), and (2.3), we show that

$$\mathcal{D}_{\infty,\infty}(0,4;t) = P_{L,\zeta}(u,v,t)\frac{(1-2^{2it-s_-})(1-2^{-2it-s_+})}{4(1-2^{-2s_-})(1-2^{-2s_+})}. \tag{5.7}$$

Finally, it follows from (5.6), (5.7), and (5.3) that (5.4) holds. □

Lemma 5.2. *For $\Re(u) > 1/2$, $\Re(v) = 0$, we have*

$$\sum_{m,n=1}^{\infty} \frac{1}{m^{s_-} n^{s_+}} C_{\infty,0}(m^2, n^2, 4; \psi) = \frac{1}{4\pi} \int_{-\infty}^{\infty} \frac{\psi_D(t)\sinh(\pi t)}{t\cosh(\pi t)} \frac{-iP_{L,\zeta}(u,v,t)F_2(t)}{4(1-2^{-2s_-})(1-2^{-2s_+})}\, dt, \tag{5.8}$$

where

$$F_2(t) = (1-2^{-2it-s_-})(1-2^{-2it-s_+})2^{2it} + (1-2^{2it-s_-})(1-2^{2it-s_+})2^{-2it}. \tag{5.9}$$

Proof: By Lemma 2.9 the following cusps are singular: 0, ∞. Applying (2.79), (2.77), (2.60), (2.59), and (2.3), we show that

$$\mathcal{D}_{\infty,0}(\infty, 4; t) = P_{L,\zeta}(u,v,t) \frac{-i(1-2^{-2it-s_-})(1-2^{-2it-s_+})}{2^{2-2it}(1-2^{-2s_-})(1-2^{-2s_+})}. \tag{5.10}$$

As a consequence of (2.78), (2.77), (2.60), (2.59), and (2.3), we infer

$$\mathcal{D}_{\infty,0}(0, 4; t) = P_{L,\zeta}(u,v,t) \frac{-i(1-2^{2it-s_-})(1-2^{2it-s_+})}{2^{2+2it}(1-2^{-2s_-})(1-2^{-2s_+})}. \tag{5.11}$$

Finally, substituting (5.10) and (5.11) into (5.3), we prove (5.8). □

Lemma 5.3. *For $\Re(u) > 1/2$, $\Re(v) = 0$, the following identity holds:*

$$\sum_{m,n=1}^{\infty} \frac{1}{m^{s_-} n^{s_+}} C_{\infty,0}(m^2, n^2, 16; \psi) = \frac{1}{4\pi} \int_{-\infty}^{\infty} \frac{\psi_D(t)\sinh(\pi t)}{t\cosh(\pi t)} \frac{-iP_{L,\zeta}(u,v,t)F_3(t)\, dt}{8(1-2^{-2s_-})(1-2^{-2s_+})}, \tag{5.12}$$

where

$$\begin{aligned} F_3(t) = &\frac{(1-2^{2it-s_-})(1-2^{2it-s_+})}{2^{2it+s_+}} + \frac{(1-2^{-2it-s_-})(1-2^{-2it-s_+})}{2^{-2it+s_+}} \\ &+ \frac{(1-2^{-2it-s_-})(1-2^{-2it-s_+})}{2^{-2it+s_-}} + \frac{(1-2^{2it-s_-})(1-2^{2it-s_+})}{2^{2it+s_-}} \\ &- \frac{(1-2^{-2it-s_-})(1-2^{-2it-s_+})}{2^{-4it+s_++s_-}} - \frac{(1-2^{2it-s_-})(1-2^{2it-s_+})}{2^{4it+s_++s_-}}. \end{aligned} \tag{5.13}$$

Proof: By Lemma 2.9 the following cusps are singular:

$$0,\ 1/2,\ 1/4,\ 1/8,\ 1/12,\ \infty.$$

As a direct consequence of (2.81) we have

$$\mathcal{D}_{\infty,0}(1/2, 16; t) = \mathcal{D}_{\infty,0}(1/8, 16; t) = 0. \tag{5.14}$$

First, consider $\mathfrak{c} = 0$. Applying (2.71), (2.74), (2.60), (2.59), and (2.3), we show that

$$\mathcal{D}_{\infty,0}(0,16;t) = P_{L,\zeta}(u,v,t)\frac{-i(1-2^{2it-s_-})(1-2^{2it-s_+})}{2^{3+s_++2it}(1-2^{-2s_-})(1-2^{-2s_+})}. \tag{5.15}$$

Second, consider $\mathfrak{c} = \infty$. Applying (2.76), (2.73), (2.60), (2.59), and (2.3), we show that

$$\mathcal{D}_{\infty,0}(\infty,16;t) = P_{L,\zeta}(u,v,t)\frac{-i(1-2^{-2it-s_-})(1-2^{-2it-s_+})}{2^{3+s_--2it}(1-2^{-2s_-})(1-2^{-2s_+})}. \tag{5.16}$$

Third, consider $\mathfrak{c} = 1/(4\Upsilon)$ for $\Upsilon = 1, 3$. It follows from (2.75) and (2.72) that

$$\sum_{m,n=1}^{\infty} \frac{1}{m^{s_-+2it}n^{s_+-2it}} \overline{\phi_{\infty,1/4\Upsilon}(m^2, 1/2+it, \chi_{-4})}\phi_{0,1/4\Upsilon}(n^2, 1/2+it, \chi_{-4})$$
$$= \frac{\chi_{-4}(-\Upsilon)}{16L(\chi_{-4}, 1+2it)L(\chi_{-4}, 1-2it)} \times \sum_{m,n=1}^{\infty} \frac{\sigma_{2it}(\chi_{-4}; m^2)\sigma_{-2it}(\chi_{-4}; n^2)}{m^{s_-+2it}n^{s_+-2it}} e\left(\frac{(m^2+n^2)\Upsilon}{4}\right).$$

If m and n have the same parity, then

$$\sum_{\Upsilon=1,3} \chi_{-4}(-\Upsilon)e\left(\frac{(m^2+n^2)\Upsilon}{4}\right) = 0.$$

If the parity of m and n differ, then

$$\sum_{\Upsilon=1,3} \chi_{-4}(-\Upsilon)e\left(\frac{(m^2+n^2)\Upsilon}{4}\right) = -2i.$$

As a result,

$$\mathcal{D}_{\infty,0}(1/4,16;t) + \mathcal{D}_{\infty,0}(1/12,16;t) = \frac{-i}{8L(\chi_{-4}, 1+2it)L(\chi_{-4}, 1-2it)}$$
$$\times \left(\sum_{m\equiv 1(2)} \frac{\sigma_{2it}(\chi_{-4}; m^2)}{m^{s_-+2it}} \sum_{n\equiv 0(2)} \frac{\sigma_{-2it}(\chi_{-4}; n^2)}{n^{s_+-2it}} + \sum_{m\equiv 0(2)} \frac{\sigma_{2it}(\chi_{-4}; m^2)}{m^{s_-+2it}} \sum_{n\equiv 1(2)} \frac{\sigma_{-2it}(\chi_{-4}; n^2)}{n^{s_+-2it}} \right).$$

Using (2.2), we find that

$$\begin{aligned} \sum_{m\equiv 1(2)} \frac{\sigma_{2it}(\chi_{-4}; m^2)}{m^{s_-+2it}} &= \sum_{m=1}^{\infty} \frac{\sigma_{2it}(\chi_{-4}; m^2)}{m^{s_-+2it}} - \sum_{m\equiv 0(2)} \frac{\sigma_{2it}(\chi_{-4}; m^2)}{m^{s_-+2it}} \\ &= (1-2^{-s_--2it}) \sum_{m=1}^{\infty} \frac{\sigma_{2it}(\chi_{-4}; m^2)}{m^{s_-+2it}}. \end{aligned} \tag{5.17}$$

The sum over odd n can be treated similarly. Applying (2.3) in order to compute the sums over m and n, we derive the following expression:

(5.18)
$$\mathcal{D}_{\infty,0}(1/4,16;t)+\mathcal{D}_{\infty,0}(1/12,16;t)=P_{L,\zeta}(u,v,t)\frac{-i(1-2^{2it-s_-})(1-2^{-2it-s_+})}{2^3(1-2^{-2s_-})(1-2^{-2s_+})}$$
$$\times\left(\frac{1-2^{-s_--2it}}{2^{s_+-2it}}+\frac{1-2^{-s_++2it}}{2^{s_-+2it}}\right).$$

Now the required identity (5.12) follows from (5.14), (5.15), (5.16), (5.18), and (5.3). ☐

Lemma 5.4. *For $\Re(u)>1/2$, $\Re(v)=0$, we have*

(5.19)
$$\sum_{m,n=1}^{\infty}\frac{1}{m^{s_-}n^{s_+}}C_{\infty,0}(m^2,n^2,64;\psi)$$
$$=\frac{1}{4\pi}\int_{-\infty}^{\infty}\frac{\psi_D(t)\sinh(\pi t)}{t\cosh(\pi t)}\frac{-iP_{L,\zeta}(u,v,t)F_4(t)\,dt}{16(1-2^{-2s_-})(1-2^{-2s_+})},$$

where

(5.20)
$$F_4(t)=(1-2^{2it-s_-})(1-2^{2it-s_+})(1-2^{-2it-s_-})(1-2^{-2it-s_+})$$
$$+\frac{(1-2^{-2it-s_-})(1-2^{2it-s_-})(1-2^{2it-s_+})}{2^{2it+2s_+}}$$
$$+\frac{(1-2^{-2it-s_-})(1-2^{2it-s_-})(1-2^{-2it-s_+})}{2^{-2it+2s_+}}$$
$$+\frac{(1-2^{-2it-s_+})(1-2^{2it-s_+})(1-2^{2it-s_-})}{2^{2it+2s_-}}$$
$$+\frac{(1-2^{-2it-s_+})(1-2^{2it-s_+})(1-2^{-2it-s_-})}{2^{-2it+2s_-}}$$
$$+\frac{(1-2^{2it-s_-})(1-2^{2it-s_+})}{2^{2it+s_++s_-}}+\frac{(1-2^{-2it-s_-})(1-2^{-2it-s_+})}{2^{-2it+s_++s_-}}.$$

Proof: By Lemma 2.9 the following cusps are singular:

$$0,\ 1/2,\ 1/(4\Upsilon_4),\ 1/(8\Upsilon_8),\ 1/(16\Upsilon_4),\ 1/32,\ \infty,$$

where $\Upsilon_4=1,3$ and $\Upsilon_8=1,3,5,7$. Let us split each of the sums over m and n into two parts depending on the parity of the summation variable.

Then

$$\mathcal{D}_{\infty,0}(\mathfrak{c},64;t)=\mathcal{E}_{11}(\mathfrak{c};t)+\mathcal{E}_{10}(\mathfrak{c};t)+\mathcal{E}_{01}(\mathfrak{c};t)+\mathcal{E}_{00}(\mathfrak{c};t), \tag{5.21}$$

where

$$\mathcal{E}_{kj}(\mathfrak{c};t)=\sum_{m\equiv k(2)}\sum_{n\equiv j(2)}\frac{\overline{\phi_{\infty,\mathfrak{c}}(m^2,1/2+it,\chi_{-4})}\phi_{0,\mathfrak{c}}(n^2,1/2+it,\chi_{-4})}{m^{s_-+2it}n^{s_+-2it}}.$$

Let us compute $\mathcal{E}_{11}(\mathfrak{c};t)$. Since the summation is taken over odd n, Lemma 2.11 implies that

$$\mathcal{E}_{11}(0;t)=\mathcal{E}_{11}(1/2;t)=\mathcal{E}_{11}(1/(4\Upsilon_4);t)=0. \tag{5.22}$$

Since the summation is taken over odd m, Lemma 2.10 shows that

$$\mathcal{E}_{11}(1/(16\Upsilon_4);t)=\mathcal{E}_{11}(1/32;t)=\mathcal{E}_{11}(\infty;t)=0. \tag{5.23}$$

Therefore, we are left to compute

$$\sum_{\Upsilon_8=1,3,5,7}\mathcal{E}_{11}(1/(8\Upsilon_8);t).$$

Next, we apply (2.63), (2.68), and (2.59) for evaluation of the Fourier coefficients. In order to compute the sum over Υ_8 we use the equality

$$\sum_{\Upsilon_8=1,3,5,7}\chi_{-4}(-\Upsilon_8)e\left(\frac{\Upsilon_8}{4}\right)=-4i.$$

The sums over m and n can be treated using (5.17) and (2.3). As a result,

$$\begin{aligned}\sum_{\Upsilon_8=1,3,5,7}&\mathcal{E}_{11}(1/(8\Upsilon_8);t)\\ &=P_{L,\zeta}(u,v,t)\frac{-i(1-2^{-2it-s_-})(1-2^{-2it-s_+})(1-2^{2it-s_-})(1-2^{2it-s_+})}{16(1-2^{-2s_-})(1-2^{-2s_+})}.\end{aligned} \tag{5.24}$$

Combining (5.22), (5.23), and (5.24), we have

$$\begin{aligned}\sum_{\mathfrak{c}\text{ sing.}}&\mathcal{E}_{11}(\mathfrak{c};t)\\ &=P_{L,\zeta}(u,v,t)\frac{-i(1-2^{-2it-s_-})(1-2^{-2it-s_+})(1-2^{2it-s_-})(1-2^{2it-s_+})}{16(1-2^{-2s_-})(1-2^{-2s_+})}.\end{aligned} \tag{5.25}$$

Now let us compute $\mathcal{E}_{10}(\mathfrak{c};t)$. As a direct consequence of (2.80), we know that

$$\mathcal{E}_{10}(1/2;t)=\mathcal{E}_{10}(1/32;t)=0. \tag{5.26}$$

Since in $\mathcal{E}_{10}(\mathfrak{c};t)$ the summation is taken over odd m, then it follows from (2.64) and (2.65) that

$$\mathcal{E}_{10}(1/(16\Upsilon_4);t)=\mathcal{E}_{10}(\infty;t)=0. \tag{5.27}$$

Applying (2.61), (2.66), (2.59), (2.60), (5.17), and (2.3), we infer

$$\mathcal{E}_{10}(0;t)=P_{L,\varsigma}(u,v,t)\frac{-i(1-2^{-2it-s_-})(1-2^{2it-s_-})(1-2^{2it-s_+})}{16(1-2^{-2s_-})(1-2^{-2s_+})2^{2it+2s_+}}. \tag{5.28}$$

Next, we compute

$$\sum_{\Upsilon_4=1,3}\mathcal{E}_{10}(1/(4\Upsilon_4);t).$$

In view of (2.62) and (2.67), it is required to work with the following sum:

$$\sum_{\Upsilon_4=1,3}\chi_{-4}(-\Upsilon_4)e\left(\frac{\Upsilon_4 m^2}{4}\right)e\left(\frac{\Upsilon_4 n^2}{16}\right).$$

Note that m is odd and n is even. If $n\equiv 2 \pmod 4$, then

$$\sum_{\Upsilon_4=1,3}\chi_{-4}(-\Upsilon_4)e\left(\frac{\Upsilon_4 m^2}{4}\right)e\left(\frac{\Upsilon_4 n^2}{16}\right)=\sum_{\Upsilon_4=1,3}\chi_{-4}(-\Upsilon_4)e\left(\frac{\Upsilon_4}{2}\right)=0.$$

If $n\equiv 0 \pmod 4$, then

$$\sum_{\Upsilon_4=1,3}\chi_{-4}(-\Upsilon_4)e\left(\frac{\Upsilon_4 m^2}{4}\right)e\left(\frac{\Upsilon_4 n^2}{16}\right)=\sum_{\Upsilon_4=1,3}\chi_{-4}(-\Upsilon_4)e\left(\frac{\Upsilon_4}{4}\right)=-2i.$$

Using this as well as (2.62), (2.67), (2.59), (5.17), and (2.3), we conclude that

$$\sum_{\Upsilon_4=1,3}\mathcal{E}_{10}(1/(4\Upsilon_4);t) = P_{L,\varsigma}(u,v,t)\frac{-i(1-2^{-2it-s_-})(1-2^{2it-s_-})(1-2^{-2it-s_+})}{16(1-2^{-2s_-})(1-2^{-2s_+})2^{-2it+2s_+}}. \tag{5.29}$$

Consider

$$\sum_{\Upsilon_8=1,3,5,7}\mathcal{E}_{10}(1/(8\Upsilon_8);t).$$

By (2.63) and (2.68) it is required to compute the sum

$$\sum_{\Upsilon_8=1,3,5,7}\chi_{-4}(-\Upsilon_8)e\left(\frac{\Upsilon_8(m^2+n^2)}{8}\right).$$

We know that m is odd and n is even. Note that for odd m we have $e\left(\frac{\Upsilon_8 m^2}{8}\right)=e\left(\frac{\Upsilon_8}{8}\right)$. If $n\equiv 0 \pmod 4$, then

$$\sum_{\Upsilon_8=1,3,5,7}\chi_{-4}(-\Upsilon_8)e\left(\frac{\Upsilon_8(m^2+n^2)}{8}\right)=\sum_{\Upsilon_8=1,3,5,7}\chi_{-4}(-\Upsilon_8)e\left(\frac{\Upsilon_8}{8}\right)=0.$$

If $n \equiv 2 \pmod 4$, then

$$\sum_{\Upsilon_8=1,3,5,7} \chi_{-4}(-\Upsilon_8) e\left(\frac{\Upsilon_8(m^2+n^2)}{8}\right) = -\sum_{\Upsilon_8=1,3,5,7} \chi_{-4}(-\Upsilon_8) e\left(\frac{\Upsilon_8}{8}\right) = 0.$$

Therefore,

$$\sum_{\Upsilon_8=1,3,5,7} \mathcal{E}_{10}(1/(8\Upsilon_8);t) = 0. \tag{5.30}$$

Combining (5.26), (5.27), (5.28), (5.29), and (5.30), we have

$$\sum_{\mathfrak{c} \text{ sing.}} \mathcal{E}_{10}(\mathfrak{c};t) \tag{5.31}$$
$$= P_{L,\zeta}(u,v,t) \frac{-i(1-2^{-2it-s_-})(1-2^{2it-s_-})}{16(1-2^{-2s_-})(1-2^{-2s_+})} \left(\frac{(1-2^{-2it-s_+})}{2^{-2it+2s_+}} + \frac{(1-2^{2it-s_+})}{2^{2it+2s_+}} \right).$$

Let us compute $\mathcal{E}_{01}(\mathfrak{c};t)$. According to (2.80)

$$\mathcal{E}_{01}(1/2;t) = \mathcal{E}_{01}(1/32;t) = 0. \tag{5.32}$$

Since in $\mathcal{E}_{01}(\mathfrak{c};t)$ the summation is taken over odd n, then by (2.66) and (2.67)

$$\mathcal{E}_{01}(0;t) = \mathcal{E}_{10}(1/(4\Upsilon_4);t) = 0. \tag{5.33}$$

Similarly to (5.30) we show that

$$\sum_{\Upsilon_8=1,3,5,7} \mathcal{E}_{01}(1/(8\Upsilon_8);t) = 0. \tag{5.34}$$

We now proceed to evaluate

$$\sum_{\Upsilon_4=1,3} \mathcal{E}_{01}(1/(16\Upsilon_4);t).$$

According to (2.64) and (2.69), it is required to compute the sum

$$\sum_{\Upsilon_4=1,3} \chi_{-4}(-\Upsilon_4) e\left(\frac{\Upsilon_4 m^2}{16}\right) e\left(\frac{\Upsilon_4 n^2}{4}\right).$$

We know that m is even and n odd. If $m \equiv 2 \pmod 4$, then

$$\sum_{\Upsilon_4=1,3} \chi_{-4}(-\Upsilon_4) e\left(\frac{\Upsilon_4 m^2}{16}\right) e\left(\frac{\Upsilon_4 n^2}{4}\right) = \sum_{\Upsilon_4=1,3} \chi_{-4}(-\Upsilon_4) e\left(\frac{\Upsilon_4}{2}\right) = 0.$$

If $m \equiv 0 \pmod 4$, then

$$\sum_{\Upsilon_4=1,3} \chi_{-4}(-\Upsilon_4) e\left(\frac{\Upsilon_4 m^2}{16}\right) e\left(\frac{\Upsilon_4 n^2}{4}\right) = \sum_{\Upsilon_4=1,3} \chi_{-4}(-\Upsilon_4) e\left(\frac{\Upsilon_4}{4}\right) = -2i.$$

Using this in combination with (2.64), (2.69), (2.59), (5.17), and (2.3), we conclude that

$$(5.35)\quad \sum_{\Upsilon_4=1,3}\mathcal{E}_{01}(1/(4\Upsilon_4);t) = P_{L,\zeta}(u,v,t)\frac{-i(1-2^{2it-s_-})(1-2^{2it-s_+})(1-2^{-2it-s_+})}{16(1-2^{-2s_-})(1-2^{-2s_+})2^{2it+2s_-}}.$$

Applying (2.65), (2.70), (2.59), (2.60), (5.17), and (2.3), we have

$$(5.36)\quad \mathcal{E}_{01}(\infty;t) = P_{L,\zeta}(u,v,t)\frac{-i(1-2^{-2it-s_-})(1-2^{2it-s_+})(1-2^{-2it-s_+})}{16(1-2^{-2s_-})(1-2^{-2s_+})2^{-2it+2s_-}}.$$

Summation of (5.32), (5.33), (5.34), (5.35), and (5.36) gives

$$(5.37)\quad \sum_{\mathfrak{c}\text{ sing.}}\mathcal{E}_{10}(\mathfrak{c};t) = P_{L,\zeta}(u,v,t)\frac{-i(1-2^{-2it-s_+})(1-2^{2it-s_+})}{16(1-2^{-2s_-})(1-2^{-2s_+})}\left(\frac{(1-2^{-2it-s_-})}{2^{-2it+2s_-}}+\frac{(1-2^{2it-s_-})}{2^{2it+2s_-}}\right).$$

Let us compute $\mathcal{E}_{00}(\mathfrak{c};t)$. According to (2.80)

$$(5.38)\quad \mathcal{E}_{00}(1/2;t) = \mathcal{E}_{00}(1/32;t) = 0.$$

It follows from (2.61), (2.66), (2.59), (2.60), and (2.3) that

$$(5.39)\quad \mathcal{E}_{00}(0;t) = P_{L,\zeta}(u,v,t)\frac{-i(1-2^{2it-s_-})(1-2^{2it-s_+})}{16(1-2^{-2s_-})(1-2^{-2s_+})2^{4it+2s_++s_-}}.$$

By (2.65), (2.70), (2.59), (2.60), and (2.3) we have

$$(5.40)\quad \mathcal{E}_{00}(\infty;t) = P_{L,\zeta}(u,v,t)\frac{-i(1-2^{-2it-s_-})(1-2^{-2it-s_+})}{16(1-2^{-2s_-})(1-2^{-2s_+})2^{-4it+2s_-+s_+}}.$$

We proceed to evaluate

$$\sum_{\Upsilon_8=1,3,5,7}\mathcal{E}_{00}(1/(8\Upsilon_8);t).$$

By (2.63) and (2.68) we need to consider the sum

$$\sum_{\Upsilon_8=1,3,5,7}\chi_{-4}(-\Upsilon_8)e\left(\frac{\Upsilon_8(m^2+n^2)}{8}\right)$$

provided that m and n are both even. In this case

$$\sum_{\Upsilon_8=1,3,5,7}\chi_{-4}(-\Upsilon_8)e\left(\frac{\Upsilon_8(m^2+n^2)}{8}\right)=0,$$

and therefore,

$$(5.41)\quad \sum_{\Upsilon_8=1,3,5,7}\mathcal{E}_{00}(1/(8\Upsilon_8);t) = 0.$$

Let us compute

$$\sum_{\Upsilon_4=1,3} \mathcal{E}_{00}(1/(4\Upsilon_4);t).$$

By (2.62) and (2.67), it is required to study the sum

$$\sum_{\Upsilon_4=1,3} \chi_{-4}(-\Upsilon_4) e\left(\frac{\Upsilon_8 m^2}{4}\right) e\left(\frac{\Upsilon_8 n^2}{16}\right)$$

provided that both m and n are even. If $n \equiv 0 \pmod 4$, then

$$\sum_{\Upsilon_4=1,3} \chi_{-4}(-\Upsilon_4) e\left(\frac{\Upsilon_8 m^2}{4}\right) e\left(\frac{\Upsilon_8 n^2}{16}\right) = 0.$$

If $n \equiv 2 \pmod 4$, then

$$\sum_{\Upsilon_4=1,3} \chi_{-4}(-\Upsilon_4) e\left(\frac{\Upsilon_8 m^2}{4}\right) e\left(\frac{\Upsilon_8 n^2}{16}\right) = -2i.$$

Combining this with (2.59) and (2.3), we show that

$$\sum_{\Upsilon_4=1,3} \mathcal{E}_{00}(1/(4\Upsilon_4);t) = P_{L,\zeta}(u,v,t) \frac{-i(1-2^{2it-s_-})(1-2^{2it-s_+})(1-2^{-2it-s_+})}{16(1-2^{-2s_-})(1-2^{-2s_+})2^{2it+s_-+s_+}}. \tag{5.42}$$

Similarly,

$$\sum_{\Upsilon_4=1,3} \mathcal{E}_{00}(1/(16\Upsilon_4);t) = P_{L,\zeta}(u,v,t) \frac{-i(1-2^{-2it-s_+})(1-2^{2it-s_-})(1-2^{-2it-s_-})}{16(1-2^{-2s_-})(1-2^{-2s_+})2^{-2it+s_-+s_+}}. \tag{5.43}$$

Summing (5.38), (5.39), (5.40), (5.41), (5.42), and (5.43), we infer

$$\sum_{\mathfrak{c}\text{ sing.}} \mathcal{E}_{00}(\mathfrak{c};t) = P_{L,\zeta}(u,v,t) \frac{-i}{16(1-2^{-2s_-})(1-2^{-2s_+})} \times \left(\frac{(1-2^{2it-s_-})(1-2^{2it-s_+})}{2^{2it+s_-+s_+}} + \frac{(1-2^{-2it-s_-})(1-2^{-2it-s_+})}{2^{-2it+s_-+s_+}} \right). \tag{5.44}$$

Substituting (5.25), (5.31), (5.37), (5.44) into (5.21), we prove (5.19). This completes the proof. □

Substituting (5.4), (5.8), (5.12), and (5.19) into (5.2), and using (5.1), we prove the next lemma.

Lemma 5.5. *For $\Re(u)>1/2$, $\Re(v)=0$, the following identity holds:*

$$\mathcal{C}(u,v)=\frac{L(\chi_{-4},s_+)L(\chi_{-4},s_-)}{(2\pi)^{1-2u}\pi^{2u}(1-2^{-2s_-})(1-2^{-2s_+})}\mathfrak{M}^{\text{con}}(F;u,v), \tag{5.45}$$

where $\mathfrak{M}^{\text{con}}(F;u,v)$ is defined by (1.6)*, and*

$$F(t)=F_1(t)+2^{-1-2u}F_2(t)+(1-2^{-1/2-u+v}-2^{-1/2-u-v})F_3(t)+F_4(t), \tag{5.46}$$

where $F_j(t)$ are defined by (5.5), (5.9), (5.13)*, and* (5.20).

Now using (5.45) we can obtain an expression for $\mathcal{C}(u,v)$ which is valid in the region $0\le\Re(u)<1/2$, $\Re(v)=0$.

Lemma 5.6. *For $0\le\Re(u)<1/2$, $\Re(v)=0$, the following identity holds:*

$$\mathcal{C}(u,v)=\mathcal{S}_-(u,v)+\mathcal{S}_+(u,v)+\frac{L(\chi_{-4},s_+)L(\chi_{-4},s_-)}{(2\pi)^{1-2u}\pi^{2u}(1-2^{-2s_-})(1-2^{-2s_+})}\mathfrak{M}^{\text{con}}(u,v), \tag{5.47}$$

where $\mathfrak{M}^{\text{con}}(F;u,v)$ is defined by (1.6)*, and*

$$\mathcal{S}_-(u,v)=\frac{-iF((1-s_-)/(2i))}{2(2\pi)^{1-2u}\pi^{2u}(1-2^{-2s_-})(1-2^{-2s_+})}\frac{\psi_D(t)\sinh(\pi t)}{t\cosh(\pi t)}\bigg|_{2it=1-s_-}\times\frac{L(\chi_{-4},s_+)}{L(\chi_{-4},2-s_-)}\zeta(1+s_+-s_-)\zeta(2s_--1)\zeta(s_-+s_+-1), \tag{5.48}$$

$$\mathcal{S}_+(u,v)=\frac{-iF((1-s_+)/(2i))}{2(2\pi)^{1-2u}\pi^{2u}(1-2^{-2s_-})(1-2^{-2s_+})}\frac{\psi_D(t)\sinh(\pi t)}{t\cosh(\pi t)}\bigg|_{2it=1-s_+}\times\frac{L(\chi_{-4},s_-)}{L(\chi_{-4},2-s_+)}\zeta(1+s_--s_+)\zeta(2s_+-1)\zeta(s_-+s_+-1), \tag{5.49}$$

$$F\left(\frac{1-s_-}{2i}\right)=-2^{-4u-2v-2}((3\cdot2^{2u+1}-2)2^{4v}+(3\cdot2^{2u}-5\cdot2^{4u+1}-1)2^{2v}+2^{2u}+2^{4u+1}),$$

$$F\left(\frac{1-s_+}{2i}\right)=-2^{-4u-2v-2}((2^{2u}+2^{4u+1})2^{4v}+(3\cdot2^{2u}-5\cdot2^{4u+1}-1)2^{2v}+3\cdot2^{2u+1}-2).$$

Proof: In the integral (1.6) we make the change of variables $z = 2it$, getting

$$\mathcal{C}(u,v) = \frac{L(\chi_{-4},s_+)L(\chi_{-4},s_-)}{(2\pi)^{1-2u}\pi^{2u}(1-2^{-2s_-})(1-2^{-2s_+})}\frac{-1}{8\pi} \times \int_{-\infty}^{\infty} \frac{\psi_D(z/(2i))\sinh(\pi z/(2i))}{z/(2i)\cosh(\pi z/(2i))} \times \frac{\zeta(s_+ + z)\zeta(s_+ - z)\zeta(s_- + z)\zeta(s_- - z)}{L(\chi_{-4},1+z)L(\chi_{-4},1-z)}F(z/(2i))\,dt. \tag{5.50}$$

Note that the product of the Riemann zeta functions in (5.50) has four poles (see (1.3)) at the points

$$z = 1/2 - u \pm v, \quad z = 1/2 + u \pm v. \tag{5.51}$$

Changing $\Re(u)$ from $\Re(u) > 1/2$ to $\Re(u) < 1/2$, these poles "jump" through the line of integration, resulting in additional residues (see [**2**, Lemma 6.4]). Computing the residues at the points (5.51) and using the fact that $F(t)$ is an even function, we prove (5.47). □

6. Proof of Theorem 1.1 and concluding remarks

Consider

$$\mathcal{FMT}(u,v) := R(u,v) + R(u,-v) + \mathcal{S}_-(u,v) + \mathcal{S}_+(u,v) - \mathcal{S}(u,v) - \mathcal{S}(u,-v), \tag{6.1}$$

where $R(u,v)$ is defined by (3.2), $\mathcal{S}_{\pm}(u,v)$ is given by (5.48) and (5.49), and

$$\mathcal{S}(u,v) = 2h\left(\frac{1/2-u+v}{2i}\right)\frac{\zeta(1/2+u+v)\zeta(2u-2v)\zeta(1+2v)\zeta(2u)}{\zeta(3/2-u+v)}. \tag{6.2}$$

Theorem 6.1. *For $0 \le \Re(u) < 1/2$, $\Re(v) = 0$, the following formula holds:*

$$\begin{aligned}\mathcal{M}_2(h;u,v) + \mathcal{M}_4^{(c)}(h;u,v) &= \mathcal{FMT}(u,v) \\ &+ \frac{L(\chi_{-4},s_+)L(\chi_{-4},s_-)}{(2\pi)^{1-2u}\pi^{2u}(1-2^{-2s_-})(1-2^{-2s_+})}\mathfrak{M}^{\mathrm{con}}(F,u,v) \\ &+ \frac{(1-2^{-2s_-})^{-1}(1-2^{-2s_+})^{-1}}{(2\pi)^{1-2u}\pi^{2u}} \\ &\times\Big(-4i\mathfrak{M}_{\infty,\infty}(4,u,v) + 2^{1-2u}\mathfrak{M}_{\infty,0}(4,u,v) \\ &+(8-2^{3-s_-}-2^{3-s_+})\mathfrak{M}_{\infty,0}(16,u,v) + 16\mathfrak{M}_{\infty,0}(64,u,v)\Big),\end{aligned} \tag{6.3}$$

where $\mathfrak{M}_{\mathfrak{ab}}(N,u,v)$, $\mathfrak{M}^{\mathrm{con}}(F,u,v)$ are defined by (1.5) *and* (1.6)*, respectively.*

Proof: Applying the Kuznetsov trace formula (2.40) to (3.5), using the expressions (2.45), (2.46), (2.47), (2.48), and finally applying Lemma 2.8 we have that for

$$1<\Re(u)<3/2,\quad \Re(v)=0$$

the following identity holds:

$$\begin{aligned}\mathcal{M}_2(h;u,v)+\mathcal{M}_4^{(c)}(h;u,v)&=R(u,v)+R(u,-v)\\&+\frac{L(\chi_{-4},s_+)L(\chi_{-4},s_-)}{(2\pi)^{1-2u}\pi^{2u}(1-2^{-2s_-})(1-2^{-2s_+})}\mathfrak{M}^{\mathrm{con}}(F,u,v)\\&+\frac{(1-2^{-2s_-})^{-1}(1-2^{-2s_+})^{-1}}{(2\pi)^{1-2u}\pi^{2u}}\\&\times\Big(-4i\mathfrak{M}_{\infty,\infty}(4,u,v)+2^{1-2u}\mathfrak{M}_{\infty,0}(4,u,v)\\&+(8-2^{3-s_-}-2^{3-s_+})\mathfrak{M}_{\infty,0}(16,u,v)+16\mathfrak{M}_{\infty,0}(64,u,v)\Big).\end{aligned}\tag{6.4}$$

From the estimates (4.4) and (4.10) we conclude that the summands $\mathfrak{M}_{\mathfrak{ab}}(N,u,v)$ on the right-hand side of (6.4) are holomorphic functions of variables u and v. In order to continue analytically the formula (6.4) to the region $0\le\Re(u)<1/2$ we can apply (5.47). Similarly, it is possible to replace $\mathcal{M}_4^{(c)}(h;u,v)$ by $\mathcal{M}_4^{(c)}(h;u,v)+\mathcal{S}(u,v)+\mathcal{S}(u,-v)$. The function $\mathcal{FMT}(u,v)$ is holomorphic in u and v since the left-hand side of (6.3), as well as all summands (except $\mathcal{FMT}(u,v)$) on the right-hand side of (6.3), are holomorphic in both variables u and v. □

The most difficult and most important case is $u=v=0$, which we discuss in more detail. It would be natural to expect that $\mathcal{FMT}(0,0)$ contains the main term for the moment $\mathcal{M}_2(h;u,v)$. Unfortunately, as we show below, this is not the case. In this sense, $\mathcal{FMT}(0,0)$ can be called a "fake main term".

It is known [**5**, Theorem 1.3] that if $h(r)$ is a smooth characteristic function of the interval $(T,2T)$, then

$$\mathcal{M}_2(h;0,0)\asymp T^2\log^3 T.\tag{6.5}$$

Lemma 6.2. *The following asymptotic formula holds:*

$$\mathcal{FMT}(0,0)=\sum_{l\in\mathcal{L}}c(l)a_l+O(T),\tag{6.6}$$

where a_l are some absolute constants and $c(l)$ are defined by (2.22), (2.28), (2.29).

Proof: Using (6.2) and properties of the Riemann zeta function, we have

$$\lim_{u,v\to(0,0)}(\mathcal{S}(u,v)+\mathcal{S}(u,-v)) = 2\frac{\zeta(1/2)\zeta^2(0)}{\zeta(3/2)} \times\left(\frac{h'(1/(4i))}{2i}+\left(2\gamma+\frac{\zeta'(1/2)}{\zeta(1/2)}-2\frac{\zeta'(0)}{\zeta(0)}-\frac{\zeta'(3/2)}{\zeta(3/2)}\right)h\left(\frac{1}{4i}\right)\right).$$

Consequently, the summands $\mathcal{S}(u,v)+\mathcal{S}(u,-v)$ have a negligibly small contribution to $\mathcal{FMT}(0,0)$. Using (3.2), (5.48), (5.49), (4.8), (4.7), and some properties of the Riemann zeta function, it can be shown that

$$\lim_{u,v\to(0,0)}(R(u,v)+R(u,-v)+\mathcal{S}_-(u,v)+\mathcal{S}_+(u,v)) = \sum_{j=0}^{3} d_1(j)\hat{\Phi}_{\mathcal{N}}^{(j)}(1/2)+d_2I_0(0,0)+d_3\frac{\partial}{\partial v}I_0(0,v)\Big|_{v=0},$$

where $d_1(j)$, d_2, d_3 are some absolute constants, and $I_0(u,v)$ is defined by (4.6). Therefore,

$$\mathcal{FMT}(0,0) = \sum_{j=0}^{3} d_1(j)\hat{\Phi}_{\mathcal{N}}^{(j)}(1/2)+d_2I_0(0,0)+d_3\frac{\partial}{\partial v}I_0(0,v)\Big|_{v=0}+O(T^{-A}).$$

It follows from (4.6), (2.23), and (2.27) that

$$I_0(0,0)=\frac{1}{\pi^3}\int_0^\infty r\tanh(\pi r)h(r)\int_{(\epsilon)}\Gamma(s/2-ir)\Gamma(s/2+ir)\Gamma^2(1/2-s) \times\Gamma\left(\frac{s+1/2}{2}\right)\Gamma\left(\frac{s-1/2}{2}\right)(1+\sin(\pi s))\sin(\pi s)2^{2s}\,ds\,dr+\sum_{l\in\mathcal{L}}c(l)b_1(l),$$

where $b_1(l)$ are some constants independent of T. Estimating the absolute values of the integral over s by using Stirling's formula (4.3), we show that this integral can be bounded by $(1+|r|)^{-1}$. As a result,

$$I_0(0,0)=\sum_{l\in\mathcal{L}}c(l)b_1(l)+O(T).$$

Similarly,

$$\frac{\partial}{\partial v}I_0(0,v)\Big|_{v=0}=\sum_{l\in\mathcal{L}}c(l)b_2(l)+O(T).$$

Therefore,

$$\mathcal{FMT}(0,0)=\sum_{j=0}^{3} d_1(j)\hat{\Phi}_{\mathcal{N}}^{(j)}(1/2)+\sum_{l\in\mathcal{L}}c(l)b_3(l)+O(T). \tag{6.7}$$

It follows from (2.26) that

$$\sum_{j=0}^{3} d_1(j)\hat{\Phi}_{\mathcal{N}}^{(j)}(1/2) = \sum_{j=0}^{3} b_4(l)c(l) + \sum_{j=0}^{3} d_1(j)\hat{\phi}^{(j)}(1/2). \tag{6.8}$$

Applying (2.27), we show that

$$\sum_{j=0}^{3} d_1(j)\hat{\phi}^{(j)}(1/2) = O(T^{-A}). \tag{6.9}$$

Substituting (6.8) and (6.9) into (6.7), we complete the proof of (6.6). □

From (2.28) and (2.29) we derive that $c(l) = P_{l,2\mathcal{N}-1}(T)$, where $P_{l,2\mathcal{N}-1}(x)$ is a polynomial of degree $2\mathcal{N}-1$. Consequently, (6.6) implies that

$$\mathcal{FMT}(0,0) = Q_{2\mathcal{N}-1}(T) + O(T), \tag{6.10}$$

where $Q_{2\mathcal{N}-1}(x)$ is some polynomial of degree not exceeding $2\mathcal{N}-1$. Comparing (6.10) to (6.5), we see that $\mathcal{FMT}(0,0)$ is not a suitable candidate (even taking into account the possible vanishing of the leading coefficients of the polynomial $Q_{2\mathcal{N}-1}(x)$) for the role of the main term in the asymptotic formula for the second moment $\mathcal{M}_2(h;0,0)$ because the expression in (6.10) does not contain powers of $\log T$.

We remark that the coefficients $c(l)$ arise in all summands on the right-hand side of (1.7). Applying (4.5) and (4.10), at the price of a negligibly small error term we can truncate the summations in (1.5) up to $k, t_f \ll T^A$ and integration in (1.6) up to $|t| \ll T^A$ (here $A > 0$ is a sufficiently large number).

In the remaining finite sums and in the integral, we can separate the parts containing $c(l)$. To this end, we substitute (2.23) into (4.4) and (4.9), and separate from the functions $\psi_D(t)$, $\psi_H(k)$ parts containing $c(l)$, namely

$$\psi_D(t) = \phi_D(t) + \sum_{l\in\mathcal{L}} c(l)\psi_D(l;t), \quad \psi_H(k) = \phi_H(k) + \sum_{l\in\mathcal{L}} c(l)\psi_H(l;k).$$

It can be shown that the functions $\psi_D(l;t)$, $\psi_H(l;k)$ are of very rapid decay in t and k. Accordingly, the moments (1.5) and (1.6) with the functions $\psi_D(l;t)$, $\psi_H(l;k)$ are bounded by some absolute constants (without logarithmic growth in T). Therefore, these expressions do not qualify to be a main term in the asymptotic formula for $\mathcal{M}_2(h;0,0)$.

To sum up, the summands from the right-hand side of (1.7), arising after the application of the regularized Kuznetsov trace formula (2.24), do not contain a main term for $\mathcal{M}_2(h;0,0)$. Therefore, it is reasonable to ask if it is possible to use the standard version of the Kuznetsov trace formula instead of the regularized one.

For example, in the case of Maass form L-functions, the answer to this question is yes. However, for the standard version to work it is required to apply a trick and study, instead of the moments

$$M_{gen}(n) = \sum_j \alpha_j L^n(u_j, 1/2)h(t_j),$$

the following modification:

$$M_{sign}(n) = \sum_j \epsilon_j \alpha_j L^n(u_j, 1/2)h(t_j),$$

where $\epsilon_j = \pm 1$ is a sign of Maass form u_j. Since $L(u_j, 1/2) = 0$ for $\epsilon_j = -1$, these two moments coincide! It seems that this trick with ϵ_j was first applied by Motohashi [**35**] to study the second moment of Maass form L-functions. Later Kuznetsov ([**30**]) used the same technique to prove an asymptotic formula for the fourth moment of Maass form L-functions. However, this approach does not work for

$$\sum_j h(t_j)\alpha_j L^2(\operatorname{sym}^2 u_j, 1/2).$$

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