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DYADIC LOWER LITTLE BMO ESTIMATES

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Abstract: We characterize dyadic little BMO via the boundedness of the tensor commutator with a single well-chosen dyadic shift. It is shown that several proof strategies work for this problem, both in the unweighted case and with Bloom weights. Moreover, we address the flexibility of one of our methods.

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Notation

$\mathbf{1}_E$	characteristic function of a set E;
$\mathrm{d}x$	integration with respect to Lebesgue measure;
$\lvert E\rvert$	d-dimensional Lebesgue measure of a measurable set $E \subseteq \mathbb{R}^d$;
$\langle f\rangle_E$	average with respect to Lebesgue measure, $\langle f\rangle_E := \frac{1}{\lvert E\rvert}\int_E f(x)\mathrm{d}x$;
$L^p(w)$	weighted Lebesgue space, $\lVert f\rVert^p_{L^p(w)} := \int_{\mathbb{R}^d} \lvert f(x)\rvert^p w(x)\mathrm{d}x$;
(f,g)	usual L^2-pairing, $(f,g) := \int f(x)\overline{g(x)}\mathrm{d}x$;
$w(E)$	Lebesgue integral of a weight w over a set E, $w(E) := \int_E w(x)\mathrm{d}x$;
p'	Hölder conjugate exponent to p, $1/p + 1/p' = 1$;
$\mathcal{D}$	family of all dyadic intervals in $\mathbb{R}$;
$\mathcal{D}(E)$	family of all dyadic intervals $I \in \mathcal{D}$ contained in a subset E of $\mathbb{R}$;
$\boldsymbol{\mathcal{D}}$	family of all dyadic rectangles in $\mathbb{R}^2$;
$\boldsymbol{\mathcal{D}}(E)$	family of all dyadic rectangles $R \in \boldsymbol{\mathcal{D}}$ contained in a subset E of $\mathbb{R}^2$;
I_-, I_+	left, right respectively half of an interval $I \in \mathcal{D}$;
I'	sibling in $\mathcal{D}$ of an interval $I \in \mathcal{D}$;
$\hat{I}$	parent in $\mathcal{D}$ of an interval $I \in \mathcal{D}$;

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h_I — L^2-normalized (cancellative) Haar function for an interval $I \in \mathcal{D}$, $h_I := \frac{\mathbf{1}_{I_+} - \mathbf{1}_{I_-}}{\sqrt{|I|}}$;

f_I — Haar coefficient of a function $f \in L^1_{\text{loc}}(\mathbb{R})$, $f_I := (f, h_I)$, $I \in \mathcal{D}$;

h_R — L^2-normalized (bicancellative) Haar function for a rectangle $R \in \boldsymbol{\mathcal{D}}$, $h_R := h_I \otimes h_J$, where $R = I \times J$;

f_R — usual (biparameter) Haar coefficient of a function $f \in L^1_{\text{loc}}(\mathbb{R}^2)$, $f_R := (f, h_R)$, $R \in \boldsymbol{\mathcal{D}}$.

1. Introduction

Let us denote by $\{h_I : I \in \mathcal{D}\}$ the Haar base on $\mathbb{R}$ and let S be the operator densely defined by

$$h_{I_-} \mapsto -h_{I_+} \text{ and } h_{I_+} \mapsto h_{I_-}.$$

This shift is different from the classical one in [**15**]. It is time faithful and has other nice properties, in particular, it is an excellent model for the Hilbert transform.

We will work in the two-parameter space $L^2(\mathbb{R}^2)$ and we denote by S_i the shift operator acting in variable i. In this note, we mainly consider the commutator with the tensor product $S \otimes S = S_1 S_2$ as follows:

$$C_b = [S_1 S_2, b].$$

It is bounded in L^2 if and only if the symbol b is in dyadic little BMO. More precisely,

Theorem 1.1. *There holds*

$$\|b\|_{\text{bmo}} \lesssim \|C_b\|_{L^2 \to L^2} \lesssim \|b\|_{\text{bmo}}$$

with constants independent of the symbol. The lower norm estimate means that

$$\exists C > 0 : \forall b \in \text{bmo } \exists f \in L^2 : \|C_b f\|_{L^2} \geqslant C \|b\|_{\text{bmo}} \|f\|_{L^2},$$

while the upper estimate means, as usual, that

$$\exists C : \forall b \in \text{bmo } \forall f \in L^2 : \|C_b f\|_{L^2} \leqslant C \|b\|_{\text{bmo}} \|f\|_{L^2}.$$

The theorem holds also with exponents $1 < p < \infty$ and in the Bloom setting. Moreover, we show that these estimates hold for a certain class of dyadic shifts.

We will provide two proofs of the lower bounds for the commutator with $S_1 S_2$ and refer for example to [**13**] and [**8**] for the upper bound. One proof strategy uses an explicit calculation of the kernel and a modification of the argument of [**4**] and the other passes via a direct testing on a part of the symbol.

This result is not a surprise; it is well known for the Hilbert transform, where the most elegant argument that is known to us uses Toeplitz operators [**5**] and the elementary characterization of their boundedness to deduce the lower little BMO estimate from the one-parameter result, Nehari's theorem [**12**]. A real-analytic proof relying on the explicit BMO expression can be found in [**13**], extending a one-parameter real-variable argument brought forward by [**4**]. As it turns out and as is well known to experts, the argument by [**4**] relying on the kernel expression can give lower commutator estimates in various settings. It is somewhat surprising that it can be used for some dyadic operators as well, because the original argument relied heavily on the particular form of the kernels of Hilbert or Riesz transforms and an identity for spherical harmonics.

Of independent interest is a direct argument by testing on the symbol. It also extends trivially to the rectangular BMO case of the iterated commutator. Notice that for the rectangular BMO norm we do not have John–Nirenberg inequalities [**2**], therefore the exponent 2 in the definition is highly relevant. This case was treated by [**5**] for the Hilbert transform also via direct testing.

Holmes, Treil, and Volberg ([**9**]) have proved a similar result for a different dyadic shift. Blasco and Pott ([**2**]) have an interesting result in the product BMO setting, requiring boundedness of a commutator with a large class of dyadic multiplier operators.

Our proof strategies extend to the weighted Bloom case, in which we demonstrate lower estimates and commutator characterizations as well. An interesting result in the Bloom product BMO setting can be found in [**11**].

Our arguments, inspired by [**4**], are somewhat flexible and we give certain criteria under which we have a lower bound for a shift operator. These include the one considered in [**9**] and our S_1S_2 in this note, showing that both these groups have selected their operators wisely.

Definition 1.2. A function $b(x)$ is in the dyadic BMO space if

$$\|b\|_{1,\mathrm{BMO}} = \sup_I \frac{1}{|I|}\int_I |b(x) - \langle b\rangle_I| \mathrm{d}x$$

is finite. The supremum runs over dyadic intervals.

Since John–Nirenberg holds for this space, we may define the equivalent norm for $1 < p < \infty$

$$\|b\|_{p,\mathrm{BMO}}^p = \sup_I \frac{1}{|I|}\int_I |b(x) - \langle b\rangle_I|^p \mathrm{d}x.$$

Definition 1.3. A function $b(x_1, x_2)$ is in the dyadic little BMO space if

$$\|b\|_{1,\mathrm{bmo}} = \sup_R \frac{1}{|R|} \int_R |b(x_1, x_2) - \langle b \rangle_R| \mathrm{d}x_1 \mathrm{d}x_2$$

is finite. The supremum runs over all dyadic rectangles.

Since John–Nirenberg holds for this space, we may define the equivalent norm

$$\|b\|_{p,\mathrm{bmo}}^p = \sup_R \frac{1}{|R|} \int_R |b(x_1, x_2) - \langle b \rangle_R|^p \mathrm{d}x_1 \mathrm{d}x_2.$$

Definition 1.4. A function $b(x_1, x_2)$ is in the dyadic rectangular BMO space if

$$\|b\|_{\mathrm{BMO_{rec}}}^2 = \sup_R \frac{1}{|R|} \int_R |b(x_1, x_2) - \langle b \rangle_{R_2}(x_1) - \langle b \rangle_{R_1}(x_2) + \langle b \rangle_R|^2 \mathrm{d}x_1 \mathrm{d}x_2$$

is finite. The supremum runs over all dyadic rectangles.

Little BMO can also be realized as a function belonging uniformly to BMO in each variable separately. The rectangular BMO space can be realized via a convexity argument as the probabilistic BMO space, where admissible stopping times are restricted to be tensor products of one-parameter dyadic stopping times. See [**1**] for the definition of the probabilistic BMO space in two parameters.

2. Remarks on one parameter

2.1. Lower BMO estimate in one parameter via testing. In this section we prove the following lower estimate by testing the commutator on an appropriate test function:

Theorem 2.1. *There holds the one-parameter two-sided estimate*

$$\|b\|_{\mathrm{BMO}} \lesssim \|C_b\|_{L^2 \to L^2} \lesssim \|b\|_{\mathrm{BMO}}.$$

Proof: For any dyadic interval I with parent $\hat{I}$, we will provide a lower estimate for

$$\|S(b\mathbf{1}_I) - bS(\mathbf{1}_I)\|_{L^2(\hat{I})},$$

which is bounded above by the full L^2 norm of the commutator. It will be useful to know how S acts on characteristic functions. There holds

$$\begin{aligned} S(\mathbf{1}_I) &= S\left(\sum_L (\mathbf{1}_I, h_L) h_L\right) \\ &= S\left(\sum_{L: L \supsetneq I} (\mathbf{1}_I, h_L) h_L\right). \end{aligned}$$

A simple but important observation is that $S(\mathbf{1}_I)$ is supported outside of $\hat{I}$. Indeed, for intervals $L \supsetneq I$ observe that $S(h_L)$ is supported on the dyadic sibling L' of L and therefore has no support on I or I'. Let us now consider the local part $P_I b$ defined as

$$P_I b = \sum_{K \in \mathcal{D}(I)} b_K h_K.$$

We calculate

$$S(P_I b \cdot \mathbf{1}_I) = S\Big(\sum_{K \in \mathcal{D}(I)} b_K h_K \cdot \mathbf{1}_I\Big) = S\Big(\sum_{K \in \mathcal{D}(I)} b_K h_K\Big).$$

Then we calculate

$$P_I b \cdot S(\mathbf{1}_I) = \sum_{K \in \mathcal{D}(I)} b_K h_K \left[S\Big(\sum_{L : L \supsetneq I} (\mathbf{1}_I, h_L) h_L\Big)\right] = 0.$$

Indeed, $S(\mathbf{1}_I)$ has no support on I but $P_I b$ is only supported on I. Let us now consider the outer part

$$P_{I^c} b = \sum_{K \notin \mathcal{D}(I)} b_K h_K.$$

Observe that $P_{I^c} b$ is constant on I: if $K \cap I = \varnothing$, that constant is 0. If $I \subsetneq K$, then h_K is constant on I. Therefore $S(P_{I^c} b \cdot \mathbf{1}_I) = P_{I^c} b(I) S(\mathbf{1}_I)$ and this part does not have a contribution on $\hat{I}$. Likewise $P_{I^c} b S(\mathbf{1}_I)$ has no contribution on $\hat{I}$. Gathering the information, we have

$$\begin{aligned} \|[S,b]\mathbf{1}_I\|^2_{L^2(\hat{I})} &= \|[S, P_I b]\mathbf{1}_I + [S, P_{I^c} b]\mathbf{1}_I\|^2_{L^2(\hat{I})} \\ &= \left\| S\Big(\sum_{K \in \mathcal{D}(I)} b_K h_K\Big)\right\|^2_{L^2(\hat{I})} \\ &= \int_I |b - \langle b \rangle_I|^2. \end{aligned}$$

Here we have used the fact that the support of $S(P_I b)$ is contained in $\hat{I}$ and that S is an isometry in L^2. These considerations tell us that the lower BMO estimate is seen when testing on $\mathbf{1}_I$ and taking the supremum in I.

$$\|b\|_{\mathrm{BMO}} \leqslant \sup_I \left\| [S,b] \frac{\mathbf{1}_I}{|I|^{1/2}} \right\|_{L^2}.$$

The upper estimate is similar to the one for the classical Haar shift and is omitted. □

3. Two-parameter implications

3.1. Lower little BMO estimate via testing.

Theorem 3.1. *There holds the two-parameter two-sided estimate*

$$\|b\|_{\mathrm{bmo}} \lesssim \|C_b\|_{L^2\to L^2} \lesssim \|b\|_{\mathrm{bmo}}.$$

Proof: Let us write for $R \in \boldsymbol{\mathcal{D}}$, $R = R_1 \times R_2$, define the domain $\check{R} = (\hat{R}_1 \times \mathbb{R}) \cup (\mathbb{R} \times \hat{R}_2)$ and define $\mathcal{R}^c = \{K \in \boldsymbol{\mathcal{D}} : R_i \subsetneqq K_i \text{ for } i = 1, 2\}$. Then, let

$$P_{\mathcal{R}} b = \sum_{K\in\mathcal{R}} b_K h_K \text{ and } P_{\mathcal{R}^c} b = \sum_{K\in\mathcal{R}^c} b_K h_K.$$

Observe that the sum over $K \in \mathcal{R}$ can be split as

$$P_{\mathcal{R}} b = \sum_{K\in\boldsymbol{\mathcal{D}}} b_K h_K - \sum_{K_1\notin\mathcal{D}(R_1)} \sum_{K_2\notin\mathcal{D}(R_2)} b_{K_1\times K_2} h_{K_1\times K_2}.$$

The first term in the previous expression is just $b(x, y)$, while for $(x, y) \in R$, the second term is just $\langle b\rangle_R$ (it is enough to apply the one-parameter argument on each variable separately). In other words, we have that

$$P_{\mathcal{R}} b(x, y) = b(x, y) - \langle b\rangle_R \tag{3.1}$$

for all $(x, y) \in R$. We test on $\mathbf{1}_R$, split the commutator $C_b = C_{P_{\mathcal{R}} b} + C_{P_{\mathcal{R}^c} b}$, and integrate only in $\check{R}$. Observe that $(P_{\mathcal{R}} b + P_{\mathcal{R}^c} b)S_1S_2(\mathbf{1}_R)$ has no contribution on $\check{R}$. Likewise, observe that when $K \in \mathcal{R}^c$, then h_K is constant on R and $S_1S_2(h_K\mathbf{1}_R) = h_K(R)S_1S_2(\mathbf{1}_R)$ with no contribution on $\check{R}$. The remaining term is $S_1S_2(P_{\mathcal{R}} b\mathbf{1}_R)$. Now, we observe that the support of $S_1S_2(P_{\mathcal{R}} b\mathbf{1}_R)$ lies in $\check{R}$. To this end, we only need to check the action of the shifts on functions $h_K\mathbf{1}_R$ with $K \in \mathcal{R}$. Indeed, when $K \cap R = \emptyset$, then $h_K\mathbf{1}_R = 0$. The $K \in \mathcal{R}$ that remain have $K_i \subseteq R_i$ for either $i = 1$ or $i = 2$. Let us assume that $K_1 \subseteq R_1$, as the other case follows the same argument. It holds that $S_1(h_K\mathbf{1}_R)$ has support on $\hat{K}_1 \times \mathbb{R}$ and so $S_1S_2(h_K\mathbf{1}_R)$ has support on $\check{R}$. Finally, $P_{\mathcal{R}} b\mathbf{1}_R = \mathbf{1}_R(b - \langle b\rangle_R)$, which follows from (3.1).

Gathering the information, we have

$$\begin{aligned}\|[S_1S_2, b]\mathbf{1}_R\|^2_{L^2(\check{R})} &= \|S_1S_2(P_{\mathcal{R}} b\mathbf{1}_R)\|^2_{L^2(\check{R})}\\ &= \|S_1S_2(P_{\mathcal{R}} b\mathbf{1}_R)\|^2_{L^2}\\ &= \int_R |b(x_1, x_2) - \langle b\rangle_R|^2 \mathrm{d}x_1\mathrm{d}x_2. \qquad \square\end{aligned}$$

3.2. Lower rectangular BMO estimate for iterated commutator. In this section we prove a lower rectangular BMO estimate by testing the iterated commutator $C_b^{\mathrm{it}} = [S_1, [S_2, b]] = [S_2, [S_1, b]]$ on $\mathbf{1}_R$.

Theorem 3.2. *There holds*

$$\|b\|_{\mathrm{BMO}_{\mathrm{rec}}} \lesssim \|C_b^{\mathrm{it}}\|_{L^2 \to L^2}.$$

The proof is similar to the one-parameter case above. Let $R = R_1 \times R_2$ be a dyadic rectangle.

We calculate $C_b^{\mathrm{it}} \mathbf{1}_R$. For that, introduce $\varphi = [S_1, b](\mathbf{1}_{R_1})$ and observe that

$$[S_1, b](\mathbf{1}_R) = [S_1, b](\mathbf{1}_{R_1})\mathbf{1}_{R_2} = \varphi \mathbf{1}_{R_2},$$

and

$$[S_1, b](S_2(\mathbf{1}_R)) = [S_1, b](\mathbf{1}_{R_1}) S_2(\mathbf{1}_{R_2}) = \varphi S_2(\mathbf{1}_{R_2}).$$

It follows that

$$C_b^{\mathrm{it}} \mathbf{1}_R = S_2([S_1, b](\mathbf{1}_R)) - [S_1, b](S_2(\mathbf{1}_R)) = S_2(\varphi \mathbf{1}_{R_2}) - \varphi S_2(\mathbf{1}_{R_2}).$$

Now we integrate the commutator, using the facts learned in the one-parameter case.

$$\begin{aligned}
&\int_{\widehat{R_1}} \int_{\widehat{R_2}} |[S_2, \varphi]\mathbf{1}_{R_2}|^2 \mathrm{d}x_2 \mathrm{d}x_1 \\
&\quad = \int_{\widehat{R_1}} \int_{R_2} |\varphi - \langle \varphi \rangle_{R_2}|^2 \mathrm{d}x_2 \mathrm{d}x_1 \\
&\quad = \int_{\widehat{R_1}} \int_{R_2} |[S_1, b]\mathbf{1}_{R_1} - \langle [S_1, b]\mathbf{1}_{R_1} \rangle_{R_2}|^2 \mathrm{d}x_2 \mathrm{d}x_1 \\
&\quad = \int_{\widehat{R_1}} \int_{R_2} |[S_1, b - \langle b \rangle_{R_2}]\mathbf{1}_{R_1}|^2 \mathrm{d}x_2 \mathrm{d}x_1 \\
&\quad = \int_{R_2} \int_{R_1} |b - \langle b \rangle_{R_2} - \langle b - \langle b \rangle_{R_2} \rangle_{R_1}|^2 \mathrm{d}x_1 \mathrm{d}x_2 \\
&\quad = \int_{R_2} \int_{R_1} |b(x_1, x_2) - \langle b \rangle_{R_2}(x_1) - \langle b \rangle_{R_1}(x_2) + \langle b \rangle_R|^2 \mathrm{d}x_1 \mathrm{d}x_2.
\end{aligned}$$

Taking the supremum over R delivers the lower rectangular BMO estimate.

4. Lower estimates via the kernel

In this section we work in L^p. In the light of the upper estimates, this is not needed in the unweighted case, but will be of value in the weighted case later on.

4.1. Kernel representation. First, we establish the kernel of the operator S_1S_2. We recall that in one parameter

$$Sf := \sum_{I\in\mathcal{D}} (f_{I_+}h_{I_-} - f_{I_-}h_{I_+}).$$

Thus in two parameters (now writing $R = I \times J$ to avoid indices)

$$(S\otimes S)f = S_1S_2f = \sum_{I,J\in\mathcal{D}} \sum_{\varepsilon,\delta\in\{-,+\}} \varepsilon\delta f_{I_\varepsilon\times J_\delta} h_{I_{-\varepsilon}\times J_{-\delta}}.$$

The operator S_1S_2 has a formal kernel, namely,

$$S_1S_2f(x) = \int_{\mathbb{R}^2} \mathcal{K}(x,y)f(y)\mathrm{d}y,$$

where

$$\mathcal{K}(x,y) := \sum_{I,J\in\mathcal{D}} \sum_{\varepsilon,\delta\in\{-,+\}} \varepsilon\delta h_{I_\varepsilon\times J_\delta}(y) h_{I_{-\varepsilon}\times J_{-\delta}}(x), \tag{4.1}$$

for all $x = (x_1,x_2), y = (y_1,y_2) \in \mathbb{R}^2$ with $x_i \neq y_i$, $i = 1,2$, and $\mathcal{K}(x,y) := 0$ if $x_1 = y_1$ or $x_2 = y_2$. The kernel $\mathcal{K}$ given by formula (4.1) is well defined pointwise. In fact, for each $x,y \in \mathbb{R}^2$ with $x_i \neq y_i$, $i = 1,2$, there exists at most one dyadic rectangle $I \times J$ such that $\sum_{\varepsilon,\delta\in\{-,+\}} \varepsilon\delta h_{I_\varepsilon\times J_\delta}(y)h_{I_{-\varepsilon}\times J_{-\delta}}(x) \neq 0$, and then exactly one of the products $h_{I_\varepsilon\times J_\delta}(y)h_{I_{-\varepsilon}\times J_{-\delta}}(x), \varepsilon,\delta \in \{-,+\}$ is nonzero ($I \times J$ is then the minimal dyadic rectangle containing both x and y). In particular, for all $I,J \in \mathcal{D}$ and for all $x,y \in I\times J$ with $x_i \neq y_i$, $i = 1,2$, we have $\mathcal{K}(x,y) \neq 0$ and

$$\mathcal{K}(x,y) = \sum_{\varepsilon,\delta\in\{-,+\}} \sum_{\substack{K\in\mathcal{D}(I)\\ L\in\mathcal{D}(J)}} \varepsilon\delta h_{K_\varepsilon\times L_\delta}(y)h_{K_{-\varepsilon}\times L_{-\delta}}(x).$$

For all $x \in \mathbb{R}^2$, set

$$\begin{aligned} A_x &:= \{y \in \mathbb{R}^2 :\ \mathcal{K}(x,y) \neq 0\} \\ &= \{y \in \mathbb{R}^2 : x_i \neq y_i,\ i = 1,2,\ \exists I,J \in \mathcal{D} \text{ such that } x,y \in I\times J\}. \end{aligned}$$

Then the previous observations imply that for all $I,J \in \mathcal{D}$ and for *all* $x,y \in \mathbb{R}^2$ with $x_i \neq y_i$, $i = 1,2$, we have

$$\begin{aligned} &\mathbf{1}_{I\times J}(x)\cdot\mathbf{1}_{I\times J}(y)\cdot\frac{\mathbf{1}_{A_x}(y)}{\mathcal{K}(x,y)} \\ &\qquad = \sum_{\substack{K\in\mathcal{D}(I)\\ L\in\mathcal{D}(J)}} \sum_{\varepsilon,\delta\in\{-,+\}} \varepsilon\delta\frac{\mathbf{1}_{K_\varepsilon\times L_\delta}(y)}{h_{K_\varepsilon\times L_\delta}(y)}\cdot\frac{\mathbf{1}_{K_{-\varepsilon}\times L_{-\delta}}(x)}{h_{K_{-\varepsilon}\times L_{-\delta}}(x)}, \end{aligned} \tag{4.2}$$

where we adopt the convention $\frac{0}{0} = 0$.

4.2. A heuristic argument. We want to prove that

$$\left(\frac{1}{|I\times J|}\int_{I\times J}|b(x)-\langle b\rangle_{I\times J}|^p\mathrm{d}x\right)^{1/p} \lesssim \|[S_1S_2,b]\|_{L^p(\mathbb{R}^2)\to L^p(\mathbb{R}^2)},\quad \forall I,J\in\mathcal{D}. \tag{4.3}$$

Note that formally

$$[S_1S_2,b]f(x)=\int_{\mathbb{R}^2}(b(x)-b(y))\mathcal{K}(x,y)f(y)\mathrm{d}y.$$

Let $I,J\in\mathcal{D}$ be arbitrary. Fix $x\in I\times J$. Then

$$(I\times J)\setminus((\{x_1\}\times\mathbb{R})\cup(\mathbb{R}\times\{x_2\}))\subseteq A_x.$$

Therefore, we can formally write

$$\begin{aligned}|I\times J|\mathbf{1}_{I\times J}(x)(b(x)-\langle b\rangle_{I\times J}) &= \mathbf{1}_{I\times J}(x)\int_{I\times J}(b(x)-b(y))\mathrm{d}y\\ &=\mathbf{1}_{I\times J}(x)\int_{A_x}(b(x)-b(y))\mathbf{1}_{I\times J}(y)\mathrm{d}y\\ &=\mathbf{1}_{I\times J}(x)\int_{\mathbb{R}^2}(b(x)-b(y))\mathbf{1}_{I\times J}(y)\mathcal{K}(x,y)\frac{\mathbf{1}_{A_x}(y)}{\mathcal{K}(x,y)}\mathrm{d}y\\ &=[S_1S_2,b]\left(\mathbf{1}_{I\times J}(x)\cdot\mathbf{1}_{I\times J}\cdot\frac{\mathbf{1}_{A_x}}{\mathcal{K}(x,\cdot)}\right)(x)\\ &=\sum_{\substack{K\in\mathcal{D}(I)\\ L\in\mathcal{D}(J)}}\sum_{\varepsilon,\delta\in\{-,+\}}\varepsilon\delta[S_1S_2,b]\left(\frac{\mathbf{1}_{K_\varepsilon\times L_\delta}}{h_{K_\varepsilon\times L_\delta}}\right)(x)\cdot\frac{\mathbf{1}_{K_{-\varepsilon}\times L_{-\delta}}(x)}{h_{K_{-\varepsilon}\times L_{-\delta}}(x)}.\end{aligned}$$

Thus, taking the $L^p_x(\mathbb{R}^2)$ norms of both sides, writing K' for the sibling of a dyadic interval K, and using the triangle inequality we deduce

$$\begin{aligned}&\||I\times J|\mathbf{1}_{I\times J}(x)(b(x)-\langle b\rangle_J)\|_{L^p_x(\mathbb{R})}\\ &\leq\|[S_1S_2,b]\|_{L^p(\mathbb{R}^2)\to L^p(\mathbb{R}^2)}\sum_{\substack{K\in\mathcal{D}(I)\setminus\{I\}\\ L\in\mathcal{D}(J)\setminus\{J\}}}\left\|\frac{\mathbf{1}_{K\times L}(x)}{h_{K\times L}(x)}\right\|_{L^\infty_x(\mathbb{R}^2)}\left\|\frac{\mathbf{1}_{K'\times L'}}{h_{K'\times L'}}\right\|_{L^p(\mathbb{R}^2)}.\end{aligned}$$

We have

$$\left\|\frac{\mathbf{1}_{K\times L}(x)}{h_{K\times L}(x)}\right\|_{L^\infty_x(\mathbb{R}^2)}=\sqrt{|K\times L|},$$

$$\left\|\frac{\mathbf{1}_{K'\times L'}}{h_{K'\times L'}}\right\|_{L^p(\mathbb{R}^2)}=|K'\times L'|^{\frac{1}{2}+\frac{1}{p}}=|K\times L|^{\frac{1}{2}+\frac{1}{p}}.$$

Thus

$$\sum_{\substack{K\in\mathcal{D}(I)\setminus\{I\}\\ L\in\mathcal{D}(J)\setminus\{J\}}}\left\|\frac{\mathbf{1}_{K\times L}(x)}{h_{K\times L}(x)}\right\|_{L^\infty_x(\mathbb{R}^2)}\left\|\frac{\mathbf{1}_{K_s\times L_s}(y)}{h_{K_s\times L_s}(y)}\right\|_{L^p_y(\mathbb{R}^2)}$$
$$=\sum_{\substack{K\in\mathcal{D}(I)\setminus\{I\}\\ L\in\mathcal{D}(J)\setminus\{J\}}}|K\times L|^{1+\frac{1}{p}}=\Bigg(\sum_{\substack{K\in\mathcal{D}(I)\\ K\neq I}}|K|^{1+\frac{1}{p}}\Bigg)\Bigg(\sum_{\substack{L\in\mathcal{D}(J)\\ L\neq J}}|L|^{1+\frac{1}{p}}\Bigg).$$

We have

$$\sum_{\substack{K\in\mathcal{D}(I)\\ K\neq I}}|K|^{1+\frac{1}{p}}=|I|^{1+\frac{1}{p}}\sum_{n=1}^{\infty}2^n\cdot 2^{-n(1+\frac{1}{p})}=c_p|I|^{1+\frac{1}{p}},$$

and similarly for J. Thus

$$|I\times J|\Bigg(\int_{I\times J}|b(x)-\langle b\rangle_J|^p\mathrm{d}x\Bigg)^{1/p}\le\|[S_1S_2,b]\|_{L^p(\mathbb{R}^2)\to L^p(\mathbb{R}^2)}c_p^2|I\times J|^{1+\frac{1}{p}},$$

which immediately implies (4.3) (with constant c_p^2).

4.3. A rigorous argument.

4.3.1. Truncated kernel. For all positive integers n, denote by $\mathcal{K}_n$ the truncated kernel given pointwise by

$$\mathcal{K}_n(x,y):=\sum_{\varepsilon,\delta\in\{-,+\}}\sum_{\substack{I,J\in\mathcal{D}\\ 2^{-n}\le|I|,|J|\le 2^n}}\varepsilon\delta h_{I_\varepsilon\times J_\delta}(y)h_{I_{-\varepsilon}\times J_{-\delta}}(x),\tag{4.4}$$

for all $x,y\in\mathbb{R}^2$ with $x_i\neq y_i$, $i=1,2$, and $\mathcal{K}_n(x,y):=0$ whenever $x_1=y_1$ or $x_2=y_2$. Note that for every fixed $x\in\mathbb{R}^2$, the sum in (4.4) is always finite. For all $x\in\mathbb{R}^2$, set

$$A_{n,x}:=\{y\in\mathbb{R}^2:\ \mathcal{K}_n(x,y)\neq 0\}.$$

Clearly $A_{n,x}\subseteq A_{n+1,x}$ and $\bigcup_{n=0}^\infty A_{n,x}=A_x$. Note that $\mathcal{K}(x,y)=\mathcal{K}_n(x,y)$, for all $y\in A_{n,x}$ and for all $x\in\mathbb{R}^2$, since for each fixed $x,y\in\mathbb{R}^2$ there exists at most one $I\times J$ having a nonzero contribution in the sum in (4.1). It follows by (4.2) that for all $I,J\in\mathcal{D}$ and for *all* $x,y\in\mathbb{R}^2$ with $x_i\neq y_i$, $i=1,2$, we have

$$\mathbf{1}_{I\times J}(x)\cdot\mathbf{1}_{I\times J}(y)\cdot\frac{\mathbf{1}_{A_{n,x}}(y)}{\mathcal{K}_n(x,y)}$$
$$=\sum_{\substack{K\in\mathcal{D}(I)\\ L\in\mathcal{D}(J)\\ 2^{-n}\le|K|,|L|\le 2^n}}\sum_{\varepsilon,\delta\in\{-,+\}}\varepsilon\delta\frac{\mathbf{1}_{K_\varepsilon\times L_\delta}(y)}{h_{K_\varepsilon\times L_\delta}(y)}\cdot\frac{\mathbf{1}_{K_{-\varepsilon}\times L_{-\delta}}(x)}{h_{K_{-\varepsilon}\times L_{-\delta}}(x)}.$$

4.3.2. Truncated operators. Fix a positive integer n. Clearly, the operator T_n given pointwise *everywhere* by

$$\begin{aligned} T_n f(x) &:= \int_{\mathbb{R}^2} \mathcal{K}_n(x,y) f(y) \mathrm{d}y \\ &= \sum_{\varepsilon,\delta\in\{-,+\}} \sum_{\substack{I,J\in\mathcal{D} \\ 2^{-n}\leq |I|,|J|\leq 2^n}} \varepsilon\delta f_{I_\varepsilon\times J_\delta} h_{I_{-\varepsilon}\times J_{-\delta}}(x), \end{aligned}$$

for $x\in\mathbb{R}^2$ and $f\in L^1_{\mathrm{loc}}(\mathbb{R}^2)$, is well defined. In fact, for every fixed $x\in\mathbb{R}^2$, the sum in the last line above is finite.

4.3.3. Computations. Now let $I,J\in\mathcal{D}$ be arbitrary. Notice that for all $x\in I\times J$, we have $(I\times J)\setminus((\{x_1\}\times\mathbb{R})\cup(\mathbb{R}\times\{x_2\}))\subseteq A_x$. Therefore, for *all* $x\in\mathbb{R}^2$ we have

$$\begin{aligned} &|I\times J|\mathbf{1}_{I\times J}(x)(b(x)-\langle b\rangle_{I\times J}) = \mathbf{1}_{I\times J}(x)\int_{I\times J}(b(x)-b(y))\mathrm{d}y \\ &= \mathbf{1}_{I\times J}(x)\int_{A_x}(b(x)-b(y))\mathbf{1}_{I\times J}(y)\mathrm{d}y \\ &= \lim_{n\to\infty}\mathbf{1}_{I\times J}(x)\int_{A_{n,x}}(b(x)-b(y))\mathbf{1}_{I\times J}(y)\mathrm{d}y \\ &= \lim_{n\to\infty}\mathbf{1}_{I\times J}(x)\int_{\mathbb{R}^2}(b(x)-b(y))\mathbf{1}_{I\times J}(y)\mathcal{K}_n(x,y)\frac{\mathbf{1}_{A_{n,x}}(y)}{\mathcal{K}_n(x,y)}\mathrm{d}y \\ &= \lim_{n\to\infty}[T_n,b]\left(\mathbf{1}_{I\times J}(x)\cdot\mathbf{1}_{I\times J}\cdot\frac{\mathbf{1}_{A_{n,x}}}{\mathcal{K}_n(x,\cdot)}\right)(x) \\ &= \lim_{n\to\infty}\left[\sum_{\substack{K\in\mathcal{D}(I)\\ L\in\mathcal{D}(J)\\ 2^{-n}\leq|K|,|L|\leq 2^n}}\sum_{\varepsilon,\delta\in\{-,+\}}\varepsilon\delta[T_n,b]\left(\frac{\mathbf{1}_{K_\varepsilon\times L_\delta}}{h_{K_\varepsilon\times L_\delta}}\right)(x)\cdot\frac{\mathbf{1}_{K_{-\varepsilon}\times L_{-\delta}}(x)}{h_{K_{-\varepsilon}\times L_{-\delta}}(x)}\right]. \end{aligned}$$

Note that all sums in the last line are finite, so we have no convergence issues. Notice now that for all $K,L\in\mathcal{D}$ with $2^{-n}\leq|K|,|L|\leq 2^n$ and for any $\varepsilon,\delta\in\{+,-\}$ we have

$$\begin{aligned} \mathbf{1}_{K_{-\varepsilon}\times L_{-\delta}}T_n(f\mathbf{1}_{K_\varepsilon\times L_\delta}) &= \varepsilon\delta f_{K_\varepsilon\times L_\delta}h_{K_{-\varepsilon}\times L_{-\delta}} \\ &= \mathbf{1}_{K_{-\varepsilon}\times L_{-\delta}}S_1S_2(f\mathbf{1}_{K_\varepsilon\times L_\delta}) \quad \text{a.e.,} \end{aligned}$$

which clearly implies

$$\mathbf{1}_{K_{-\varepsilon}\times L_{-\delta}}[T_n,b](f\mathbf{1}_{K_\varepsilon\times L_\delta}) = \mathbf{1}_{K_{-\varepsilon}\times L_{-\delta}}[S_1S_2,b](f\mathbf{1}_{K_\varepsilon\times L_\delta}) \quad \text{a.e.}$$

Since $\mathcal{D}$ is a *countable* set, it follows that for all positive integers n there exists a Borel set E_n in $\mathbb{R}^2$ of zero measure, such that for all $K, L \in \mathcal{D}$, for any $\varepsilon, \delta \in \{+,-\}$, and for any $x \in \mathbb{R}^2 \setminus E_n$, there holds

$$\frac{\mathbf{1}_{K_{-\varepsilon}\times L_{-\delta}}(x)}{h_{K_{-\varepsilon}\times L_{-\delta}}(x)}[T_n, b]\left(\frac{\mathbf{1}_{K_\varepsilon\times L_\delta}}{h_{K_\varepsilon\times L_\delta}}\right)(x) = \frac{\mathbf{1}_{K_{-\varepsilon}\times L_{-\delta}}(x)}{h_{K_{-\varepsilon}\times L_{-\delta}}(x)}[S_1S_2, b]\left(\frac{\mathbf{1}_{K_\varepsilon\times L_\delta}}{h_{K_\varepsilon\times L_\delta}}\right)(x).$$

Set $E := \bigcup_{n=1}^{\infty} E_n$. Then, E is a set of zero measure in $\mathbb{R}$, and we have

$$|I\times J|\mathbf{1}_{I\times J}(x)(b(x) - \langle b\rangle_{I\times J}) = \lim_{n\to\infty}\left[\sum_{\substack{K\in\mathcal{D}(I)\\ L\in\mathcal{D}(J)\\ 2^{-n}\le|K|,|L|\le 2^n}}\ \sum_{\varepsilon,\delta\in\{-,+\}}\varepsilon\delta[S_1S_2, b]\left(\frac{\mathbf{1}_{K_\varepsilon\times L_\delta}}{h_{K_\varepsilon\times L_\delta}}\right)(x)\cdot\frac{\mathbf{1}_{K_{-\varepsilon}\times L_{-\delta}}(x)}{h_{K_{-\varepsilon}\times L_{-\delta}}(x)}\right],$$

for all $x \in \mathbb{R}^2 \setminus E$, and therefore for *almost every* $x \in \mathbb{R}^2$. Hence, taking the $L^p_x(\mathbb{R}^2)$ norm of both sides in the last display, using Fatou's lemma (to pass the limit outside the $L^p_x(\mathbb{R}^2)$ norm), and then working exactly as in the heuristic argument above, we again conclude (4.3) (with the same constant c_p^2 as in the heuristic argument).

5. Weighted estimates

We extend the unweighted bound to a weighted bound at little extra effort. Originally, this type of weighted inequality was introduced by [**3**] and has found new interest, modern proofs, and notable generalizations in [**6**], [**7**]. Meanwhile, a number of articles have been written on the subject of weighted commutators, but most concern upper estimates rather than characterizations.

Here and in what follows, recall that a weight w on $\mathbb{R}$ is a locally integrable function on $\mathbb{R}$ which is positive almost everywhere. We say that a weight w on $\mathbb{R}$ is an A_p weight, for $1 < p < \infty$, if

$$[w]_{A_p} := \sup_I\left(\frac{1}{|I|}\int_I w(x)\mathrm{d}x\right)\left(\frac{1}{|I|}\int_I (w(x))^{-1/(p-1)}\mathrm{d}x\right)^{p-1} < \infty, \tag{5.1}$$

where the supremum ranges over all finite intervals $I \subseteq \mathbb{R}$. Similarly, we say that a weight w is a dyadic A_p weight, for $1 < p < \infty$, if the supremum in condition (5.1) ranges only over dyadic intervals $I \in \mathcal{D}$. These definitions also extend to the biparameter case. Specifically, a weight w on $\mathbb{R}^2$ is a locally integrable function on $\mathbb{R}^2$ which is positive almost everywhere. Also, we say that a weight w on $\mathbb{R}^2$ is a biparameter A_p weight, for $1<p<\infty$, if

$$(5.2)\quad [w]_{A_p} := \sup_R \left(\frac{1}{|R|} \int_R w(x) \mathrm{d}x \right) \left(\frac{1}{|R|} \int_R (w(x))^{-1/(p-1)} \mathrm{d}x \right)^{p-1} < \infty,$$

where the supremum ranges over all finite rectangles $R \subseteq \mathbb{R}^2$ of positive measure with sides parallel to the axes. As before, we say that a weight w is a dyadic biparameter A_p weight, for $1<p<\infty$, if the supremum in condition (5.2) ranges only over dyadic rectangles R.

Theorem 5.1. *There holds for dyadic biparameter A_p weights μ, λ, $1<p<\infty$,*

$$\|b\|_{p,\mathrm{bmo}(\mu,\lambda)} \lesssim \|C_b\|_{L^p(\mu)\to L^p(\lambda)} \lesssim \|b\|_{p,\mathrm{bmo}(\mu,\lambda)}$$

with constants independent of the symbol.

Recall that the weighted little BMO space can be equivalently defined, as in [**8**], by the norm

$$\|b\|_{p,\mathrm{bmo}(\mu,\lambda)} = \sup_R \left(\frac{1}{\mu(R)} \int_R |b(x) - \langle b \rangle_R|^p \lambda(x) \mathrm{d}x \right)^{1/p}$$

for $1<p<\infty$.

See also [**8**] for the upper estimates.

5.1. Weighted estimate by testing. Most lower norm estimates in the Bloom setting have been obtained via a use of the argument in [**4**]; see for example [**6**], [**7**], and [**8**]. In [**11**] a different argument was needed to give lower norm estimates in a product setting.

In this section we demonstrate that the lower bounds can be obtained by testing. We work with the exponent $p=2$; the argument works for other p. We show the argument in one parameter and point out the modifications needed to pass to the little BMO case in L^p and the rectangular BMO case in L^2 for the iterated commutator.

Proof of Theorem 5.1: As before, for any dyadic interval I with parent $\hat{I}$, we will provide a lower estimate for

$$\|S(b\mathbf{1}_I) - bS(\mathbf{1}_I)\|_{L^2(\hat{I},\lambda)},$$

which is bounded above by $\|[S,b]\|_{L^2(\mu)\to L^2(\lambda)} \|\mathbf{1}_I\|_{L^2(\mu)}$. We have

$$\begin{aligned}
\|[S,b]\mathbf{1}_I\|^2_{L^2(\hat{I},\lambda)} &= \|[S,P_I b]\mathbf{1}_I + [S,P_{I^c} b]\mathbf{1}_I\|^2_{L^2(\hat{I},\lambda)} \\
&= \left\| S\left(\sum_{K\in\mathcal{D}(I)} b_K h_K \right) \right\|^2_{L^2(\hat{I},\lambda)}.
\end{aligned}$$

Using the A_2 characteristic of λ and the fact that $S\colon L^2(\lambda) \to L^2(\lambda)$ is bounded, we obtain also a lower bound $\|f\|_{L^2(\lambda)} \lesssim \|Sf\|_{L^2(\lambda)}$ by using the fact that $S^2 = -\operatorname{Id}$. Therefore

$$\Big\| S\Big(\sum_{K\in\mathcal{D}(I)} b_K h_K \Big) \Big\|^2_{L^2(\hat{I},\lambda)} \gtrsim \Big\| \sum_{K\in\mathcal{D}(I)} b_K h_K \Big\|^2_{L^2(\hat{I},\lambda)}$$
$$\gtrsim \int_I |b(x) - \langle b\rangle_I|^2 \lambda(x)\mathrm{d}x.$$

Note that

$$\int_I |b(x) - \langle b\rangle_I|^2 \lambda(x)\mathrm{d}x = \frac{1}{\mu(I)} \int_I |b(x) - \langle b\rangle_I|^2 \lambda(x)\mathrm{d}x \cdot \mu(I),$$

so these considerations tell us that the lower BMO estimate is seen when testing on $\mathbf{1}_I$ and taking the supremum in I. □

The same considerations hold true for the tensor commutator $\mathcal{C}_b$ using the pointwise identities in Subsection 3.1.

Notice also that the same reasoning, following the above calculation in Subsection 3.2, provides a weighted lower bound for the iterated commutator in terms of rectangular Bloom BMO defined by

$$\|b\|^2_{\mathrm{BMO}(\mu,\lambda)_{\mathrm{rect}}} = \sup_R \frac{1}{\mu(R)} \int_R \Big| \sum_{K\in\boldsymbol{\mathcal{D}}(R)} b_K h_K \Big|^2 \lambda,$$

where the supremum runs over dyadic rectangles R.

5.2. Weighted estimate in L^p using the kernel.

Theorem 5.2. *There holds for dyadic biparameter A_p weights μ, λ, $1 < p < \infty$,*

$$\|b\|_{p,\mathrm{bmo}(\mu,\lambda)} \lesssim \|C_b\|_{L^p(\mu)\to L^p(\lambda)} \lesssim \|b\|_{p,\mathrm{bmo}(\mu,\lambda)}$$

with constants independent of the symbol.

5.2.1. A heuristic argument. Let $1 < p < \infty$, and let μ, λ be any dyadic biparameter A_p weights on $\mathbb{R}^2$. We want to prove that

$$\left(\frac{1}{\mu(I\times J)} \int_{I\times J} |b(x) - \langle b\rangle_{I\times J}|^p \lambda(x)\mathrm{d}x \right)^{1/p} \lesssim \|[S_1S_2, b]\|_{L^p(\mu)\to L^p(\lambda)}, \tag{5.3}$$

for all $I, J \in \mathcal{D}$.

Let $I, J \in \mathcal{D}$ be arbitrary. Identically to the unweighted case we formally have

$$|I \times J| \mathbf{1}_{I\times J}(x)(b(x) - \langle b \rangle_{I\times J}) = \sum_{\substack{K\in\mathcal{D}(I)\\ L\in\mathcal{D}(J)}} \sum_{\varepsilon,\delta\in\{-,+\}} \varepsilon\delta[S_1S_2, b]\left(\frac{\mathbf{1}_{K_\varepsilon\times L_\delta}}{h_{K_\varepsilon\times L_\delta}}\right)(x) \cdot \frac{\mathbf{1}_{K_{-\varepsilon}\times L_{-\delta}}(x)}{h_{K_{-\varepsilon}\times L_{-\delta}}(x)}.$$

Thus, taking the $L^p_x(\lambda)$ norms of both sides and then using the triangle inequality we deduce

$$\| |I \times J| \mathbf{1}_{I\times J}(x)(b(x) - \langle b \rangle_{I\times J}) \|_{L^p_x(\lambda)} \leq \|[S_1S_2, b]\|_{L^p(\mu)\to L^p(\lambda)} \sum_{\substack{K\in\mathcal{D}(I)\setminus\{I\}\\ L\in\mathcal{D}(J)\setminus\{J\}}} \left\| \frac{\mathbf{1}_{K\times L}(x)}{h_{K\times L}(x)} \right\|_{L^\infty_x(\mathbb{R}^2)} \left\| \frac{\mathbf{1}_{K'\times L'}}{h_{K'\times L'}} \right\|_{L^p(\mu)},$$

where we recall that K' denotes the sibling of a dyadic interval K. We have

$$\left\| \frac{\mathbf{1}_{K\times L}(x)}{h_{K\times L}(x)} \right\|_{L^\infty_x(\mathbb{R}^2)} = \sqrt{|K \times L|},$$

$$\left\| \frac{\mathbf{1}_{K'\times L'}}{h_{K'\times L'}} \right\|_{L^p(\mu)} = |K \times L|^{\frac{1}{2}} (\mu(K' \times L'))^{\frac{1}{p}}.$$

Thus

$$\sum_{\substack{K\in\mathcal{D}(I)\setminus\{I\}\\ L\in\mathcal{D}(J)\setminus\{J\}}} \left\| \frac{\mathbf{1}_{K\times L}(x)}{h_{K\times L}(x)} \right\|_{L^\infty_x(\mathbb{R}^2)} \left\| \frac{\mathbf{1}_{K'\times L'}}{h_{K'\times L'}} \right\|_{L^p(\mu)} = \sum_{\substack{K\in\mathcal{D}(I)\setminus\{I\}\\ L\in\mathcal{D}(J)\setminus\{J\}}} |K \times L| (\mu(K' \times L'))^{\frac{1}{p}}.$$

We have

$$\begin{aligned}\sum_{\substack{K\in\mathcal{D}(I)\setminus\{I\}\\ L\in\mathcal{D}(J)\setminus\{J\}}} &|K\times L|(\mu(K'\times L'))^{\frac{1}{p}}\\ &=\sum_{n,m=1}^{\infty}\sum_{\substack{K\in\operatorname{ch}_n(I)\\ L\in\operatorname{ch}_m(J)}} 2^{-n-m}|I\times J|(\mu(K'\times L'))^{1/p}\\ &\leq \sum_{n,m=1}^{\infty}2^{-n-m}|I\times J|\Bigg(\sum_{\substack{K\in\operatorname{ch}_n(I)\\ L\in\operatorname{ch}_m(J)}}1\Bigg)^{1/p'}\Bigg(\sum_{\substack{K\in\operatorname{ch}_n(I)\\ L\in\operatorname{ch}_m(J)}}\mu(K'\times L')\Bigg)^{1/p}\\ &=c_p^2|I\times J|(\mu(I\times J))^{1/p}.\end{aligned}$$

Thus

$$\begin{aligned}|I\times J|\bigg(\int_{I\times J}|b(x)-\langle b\rangle_{I\times J}|^p\lambda(x)\mathrm{d}x\bigg)^{1/p}&\\ \leq \|[S_1S_2,b]\|_{L^p(\mu)\to L^p(\lambda)}c_p^2&|I\times J|(\mu(I\times J))^{1/p},\end{aligned}$$

which immediately implies (5.3) (with constant c_p^2).

5.2.2. A rigorous argument. Let $I,J\in\mathcal{D}$ be arbitrary. Identically to the unweighted case we have for *almost every* $x\in\mathbb{R}^2$

$$\begin{aligned}&|I\times J|\mathbf{1}_{I\times J}(x)(b(x)-\langle b\rangle_{I\times J})\\ &=\lim_{n\to\infty}\Bigg[\sum_{\substack{K\in\mathcal{D}(I)\\ L\in\mathcal{D}(J)\\ 2^{-n}\leq|K|,|L|\leq 2^n}}\ \sum_{\varepsilon,\delta\in\{-,+\}}\varepsilon\delta[S_1S_2,b]\bigg(\frac{\mathbf{1}_{K_\varepsilon\times L_\delta}}{h_{K_\varepsilon\times L_\delta}}\bigg)(x)\cdot\frac{\mathbf{1}_{K_{-\varepsilon}\times L_{-\delta}}(x)}{h_{K_{-\varepsilon}\times L_{-\delta}}(x)}\Bigg].\end{aligned}$$

Therefore, taking the $L^p_x(\lambda)$ norm of both sides, using Fatou's lemma (to pass the limit outside the $L^p_x(\lambda)$ norm), and then working exactly as in the heuristic weighted argument, we again conclude (5.3) (with the same constant c_p^2 as in the heuristic argument).

6. Lower estimates via the kernel for more general shifts

In this section we show that the variant of the classical argument due to Coifman–Rochberg–Weiss [**4**] that was used above to obtain lower bounds for commutators involving the shift S can also handle slightly more general shifts, including the one considered in [**9**].

6.1. Setup. Let i, j be nonnegative integers. Let T be a Haar shift on the real line of complexity (i,j), i.e. the action of T on (suitable) functions f on the real line is given by

$$Tf = \sum_{I\in\mathcal{D}} \sum_{\substack{K\in\mathrm{ch}_i(I)\\ L\in\mathrm{ch}_j(I)}} c^I_{KL} f_K h_L,$$

where the *shift coefficients* c^I_{KL} are complex numbers satisfying the bound

$$|c^I_{KL}| \leq 2^{-(i+j)/2},$$

for all $K\in\mathrm{ch}_i(I)$, $L\in\mathrm{ch}_j(I)$, and for all $I\in\mathcal{D}$. This definition of Haar shifts follows [**14**, p. 34]. The operator T has a formal kernel. Specifically, one can write

$$Tf(x) = \int_{\mathbb{R}} \mathcal{K}(x,y)f(y)\mathrm{d}y,$$

where

$$\mathcal{K}(x,y) := \sum_{I\in\mathcal{D}} \sum_{\substack{K\in\mathrm{ch}_i(I)\\ L\in\mathrm{ch}_j(I)}} c^I_{KL} h_K(y)h_L(x), \tag{6.1}$$

for all $x,y\in\mathbb{R}$ with $x\neq y$, and $\mathcal{K}(x,x):=0$. While the kernel representation for the shift T is formal, the kernel is well defined pointwise. Indeed, let $x,y\in\mathbb{R}$ with $x\neq y$ be arbitrary. If there is no dyadic interval containing both x and y, then $\mathcal{K}(x,y)=0$. Now assume that there is a dyadic interval containing both x and y, and let J be the minimal such dyadic interval. Let $(J_n)_{n=1}^{\infty}$ be the strictly increasing sequence of all dyadic ancestors of J, so that

$$J =: J_0 \subsetneq J_1 \subsetneq J_2 \subsetneq J_3 \subsetneq \cdots$$

Then, we have

$$\sum_{I\in\mathcal{D}} \sum_{\substack{K\in\mathrm{ch}_i(I)\\ L\in\mathrm{ch}_j(I)}} |c^I_{KL} h_K(y)h_L(x)| = \sum_{n=0}^{\infty} \sum_{\substack{K\in\mathrm{ch}_i(J_n)\\ L\in\mathrm{ch}_j(J_n)}} |c^{J_n}_{KL} h_K(y)h_L(x)|$$

$$\leq 2^{-(i+j)/2}\cdot 2^{(i+j)/2}\cdot \sum_{n=0}^{\infty}\frac{1}{|J_n|} = \frac{2}{|J|}.$$

Since the Haar function over an interval is constant in the dyadic children of that interval, the above computation shows that we can write

$$\mathcal{K}(x,y) = \sum_{I\in\mathcal{D}} \sum_{\substack{K\in \mathrm{ch}_{i+1}(I)\\ L\in \mathrm{ch}_{j+1}(I)}} a^I_{KL} \mathbf{1}_K(y)\mathbf{1}_L(x), \quad \forall x,y\in\mathbb{R},\ x\neq y, \tag{6.2}$$

where the *coefficients* a^I_{KL} satisfy

$$|a^I_{KL}| \leq \frac{2}{|I|},$$

and also $a^I_{KL} = 0$ whenever there is $J \in \mathrm{ch}(I)$ with $K \cup L \subseteq J$. Note that the only dyadic interval in the first sum for which we have nonzero terms is the minimal dyadic interval containing both x and y (if there is one), so that for each fixed $x, y \in \mathbb{R}$ with $x \neq y$, there is at most one nonzero term in the sum in (6.2). Specifically, if I is the minimal dyadic interval containing both x and y, K is the unique interval in $\mathrm{ch}_{i+1}(I)$ containing y, and L is the unique interval in $\mathrm{ch}_{j+1}(I)$ containing x, then $\mathcal{K}(x,y) = a^I_{KL}$.

6.2. A nondegeneracy condition. Now we assume the following nondegeneracy condition: there exists some constant $c > 0$ (depending only on i, j), such that whenever there is no $J \in \mathrm{ch}(I)$ such that $K \cup L \subseteq J$, then there holds

$$|a^I_{KL}| \geq \frac{1}{c|I|}. \tag{6.3}$$

Then, in particular, we deduce that if $x \neq y$ and there exists some dyadic interval containing both x and y, then $\mathcal{K}(x,y) \neq 0$, in fact,

$$\frac{1}{c|I|} \leq |\mathcal{K}(x,y)| \leq \frac{2}{|I|},$$

where I is the minimal dyadic interval containing both x and y. In Subsection 6.5 below we give representative examples of classes of Haar shifts for which condition (6.3) is satisfied.

6.3. The inverse kernel. For all $x \in \mathbb{R}$, set

$$A_x := \{y \in \mathbb{R}\setminus\{x\} : \mathcal{K}(x,y) \neq 0\}.$$

Note that for all $x \in \mathbb{R}$ and for all $J \in \mathcal{D}$ we have $J \setminus \{x\} \subseteq A_x$. Then, we can write

$$\frac{\mathbf{1}_{A_x}(y)}{\mathcal{K}(x,y)} = \sum_{I\in\mathcal{D}} \sum_{\substack{K\in \mathrm{ch}_{i+1}(I)\\ L\in \mathrm{ch}_{j+1}(I)}} b^I_{KL} \mathbf{1}_K(y)\mathbf{1}_L(x), \quad \forall x,y\in\mathbb{R},\ x\neq y,$$

where

$$|b^I_{KL}| \leq c|I|,$$

and also $b^I_{KL} = 0$ whenever there is $J \in \mathrm{ch}(I)$ with $K \cup L \subseteq J$. Moreover, as before, this means that the only relevant dyadic interval in the first sum is the minimal dyadic interval I containing both x and y (if there is one). In particular, for all $J \in \mathcal{D}$, we have the localized version

$$\frac{\mathbf{1}_J(x) \cdot \mathbf{1}_J(y) \cdot \mathbf{1}_{A_x}(y)}{\mathcal{K}(x,y)} = \sum_{I \in \mathcal{D}(J)} \sum_{\substack{K \in \mathrm{ch}_{i+1}(I) \\ L \in \mathrm{ch}_{j+1}(I)}} b^I_{KL} \mathbf{1}_K(y) \mathbf{1}_L(x), \qquad \forall x, y \in \mathbb{R},\ x \neq y.$$

Note that we have adopted everywhere the convention $\frac{0}{0} = 0$.

6.4. Lower BMO bounds. Let $1 < p < \infty$, and let μ, λ be dyadic A_p weights on $\mathbb{R}$. Let $J \in \mathcal{D}$. We will show that

$$\left(\frac{1}{\mu(J)} \int_J |b(x) - \langle b \rangle_I|^p \lambda(x) \mathrm{d}x \right)^{1/p} \leq c c_p 2^{\frac{i+1}{p'}} \|[T, b]\|_{L^p(\mu) \to L^p(\lambda)}.$$

Let us first give a heuristic argument. We have

$$\begin{aligned}
|J| \mathbf{1}_J(x)(b(x) - \langle b \rangle_J) &= \mathbf{1}_J(x) \int_J (b(x) - b(y)) \mathrm{d}y \\
&= \mathbf{1}_J(x) \int_{A_x} (b(x) - b(y)) \mathbf{1}_J(y) \mathrm{d}y \\
&= \mathbf{1}_J(x) \int_{\mathbb{R}} (b(x) - b(y)) \mathbf{1}_J(y) \mathcal{K}(x,y) \frac{\mathbf{1}_{A_x}(y)}{\mathcal{K}(x,y)} \mathrm{d}y \\
&= [T, b] \left(\mathbf{1}_J(x) \cdot \mathbf{1}_J(y) \cdot \frac{\mathbf{1}_{A_x}(y)}{\mathcal{K}(x, \cdot)} \right)(x) \\
&= \sum_{I \in \mathcal{D}(J)} \sum_{\substack{K \in \mathrm{ch}_{i+1}(I) \\ L \in \mathrm{ch}_{j+1}(I)}} b^I_{KL} [T, b](\mathbf{1}_K)(x) \mathbf{1}_L(x).
\end{aligned}$$

Taking absolute values and using the triangle inequality as well as the nondegeneracy assumption for the coefficients b^I_{KL} we get

$$\begin{aligned}
|J| \mathbf{1}_J(x) |b(x) - \langle b \rangle_J| &\leq c \sum_{I \in \mathcal{D}(J)} |I| \sum_{\substack{K \in \mathrm{ch}_{i+1}(I) \\ L \in \mathrm{ch}_{j+1}(I)}} |[T, b](\mathbf{1}_K)(x)| \mathbf{1}_L(x) \\
&= c \sum_{I \in \mathcal{D}(J)} |I| \left(\sum_{K \in \mathrm{ch}_{i+1}(I)} |[T, b](\mathbf{1}_K)(x)| \right) \left(\sum_{L \in \mathrm{ch}_{j+1}(I)} \mathbf{1}_L(x) \right) \\
&= c \sum_{I \in \mathcal{D}(J)} |I| \sum_{K \in \mathrm{ch}_{i+1}(I)} |[T, b](\mathbf{1}_K)(x)| \mathbf{1}_I(x).
\end{aligned}$$

Thus, taking $L^p(\lambda)$ norms and using the triangle inequality (for $L^p(\lambda)$ norms), we get

$$|J|^p\left(\int_J |b(x)-\langle b\rangle_J|^p\lambda(x)\mathrm{d}x\right)^{1/p} \leq c\sum_{I\in\mathcal{D}(J)}|I|\sum_{K\in\mathrm{ch}_{i+1}(I)}\|[T,b](\mathbf{1}_K)\mathbf{1}_I\|_{L^p(\lambda)}$$
$$\leq c\sum_{I\in\mathcal{D}(J)}|I|\sum_{K\in\mathrm{ch}_{i+1}(I)}\|[T,b]\|_{L^p(\mu)\to L^p(\lambda)}(\mu(K))^{1/p}.$$

We have

$$\sum_{I\in\mathcal{D}(J)}|I|\sum_{K\in\mathrm{ch}_{i+1}(I)}(\mu(K))^{1/p}$$
$$\leq \sum_{I\in\mathcal{D}(J)}|I|\Big(\sum_{K\in\mathrm{ch}_{i+1}(I)}\mu(K)\Big)^{1/p}\Big(\sum_{K\in\mathrm{ch}_{i+1}(I)}1\Big)^{1/p'}$$
$$=2^{\frac{i+1}{p'}}\sum_{I\in\mathcal{D}(J)}|I|(\mu(I))^{1/p}=2^{\frac{i+1}{p'}}\sum_{n=0}^{\infty}2^{-n}|J|\sum_{I\in\mathrm{ch}_n(J)}(\mu(I))^{1/p}$$
$$\leq 2^{\frac{i+1}{p'}}\sum_{n=0}^{\infty}2^{-n}|J|\Big(\sum_{I\in\mathrm{ch}_n(J)}\mu(I)\Big)^{1/p}\Big(\sum_{I\in\mathrm{ch}_n(J)}1\Big)^{1/p'}$$
$$=2^{\frac{i+1}{p'}}|J|(\mu(J))^{1/p}\sum_{n=0}^{\infty}2^{-n/p}=c_p2^{\frac{i+1}{p'}}|J|(\mu(J))^{1/p}.$$

The claim follows.

To make the above argument formal, one has just to truncate the kernel, similarly to the case of the shift S treated above, with the only difference that in the present more general case one has to truncate (6.2) instead of (6.1).

A biparameter variant of the previous argument also gives lower little BMO bounds for the commutator $[T\otimes T,b]$ (under the same nondegeneracy condition as in the one-parameter case), similarly to the case of the commutator $[S\otimes S,b]$ treated above. In fact, a biparameter variant of the above argument also yields lower little BMO bounds for commutators of the form $[\mathbb{S},b]$, where $\mathbb{S}$ is an arbitrary biparameter shift $\mathbb{S}$ (not necessarily of tensor type) satisfying a biparameter nondegeneracy condition analogous to (6.3).

6.5. Cases where the nondegeneracy condition (6.3) is satisfied. Here we give two representative special cases in which the nondegeneracy condition (6.3) is satisfied.

6.5.1. Purely mixing shifts of complexity (i, i) with mildly varying coefficients. Take the shift T to have complexity (i, i), so that it has the form

$$Tf = 2^{-i} \sum_{I \in \mathcal{D}} \sum_{\substack{K \in \mathrm{ch}_i(I) \\ L \in \mathrm{ch}_i(I)}} c^I_{KL} f_K h_L.$$

Moreover, assume that this shift is "purely mixing" in the sense that for any $I \in \mathcal{D}$ and any $K \in \mathrm{ch}_i(I)$ it holds that $c^I_{KK} = 0$, and that all other coefficients vary mildly in the sense that there exists $b \in [1, 2^i/(2^i - 1))$ such that

$$1 \leq |c^I_{KL}| \leq b$$

for all $I \in \mathcal{D}$ and all $K, L \in \mathrm{ch}_i(I)$ with $K \neq L$. In this case, the kernel for this shift is of the form

$$\mathcal{K}(x, y) = 2^{-i} \sum_{I \in \mathcal{D}} \sum_{\substack{K \in \mathrm{ch}_i(I) \\ L \in \mathrm{ch}_i(I)}} c^I_{KL} h_K(y) h_L(x),$$

with the previous conditions on the coefficients c^I_{KL} of Tf.

To check the nondegeneracy condition (6.3) for this shift, we estimate directly the coefficients a^I_{KL} of the kernel $\mathcal{K}$ in (6.2). Observe that for $I \in \mathcal{D}$ and $K, L \in \mathrm{ch}_{i+1}(I)$ such that $K \cup L$ is not contained in a dyadic child of I, and for $x \in L$ and $y \in K$, we have that $\mathcal{K}(x, y) = a^I_{KL}$. Thus, fix I, K, L dyadic intervals as before and $x \in L$ and $y \in K$, and take $(I_n)_{n=1}^{\infty}$ to be the increasing sequence of dyadic ancestors of $I_0 := I$. Since $c^J_{PP} = 0$ for any $J \in \mathcal{D}$ and $P \in \mathrm{ch}_i(J)$, we have that

$$\mathcal{K}(x, y) = 2^{-i} \sum_{n=0}^{i-1} \sum_{\substack{K \in \mathrm{ch}_i(I_n) \\ L \in \mathrm{ch}_i(I_n)}} c^{I_n}_{KL} h_K(y) h_L(x).$$

For this particular x and y we can choose a sequence of signs $(\varepsilon_n)_{n=0}^{i-1}$ in $\{-1, 1\}$ such that

$$\mathcal{K}(x, y) = 2^{-i} \sum_{n=0}^{i-1} \frac{\varepsilon_n c_n}{2^{-i}|I_n|} = \sum_{n=0}^{i-1} \frac{\varepsilon_n c_n}{2^n |I|},$$

where c_n are complex numbers such that

$$1 \leq |c_n| \leq b$$

for $n = 0, \ldots, i-1$. Observe that

$$\left| \sum_{n=1}^{i-1} \frac{\varepsilon_n c_n}{2^n |I|} \right| \leq \frac{b}{|I|} \sum_{n=1}^{i-1} 2^{-n} = \frac{(2^{i-1} - 1)b}{2^{i-1}|I|}$$

(here we take the sum to be just 0 if $i = 1$). Thus, by the triangle inequality, we get that

$$|\mathcal{K}(x,y)| = \left| \frac{\varepsilon_0 c_0}{|I|} + \sum_{n=1}^{i-1} \frac{\varepsilon_n c_n}{2^n |I|} \right| \geq \left(\frac{1}{|I|} - \frac{(2^{i-1}-1)b}{2^{i-1}|I|} \right) = \left(1 - \frac{(2^{i-1}-1)b}{2^{i-1}} \right) \frac{1}{|I|}$$

and the condition on b yields the nondegeneracy condition (6.3).

6.5.2. "Sliced" shifts with mildly varying coefficients. Let $\mathcal{D}_e$ be the family of all even dyadic intervals, i.e.

$$\mathcal{D}_e := \{I \in \mathcal{D} : \log_2(|I|) \text{ is even}\}.$$

Assume that the shift T is "sliced" and that its coefficients do not vary too much, in the sense that

$$Tf = 2^{-(i+j)/2} \sum_{I \in \mathcal{D}_e} \sum_{\substack{K \in \mathrm{ch}_i(I) \\ L \in \mathrm{ch}_j(I)}} c^I_{KL} f_K h_L,$$

where there exists $b \in [1,3)$ such that

$$1 \leq |c^I_{KL}| \leq b,$$

for all $K \in \mathrm{ch}_i(I)$, $L \in \mathrm{ch}_j(I)$, and for all $I \in \mathcal{D}_e$. In particular, $c^I_{KL} = 0$, for all $K \in \mathrm{ch}_i(I)$, $L \in \mathrm{ch}_j(I)$, and for all $I \in \mathcal{D} \setminus \mathcal{D}_e$. Note that

$$\mathcal{K}(x,y) = 2^{-(i+j)/2} \sum_{I \in \mathcal{D}_e} \sum_{\substack{K \in \mathrm{ch}_i(I) \\ L \in \mathrm{ch}_j(I)}} c^I_{KL} h_K(y) h_L(x).$$

We estimate directly the coefficients a^I_{KL} in (6.2).

Let $I \in \mathcal{D}$, and $K \in \mathrm{ch}_{i+1}(I)$, $L \in \mathrm{ch}_{j+1}(I)$ such that there is no dyadic child of I containing both K and L. Pick $x \in L$ and $y \in K$. Then, it is clear that $a^I_{KL} = \mathcal{K}(x,y)$.

Let us estimate $\mathcal{K}(x,y)$. Let $(I_n)_{n=1}^{\infty}$ be the strictly increasing sequence of all dyadic ancestors of $I_0 := I$. Then, it is clear that there exist a sequence $(\varepsilon_n)_{n=0}^{\infty}$ in $\{-1,0,1\}$ and a sequence $(c_n)_{n=0}^{\infty}$ of complex numbers such that

$$\mathcal{K}(x,y) = \sum_{n=0}^{\infty} \frac{\varepsilon_n c_n}{|I_n|},$$

and

$$1 \leq |c_n| \leq b, \quad \forall n = 0,1,2,\dots,$$

and $\varepsilon_n \in \{-1,1\}$ if $I_n \in \mathcal{D}_e$, while $\varepsilon_n = 0$ if $I_n \in \mathcal{D} \setminus \mathcal{D}_e$, for all $n = 0,1,2,\ldots$ We now distinguish two cases.

Case 1. Assume that $I \in \mathcal{D}_e$. Then

$$\mathcal{K}(x,y) = \frac{\varepsilon_0 c_0}{|I|} + \sum_{n=1}^{\infty} \frac{\varepsilon_{2n} c_{2n}}{|I_{2n}|}.$$

We notice that

$$\left|\sum_{n=1}^{\infty} \frac{\varepsilon_{2n} c_{2n}}{|I_{2n}|}\right| \leq \sum_{n=1}^{\infty} \frac{b}{|I_{2n}|} = \sum_{n=1}^{\infty} \frac{b}{2^{2n}|I|} = \frac{b}{3|I|},$$

therefore by the triangle inequality we deduce

$$|\mathcal{K}(x,y)| \geq \left(1 - \frac{b}{3}\right)\frac{1}{|I|}.$$

Case 2. Assume that $J \in \mathcal{D} \setminus \mathcal{D}_e$. Then

$$\mathcal{K}(x,y) = \frac{\varepsilon_1 c_1}{|I_1|} + \sum_{n=1}^{\infty} \frac{\varepsilon_{2n+1} c_{2n+1}}{|I_{2n+1}|}.$$

We notice that

$$\left|\sum_{n=1}^{\infty} \frac{\varepsilon_{2n+1} c_{2n+1}}{|I_{2n+1}|}\right| \leq \sum_{n=1}^{\infty} \frac{b}{|I_{2n+1}|} = \sum_{n=1}^{\infty} \frac{b}{2^{2n}|I_1|} = \frac{b}{3|I_1|},$$

therefore by the triangle inequality we deduce

$$|\mathcal{K}(x,y)| \geq \left(1 - \frac{b}{3}\right)\frac{1}{|I_1|} = \frac{1}{2}\left(1 - \frac{b}{3}\right)\frac{1}{|I|},$$

concluding the proof.

6.6. A question. The nondegeneracy condition (6.3) might be considered a bit too strong from a more general point of view, and especially from the point of view of Calderón–Zygmund operators. A far weaker and perhaps more natural nondegeneracy condition is the following. There exists some $c > 0$ (depending on i and j), such that for all $I \in \mathcal{D}$, for all $K \in \operatorname{ch}_{i+1}(I)$, there exists *some* $L \in \operatorname{ch}_{j+1}(I)$ (depending on K), such that

$$|a^I_{KL}| \geq \frac{1}{c|I|}.$$

This is equivalent to saying that for all $y \in \mathbb{R}$ and for all dyadic intervals I containing y, there exists $x \in I$, such that I is the minimal dyadic interval containing both x and y and $|\mathcal{K}(x,y)| \geq \frac{1}{c|I|}$. This is the direct dyadic analog of the nondegeneracy condition for Calderón–Zygmund operators considered by Hytönen in [**10**], where it is shown that it is sufficient for some lower BMO bounds for commutators with Calderón–Zygmund operators. Note that the proofs of lower BMO bounds in [**10**] depend heavily on variants of weak factorization. It is not immediately clear to us whether our methods can be adapted to cover shifts that satisfy only this much weaker nondegeneracy condition.

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