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NOTES ON COMPACTNESS IN L^p-SPACES ON LOCALLY COMPACT GROUPS

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Abstract: The main goal of the paper is to provide new insight into compactness in L^p-spaces on locally compact groups. The article begins with a brief historical overview and the current state of literature regarding the topic. Subsequently, we "take a step back" and investigate the Arzelà–Ascoli theorem on a non-compact domain together with one-point compactification. The main idea comes in Section 3, where we introduce the "L^p-properties" (L^p-boundedness, L^p-equicontinuity, and L^p-equivanishing) and study their "behaviour under convolution". The paper proceeds with an analysis of Young's convolution inequality, which plays a vital role in the final section. During the "grand finale", all the pieces of the puzzle are brought together as we lay down a new approach to compactness in L^p-spaces on locally compact groups.

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1. Introduction

Let us begin with a brief historical overview (see [**27**]): Andrey Kolmogorov was arguably the first person who succeeded in characterizing relatively compact families in $L^p(\mathbb{R}^n)$, where $1 < p < \infty$ and all functions are supported in a common bounded set (see [**34**]). A year later (in 1932) Jacob David Tamarkin got rid of the second restriction and in 1933 Marcel Riesz, a younger brother of Frigyes Riesz, proved the general case (see [**43, 50**]):

Theorem 1.1. *Let $1 \leqslant p < \infty$. A family $\mathcal{F} \subset L^p(\mathbb{R}^n)$ is relatively compact if and only if*

- *$\mathcal{F}$ is L^p-bounded, i.e., there exists $M > 0$ such that*

$$\forall_{f\in\mathcal{F}}\, \|f\|_p \leqslant M,$$

- *$\mathcal{F}$ is L^p-equicontinuous, i.e., for every $\varepsilon > 0$ there exists $\delta > 0$ such that*

$$\forall_{\substack{|y|<\delta \\ f\in\mathcal{F}}} \int_{\mathbb{R}^n} |f(x+y) - f(x)|^p \, dx \leqslant \varepsilon,$$

- *$\mathcal{F}$ is L^p-equivanishing, i.e., for every $\varepsilon > 0$ there exists $R > 0$ such that*
$$\forall_{f\in\mathcal{F}} \int_{|x|>R} |f(x)|^p \, dx \leqslant \varepsilon.$$

In 1940, André Weil published a book, in which he demonstrated that Theorem 1.1 holds true even if we replace $\mathbb{R}^n$ with a locally compact Hausdorff group G (see [**51**, pp. 53–54]):

Theorem 1.2. *Let G be a locally compact Hausdorff group. A family $\mathcal{F} \subset L^p(G)$ is relatively compact if and only if*

- *$\mathcal{F}$ is L^p-bounded,*
- *$\mathcal{F}$ is L^p-equicontinuous, i.e., for every $\varepsilon > 0$ there exists an open neighbourhood U_e of the neutral element e such that*
$$\forall_{\substack{y\in U_e\\ f\in\mathcal{F}}} \int_G |f(xy) - f(x)|^p \, dx \leqslant \varepsilon,$$
- *$\mathcal{F}$ is L^p-equivanishing, i.e., for every $\varepsilon > 0$ there exists $K \Subset G$ (which means that K is a compact subset of G) such that*
$$\forall_{f\in\mathcal{F}} \int_{G\setminus K} |f(x)|^p \, dx \leqslant \varepsilon.$$

Weil's book is oftentimes cited as a reference source when it comes to characterizing relatively compact families in L^p-spaces (see [**14, 20, 21, 23, 27**]). However, the argument of the French mathematician is (at least in the author's opinion) rather difficult to follow − the exposition is very terse, avoids technical details, and leaves much of the work to the reader. The fact that the book is written in French does not make matters easier (at least for the new generation of mathematicians, who treat English rather than French or German as the lingua franca of modern science). The need to overcome these inconveniences was the primary motivation behind this article.

One might get the misleading impression that the topic of compactness in L^p-spaces is outdated. This could not be further from the truth and we shall invoke a handful of contemporary papers to prove that the subject is still an active part of mathematical research. A brilliant place to start is the article published in 2010 by Harald Hanche-Olsen and Helge Holden (see [**27**]), which over the years has gained a reputation as a "cult classic" in the field. Eight years after their initial publication, the authors joined forces with Eugenia Malinnikova and published a sort of sequel to their previous paper (see [**28**]). In the meantime (in 2014 to be precise), Przemysław Górka and Anna Macios published an analysis of the Kolmogorov–Riesz theorem on metric spaces (see [**21**]). Górka continued the research on compactness and around 2016 collaborated with Humberto Rafeiro (see [**23**]). Three years later, Przemysław Górka and Paweł Pośpiech studied compactness in Banach function spaces (see [**22**]), which in turn inspired the article by Weichao Guo and Guoping Zhao (see [**24**]).

At this point one may still wonder what the "motive" is for studying compactness in Lebesgue spaces defined on locally compact groups. We can even ask more broadly: why bother with harmonic analysis beyond $\mathbb{R}$? Surprisingly, the literature is suspiciously silent on that matter. Even the big names in the field like Anton Deitmar, Siegfried Echterhoff, Edwin Hewitt, Eberhard Kaniuth, or Kenneth A. Ross simply *assume* in their monographs that generalizing the results from classical harmonic analysis is interesting and work onwards with that assumption unquestioned. This might feel like a "cop-out answer", so let us quote Gerald B. Folland's preface to his monograph [**15**]: "*(...) one may ask what is the excuse for a new book on the abstract theory at this time. Well, in the first place, I submit that the material presented here is beautiful. I fell in love with it as a student, and this book is the fulfillment of a long-held promise to myself to return to it.*" This is exactly how we feel about studying compactness in Lebesgue spaces on locally compact groups – it is a beautiful subject that occupied great minds in the past and the opportunity to add even a small brick to this mathematical theory is a privilege. Folland goes on to say: "*(...) the abstract theory is still an indispensable foundation for the study of concrete cases; it shows what the general picture should look like and provides a number of results that are useful again and again.*" We fully concur with this view – studying Lebesgue space on locally compact groups rather than on special cases like $\mathbb{R}$ or $\mathbb{Z}$ enables us to catch a glimpse of the big scheme of things and unravel the fundamental principles governing L^p-spaces. The special cases subsequently follow from these underlying laws of mathematics (see Corollaries 5.3, 5.4, 5.5, 5.6, and 5.7 at the end of the paper).

After this brief summary of the current state of the literature and the presentation of a "motive" for addressing the subject, let us outline the structure of the paper. Section 2 is devoted to the Arzelà–Ascoli theorem and we painstakingly document that the classical version of this theorem (for $C(X)$, X being a compact space) is very well established in the literature. However, the version for $C_0(X)$, X being a locally compact space, is almost non-existent in the literature (though it is considered well-known "folklore"). The point of the section is to employ the concept of one-point compactification in order to lay out a detailed proof of the

Arzelà–Ascoli theorem for $C_0(X)$, where X is a locally compact space. Thus, we close the gap in the literature.

Section 3 commences with a short summary of the Haar measure. Subsequently, we introduce three L^p-properties: L^p-boundedness, L^p-equicontinuity, and L^p-equivanishing. Theorems 3.1, 3.2, and 3.4 demonstrate how these L^p-properties are "passed down" from $\mathcal{F} \subset L^p(G)$, G being a locally compact group, to $\mathcal{F} \star \phi \subset C_0(G)$, where $\phi \in C_c(G)$.

Section 4 is divided into two parts: the first one pivots around proving that $\mathcal{F} \star \phi$ is "not far away" (for an appropriate choice of $\phi \in C_c(G)$) from $\mathcal{F}$ (see Theorem 4.2), while the subject of the second part is Young's convolution inequality. Our goal is to prove what is currently known solely in mathematical folklore and does not exist in the literature (as far as we are aware).

Finally, Section 5 brings together all the "pieces of the puzzle" from previous sections. In the climax of the paper (see Theorem 5.2) we provide an elegant and natural proof of the characterization of relatively compact families in $L^p(G)$. To the best of our knowledge, our approach is novel and thus sheds new light on the issues of compactness in L^p-spaces on locally compact groups.

2. Arzelà–Ascoli theorem and one-point compactification

The current section focuses on proving a version of the Arzelà–Ascoli theorem for $C_0(X)$, where X is a locally compact space. One might argue that in view of our final destination point a new version of the Arzelà–Ascoli theorem is redundant. This perspective is partially true and such a theorem is a case of "overkill" as we could (with a little effort) stick to the theory of compactness in $C_c(X)$ in the sequel. However, we consciously take this detour and shall briefly present the motivation for making such a decision.

To begin with, if $\mathcal{F} \subset L^p(G)$, then $\mathcal{F} \star \phi \subset C^b(G)$ for every $\phi \in C_c(G)$ (see the proof of Theorem 3.1 in the next section). This means that the convolution "embeds" L^p-families into a Banach space $C^b(G)$. Furthermore, it turns out that if $\mathcal{F}$ is L^p-equivanishing, then $\mathcal{F}\star\phi$ (where $\phi(e) \neq 0$) is a subset of a Banach space $C_0(G)$ (see Theorem 3.4). Thus, it is very natural to ask: what do relatively compact families in $C_0(G)$ look like?

Another (possibly stronger) reason for focusing on the Arzelà–Ascoli theorem for $C_0(X)$, where X is a locally compact space, is that the result is interesting in itself. As we explain in the course of the current section, a C_0-version of the Arzelà–Ascoli theorem is known in mathematical folklore but lacks a comprehensive exposition in the literature. Even worse – at some point we came across a version of the Arzelà–Ascoli theorem that is palpably wrong! This false claim in Kaniuth's monograph (see [**31**, Theorem A.1.4]) reads as follows:

Claim. Let X be a locally compact space and $\mathcal{F} \subset C_0(X)$. Suppose that $\mathcal{F}$ satisfies the following two conditions:

- $\mathcal{F}$ is pointwise bounded,
- $\mathcal{F}$ is equicontinuous at every point.

Then $\mathcal{F}$ is relatively compact in $(C_0(X), \|\cdot\|_\infty)$.

If this claim were true, then it would work in particular for $X = \mathbb{Z}$. We observe that for such a choice of X the equicontinuity condition becomes obsolete – every family $\mathcal{F} \subset C_0(\mathbb{Z})$ is equicontinuous. Thus it suffices to consider a sequence of characteristic functions $\mathcal{F} = (\mathbb{1}_{\{n\}})_{n\in\mathbb{N}}$, which is obviously pointwise bounded. However, this sequence does not contain any convergent subsequence, so $\mathcal{F}$ cannot be relatively compact (contrary to what the claim asserts)! This demonstrates that Kaniuth's claim cannot be true.

We feel that the above reasoning is a sufficient motivation for pursuing a C_0 version of the Arzelà–Ascoli theorem, even if this is a slight detour from our final objective.

The classical version of the Arzelà–Ascoli theorem characterizes relatively compact families in $C(X)$, the space of continuous functions on a compact space X (we assume that *every* topological space in this paper is Hausdorff). Such a version of the result appears in many sources throughout the literature (see for instance [**16**, Theorem 4.43], [**33**, Corollary 10.49], or [**45**, Theorem A5]) although some authors assume additionally that the space X is metric for "simplicity of the proof" (see for instance [**4**, Theorem 4.25], [**6**, Theorem 23.2], [**12**, Theorem 6.3.1], or [**44**, Theorem 11.28]):

Theorem 2.1 (Classical version of the Arzelà–Ascoli theorem)**.** *Let X be a compact space. The family $\mathcal{F} \subset C(X)$ is relatively compact if and only if*

- *$\mathcal{F}$ is pointwise bounded, i.e., for every $x \in X$ there exists $M_x > 0$ such that*

$$\forall_{f\in\mathcal{F}}\, |f(x)| \leqslant M_x,$$

- *$\mathcal{F}$ is equicontinuous at every point, i.e., for every $x \in X$ and $\varepsilon > 0$ there exists an open neighbourhood U_x of x such that*

$$\forall_{\substack{y\in U_x\\ f\in\mathcal{F}}} |f(y) - f(x)| \leqslant \varepsilon.$$

Although the importance of this theorem is difficult to overestimate, it is not general enough for our further purposes. What we require is a characterization of relatively compact families in $C_0(X)$, the space of continuous functions on a locally compact space X which vanish at infinity, i.e., for every $\varepsilon > 0$ there exists a compact set K in X (we denote this situation by $K \Subset X$) such that

$$\forall_{x\in X\setminus K} |f(x)| \leqslant \varepsilon.$$

Browsing through the literature, one may stumble upon a theorem, which at first glance appears to be what we need (see [**13**, Theorem 3.4.20], [**32**, Theorem 17], or [**36**, Theorem 47.1]):

Theorem 2.2 (Arzelà–Ascoli theorem for $C_0(X)$ with the compact-open topology)**.** *Let X be a locally compact space. The family $\mathcal{F} \subset C_0(X)$ is relatively compact in the compact-open topology if and only if*

- *$\mathcal{F}$ is pointwise bounded,*
- *$\mathcal{F}$ is equicontinuous at every point.*

However, a careful reader immediately recognizes that the theorem changes the "rules of the game" significantly. Instead of the supremum-norm topology, Theorem 2.2 uses the weaker compact-open topology. Both topologies coincide in $C(X)$ if X is compact, but they are essentially different in $C_0(X)$ if X is only locally compact (in fact, compact-open topology is not normalizable).

Our question is this: how do we get rid of this "splinter" in the form of the switch of topologies? Is there a version of the Arzelà–Ascoli theorem for $C_0(X)$ in which the supremum-norm topology is *not* replaced with the compact-open topology (or any other topology for that matter)? Surprisingly, the literature is rather scarce in this respect. One possible reference is Exercise 17 on p. 182 in John B. Conway's monograph (see [**7**]). However, Conway does not provide a solution to his exercise, leaving this task to the reader. Another possible reference is the monograph by Constantin Corduneanu (see [**8**, p. 62]). Unlike Conway, Corduneanu does provide a proof, but (in our opinion) he does not provide an insight into *why* the theorem is true. The author (Corduneanu) basically takes a sequence from a family in $C_0(X)$ and repeatedly chooses some subsequences (obviously using the assumptions along the way) until arriving at a convergent subsequence. This demonstrates the relative compactness of the family in question but still leaves the reader wondering "why is this theorem true?". Our goal is not only to see that the theorem *is* true, but also to understand *why* it is so.

To begin with, recall that for any locally compact space X (please bear in mind that we always assume topological spaces to be Hausdorff), there exists a compact space X_∞, called the *one-point compactification* of X, such that

- X is a subspace of X_∞,
- the closure of X is X_∞,
- $X_\infty \backslash X$ is a singleton or an empty set (see [**13**, Theorem 3.5.11], [**36**, Theorem 29.1], or [**39**, Proposition 1.7.3]).

The topology of X_∞ allows for a simple description – it consists of open sets in X (the topology of X) plus all sets of the form $X_\infty \backslash K$, where $K \Subset X$ (as we agreed earlier, this means that K is compact in X).

If X happens to be compact itself, then $X = X_\infty$. Otherwise, $X_\infty \backslash X$ is a singleton, whose element is often denoted by ∞ (hence the notation "X_∞"). A common example of the one-point compactification is $\mathbb{C}_\infty$, which arises in the field of complex analysis as a natural domain for Möbius transformations. It turns out that $\mathbb{C}_\infty$ is homeomorphic to a sphere S^2, called the *Riemann sphere*, and the element ∞ may be regarded as the "north pole" of that sphere (see [**18**, p. 38], [**19**, p. 10], or [**48**, pp. 88–89]).

At this point we are ready to prove the following result:

Theorem 2.3 (Arzelà–Ascoli theorem for $C_0(X)$ with the supremum-norm topology)**.** *Let X be a locally compact space. The family $\mathcal{F} \subset C_0(X)$ is relatively compact in the supremum-norm topology if and only if*

- *$\mathcal{F}$ is pointwise bounded,*
- *$\mathcal{F}$ is equicontinuous at every point,*
- *$\mathcal{F}$ is equivanishing, i.e., for every $\varepsilon > 0$ there exists $K \Subset X$ such that*

$$\forall_{\substack{x \notin K \\ f \in \mathcal{F}}} |f(x)| \leqslant \varepsilon. \tag{2.1}$$

Proof: For every $f \in C_0(X)$ we define a function $\Psi_f \colon X_\infty \longrightarrow \mathbb{C}$ such that $\Psi_f|_X = f$ and $\Psi_f(\infty) = 0$. Obviously, every Ψ_f is continuous at each element $x \in X$. Furthermore, functions Ψ_f are also continuous at ∞: for a fixed $\varepsilon > 0$ we choose $K \Subset X$ such that (2.1) is satisfied and put $U_\infty := X_\infty \backslash K$, which is an open neighbourhood of ∞. Consequently, (2.1) corresponds to

$$\forall_{x \in U_\infty} |\Psi_f(x) - \Psi_f(\infty)| \leqslant \varepsilon,$$

which demonstrates that $\Psi_f \in C(X_\infty)$ for every $f \in C_0(X)$.

The crucial observation is that for every $f \in C_0(X)$ we have

$$\sup_{x\in X} |f(x)| = \sup_{x\in X_\infty} |\Psi_f(x)|,$$

so the function $\Psi\colon C_0(X) \longrightarrow C(X_\infty)$, given by $\Psi(f) := \Psi_f$, is an isometry. It is thus clear that the following conditions are equivalent:

- $\mathcal{F}$ is relatively compact in $C_0(X)$,
- $\Psi(\mathcal{F})$ is relatively compact in $C(X_\infty)$,
- $\Psi(\mathcal{F})$ is pointwise bounded and equicontinuous at every point $x \in X_\infty$ (by the classical Arzelà–Ascoli theorem).

Pointwise boundedness of $\Psi(\mathcal{F})$ is equivalent to pointwise boundedness of $\mathcal{F}$ (since for every $\Psi_f \in \Psi(\mathcal{F})$ we have $\Psi_f(\infty) = 0$), while the equicontinuity of $\Psi(\mathcal{F})$ at $x \in X$ is equivalent to equicontinuity of $\mathcal{F}$ at this element. Last but not least, the equicontinuity of $\Psi(\mathcal{F})$ at ∞ means that for every $\varepsilon > 0$ there exists $K \Subset X$ such that

$$\forall_{\substack{x\in X_\infty\setminus K\\ F\in\Psi(\mathcal{F})}} |F(x) - F(\infty)| \leqslant \varepsilon,$$

which is equivalent to equivanishing of $\mathcal{F}$. This concludes the proof. $\square$

3. L^p-properties

As the title of the section suggests, we will currently focus on certain L^p-properties and the way they behave under convolution. Before we commence, however, let us state an assumption which we work under until the end of the paper:

G is a locally compact (Hausdorff) group.

It turns out (see [**10**, Chapter 1.3], [**15**, Chapter 2.2], [**29**, Section 15], or [**51**, Chapter 2]) that any such group G admits a *Haar measure* μ, i.e., a non-zero, Borel measure which is

- finite on compact sets,
- *inner regular*, i.e., for every open set U we have

$$\mu(U) = \sup\{\mu(K) : K \subset U,\ K\text{-compact}\},$$

- *outer regular*, i.e., for every Borel set A we have

$$\mu(A) = \inf\{\mu(U) : A \subset U,\ U\text{-open}\},$$

- *left-invariant*, i.e., for every $x \in G$ and Borel set A we have $\mu(xA) = \mu(A)$.

Haar measure is not determined uniquely, but is "unique up to a positive constant", i.e., if μ_1, μ_2 are two (left) Haar measures on G, then there exists a constant $c > 0$ such that $\mu_1 = c \cdot \mu_2$. Furthermore, similarly to the *left* Haar measure, one can also prove the existence (and uniqueness up to positive constant) of the *right* Haar measure. In general, these two objects need not coincide on G – if they do, we say that G is a *unimodular group*. The family of unimodular groups is quite vast and contains all locally compact *abelian* groups as well as all *compact* groups.

As far as history is concerned, the construction of the Haar measure is commonly attributed to Alfréd Haar, although the contributions of Henri Cartan and André Weil deserve recognition as well (see [**5, 25**] or [**51**, Chapter 2]). For a detailed account of the subject we refer the reader to Diestel and Spalsbury's monograph (see [**11**]).

From this point onwards

we assume that $p \geqslant 1$ and p' is its Hölder conjugate.

We say that a family $\mathcal{F} \subset L^p(G)$ is L^p*-bounded* if there exists $M > 0$ such that

$$\forall_{f \in \mathcal{F}} \, \|f\|_p \leqslant M.$$

Secondly, a family $\mathcal{F} \subset L^p(G)$ is called L^p*-equicontinuous* if for every $\varepsilon > 0$ there exists an open neighbourhood U_e of the neutral element $e \in G$ such that

$$\forall_{f \in \mathcal{F}} \sup_{x \in U_e} \|L_x f - f\|_p \leqslant \varepsilon \quad \text{and} \quad \sup_{x \in U_e} \|R_x f - f\|_p \leqslant \varepsilon,$$

where $L_x f(y) := f(xy)$ and $R_x f(y) := f(yx)$ are the *left* and *right shift operators*, respectively. Thirdly, $\mathcal{F} \subset L^p(G)$ is said to be L^p*-equivanishing* if for every $\varepsilon > 0$ there exists $K \Subset G$ such that

$$\forall_{f \in \mathcal{F}} \int_{G \setminus K} |f(y)|^p \, dy \leqslant \varepsilon.$$

Our aim is to demonstrate that all three properties mentioned above (L^p-boundedness, L^p-equicontinuity, and L^p-equivanishing) are "inherited" when the family $\mathcal{F} \subset L^p(G)$ is convolved with a function $\phi \in C_c(G)$, i.e., a continuous function with compact support. Apart from $C_c(G)$ we will use the notation $C^b(G)$, which stands for the Banach space (with supremum norm) of continuous and bounded functions on G.

Theorem 3.1. *Let $\mathcal{F} \subset L^p(G)$ be L^p-bounded. If $\phi \in C_c(G)$, then $\mathcal{F} \star \phi \subset C^b(G)$ is bounded.*

Proof: Let $M > 0$ be an L^p-bound on the family $\mathcal{F}$. We divide the proof into two steps.

Step 1. Proving that $f \star \phi$ is continuous for every $f \in \mathcal{F}$.

Fix $\varepsilon > 0$, $x_* \in G$, and suppose that $p > 1$. By Proposition 2.41 in [**15**], there exists a symmetric open neighbourhood U_e of the neutral element such that

$$\forall_{x \in U_e} \left(\int_G |\phi \circ \iota(xy) - \phi \circ \iota(y)|^{p'} \, dy \right)^{\frac{1}{p'}} \leqslant \frac{\varepsilon}{M}, \tag{3.1}$$

where $\iota \colon G \longrightarrow G$ stands for the *inverse function*, i.e., $\iota(x) := x^{-1}$. For $x \in x_* U_e$ and $f \in \mathcal{F}$ we have

$$\begin{aligned}
|f \star \phi(x) - f \star \phi(x_*)| &= \left| \int_G f(y) \cdot \phi(y^{-1}x) - f(y) \cdot \phi(y^{-1}x_*) \, dy \right| \\
&\overset{\text{Hölder ineq.}}{\leqslant} \|f\|_p \cdot \left(\int_G |\phi(y^{-1}x) - \phi(y^{-1}x_*)|^{p'} \, dy \right)^{\frac{1}{p'}} \\
&\overset{y \mapsto x_* y}{\leqslant} M \cdot \left(\int_G |\phi(y^{-1}x_*^{-1}x) - \phi(y^{-1})|^{p'} \, dy \right)^{\frac{1}{p'}} \\
&= M \cdot \left(\int_G |\phi \circ \iota(x^{-1}x_* y) - \phi \circ \iota(y)|^{p'} \, dy \right)^{\frac{1}{p'}} \\
&\overset{(3.1)}{\leqslant} \varepsilon.
\end{aligned}$$

This proves that $\mathcal{F} \star \phi$ is a family of continuous functions if $p > 1$.

For $p = 1$ let us again fix $\varepsilon > 0$ and $x_* \in G$. By Lemma 1.3.6 in [**10**], function ϕ is uniformly continuous, so there exists a symmetric open neighbourhood V_e of the neutral element such that

$$\forall_{x \in x_* V_e} \sup_{y \in G} |\phi(y^{-1}x) - \phi(y^{-1}x_*)| \leqslant \frac{\varepsilon}{M}. \tag{3.2}$$

Consequently, for $x \in x_* V_e$ and $f \in \mathcal{F}$ we have

$$\begin{aligned}
|f \star \phi(x) - f \star \phi(x_*)| &= \left| \int_G f(y) \cdot \phi(y^{-1}x) - f(y) \cdot \phi(y^{-1}x_*) \, dy \right| \\
&\leqslant \sup_{y \in G} |\phi(y^{-1}) - \phi(y^{-1}x_*)| \cdot \int_G |f(y)| \, dy \\
&\overset{(3.2)}{\leqslant} \frac{\varepsilon}{M} \cdot \|f\|_1 \leqslant \varepsilon.
\end{aligned}$$

This demonstrates that $\mathcal{F} \star \phi$ is a family of continuous functions if $p = 1$.

Step 2. Proving that the family $\mathcal{F}\star\phi$ is bounded in the supremum norm.

Suppose that $p>1$ and put $K:=\iota(\operatorname{supp}(\phi))$. We observe that

$$\forall_{\substack{x\in G\\ f\in\mathcal{F}}}\left|\int_G f(y)\cdot\phi(y^{-1}x)\,dy\right| \overset{\text{Hölder ineq.}}{\leqslant} \|f\|_p\cdot\left(\int_G|\phi(y^{-1}x)|^{p'}\,dy\right)^{\frac{1}{p'}}$$

$$\leqslant M\cdot\|\phi\|_\infty\cdot\mu(xK)^{\frac{1}{p'}}=M\cdot\|\phi\|_\infty\cdot\mu(K)^{\frac{1}{p'}},$$

where the second inequality stems from the fact that if

$$y^{-1}x\notin\operatorname{supp}(\phi)\iff y\notin xK,$$

then $\phi(y^{-1}x)=0$. We conclude that $\mathcal{F}\star\phi\subset C^b(G)$ is bounded if $p>1$.

For $p=1$, the reasoning is even simpler:

$$\forall_{\substack{x\in G\\ f\in\mathcal{F}}}\left|\int_G f(y)\cdot\phi(y^{-1}x)\,dy\right|\leqslant\int_G|f(y)|\,dy\cdot\|\phi\|_\infty\leqslant M\cdot\|\phi\|_\infty.$$

This concludes the proof. □

Going off on a tangent for a brief moment, one may wonder whether $\mu(\iota(\operatorname{supp}(\phi)))$ (which appears in the second step of the proof above) is equal to $\mu(\operatorname{supp}(\phi))$. In general, this is not the case! However if (and only if) the group G is unimodular (left and right Haar measures coincide), then for every measurable set A we have $\mu(A)=\mu(\iota(A))$ (see [**38**, p. 1112]).

Theorem 3.2. *Let $\mathcal{F}\subset L^p(G)$ be L^p-equicontinuous. If $\phi\in C_c(G)$, then $\mathcal{F}\star\phi$ is equicontinuous.*

Proof: Firstly, suppose that $p>1$, $\varepsilon>0$, $x_*\in G$, and $K:=\iota(\operatorname{supp}(\phi))$. By L^p-equicontinuity of the family $\mathcal{F}$, let U_e be a symmetric open neighbourhood of the neutral element such that

$$\forall_{f\in\mathcal{F}}\sup_{x\in U_e}\|L_xf-f\|_p\leqslant\frac{\varepsilon}{\|\phi\|_\infty\cdot\mu(K)^{\frac{1}{p'}}}. \tag{3.3}$$

Observe that for every $x\in G$ and $f\in\mathcal{F}$ we have

$$f\star\phi(x)=\int_G f(y)\cdot\phi(y^{-1}x)\,dy\overset{y\mapsto xy}{=}\int_G f(xy)\cdot\phi(y^{-1})\,dy. \tag{3.4}$$

Consequently, for every $x \in x_* U_e$ and $f \in \mathcal{F}$ we have

$$\begin{aligned}
|f \star \phi(x) - f \star \phi(x_*)| &\overset{(3.4)}{\leqslant} \int_G |f(xy) - f(x_* y)| \cdot |\phi(y^{-1})| \, dy \\
&\overset{\text{Hölder ineq.}}{\leqslant} \left(\int_G |f(xy) - f(x_* y)|^p \, dy \right)^{\frac{1}{p}} \cdot \left(\int_G |\phi(y^{-1})|^{p'} \, dy \right)^{\frac{1}{p'}} \\
&\leqslant \left(\int_G |f(xy) - f(x_* y)|^p \, dy \right)^{\frac{1}{p}} \cdot \|\phi\|_\infty \cdot \mu(K)^{\frac{1}{p'}} \\
&\overset{y \mapsto x_*^{-1} y}{=} \left(\int_G |f(x x_*^{-1} y) - f(y)|^p \, dy \right)^{\frac{1}{p}} \cdot \|\phi\|_\infty \cdot \mu(K)^{\frac{1}{p'}} \\
&\overset{(3.3)}{\leqslant} \varepsilon,
\end{aligned}$$

which ends the proof if $p > 1$.

For $p = 1$ let us again fix $\varepsilon > 0$ and $x_* \in G$. By L^1-equicontinuity of the family $\mathcal{F}$, let U_e be a symmetric open neighbourhood of the neutral element such that

$$\forall_{f \in \mathcal{F}} \sup_{x \in U_e} \|L_x f - f\|_1 \leqslant \frac{\varepsilon}{\|\phi\|_\infty}. \tag{3.5}$$

For every $x \in x_* U_e$ and $f \in \mathcal{F}$, we obtain

$$\begin{aligned}
|f \star \phi(x) - f \star \phi(x_*)| &\overset{(3.4)}{\leqslant} \int_G |f(xy) - f(x_* y)| \cdot |\phi(y^{-1})| \, dy \\
&\leqslant \|\phi\|_\infty \cdot \int_G |f(xy) - f(x_* y)| \, dy \\
&\overset{y \mapsto x_*^{-1} y}{=} \|\phi\|_\infty \cdot \int_G |f(x x_*^{-1} y) - f(y)| \, dy \overset{(3.5)}{\leqslant} \varepsilon,
\end{aligned}$$

which ends the proof. □

Before we demonstrate how L^p-equivanishing behaves "under convolution", we note the following result:

Lemma 3.3. *If K_1 and K_2 are compact subsets of G, then there exists a compact set D such that*

$$\forall_{x \notin D} \, x K_1 \cap K_2 = \emptyset. \tag{3.6}$$

Proof: We put $D := K_2 \cdot K_1^{-1}$. It is obviously compact by the continuity of the group operations and the compactness of K_1 and K_2. To prove that (3.6) is true, suppose that $z \in x K_1 \cap K_2$ for some $x \notin D$. In particular, $z \in K_2$ and $z = xy$ for some $y \in K_1$. This means that

$$x = z y^{-1} \in K_2 \cdot K_1^{-1} = D,$$

which is a contradiction. This concludes the proof. □

Theorem 3.4. *Let $\mathcal{F} \subset L^p(G)$ be L^p-equivanishing. If $\phi \in C_c(G)$ is such that $\phi(e) \neq 0$ (e is the neutral element of G), then $\mathcal{F} \star \phi$ is equivanishing.*

Proof: Firstly, suppose that $p > 1$. We fix $\varepsilon > 0$ and choose $K \Subset G$ such that

$$\forall_{f \in \mathcal{F}} \left(\int_{G \setminus K} |f|^p \, d\mu \right)^{\frac{1}{p}} \leqslant \frac{\varepsilon}{\|\phi\|_\infty \cdot \mu(\iota(\operatorname{supp}(\phi)))^{\frac{1}{p'}}}. \tag{3.7}$$

Let us denote $U := \iota(\{\phi \neq 0\})$, which is an open and relatively compact neighbourhood of the neutral element with $\overline{U} = \iota(\operatorname{supp}(\phi))$. By Lemma 3.3, there exists $D \Subset G$ such that

$$\forall_{x \notin D} \, x\overline{U} \cap K = \emptyset. \tag{3.8}$$

For $x \notin D$ and $f \in \mathcal{F}$ we have

$$\begin{aligned} |f \star \phi(x)| &\leqslant \int_G |f(y) \cdot \phi(y^{-1}x)| \, dy \leqslant \|\phi\|_\infty \cdot \int_{x\overline{U}} |f| \, d\mu \\ &\overset{\text{Hölder ineq.}}{\leqslant} \|\phi\|_\infty \cdot \left(\int_{x\overline{U}} |f|^p \, d\mu \right)^{\frac{1}{p}} \cdot \mu(x\overline{U})^{\frac{1}{p'}} \\ &\overset{(3.8)}{\leqslant} \|\phi\|_\infty \cdot \left(\int_{G \setminus K} |f|^p \, d\mu \right)^{\frac{1}{p}} \cdot \mu(x\overline{U})^{\frac{1}{p'}} \overset{(3.7)}{\leqslant} \varepsilon, \end{aligned}$$

which ends the proof if $p > 1$.

For $p = 1$ let us again fix $\varepsilon > 0$ and choose $K \Subset G$ such that

$$\forall_{f \in \mathcal{F}} \int_{G \setminus K} |f| \, d\mu \leqslant \frac{\varepsilon}{\|\phi\|_\infty}. \tag{3.9}$$

Furthermore, we choose U and D as previously. For every $f \in \mathcal{F}$ and $x \notin D$ we have

$$\begin{aligned} |f \star \phi(x)| &\leqslant \int_G |f(y) \cdot \phi(y^{-1}x)| \, dy \\ &\leqslant \|\phi\|_\infty \cdot \int_{x\overline{U}} |f| \, d\mu \\ &\overset{(3.8)}{\leqslant} \|\phi\|_\infty \cdot \int_{G \setminus K} |f| \, d\mu \overset{(3.9)}{\leqslant} \varepsilon. \end{aligned}$$

This concludes the proof. $\square$

4. Approximation theorem and Young's convolution inequality

The first part of the current section is devoted to demonstrating that the family $\mathcal{F} \star \phi$ (for a suitably chosen $\phi \in C_c(G)$) approximates the family $\mathcal{F}$ in the L^p-norm. To this end, let us recall Minkowski's integral inequality (see [**16**, Theorem 6.19]):

Theorem 4.1. *Let X, Y be σ-finite measure spaces and let $1 \leqslant p < \infty$. If $F\colon X \times Y \longrightarrow \mathbb{C}$ is a measurable function, then*

$$\left(\int_X \left(\int_Y |F(x,y)|\,dy\right)^p dx\right)^{\frac{1}{p}} \leqslant \int_Y \left(\int_X |F(x,y)|^p\,dx\right)^{\frac{1}{p}} dy. \tag{4.1}$$

To a certain degree, the approximation theorem established below resembles Proposition 2.42 in Folland's monograph (see [**15**, p. 53]). However, Folland's proposition focuses on a single function f whereas our result deals with an L^p-equicontinuous family $\mathcal{F} \subset L^p(G)$:

Theorem 4.2. *If $\mathcal{F} \subset L^p(G)$ is L^p-equicontinuous, then for every $\varepsilon > 0$ there exists a function $\phi \in C_c(G)$ such that*

$$\forall_{f\in\mathcal{F}}\ \|f \star \phi - f\|_p \leqslant \varepsilon.$$

Proof: Fix $\varepsilon > 0$ and let U_e be the open neighbourhood of the neutral element such that

$$\forall_{f\in\mathcal{F}} \sup_{y\in U_e} \|R_y f - f\|_p \leqslant \varepsilon. \tag{4.2}$$

We fix $f \in \mathcal{F}$ and choose f_B to be a Borel-measurable function such that $f = f_B$ almost everywhere. Since G is a Tychonoff space, we can pick $\phi \in C_c(G)$ such that

- $\phi(e) \neq 0$, $\phi \geqslant 0$,
- $\int_G \phi \circ \iota\, d\mu = 1$,
- $\mathrm{supp}(\phi \circ \iota) \subset U_e$.

For every $x \in G$ we have

$$\begin{aligned} f \star \phi(x) - f(x) &= \int_G f(y) \cdot \phi(y^{-1}x)\,dy - f(x) \cdot \int_G \phi(y^{-1})\,dy \\ &= \int_G (f(xy) - f(x)) \cdot \phi(y^{-1})\,dy \\ &= \int_G (f_B(xy) - f_B(x)) \cdot \phi(y^{-1})\,dy. \end{aligned}$$

We put

$$F(x,y) := (f_B(xy) - f_B(x)) \cdot \phi(y^{-1}),$$

which is Borel-measurable as a composition of the following Borel-measurable functions:

$$\begin{aligned} F_1\colon (x,y) &\longmapsto (x,y,y),\\ F_2\colon (x,y,z) &\longmapsto (x,y,z^{-1}),\\ F_3\colon (x,y,z) &\longmapsto (x,xy,\phi(z)),\\ F_4\colon (x,y,z) &\longmapsto (f_B(x),f_B(y),z),\\ F_5\colon (x,y,z) &\longmapsto (y-x)z. \end{aligned}$$

Observe that if we replace f_B with f in F_4, then F does not need to be Borel-measurable (or even measurable), since a composition of a measurable function with a continuous function need not be measurable.

Furthermore, since f_B and ϕ are integrable with p-th power, then $\operatorname{supp}(f_B)$ and $\operatorname{supp}(\phi)$ are σ-compact (see [**10**, Corollary 1.3.5]). Hence, also the sets

$$(\operatorname{supp}(f_B)\cdot\operatorname{supp}(\phi))\times\iota(\operatorname{supp}(\phi)) \quad\text{and}\quad \operatorname{supp}(f_B)\times\iota(\operatorname{supp}(\phi))$$

are σ-compact. Following a series of logical implications:

$$\begin{aligned} (x,y)\in\{F\neq 0\} &\Longrightarrow f_B(xy)-f_B(x)\neq 0 \text{ and } \phi(y^{-1})\neq 0\\ &\Longrightarrow (xy\in\operatorname{supp}(f_B) \text{ or } x\in\operatorname{supp}(f_B)) \text{ and } y^{-1}\in\operatorname{supp}(\phi)\\ &\Longrightarrow (xy\in\operatorname{supp}(f_B) \text{ and } y^{-1}\in\operatorname{supp}(\phi))\\ &\qquad\qquad \text{or } (x\in\operatorname{supp}(f_B) \text{ and } y^{-1}\in\operatorname{supp}(\phi))\\ &\Longrightarrow (x,y)\in(\operatorname{supp}(f_B)\cdot\operatorname{supp}(\phi))\times\iota(\operatorname{supp}(\phi))\\ &\qquad\qquad \text{or } (x,y)\in\operatorname{supp}(f_B)\times\iota(\operatorname{supp}(\phi)) \end{aligned}$$

we conclude that $\{F\neq 0\}$ is σ-compact.

Finally, we are in a position to apply Minkowski's integral inequality:

$$\begin{aligned} \forall_{f\in\mathcal{F}}\ \|f\star\phi-f\|_p &= \|f_B\star\phi-f_B\|_p\\ &= \left(\int_G\left|\int_G (f_B(xy)-f_B(x))\cdot\phi(y^{-1})\,dy\right|^p dx\right)^{\frac{1}{p}}\\ &\overset{(4.1)}{\leqslant} \int_G\left(\int_G |f_B(xy)-f_B(x)|^p\cdot|\phi(y^{-1})|^p\,dx\right)^{\frac{1}{p}} dy\\ &= \int_G \|R_yf-f\|_p\cdot|\phi(y^{-1})|\,dy \leqslant \sup_{y\in U}\|R_yf-f\|_p \overset{(4.2)}{\leqslant} \varepsilon. \end{aligned}$$

This concludes the proof. $\square$

Young's convolution inequality, which is the second topic in the current section, was first proved for the circle group S^1 by William H. Young (see [**52**] for the original paper by Young). Young's observation was later extended to unimodular groups (see [**35**, Chapter 5]) and in the 1970s the theorem had already been referred to as the "classical inequality of Young", while T. S. Quek and Leonard Y. H. Yap called it "one of the most basic results in harmonic analysis" (see [**3**, Chapter 1] and [**41**], respectively). The "popularity burst" of the topic carried on well into the 1990s, though even nowadays some researchers publish papers related to Young's inequality (see [**1, 2, 3, 17, 37, 47**] for just a few of examples from the past as well as [**46**] for a modern treatment of the subject and its applications to convex and set-valued analysis).

As we mentioned earlier, our goal is to give a detailed proof of Young's convolution inequality on an arbitrary locally compact group. Such a result is referred to by Ole A. Nielsen [**37**], but the author does not provide any proof (or even a reference!) of this assertion. Our objective is to fill this gap in the literature. In order to formulate (and prove!) the most general version of the theorem, let us recall (see [**10**, Chapter 1.4], [**15**, Chapter 2.4], [**29**, Section 15], or [**42**, Chapter 3.3]) that the *modular function* associated with the Haar measure μ is a continuous group homomorphism $\Delta\colon G \longrightarrow (\mathbb{R}_+, \cdot)$, which satisfies $\mu(Ax) = \Delta(x)\mu(A)$ for every Borel set A and every element $x \in G$. Amongst numerous properties of the modular function we will find the following equality (see [**10**, Theorem 1.4.1(d)]) particularly useful:

$$\forall_{f\in L^1(G)} \int_G f(x^{-1}) \cdot \Delta(x^{-1})\, dx = \int_G f(x)\, dx. \tag{4.3}$$

Without further ado we proceed with Young's convolution inequality for an arbitrary locally compact group G:

Theorem 4.3. *Let $p, q, r \in [1, \infty)$ be such that*

$$\frac{1}{r} = \frac{1}{p} + \frac{1}{q} - 1. \tag{4.4}$$

For functions $f \in L^p(G)$ and $g \in L^q(G)$, the convolution $f \star (\Delta^{\frac{1}{p'}} g)$ exists almost everywhere. Moreover, $f \star (\Delta^{\frac{1}{p'}} g) \in L^r(G)$ and we have

$$\|f \star (\Delta^{\frac{1}{p'}} g)\|_r \leqslant \|f\|_p \cdot \|g\|_q. \tag{4.5}$$

Proof: Observe that it suffices to prove (4.5), which will immediately establish that $f \star (\Delta^{\frac{1}{p'}} g) \in L^r(G)$ and consequently that the convolution exists almost everywhere.

Firstly, we note some useful equalities:

$$\frac{1}{r}+\frac{1}{q'}+\frac{1}{p'}=\frac{1}{r}+\left(1-\frac{1}{q}\right)+\left(1-\frac{1}{p}\right)\overset{(4.4)}{=}1,$$

$$\left(1-\frac{p}{r}\right)q'\overset{(4.4)}{=}p\left(1-\frac{1}{q}\right)q'=p, \tag{4.6}$$

$$\left(1-\frac{q}{r}\right)p'\overset{(4.4)}{=}q\left(1-\frac{1}{p}\right)p'=q.$$

Next, we use Hölder's inequality to obtain

$$\begin{aligned}|f\star(\Delta^{\frac{1}{p'}}g)(x)| &= \left|\int_G f(y)\cdot\Delta^{\frac{1}{p'}}(y^{-1}x)\cdot g(y^{-1}x)\,dy\right| \\ &\leqslant \int_G (|f|(y)^{\frac{p}{r}}\cdot|g|(y^{-1}x)^{\frac{q}{r}})\cdot|f|(y)^{(1-\frac{p}{r})} \\ &\qquad\cdot(|g|(y^{-1}x))^{(1-\frac{q}{r})}\cdot\Delta^{\frac{1}{p'}}(y^{-1}x)\,dy \\ &\leqslant \left(\int_G |f|(y)^p\cdot|g|(y^{-1}x)^q\,dy\right)^{\frac{1}{r}}\cdot\left(\int_G |f|^{(1-\frac{p}{r})q'}\,d\mu\right)^{\frac{1}{q'}} \\ &\qquad\cdot\left(\int_G |g|(y^{-1}x)^{(1-\frac{q}{r})p'}\cdot\Delta(y^{-1}x)\,dy\right)^{\frac{1}{p'}}.\end{aligned}$$

With the aid of formulae (4.6), we are able to transform the last expression as follows:

$$\begin{aligned}&\left(\int_G |f|(y)^p\cdot|g|(y^{-1}x)^q\,dy\right)^{\frac{1}{r}}\cdot\left(\int_G |f|^p\,d\mu\right)^{\frac{1}{q'}}\cdot\left(\int_G |g|(y^{-1}x)^q\cdot\Delta(y^{-1}x)\,dy\right)^{\frac{1}{p'}} \\ &\overset{y\mapsto xy}{=}\left(\int_G |f|(y)^p\cdot|g|(y^{-1}x)^q\,dy\right)^{\frac{1}{r}}\cdot\|f\|_p^{\frac{p}{q'}}\cdot\left(\int_G |g|(y^{-1})^q\cdot\Delta(y^{-1})\,dy\right)^{\frac{1}{p'}}.\end{aligned}$$

Lastly, employing property (4.3) of the modular function, we can reduce the expression even further:

$$\begin{aligned}&\left(\int_G |f|(y)^p\cdot|g|(y^{-1}x)^q\,dy\right)^{\frac{1}{r}}\cdot\|f\|_p^{\frac{p}{q'}}\cdot\left(\int_G |g|^q\,d\mu\right)^{\frac{1}{p'}} \\ &\qquad= \left(\int_G |f|(y)^p\cdot|g|(y^{-1}x)^q\,dy\right)^{\frac{1}{r}}\cdot\|f\|_p^{\frac{p}{q'}}\cdot\|g\|_q^{\frac{q}{p'}} \\ &\qquad= (|f|^p\star|g|^q(x))^{\frac{1}{r}}\cdot\|f\|_p^{\frac{p}{q'}}\cdot\|g\|_q^{\frac{q}{p'}}.\end{aligned}$$

The computation we have just performed leads to

$$\begin{aligned}\int_G |f\star(\Delta^{\frac{1}{p'}} g)(x)|^r\,dx &\leqslant \left(\int_G |f|^p\star|g|^q(x)\,dx\right)\cdot\|f\|_p^{\frac{pr}{q'}}\cdot\|g\|_q^{\frac{qr}{p'}}\\ &= \||f|^p\star|g|^q\|_1\cdot\|f\|_p^{\frac{pr}{q'}}\cdot\|g\|_q^{\frac{qr}{p'}}\\ &\leqslant \||f|^p\|_1\cdot\||g|^q\|_1\cdot\|f\|_p^{\frac{pr}{q'}}\cdot\|g\|_q^{\frac{qr}{p'}}\\ &= \|f\|_p^{p+\frac{pr}{q'}}\cdot\|g\|_q^{q+\frac{qr}{p'}},\end{aligned}$$

where the second inequality follows from Theorem 1.6.2 in [**10**]. Taking the r-th root, we conclude that

$$\|f\star g\|_r \leqslant \|f\|_p^{\frac{p}{r}+\frac{p}{q'}}\cdot\|g\|_q^{\frac{q}{r}+\frac{q}{p'}} = \|f\|_p\cdot\|g\|_q,$$

which ends the proof. □

5. Characterization of relatively compact families in L^p-spaces

Let us begin the final section of the paper with a simple lemma:

Lemma 5.1. *Let $\mathcal{F}\subset L^p(G)$ be L^p-bounded and L^p-equicontinuous. If $\phi\in C_c(G)$ and $K\Subset G$, then $\mathcal{F}|_K\star\phi$ is relatively compact in $C_0(G)$ and*

$$\forall_{f\in\mathcal{F}}\ \mathrm{supp}(f|_K\star\phi)\subset K\cdot\mathrm{supp}(\phi).$$

Proof: Obviously, if $\mathcal{F}$ is L^p-bounded, then so is $\mathcal{F}|_K$. Furthermore, $\mathcal{F}|_K$ is trivially L^p-equivanishing, because it is supported in K. Last but not least, the L^p-equicontinuity of $\mathcal{F}$ implies the L^p-equicontinuity of $\mathcal{F}|_K$.

By Theorems 3.1, 3.2, and 3.4 we conclude that $\mathcal{F}|_K\star\phi$ is bounded, equicontinuous, and equivanishing in $C_0(G)$. By Theorem 2.3 we establish that $\mathcal{F}|_K\star\phi$ is relatively compact.

For the last part of the theorem, observe that for every $f\in\mathcal{F}$ we have

$$\forall_{x\in G}\,|f|_K\star\phi(x)| = \int_K |f(y)\cdot\phi(y^{-1}x)|\,dy.$$

The integral on the right-hand side is 0 for $x\notin K\cdot\mathrm{supp}(\phi)$, so

$$\forall_{f\in\mathcal{F}}\ \mathrm{supp}(f|_K\star\phi)\subset K\cdot\mathrm{supp}(\phi).$$

This concludes the proof. □

Before we proceed to the climax of the paper we explain how our result relates to other compactness theorems known in the mathematical literature. To begin with, Theorem 5.2 generalizes the Kolmogorov–Riesz theorem, as $\mathbb{R}^n$ is a locally compact (Hausdorff) group. The proofs of the classical Kolmogorov–Riesz theorem can be found in multiple sources, both articles [**27, 34, 43, 50**] and monographs [**4**, Theorem 4.26], or in [**40**, Theorem 1.3]. However, none of these proofs carry over to the general case of a locally compact group G.

Furthermore, we go to great lengths to render our reasoning more transparent than the terse and very "dry" argument of André Weil (see [**51**, pp. 53–54]). It is noteworthy that our proof is definitely *not* a rewrite of Weil's work, as it hinges on an entirely different and innovative idea of "L^p-property inheritance".

It is also instructive to juxtapose our results with the modern approach taken by Górka and Pośpiech in [**22**]. The authors study compactness in the realm of Banach function spaces, which include classical Lebesgue spaces, but also variable exponent Lebesgue spaces, Lorentz spaces, and Orlicz spaces. This setting is admittedly more general than ours, but the techniques used by Górka and Pośpiech force them to assume that the Haar measure is σ-finite. As we demonstrate in Theorem 5.2, such an assumption is completely redundant (in our setting). Furthermore, the paper by Górka and Pośpiech lacks an analysis of L^p-properties (a topic we discussed at length in Section 3) or a detailed proof of Young's convolution inequality (demonstrated in the previous section) which, as far as we are aware, is a novelty in the field. Last but not least, a version of the Sudakov theorem (see [**28, 49**]) presented by Górka and Pośpiech applies only to locally compact *connected* groups. As we explain in Theorem 5.2, there is a more fundamental principle lying at the heart of the Sudakov theorem, which has not yet been explored in the literature. All that being said, both Górka and Pośpiech (as well as other authors before them) deserve high praise and acclamation for their substantial contribution to the subject of compactness in function spaces.

Without further ado, we present the main result of the paper:

Theorem 5.2. *A family $\mathcal{F} \subset L^p(G)$ is relatively compact if and only if*

- *$\mathcal{F}$ is L^p-bounded,*
- *$\mathcal{F}$ is L^p-equicontinuous,*
- *$\mathcal{F}$ is L^p-equivanishing.*

Furthermore, suppose that the group G is such that for every open neighbourhood U of the neutral element there exists an element $x \in U$ such that $(x^k)_{k\in\mathbb{N}}$ is not contained in any compact set. Then the condition of L^p-boundedness is redundant.

Proof: For the entire proof, which we divide into five steps, we fix $\varepsilon > 0$.

Step 1. Relative compactness of $\mathcal{F}$ implies L^p-boundedness.

This step is immediate, as compact sets are always bounded in a metric space.

Step 2. Relative compactness of $\mathcal{F}$ implies L^p-equicontinuity.

Let $(f_k)_{k=1}^l$ be an $\frac{\varepsilon}{3}$-net for the family $\mathcal{F}$. By Proposition 2.41 in [**15**], for every $k = 1, \ldots, l$ there exists an open neighbourhood U_k of the neutral element such that

$$\sup_{x \in U_k} \|L_x f_k - f_k\|_p \leqslant \frac{\varepsilon}{3} \quad \text{and} \quad \sup_{x \in U_k} \|R_x f_k - f_k\|_p \leqslant \frac{\varepsilon}{3}. \tag{5.1}$$

Put $U := \bigcap_{k=1}^l U_k$, which is obviously an open set. Consequently, for every $f \in \mathcal{F}$ there exists $k = 1, \ldots, l$ such that

$$\begin{aligned}\sup_{x \in U} \|L_x f - f\|_p &\leqslant \sup_{x \in U} \|L_x f - L_x f_k\|_p + \sup_{x \in U} \|L_x f_k - f_k\|_p + \|f_k - f\|_p \\ &= 2\|f_k - f\|_p + \sup_{x \in U} \|L_x f_k - f_k\|_p \overset{(5.1)}{\leqslant} \varepsilon.\end{aligned}$$

An analogous reasoning works for $\|R_x f - f\|_p$. We conclude that the family $\mathcal{F}$ is L^p-equicontinuous.

Step 3. Relative compactness of $\mathcal{F}$ implies L^p-equivanishing.

Let $(f_k)_{k=1}^l$ be an $\frac{\varepsilon}{2}$-net for the family $\mathcal{F}$. For every $k = 1, \ldots, l$ there exists $K_k \Subset G$ such that

$$\int_{G \setminus K_k} |f_k|^p \, d\mu \leqslant \frac{\varepsilon}{2}. \tag{5.2}$$

Put $K := \bigcup_{k=1}^l K_k$, which is obviously a compact set. Consequently, for every $f \in \mathcal{F}$ there exists $k = 1, \ldots, l$ such that

$$\int_{G \setminus K} |f|^p \, d\mu \leqslant \int_{G \setminus K} |f - f_k|^p \, d\mu + \int_{G \setminus K} |f_k|^p \, d\mu \overset{(5.2)}{\leqslant} \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

This proves that $\mathcal{F}$ is L^p-equivanishing.

Step 4. L^p-boundedness, L^p-equicontinuity, and L^p-equivanishing imply relative compactness of $\mathcal{F}$.

Using Theorem 4.2 we pick $\phi \in C_c(G)$ such that

$$\forall_{f \in \mathcal{F}} \, \|f \star \phi - f\|_p \leqslant \frac{\varepsilon}{4}. \tag{5.3}$$

By L^p-equivanishing of $\mathcal{F}$, there exists $K \Subset G$ such that

$$\forall_{f\in\mathcal{F}}\, \|f - f|_K\|_p \leqslant \frac{\varepsilon}{4\cdot\|\Delta^{-\frac{1}{p'}}\phi\|_1}. \tag{5.4}$$

Due to Lemma 5.1 we know that $\mathcal{F}|_K \star \phi$ is relatively compact in $C_0(G)$ and that

$$\forall_{f\in\mathcal{F}}\ \operatorname{supp}(f|_K \star \phi) \subset D,$$

where $D := K\cdot\operatorname{supp}(\phi)$. By relative compactness of $\mathcal{F}|_K \star \phi$ there exists a finite sequence of functions $(g_k)_{k=1}^l \subset C_c(G)$ such that for every $f \in \mathcal{F}$ there exists $k = 1,\dots,l$ such that

$$\|f|_K \star \phi - g_k\|_\infty \leqslant \frac{\varepsilon}{4\cdot\mu(D)^{\frac{1}{p}}}. \tag{5.5}$$

Due to the inner regularity of μ there exists an open set $V \supset D$ such that

$$\mu(V\backslash D) \leqslant \left(\frac{\varepsilon}{4\cdot\max_{k=1,\dots,l}\|g_k\|_\infty}\right)^p. \tag{5.6}$$

Since every locally compact group is normal (see [**29**, Theorem 8.13]) by Urysohn's lemma (see for instance [**13**, Theorem 1.5.11], [**32**, Lemma 4], [**36**, Theorem 33.1], or [**39**, Theorem 1.5.6]) there exists a function $u\colon G \longrightarrow [0,1]$ such that

$$u|_D = 1 \quad\text{and}\quad u|_{G\backslash V} = 0. \tag{5.7}$$

Finally, for every $f \in \mathcal{F}$ there exists $k = 1,\dots,l$ such that

$$\begin{aligned}
\|f-h_k\cdot u\|_p &\leqslant \|f - f\star\phi\|_p + \|f\star\phi - f|_K\star\phi\|_p + \|f|_K\star\phi - g_k\cdot u\|_p\\
&\overset{(4.5),(5.3)}{\leqslant} \frac{\varepsilon}{4} + \|f - f|_K\|_p\cdot\|\Delta^{-\frac{1}{p'}}\phi\|_1 + \left(\int_V |f|_K\star\phi - g_k\cdot u|^p\,d\mu\right)^{\frac{1}{p}}\\
&\overset{(5.4),(5.7)}{\leqslant} \frac{\varepsilon}{2} + \left(\int_D |f|_K\star\phi - g_k|^p\,d\mu\right)^{\frac{1}{p}} + \left(\int_{V\backslash D} |g_k|^p\,d\mu\right)^{\frac{1}{p}}\\
&\leqslant \frac{\varepsilon}{2} + \|f|_K\star\phi - g_k\|_\infty\cdot\mu(D)^{\frac{1}{p}} + \|g_k\|_\infty\cdot\mu(V\backslash D)^{\frac{1}{p}} \overset{(5.5),(5.6)}{\leqslant} \varepsilon.
\end{aligned}$$

This demonstrates that $(g_k\cdot u)_{k=1}^l$ is an ε-net for $\mathcal{F}$. Thus $\mathcal{F}$ is relatively compact.

Step 5. Sudakov's part.

We will now prove that L^p-boundedness follows from L^p-equicontinuity and L^p-equivanishing under the assumption that for every open neighbourhood U of the neutral element there exists an element $x \in U$ such that $(x^k)_{k\in\mathbb{N}}$ is not contained in any compact set. By L^p-equicontinuity of $\mathcal{F}$ there exists an open neighbourhood U_e of the neutral element such that

$$\forall_{\substack{x\in U_e\\ f\in\mathcal{F}}} \left(\int_G |L_x f - f|^p \, d\mu\right)^{\frac{1}{p}} \leqslant 1. \tag{5.8}$$

Furthermore, due to L^p-equivanishing there exists $K \Subset G$ such that

$$\forall_{f\in\mathcal{F}} \left(\int_{G\setminus K} |f|^p \, d\mu\right)^{\frac{1}{p}} \leqslant 1. \tag{5.9}$$

Let $x_* \in U_e$ be an element such that $(x_*^k)_{k\in\mathbb{N}}$ is not contained in any compact set. For every $f \in \mathcal{F}$ we have

$$\begin{aligned}\left(\int_K |f|^p \, d\mu\right)^{\frac{1}{p}} &\leqslant \left(\int_K |L_{x_*} f - f|^p \, d\mu\right)^{\frac{1}{p}} + \left(\int_K |L_{x_*} f|^p \, d\mu\right)^{\frac{1}{p}} \\ &\overset{(5.8)}{\leqslant} 1 + \left(\int_G |f(x_* y)|^p \cdot \mathbb{1}_K(y)\, dy\right)^{\frac{1}{p}} = 1 + \left(\int_{x_* K} |f|^p \, d\mu\right)^{\frac{1}{p}}.\end{aligned}$$

Using an inductive reasoning we have

$$\forall_{k\in\mathbb{N}} \left(\int_K |f|^p \, d\mu\right)^{\frac{1}{p}} \leqslant k + \left(\int_{x_*^k K} |f|^p \, d\mu\right)^{\frac{1}{p}}.$$

Observe that there exists $l \in \mathbb{N}$ such that $x_*^l K \cap K = \emptyset$, since otherwise we would have $(x_k)_{k\in\mathbb{N}} \subset K \cdot K^{-1}$, contrary to our assumption. Finally, we have

$$\|f\|_p \leqslant \left(\int_K |f|^p \, d\mu\right)^{\frac{1}{p}} + \left(\int_{G\setminus K} |f|^p \, d\mu\right)^{\frac{1}{p}} \leqslant l + \left(\int_{x_*^l K} |f|^p \, d\mu\right)^{\frac{1}{p}} + 1 \overset{(5.9)}{\leqslant} l + 2,$$

which ends the proof. □

It is high time we relished the fruits of our labour, i.e., examined various locally compact groups in the context of Theorem 5.2. We commence with the easiest instances of finite groups, like the cyclic groups $\mathbb{Z}_n$, permutation groups S_n, alternating groups A_n, dihedral groups Dih_n, the quaternion group Q_8, etc.

Corollary 5.3. *Let G be a finite group. A family $\mathcal{F} \subset L^p(G)$ is relatively compact if and only if $\mathcal{F}$ is L^p-bounded.*

Proof: It suffices to note that on a finite group any family of functions is L^p-equicontinuous and L^p-equivanishing. $\square$

Next, we focus on compact groups such as the circle group S^1, the torus $S^1 \times S^1$, orthogonal groups $O(n)$, special orthogonal groups $SO(n)$, unitary groups $U(n)$, special unitary groups $SU(n)$ (see [**30**, Lemma 2.1.4] or [**26**, Example 1.3.1], for a proof of compactness of these matrix groups), etc.

Corollary 5.4. *Let G be a compact group. A family $\mathcal{F} \subset L^p(G)$ is relatively compact if and only if*

- *$\mathcal{F}$ is L^p-bounded,*
- *$\mathcal{F}$ is L^p-equicontinuous.*

Proof: Obviously, on a compact group any family of functions is L^p-equivanishing. $\square$

We shift our focus to locally compact connected groups, like the Euclidean spaces $\mathbb{R}^n$ or $\mathbb{C}^n$, general linear groups $GL(n, \mathbb{C})$ (see [**26**, Proposition 1.9], for a proof of connectedness of $GL(n, \mathbb{C})$), special linear groups $SL(n, \mathbb{R})$ and $SL(n, \mathbb{C})$ (see [**26**, Proposition 1.10], for a proof of connectedness of $SL(n, \mathbb{C})$), the Heisenberg group (see [**9**, Chapter 12]), etc.

Corollary 5.5. *Let G be a locally compact connected group. A family $\mathcal{F} \subset L^p(G)$ is relatively compact if and only if*

- *$\mathcal{F}$ is L^p-equicontinuous,*
- *$\mathcal{F}$ is L^p-equivanishing.*

Proof: It suffices to note that by Theorem 7.4 in [**29**], the connectedness of the the group G implies that for every open neighbourhood U of the neutral element there exists a non-zero element $x \in U$ such that $(x^k)_{k\in\mathbb{N}}$ is not contained in any compact subset of G. $\square$

It turns out that there are non-connected (and non-compact) groups for which the characterization of relatively compact families looks exactly like Corollary 5.5. Examples of these groups include the general linear groups $GL(n, \mathbb{R})$ (see [**26**, Proposition 1.12], for a study of connected components of $GL(n, \mathbb{R})$):

Corollary 5.6. *Let $G = GL(n, \mathbb{R})$ for some $n \in \mathbb{N}$. A family $\mathcal{F} \subset L^p(G)$ is relatively compact if and only if*

- *$\mathcal{F}$ is L^p-equicontinuous,*
- *$\mathcal{F}$ is L^p-equivanishing.*

Proof: It suffices to note that for any open neighbourhood U of the neutral element of $GL(n,\mathbb{R})$ (i.e., the identity matrix) there exists $\varepsilon > 0$ such that the matrix

$$A := \begin{pmatrix} 1+\varepsilon & 0 & \dots & \dots & 0 \\ 0 & 1 & & & \vdots \\ \vdots & & \ddots & & \vdots \\ \vdots & & & 1 & 0 \\ 0 & \dots & \dots & 0 & 1 \end{pmatrix}$$

belongs to U and since $(A^k)_{1,1} = (1+\varepsilon)^k \longrightarrow \infty$ as $k \longrightarrow \infty$, then $(A^k)_{k\in\mathbb{N}}$ is not contained in any compact set. □

As a final example let us consider the group $\mathbb{Z}$, which is neither compact nor connected. However, its behaviour is different from the general linear groups discussed above:

Corollary 5.7. *A family $\mathcal{F} \subset L^p(\mathbb{Z})$ is relatively compact if and only if*

- *$\mathcal{F}$ is L^p-bounded,*
- *$\mathcal{F}$ is L^p-equivanishing, i.e., for every $\varepsilon > 0$ there exists $l \in \mathbb{N}$ such that*
$$\forall_{x\in\mathcal{F}} \sum_{|k|>l} |x_k|^p \leqslant \varepsilon.$$

Let us emphasize that L^p-boundedness is not a redundant condition in characterizing relatively compact families in $L^p(\mathbb{Z})$. This does not contradict the final part of Theorem 5.2, because the singleton $\{0\}$ is an open neighbourhood of the neutral element (namely 0) and the sequence $(k \cdot 0)_{k\in\mathbb{N}}$ (the only possible sequence we can construct from the single element of $\{0\}$) is contained in the compact set $\{0\}$.

Furthermore, it is not difficult to construct an L^p-equivanishing family in $L^p(\mathbb{Z})$, which is not relatively compact – let $x \in L^p(\mathbb{Z})$ be such that $x_0 = 1$, $x_k = 0$ for $k \in \mathbb{Z}\backslash\{0\}$, and consider the family $\mathcal{F} = (k \cdot x)_{k\in\mathbb{N}}$. Obviously, the family $\mathcal{F}$ is L^p-equivanishing, but it is not L^p-bounded and consequently, not relatively compact. This demonstrates that the assumption of L^p-boundedness is indispensable for Corollary 5.7.

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References

[1] F. Barthe, Optimal Young's inequality and its converse: a simple proof, *Geom. Funct. Anal.* **8(2)** (1998), 234–242. DOI: 10.1007/s000390050054.

[2] W. Beckner, Inequalities in Fourier analysis, *Ann. of Math. (2)* **102(1)** (1975), 159–182. DOI: 10.2307/1970980.

[3] H. J. Brascamp and E. H. Lieb, Best constants in Young's inequality, its converse, and its generalization to more than three functions, *Advances in Math.* **20(2)** (1976), 151–173. DOI: 10.1016/0001-8708(76)90184-5.

[4] H. Brezis, *"Functional Analysis, Sobolev Spaces and Partial Differential Equations"*, Universitext, Springer, New York, 2011. DOI: 10.1007/978-0-387-70914-7.

[5] H. Cartan, Sur la mesure de Haar, *C. R. Acad. Sci. Paris* **211** (1940), 759–762.

[6] G. Choquet, *"Topology"*, Translated from the French by Amiel Feinstein, Pure and Applied Mathematics **XIX**, Academic Press, New York-London, 1966.

[7] J. B. Conway, *"A Course in Functional Analysis"*, Graduate Texts in Mathematics **96**, Springer-Verlag, New York, 1985. DOI: 10.1007/978-1-4757-3828-5.

[8] C. Corduneanu, *"Integral Equations and Stability of Feedback Systems"*, Mathematics in Science and Engineering **104**, Academic Press [Harcourt Brace Jovanovich, Publishers], New York-London, 1973.

[9] A. Deitmar, *"A First Course in Harmonic Analysis"*, Second edition, Universitext, Springer-Verlag, New York, 2005. DOI: 10.1007/0-387-27561-4.

[10] A. Deitmar and S. Echterhoff, *"Principles of Harmonic Analysis"*, Universitext, Springer, New York, 2009. DOI: 10.1007/978-0-387-85469-4.

[11] J. Diestel and A. Spalsbury, *"The Joys of Haar Measure"*, Graduate Studies in Mathematics **150**, American Mathematical Society, Providence, RI, 2014. DOI: 10.1090/gsm/150.

[12] J. Dixmier, *"General Topology"*, Translated from the French by Sterling K. Berberian, Undergraduate Texts in Mathematics, Springer-Verlag, New York, 1984. DOI: 10.1007/978-1-4757-4032-5.

[13] R. Engelking, *"General Topology"*, Translated from the Polish by the author, Monografie Matematyczne **60**, PWN—Polish Scientific Publishers, Warsaw, 1977.

[14] H. G. Feichtinger, Compactness in translation invariant Banach spaces of distributions and compact multipliers, *J. Math. Anal. Appl.* **102(2)** (1984), 289–327. DOI: 10.1016/0022-247X(84)90173-2.

[15] G. B. Folland, *"A Course in Abstract Harmonic Analysis"*, Studies in Advanced Mathematics, CRC Press, Boca Raton, FL, 1995.

[16] G. B. Folland, *"Real Analysis: Modern Techniques and Their Applications"*, Second edition, Pure and Applied Mathematics (New York), A Wiley-Interscience Publication, John Wiley & Sons, Inc., New York, 1999.

[17] J. J. F. Fournier, Sharpness in Young's inequality for convolution, *Pacific J. Math.* **72(2)** (1977), 383–397. DOI: 10.2140/PJM.1977.72.383.

[18] J. P. Gilman, I. Kra, and R. E. Rodríguez, *"Complex Analysis. In the Spirit of Lipman Bers"*, Graduate Texts in Mathematics **245**, Springer, New York, 2007. DOI: 10.1007/978-0-387-74715-6.

[19] S. Gong and Y. Gong, *"Concise Complex Analysis"*, Revised edition, World Scientific Publishing Co. Pte. Ltd., Hackensack, NJ, 2007. DOI: 10.1142/6457.

[20] P. Górka and T. Kostrzewa, Pego everywhere, *J. Algebra Appl.* **15(4)** (2016), 1650074, 3 pp. DOI: 10.1142/S0219498816500742.

[21] P. Górka and A. Macios, The Riesz–Kolmogorov theorem on metric spaces, *Miskolc Math. Notes* **15(2)** (2014), 459–465. DOI: 10.18514/mmn.2014.784.

[22] P. Górka and P. Pośpiech, Banach function spaces on locally compact groups, *Ann. Funct. Anal.* **10(4)** (2019), 460–471. DOI: 10.1215/20088752-2019-0002.

[23] P. Górka and H. Rafeiro, From Arzelà–Ascoli to Riesz–Kolmogorov, *Nonlinear Anal.* **144** (2016), 23–31. DOI: 10.1016/j.na.2016.06.004.

[24] W. Guo and G. Zhao, On relatively compact sets in quasi-Banach function spaces, *Proc. Amer. Math. Soc.* **148(8)** (2020), 3359–3373. DOI: 10.1090/proc/14963.

[25] A. Haar, Der Massbegriff in der Theorie der Kontinuierlichen Gruppen, *Ann. of Math. (2)* **34(1)** (1933), 147–169. DOI: 10.2307/1968346.

[26] B. C. Hall, *"Lie Groups, Lie Algebras, and Representations. An Elementary Introduction"*, Graduate Texts in Mathematics **222**, Springer-Verlag, New York, 2003. DOI: 10.1007/978-0-387-21554-9.

[27] H. Hanche-Olsen and H. Holden, The Kolmogorov–Riesz compactness theorem, *Expo. Math.* **28(4)** (2010), 385–394. DOI: 10.1016/j.exmath.2010.03.001.

[28] H. Hanche-Olsen, H. Holden, and E. Malinnikova, An improvement of the Kolmogorov–Riesz compactness theorem, *Expo. Math.* **37(1)** (2019), 84–91. DOI: 10.1016/j.exmath.2018.03.002.

[29] E. Hewitt and K. A. Ross, *"Abstract Harmonic Analysis. Volume I: Structure of Topological Groups Integration Theory Group Representations"*, Second edition, Grundlehren der Mathematischen Wissenschaften **115**, Springer-Verlag, Berlin-New York, 1979.

[30] J. Hilgert and K.-H. Neeb, *"Structure and Geometry of Lie Groups"*, Springer Monographs in Mathematics, Springer, New York, 2012. DOI: 10.1007/978-0-387-84794-8.

[31] E. Kaniuth, *"A Course in Commutative Banach Algebras"*, Graduate Texts in Mathematics **246**, Springer, New York, 2009. DOI: 10.1007/978-0-387-72476-8.

[32] J. L. Kelley, *"General Topology"*, Reprint of the 1955 edition [Van Nostrand, Toronto, Ont.], Graduate Texts in Mathematics **27**, Springer-Verlag, New York-Berlin, 1975.

[33] A. W. Knapp, *"Basic Real Analysis"*, Along with a Companion Volume *Advanced Real Analysis*, Cornerstones, Birkhäuser Boston, Inc., Boston, MA, 2005. DOI: 10.1007/0-8176-4441-5.

[34] A. Kolmogorov, Über Kompaktheit der Funktionenmengen bei der Konvergenz im Mittel, *Nachr. Ges. Wiss. Göttingen, Math.-Phys. Kl. (I, Math.)* **1931** (1931), 60–63.

[35] R. A. Kunze, L_p Fourier transforms on locally compact unimodular groups, *Trans. Amer. Math. Soc.* **89(2)** (1958), 519–540. DOI: 10.2307/1993198.

[36] J. R. MUNKRES, *"Topology"*, Second edition, Prentice Hall, Inc., Upper Saddle River, NJ, 2000.
[37] O. A. NIELSEN, Sharpness in Young's inequality for convolution products, *Canad. J. Math.* **46(6)** (1994), 1287–1298. DOI: 10.4153/CJM-1994-073-7.
[38] E. PAP, Some elements of the classical measure theory, in: *"Handbook of Measure Theory"*, Vol. I, North-Holland, Amsterdam, 2002, pp. 27–82. DOI: 10.1016/B978-044450263-6/50003-8.
[39] G. K. PEDERSEN, *"Analysis Now"*, Graduate Texts in Mathematics **118**, Springer-Verlag, New York, 1989. DOI: 10.1007/978-1-4612-1007-8.
[40] R. PRECUP, *"Methods in Nonlinear Integral Equations"*, Kluwer Academic Publishers, Dordrecht, 2002. DOI: 10.1007/978-94-015-9986-3.
[41] T. S. QUEK AND L. Y. H. YAP, Sharpness of Young's inequality for convolution, *Math. Scand.* **53(2)** (1983), 221–237. DOI: 10.7146/math.scand.a-12030.
[42] H. REITER AND J. D. STEGEMAN, *"Classical Harmonic Analysis and Locally Compact Groups"*, Second edition, London Mathematical Society Monographs. New Series **22**, The Clarendon Press, Oxford University Press, New York, 2000.
[43] M. RIESZ, Sur les ensembles compacts de fonctions sommables, *Acta Sci. Math. (Szeged)* **6** (1933), 136–142.
[44] W. RUDIN, *"Real and Complex Analysis"*, Third edition, McGraw-Hill Book Co., New York, 1987.
[45] W. RUDIN, *"Functional Analysis"*, Second edition, International Series in Pure and Applied Mathematics, McGraw-Hill, Inc., New York, 1991.
[46] J. SADOWSKI, Young's inequality for convolution and its applications in convex- and set-valued analysis, *J. Math. Anal. Appl.* **421(2)** (2015), 1274–1294. DOI: 10.1016/j.jmaa.2014.07.045.
[47] S. SAEKI, The L^p-conjecture and Young's inequality, *Illinois J. Math.* **34(3)** (1990), 614–627. DOI: 10.1215/ijm/1255988174.
[48] E. M. STEIN AND R. SHAKARCHI, *"Complex Analysis"*, Princeton Lectures in Analysis **2**, Princeton University Press, Princeton, NJ, 2003.
[49] V. N. SUDAKOV, Criteria of compactness in function spaces (Russian), *Uspehi Mat. Nauk (N.S.)* **12** (1957), no. 3(75), 221–224.
[50] J. D. TAMARKIN, On the compactness of the space L_p, *Bull. Amer. Math. Soc.* **38(2)** (1932), 79–84. DOI: 10.1090/S0002-9904-1932-05332-0.
[51] A. WEIL, *"L'intégration dans les groupes topologiques et ses applications"*, [This book has been republished by the author at Princeton, N. J., 1941.] Actualités Scientifiques et Industrielles **869**, Hermann & Cie, Paris, 1940.
[52] W. H. YOUNG, On the multiplication of successions of Fourier constants, *Proc. Roy. Soc. London Ser. A* **87(596)** (1912), 331–339. DOI: 10.1098/rspa.1912.0086.

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