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AN α-NUMBER CHARACTERIZATION OF L^p SPACES ON UNIFORMLY RECTIFIABLE SETS

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Abstract: We give a characterization of $L^p(\sigma)$ for uniformly rectifiable measures σ using Tolsa's α-numbers, by showing, for $1<p<\infty$ and $f\in L^p(\sigma)$, that

$$\|f\|_{L^p(\sigma)} \sim \left\| \left(\int_0^\infty (\alpha_{f\sigma}(x,r)+|f|_{x,r}\alpha_\sigma(x,r))^2 \frac{dr}{r} \right)^{\frac{1}{2}} \right\|_{L^p(\sigma)}.$$

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1. Introduction

We say a measure μ in $\mathbb{R}^n$ is *d-rectifiable* if μ is absolutely continuous with respect to a d-dimensional Hausdorff measure and we may exhaust μ-almost all of $\mathbb{R}^n$ by countably many Lipschitz graphs. It is a classical result that, for μ-a.e. $x\in\mathbb{R}^n$, the densities $\mu(B(x,r))/r^d$ stabilize as $r\to 0$ in the sense that they converge to a nonzero constant and so on small scales the measure μ scales like a d-dimensional Lebesgue measure. Furthermore, the *shape* of the measure μ also stabilizes: as we zoom in on x, if we set $\mu_{x,r}(A)=\mu(rA+x)$, then $\mu_{x,r}r^{-d}$ converges weakly to (a constant times) a d-dimensional Lebesgue measure on some d-dimensional plane.

In **[19]**, Tolsa quantified how much a uniformly rectifiable measure can deviate from resembling a planar Lebesgue measure. Recall that a measure σ is *uniformly rectifiable* (UR) if firstly it is *Ahlfors d-regular*, meaning there is $A>0$ so that

$$A^{-1}r^d \le \sigma(B(x,r)) \le Ar^d \quad \text{for all } x\in\operatorname{supp}\sigma,\ 0<r<\operatorname{diam}(\operatorname{supp}\sigma),$$

and σ has *big pieces of Lipschitz images* (BPLI), meaning there are constants $L,c>0$ so that for each $x\in\operatorname{supp}\sigma$ and $0<r<\operatorname{diam}(\operatorname{supp}\sigma)$ there is an L-Lipschitz mapping $f\colon B_d(0,r)\to B(x,r)$ so that $\sigma(f(B_d(0,r))\ge cr^n$. We say a set $E\subseteq\mathbb{R}^n$ is UR if $\mathcal{H}^d|_E$ is UR.

Before stating Tolsa's result, we will describe how he measures the planarity of a measure. First, we define a distance between measures. For two measures μ and ν and a ball B we define

$$F_B(\sigma,\nu) := \sup\left\{\left|\int \phi\, d\sigma - \int \phi\, d\nu\right| : \phi \in \operatorname{Lip}_1(B)\right\},$$

where $\operatorname{Lip}_1(B)$ is the set of 1-Lipschitz functions supported in B. This is a variant of the Wasserstein 1-distance from mass transport theory. See [**15**, Chapter 14] for a discussion about this distance.

For a (possibly real-valued) measure μ and $d \in \mathbb{N}$, if $B = B(x,r)$, we define

$$\alpha_\mu^d(x,r) = \alpha_\mu^d(B) := \frac{1}{r^{d+1}} \inf_{c\in\mathbb{R},L} F_B(\mu, c\mathcal{H}^d|_L),$$

where the infimum is taken over all $c \in \mathbb{R}$ and all d-planes $L \subset \mathbb{R}^n$. We will often omit the superscript d, as it will be fixed throughout.

Theorem 1.1 ([**19**, Theorem 1.2])**.** *An Ahlfors d-regular measure σ is UR if and only if the measure $\alpha_\sigma^d(x,r)^2\, d\sigma(x)\frac{dr}{r}$ is a Carleson measure, meaning that for all balls B centered on* $\operatorname{supp}\sigma$ *with $0 < r_B <$* $\operatorname{diam}(\operatorname{supp}\sigma)$*,*

$$\int_0^{r_B}\int_B \alpha_\sigma^d(x,r)^2\, d\sigma(x)\frac{dr}{r} \le C\sigma(B)$$

for some fixed $C > 0$.

Estimates on α-type numbers are particularly useful in studying rectifiability. From a geometric viewpoint, they give quite a lot of information about the shape of a measure. David and Semmes ([**11**]) gave an earlier characterization of UR sets in terms of a Carleson measure condition on β-numbers, which are quantities like α-numbers except they only measure the average distance of a measure to a plane, so while a measure could be very close to lying on a plane, its mass could be very unevenly distributed, resulting in a large α-number. The additional information provided by the α-numbers was crucial for the main result of [**19**], where Tolsa improved on the work in [**11**] by expanding the class of Calderón–Zygmund operators on UR sets that were known to be bounded. See also [**18**], where α-numbers are used to characterize rectifiability of sets of finite measure in terms of the existence of principal values for the Riesz transform, and [**8, 12, 10**], where they are used to study higher co-dimensional analogues of harmonic measure.

The purpose of this note is to extend Tolsa's result to measures that are not Ahlfors regular, but are given by $L^p(\sigma)$ functions, where σ is uniformly rectifiable.

Given a Radon measure σ, $f \in L^1_{\mathrm{loc}}(\sigma)$, and a ball $B = B(x,r)$ with $\sigma(B(x,r)) > 0$, set

$$f_B = f_{x,r} = \frac{\int_B f\, d\sigma}{\sigma(B)}.$$

Main theorem. *Let σ be a UR measure and $f \in L^p(\sigma)$, where $1 < p < \infty$. Then*

$$\|f\|_{L^p(\sigma)} \sim \left\| \left(\int_0^\infty (\alpha_{f\sigma}(x,r) + |f|_{x,r}\alpha_\sigma(x,r))^2 \frac{dr}{r} \right)^{\frac{1}{2}} \right\|_{L^p(\sigma)}, \tag{1.1}$$

with the implicit constant depending on p and σ.

Sharpness of the result. An interesting aspect of our result is the presence of two terms that comprise our square function. We don't know whether the result holds for general UR sets without the second term. Neither of the terms bounds the other in the pointwise sense: one could be zero while the other is nonzero. On the other hand, we don't know whether the norm of the square function involving only $\alpha_{f\sigma}$ dominates the one involving only $|f|_{x,r}\alpha_\sigma$. The reverse inequality is certainly not true, as the latter square function vanishes if σ is the Lebesgue measure on $\Sigma = \mathbb{R}^d$.

Question. *Let σ be a UR measure and $f \in L^p(\sigma)$, where $1 < p < \infty$. Do we have*

$$\|f\|_{L^p(\sigma)} \lesssim \left\| \left(\int_0^\infty \alpha_{f\sigma}(x,r)^2 \frac{dr}{r} \right)^{\frac{1}{2}} \right\|_{L^p(\sigma)} ? \tag{1.2}$$

Equivalently, is it true that

$$\left\| \left(\int_0^\infty (|f|_{x,r}\alpha_\sigma(x,r))^2 \frac{dr}{r} \right)^{\frac{1}{2}} \right\|_{L^p(\sigma)} \lesssim \left\| \left(\int_0^\infty \alpha_{f\sigma}(x,r)^2 \frac{dr}{r} \right)^{\frac{1}{2}} \right\|_{L^p(\sigma)} ?$$

The answer to the question above is obviously affirmative in the flat case, i.e. $\sigma = \mathcal{H}^d \llcorner L$ for L a d-dimensional plane. It is also positive if σ is an Ahlfors d-regular measure on a d-dimensional plane L, i.e. $\sigma = g\mathcal{H}^d \llcorner L$ for some g satisfying $A^{-1} \le g \le A$. Indeed, let $\tilde\sigma = \mathcal{H}^d \llcorner L$, so that $f\sigma = fg\tilde\sigma$. In that case, by the main theorem,

$$\begin{aligned}\|f\|_{L^p(\sigma)} \sim_A \|fg\|_{L^p(\tilde\sigma)} &\sim_p \left\| \left(\int_0^\infty \alpha_{fg\tilde\sigma}(x,r)^2 \frac{dr}{r} \right)^{\frac{1}{2}} \right\|_{L^p(\tilde\sigma)} \\ &\sim_A \left\| \left(\int_0^\infty \alpha_{f\sigma}(x,r)^2 \frac{dr}{r} \right)^{\frac{1}{2}} \right\|_{L^p(\sigma)}.\end{aligned}$$

Finally, one could show that (1.2) is true for "sufficiently flat" UR measures σ. What we mean by this is that if the constant C from Theorem 1.1 is sufficiently small, then some variant of Carleson's embedding theorem can be used[1] to show that

$$\left\| \left(\int_0^\infty (|f|_{x,r}\alpha_\sigma(x,r))^2 \frac{dr}{r} \right)^{\frac{1}{2}} \right\|_{L^p(\sigma)} \lesssim_p C\|f\|_{L^p(\sigma)} \ll \|f\|_{L^p(\sigma)}.$$

This means that the second term from the square function in (1.1) can essentially be absorbed by the left-hand side. To make this more rigorous, one should perhaps track the dependence of the implicit constants in (1.1) on the UR constants of σ with more diligence than we did. However, the implicit constants can only get *better* as σ becomes flatter, and they certainly cannot blow up as the Carleson constant C goes to 0: if σ satisfies the Carleson condition of Theorem 1.1 with some C, then it also satisfies it with constant C' for every $C' \geq C$.

Related work. While our focus has been in the Ahlfors regular setting, α-numbers have also been used to study measures in more general settings. In [**3**], it was shown that pointwise doubling measures μ were d-rectifiable on the set where the square function $\int_0^\infty \alpha_\mu(x,r)^2 \frac{dr}{r}$ was finite, resolving a question left open in [**1**]. This paper also exposed some limitations with working with α-numbers, as a counterexample showed that the same result is not true for general measures. However, the second author of this paper obtained such a generalization in [**5, 6**] using a different α-number, which measures distance between a measure and a planar measure using the Wasserstein 2-metric, which Tolsa had used earlier to give a characterization of UR measures in [**20**]. So while using the Wasserstein 2-distance allows one to get a more complete picture, the α-number in Theorem 1.1 has a more transparent definition and thus is easier to work with.

In [**16**] Orponen uses a similar square function to characterize when two measures on the real line (one being doubling) are absolutely continuous. However, among a few of the differences between the α-numbers he uses and ours, while we compare distance between a measure and a plane, his numbers compare the distance between the two measures, which is another interesting direction.

Organization of the article. In Section 2 we introduce the necessary tools and make some initial reductions. We define also Jf, a dyadic variant of the square function from the main theorem; see (2.3).

[1] For $p = 2$ use e.g. [**21**, Theorem 5.8]; for $p \neq 2$ one can show a corresponding statement by proving an appropriate good-lambda inequality, in the spirit of what we do in Section 4 (but simpler).

We show that $\|Jf\|_2 \lesssim \|f\|_2$ in Section 3. The proof uses martingale difference operators, and it is inspired by how Theorem 1.1 was originally proved; see [**19**, Section 4]. In Section 4 we use the estimate $\|Jf\|_2 \lesssim \|f\|_2$ and an appropriate good-lambda inequality to conclude that $\|Jf\|_p \lesssim \|f\|_p$ for general $1 < p < \infty$.

Finally, in Section 5 we prove $\|f\|_p \lesssim \|Jf\|_p$. To do that we use the Littlewood–Paley theory of David, Journé, and Semmes [**9**].

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2. Preliminaries

2.1. Notation. In our estimates we will write $f \lesssim g$ to denote $f \le Cg$ for some constant C (the so-called "implicit constant"). If the implicit constant depends on a parameter t, i.e. $C = C(t)$, we will write $f \lesssim_t g$. The notation $f \sim g$ and $f \sim_t g$ stands for $g \lesssim f \lesssim g$ and $g \lesssim_t f \lesssim_t f$, respectively. To make the notation lighter, we will usually not track the dependence of C on dimensions n, d, on the Ahlfors regularity constant of σ, or the parameter $1 < p < \infty$.

Given $x \in \mathbb{R}^n$ and $r > 0$ we denote by $B(x,r)$ the open ball centered at x with radius r. Conversely, given a ball B (either open or closed, and either n or d-dimensional), r_B and z_B denote the radius and the center of B, respectively.

For simplicity, we will sometimes write

$$\|f\|_p := \|f\|_{L^p(\sigma)}.$$

In the introduction we introduced the notation f_B to signify the average of f over a ball B with respect to σ. For general Borel sets $E \subset \mathbb{R}^d$ with $\sigma(E) > 0$ and $f \in L^1_{\mathrm{loc}}(\sigma)$ we will write

$$\langle f \rangle_E = \frac{\int_E f \, d\sigma}{\sigma(E)}.$$

For a finite set I we will write $\#I$ to denote the cardinality of I.

If $v, w \in \mathbb{R}^n$, then $v \cdot w$ denotes their scalar product.

Given $E, F \subset \mathbb{R}^d$, $\operatorname{dist}_H(E, F)$ stands for the Hausdorff distance between E and F.

2.2. Adjacent systems of cubes. As usual, we will work with a family of subsets of $\operatorname{supp}\sigma =: \Sigma$ that in many ways resemble the family of dyadic cubes on $\mathbb{R}^d$. For this reason we will call these sets "cubes". Many different systems of cubes have been constructed over the years, beginning with the work of David [**7**] and Christ [**4**]. In our proof it will be convenient to use adjacent systems of cubes constructed by Hytönen and Tapiola [**14**]. One should think of them as a generalization of the translated dyadic grids in $\mathbb{R}^d$, widely used to perform the "1/3 trick".

First, we will say that a family $\mathcal{D}$ of Borel subsets of Σ satisfies *the usual properties of David–Christ cubes* if $\mathcal{D} = \bigcup_{k\in\mathbb{Z}} \mathcal{D}_k$, and for each $k \in \mathbb{Z}$:

(a) For $P, Q \in \mathcal{D}_k$, $P \neq Q$, we have $\sigma(P \cap Q) = \varnothing$.

(b) The sets in $\mathcal{D}_k$ cover Σ:
$$\Sigma = \bigcup_{Q\in\mathcal{D}_k} Q.$$

(c) For each $Q \in \mathcal{D}_k$ and each $l \geq k$
$$Q = \bigcup_{P\in\mathcal{D}_l : P\subset Q} P.$$

(d) There exists $0 < \delta < 1$ (independent of k) such that each $Q \in \mathcal{D}_k$ has a center $z_Q \in Q$ satisfying
$$B\left(z_Q, \frac{\delta^k}{5}\right) \cap \Sigma \subset Q \subset B(z_Q, 3\delta^k) \cap \Sigma.$$
Consequently, as long as $\delta^k \lesssim \operatorname{diam}(\Sigma)$, we have $\sigma(Q) \sim \delta^{kd}$. Set $\ell(Q) := \delta^k$.

(e) The cubes $Q \in \mathcal{D}_k$ have thin boundaries, that is, there exists $\gamma \in (0, 1)$ such that for $\eta \in (0, 0.1)$ we have

$$\sigma(\{x \in \Sigma : \operatorname{dist}(x, Q) + \operatorname{dist}(x, \Sigma \setminus Q) < \eta\ell(Q)\}) \leq \eta^\gamma \sigma(Q). \tag{2.1}$$

Remark 2.1. Note that in the above we assume $\mathcal{D}_k$ to be defined for all $k \in \mathbb{Z}$. In the case of unbounded Σ, this translates to having arbitrarily large cubes as $k \to -\infty$. In the case of compact Σ, there exists some k_0 such that for all $k \leq k_0$ we have $\mathcal{D}_k = \{\Sigma\}$. However, in our proof we will assume that Σ is unbounded; see Lemma 2.5.

In our setting, the results [**14**, Theorem 2.9, Theorem 5.9] can be summarized as follows.

Lemma 2.2. *Let σ be a d-Ahlfors regular measure on $\mathbb{R}^n$. Then, there exist $1 \leq N < \infty$ and a small constant $0 < \delta < 0.01$, depending only on the Ahlfors regularity constants of σ, such that the following holds. Let $\Omega = \{1, \ldots, N\}$. For each $\omega \in \Omega$ we have a system of cubes $\mathcal{D}(\omega)$ satisfying the usual properties of David–Christ cubes, and additionally, for all $x \in \operatorname{supp} \sigma$ and $r > 0$ there are $\omega \in \Omega$, $k \in \mathbb{Z}$, and $Q \in \mathcal{D}_k(\omega)$ with*

$$B(x, r) \cap \operatorname{supp} \sigma \subset Q$$

and

$$\ell(Q) = \delta^k \sim_\delta r.$$

Remark 2.3. The construction in [**14**] is valid for general (geometrically) doubling metric spaces, possibly with no underlying measure space structure. The constants N and δ from Lemma 2.2 depend on the doubling constant of the metric space. Hytönen and Tapiola construct two different kinds of cubes, which they call open and closed cubes; see [**14**, Theorem 2.9]. In the above we consider closed cubes, so that properties (b), (c), and (d) follow immediately from [**14**, Theorem 2.9]. To get the property (a) one uses the fact that interiors of P and Q are disjoint by [**14**, (2.11)], and then $\sigma(\partial P) = \sigma(\partial Q) = 0$ follows from (e). To prove the thin boundaries property (e) one may adapt the proof of Christ [**4**, pp. 610–612] together with the Ahlfors regularity of σ. We omit the details.

From now on, let us fix a uniformly rectifiable measure σ, with $\Sigma = \operatorname{supp} \sigma$. Let Ω, δ, and $\mathcal{D}(\omega)$ be as in Lemma 2.2. For simplicity, in our estimates we will not track the dependence of implicit constants on δ.

For all $\omega \in \Omega$ and $Q \in \mathcal{D}_k(\omega)$ we will write

$$\mathcal{D}(Q) := \{P \in \mathcal{D}(\omega) : P \subset Q\},$$
$$\mathsf{Ch}(Q) := \mathcal{D}(Q) \cap \mathcal{D}_{k+1}(\omega).$$

The elements of $\mathsf{Ch}(Q)$ will be called children of Q, and Q will be called their parent.

Set

$$B_Q := B(z_Q, 4\ell(Q)),$$

so that $Q \subset B_Q \cap \Sigma$, and whenever $P \in \mathcal{D}(Q)$ we also have $B_P \subset B_Q$.

Fix some $\omega_0 \in \Omega$, and set

$$\mathcal{D} := \mathcal{D}(\omega_0).$$

This will be our system of reference.

Recall that for each $x \in \operatorname{supp}\sigma$ and $r > 0$ there exists $\omega \in \Omega$ and $Q \in \mathcal{D}(\omega)$ such that $B(x,r) \cap \operatorname{supp}\sigma \subset Q$ and $\ell(Q) \sim r$. Using this fact, to each $Q \in \mathcal{D}$ we assign an index $\omega(Q) \in \Omega$ in the following way. We choose $\omega(Q)$ so that there exists a cube $R = R(Q) \in \mathcal{D}(\omega(Q))$ satisfying $B_Q \cap \operatorname{supp}\sigma \subset R$ and $\ell(R) \sim \ell(Q)$. If there is more than one such ω, we simply choose one.

For any $\omega \in \Omega$ we define also $\mathcal{G}(\omega) \subset \mathcal{D}$ as the family of cubes $Q \in \mathcal{D}$ such that $\omega(Q) = \omega$. Clearly,

$$\bigcup_{\omega\in\Omega} \mathcal{G}(\omega) = \mathcal{D}.$$

2.3. $\boldsymbol{\alpha}$-numbers. In proving the main theorem, it will be more convenient to work with dyadic versions of the α-numbers. Below we will introduce the notation needed for this framework. Given a Radon measure μ we denote by $L^\mu_{x,r}$ a minimizing d-plane for $\alpha_\mu(x,r)$, and by $c^\mu_{x,r}$ the corresponding constant. They may be nonunique, in which case we just choose one of the minimizers. Set $\mathcal{P}^\mu_{x,r} = \mathcal{H}^d \llcorner L^\mu_{x,r}$ and $\mathcal{L}^\mu_{x,r} = c^\mu_{x,r}\mathcal{P}^\mu_{x,r}$. If $B = B(x,r)$, we will also write L^μ_B, c^μ_B, etc.

For $Q \in \mathcal{D}$ and a Radon measure μ we set

$$\alpha_\mu(Q) := \alpha_\mu(B_Q).$$

We will write $L^\mu_Q := L^\mu_{B_Q}$, $c^\mu_Q := c^\mu_{B_Q}$, etc.

Observe that whenever $B_1 \subset B_2$ are balls we have $\operatorname{Lip}_1(B_1) \subset \operatorname{Lip}_1(B_2)$, and so if $r_{B_1} \geq C r_{B_2}$, then

$$\begin{aligned} \alpha_\mu(B_1) &= \frac{1}{r_{B_1}^{d+1}} \inf_{c\in\mathbb{R},L} F_{B_1}(\mu, c\mathcal{H}^d \llcorner L) \\ &\leq \frac{1}{r_{B_1}^{d+1}} F_{B_1}(\mu, \mathcal{L}^\mu_{B_2}) \leq \frac{1}{r_{B_1}^{d+1}} F_{B_2}(\mu, \mathcal{L}^\mu_{B_2}) \sim_C \alpha_\mu(B_2). \end{aligned} \tag{2.2}$$

Consider the following square function:

$$J(x) = \Big(\sum_{x\in Q\in\mathcal{D}} \alpha_{f\sigma}(Q)^2 + |f|^2_{B_Q} \alpha_\sigma(Q)^2 \Big)^{1/2}. \tag{2.3}$$

The main theorem will follow from the following dyadic version:

Theorem 2.4. *Let σ be a uniformly rectifiable measure with unbounded support, and let $f \in L^p(\sigma)$ for some $1 < p < \infty$. Then*

$$\|Jf\|_{L^p(\sigma)} \sim \|f\|_{L^p(\sigma)}.$$

First, let us show why we may assume that $\operatorname{supp}\sigma$ is unbounded.

Lemma 2.5. *It suffices to prove the main theorem in the case that* $\operatorname{supp}\sigma$ *is unbounded.*

Proof: Suppose σ has compact support. Without loss of generality, we may assume $\operatorname{diam}(\operatorname{supp}\sigma) = 1$, $\operatorname{supp}\sigma \subseteq \mathbb{B} = B(0,1)$, and $L_{\mathbb{B}}^{\sigma} = \mathbb{R}^d$. Let

$$\mu = \sigma + \mathcal{P}_{\mathbb{B}}^{\sigma} \llcorner (\mathbb{R}^d \backslash 4\mathbb{B}).$$

It is not hard to show that μ is also UR. If the main theorem holds for UR measures of unbounded support, then it holds for μ. Let $f \in L^p(\sigma) \subseteq L^p(\mu)$ and let

$$\theta_\sigma^f(x,r) := \alpha_{f\sigma}(x,r) + |f|_{x,r}\alpha_\sigma(x,r),$$

so that, by the main theorem,

$$\left\| \left(\int_0^\infty \theta_\mu^f(x,r)^2 \frac{dr}{r} \right)^{1/2} \right\|_{L^p(\mu)} \sim \|f\|_{L^p(\mu)} = \|f\|_{L^p(\sigma)}. \tag{2.4}$$

Observe that

$$\theta_\sigma^f(x,r) = \theta_\mu^f(x,r) \text{ for } x \in \operatorname{supp}\sigma \text{ and } 0 < r < 2.$$

Thus,

$$\begin{aligned} \left\| \left(\int_0^2 \theta_\sigma^f(x,r)^2 \frac{dr}{r} \right)^{1/2} \right\|_{L^p(\sigma)} &= \left\| \left(\int_0^2 \theta_\mu^f(x,r)^2 \frac{dr}{r} \right)^{1/2} \right\|_{L^p(\sigma)} \\ &\le \left\| \left(\int_0^\infty \theta_\mu^f(x,r)^2 \frac{dr}{r} \right)^{1/2} \right\|_{L^p(\mu)} \overset{(2.4)}{\lesssim} \|f\|_{L^p(\sigma)}. \end{aligned}$$

Furthermore, we claim that for any $x \in \operatorname{supp}\sigma$ and $r > 2$ we have

$$\theta_\sigma^f(x,r) \lesssim r^{-d}|f|_{\mathbb{B}}. \tag{2.5}$$

Indeed, since $\operatorname{supp} f \subset \operatorname{supp}\sigma \subset \mathbb{B}$,

$$\alpha_{f\sigma}(x,r) \le \frac{1}{r^{d+1}} F_{B(x,r)}(f\sigma, 0) \le \frac{1}{r^d}\int_{\mathbb{B}} |f|\, d\sigma \sim \frac{1}{r^d}|f|_{\mathbb{B}}, \tag{2.6}$$

and also

$$|f|_{x,r}\alpha_\sigma(x,r) \lesssim \frac{1}{r^d}\int_{\mathbb{B}} |f|\, d\sigma \sim \frac{1}{r^d}|f|_{\mathbb{B}}.$$

It follows from (2.5) that

$$\int_2^\infty \theta_\sigma^f(x,r)^2 \frac{dr}{r} \lesssim \int_2^\infty |f|_{\mathbb{B}}^2 \frac{dr}{r^{2d+1}} \lesssim |f|_{\mathbb{B}}^2,$$

and so

$$\left\| \left(\int_0^\infty \theta_\sigma^f(x,r)^2 \frac{dr}{r} \right)^{1/2} \right\|_{L^p(\sigma)} \lesssim \|f\|_{L^p(\sigma)} + \int_{\mathbb{B}} |f|\, d\sigma \lesssim \|f\|_{L^p(\sigma)}.$$

To finish the proof we now need to show the reverse inequality. Notice that since f is supported on $\operatorname{supp}\sigma$, $\alpha_{f\mu}(x,r)=\alpha_{f\sigma}(x,r)$ for all $x\in\operatorname{supp}\sigma$ and $r>0$. We can argue just as in (2.6) to get that for $x\in\operatorname{supp}\mu$ and $r\geq 2$,

$$\alpha_{f\mu}(x,r)=\alpha_{f\sigma}(x,r)\leq\frac{1}{r^{d+1}}F_{B(x,r)}(f\sigma,0)\lesssim\frac{|f|_{\mathbb{B}}}{r^d}\lesssim\frac{|f|_{2\mathbb{B}}}{r^d}\alpha_\sigma(2\mathbb{B}),$$

where we also use $\alpha_\sigma(2\mathbb{B})\sim 1$.

Hence,

$$\begin{aligned}
&\int_{\mathbb{B}}\left(\int_0^\infty\alpha_{f\mu}(x,r)^2\frac{dr}{r}\right)^{\frac{p}{2}}d\mu(x)\\
&\quad\lesssim\int_{\mathbb{B}}\left(\int_0^2\alpha_{f\mu}(x,r)^2\frac{dr}{r}\right)^{\frac{p}{2}}d\mu(x)+\int_{\mathbb{B}}\left(\int_2^\infty\alpha_{f\mu}(x,r)^2\frac{dr}{r}\right)^{\frac{p}{2}}d\mu(x)\\
&\quad\lesssim\int_{\mathbb{B}}\left(\int_0^2\alpha_{f\sigma}(x,r)^2\frac{dr}{r}\right)^{\frac{p}{2}}d\mu(x)+\int_{\mathbb{B}}\left(\int_2^\infty(|f|_{2\mathbb{B}}\alpha_\sigma(2\mathbb{B}))^2\frac{dr}{r^{2d+1}}\right)^{\frac{p}{2}}d\mu(x)\\
&\quad\lesssim\int_{\mathbb{B}}\left(\int_0^2\alpha_{f\sigma}(x,r)^2\frac{dr}{r}\right)^{\frac{p}{2}}d\mu(x)+(|f|_{2\mathbb{B}}\alpha_\sigma(2\mathbb{B}))^p\\
&\quad\lesssim\int_{\mathbb{B}}\left(\int_0^\infty\theta_\sigma^f(x,r)^2\frac{dr}{r}\right)^{\frac{p}{2}}d\mu(x),
\end{aligned}$$

where we use (2.2) in the final inequality.

Furthermore, for $x\in\mathbb{R}^d\backslash 4\mathbb{B}$, if $\alpha_{f\mu}(x,r)\neq 0$, then $r\geq|x|/2$ and so

$$\begin{aligned}
&\int_{\mathbb{R}^d\backslash 4\mathbb{B}}\left(\int_0^\infty\alpha_{f\mu}(x,r)^2\frac{dr}{r}\right)^{\frac{p}{2}}d\mu(x)\\
&\qquad=\sum_{j=2}^\infty\int_{\mathbb{R}^d\cap(2^{j+1}\mathbb{B}\backslash 2^j\mathbb{B})}\left(\int_{|x|/2}^\infty(|f|_{2\mathbb{B}}\alpha_\sigma(2\mathbb{B}))^2\frac{dr}{r^{2d+1}}\right)^{\frac{p}{2}}d\mu(x)\\
&\qquad\lesssim(|f|_{2\mathbb{B}}\alpha_\sigma(2\mathbb{B}))^p\sum_{j=2}^\infty\int_{\mathbb{R}^d\cap(2^{j+1}\mathbb{B}\backslash 2^j\mathbb{B})}|x|^{-pd}\,d\mu(x)\\
&\qquad\lesssim(|f|_{2\mathbb{B}}\alpha_\sigma(2\mathbb{B}))^p\lesssim\int_{\mathbb{B}}\left(\int_0^\infty|f|_{x,r}\alpha_\sigma(x,r)^2\frac{dr}{r}\right)^{\frac{p}{2}}d\mu(x),
\end{aligned}$$

again using (2.2). Note that $\mu\llcorner\mathbb{B}=\sigma$ and $\mu(4\mathbb{B}\setminus\mathbb{B})=0$, and so the two estimates above imply

$$\left\|\left(\int_0^\infty\alpha_{f\mu}(x,r)^2\frac{dr}{r}\right)^{1/2}\right\|_{L^p(\mu)}\lesssim\left\|\left(\int_0^\infty\theta_\sigma^f(x,r)^2\frac{dr}{r}\right)^{1/2}\right\|_{L^p(\sigma)}. \tag{2.7}$$

Note that for $x \in \operatorname{supp}\sigma$ and $r < 2$ we have $\mathcal{P}^\sigma_{x,r} = \mathcal{P}^\mu_{x,r}$. For $r \geq 2$, notice that $\alpha_\sigma(2\mathbb{B}) \sim 1$, and so

$$\alpha_\mu(x,r) \leq \frac{1}{r^{d+1}} F_{B(x,r)}(\mu, \mathcal{P}^\sigma_{\mathbb{B}}) = \frac{1}{r^{d+1}} F_{B(x,r)}(\sigma, \mathcal{P}^\sigma_{\mathbb{B}} \llcorner 4B) \lesssim r^{-d} \lesssim \frac{\alpha_\sigma(2\mathbb{B})}{r^d},$$

hence

$$|f|^\mu_{x,r} \alpha_\mu(x,r) \leq |f|_{2\mathbb{B}} \frac{\alpha_\sigma(2\mathbb{B})}{r^{2d}},$$

where $|f|^\mu_{x,r} = \int_{B(x,r)} f \, d\mu / \mu(B(x,r))$. Thus, just as how we proved (2.7), we can show

$$\left\| \int_0^\infty (|f|^\mu_{x,r})^2 \alpha_\mu(x,r)^2 \frac{dr}{r} \right\|_{L^p(\mu)} \lesssim \left\| \int_0^\infty |f|^2_{x,r} \alpha_\sigma(x,r)^2 \frac{dr}{r} \right\|_{L^p(\sigma)}.$$

This, (2.7) and (2.4) imply the desired estimate:

$$\|f\|_{L^p(\sigma)} \lesssim \left\| \left(\int_0^\infty \theta^f_\sigma(x,r)^2 \frac{dr}{r} \right)^{1/2} \right\|_{L^p(\sigma)}. \qquad \square$$

Proof of the main theorem using Theorem 2.4: By Lemma 2.5, we may assume that $\operatorname{supp}\sigma = \Sigma$ is unbounded, so that Theorem 2.4 holds.

Let $x \in \Sigma$, $r > 0$. Let $k \in \mathbb{Z}$ be such that $\delta^{k+1} < r \leq \delta^k$, and let Q be a cube in $\mathcal{D}_k$ containing x. Recall that $Q \subset B(z_Q, 3\ell(Q))$. Since $r \leq \ell(Q)$, we have

$$B(x,r) \subset B(z_Q, 3\ell(Q) + r) \subset B(z_Q, 4\ell(Q)) = B_Q.$$

Hence, by (2.2),

$$\alpha_{f\sigma}(x,r) \lesssim \alpha_{f\sigma}(Q).$$

We also have $|f|_{x,r} \lesssim |f|_{B_Q}$, and so

$$|f|_{x,r} \alpha_\sigma(x,r) \lesssim |f|_{B_Q} \alpha_\sigma(Q).$$

Consequently,

$$\int_{\delta^{k+1}}^{\delta^k} (\alpha_{f\sigma}(x,r) + |f|_{x,r}\alpha_\sigma(x,r))^2 \frac{dr}{r} \lesssim \alpha_{f\sigma}(Q)^2 + |f|^2_{B_Q} \alpha_\sigma(Q)^2.$$

Summing over $k \in \mathbb{Z}$ yields

$$\int_0^\infty (\alpha_{f\sigma}(x,r) + |f|_{x,r}\alpha_\sigma(x,r))^2 \frac{dr}{r} \lesssim \sum_{x \in Q \in \mathcal{D}} \alpha_{f\sigma}(Q)^2 + |f|^2_{B_Q} \alpha_\sigma(Q)^2.$$

Similarly, for $x \in \Sigma$, $r > 0$, $\delta^{k+1} < r \leq \delta^k$, we may consider a cube $Q \in \mathcal{D}_{k+2}$ such that $x \in Q \subset B_Q \subset B(x,r)$. Mimicking the estimates above, one gets

$$\sum_{x \in Q \in \mathcal{D}} \alpha_{f\sigma}(Q)^2 + |f|^2_{B_Q} \alpha_\sigma(Q)^2 \lesssim \int_0^\infty (\alpha_{f\sigma}(x,r) + |f|_{x,r}\alpha_\sigma(x,r))^2 \frac{dr}{r}.$$

Putting the two estimates together, we get the comparability of the dyadic and continuous variants of the square function:

$$Jf(x)^2 = \sum_{x\in Q\in\mathcal{D}} \alpha_{f\sigma}(Q)^2 + |f|_{B_Q}^2\alpha_\sigma(Q)^2$$
$$\sim \int_0^\infty (\alpha_{f\sigma}(x,r) + |f|_{x,r}\alpha_\sigma(x,r))^2 \frac{dr}{r}. \qquad \square$$

Theorem 2.4 will follow from the results from the next three sections. From now on we assume that σ is a uniformly rectifiable measure with unbounded support.

3. The estimate $\|Jf\|_2 \lesssim \|f\|_2$

First, we prove the estimate $\|Jf\|_p \lesssim \|f\|_p$ in the case $p = 2$.

Proposition 3.1. *Let $f \in L^2(\sigma)$. Then*

$$\sum_{Q\in\mathcal{D}} (\alpha_{f\sigma}(Q)^2 + |f|_{B_Q}^2\alpha_\sigma(Q)^2)\ell(Q)^d \lesssim \|f\|_{L^2(\sigma)}^2.$$

Our main tool in the proof of Proposition 3.1 is the martingale difference operators associated to systems of cubes $\mathcal{D}(\omega)$.

Given $\omega \in \Omega$, $Q \in \mathcal{D}(\omega)$, and $f \in L^1_{\rm loc}(\sigma)$ we set

$$\Delta_Q f = \sum_{P\in\mathsf{Ch}(Q)} \langle f\rangle_P \mathbb{1}_P - \langle f\rangle_Q \mathbb{1}_Q.$$

Observe that all $\Delta_Q f$ have zero mean, i.e. $\int \Delta_Q f\, d\sigma = 0$.

It is well known (see e.g. [**13**, Chapter 6.4]) that given $f \in L^2(\sigma)$ and some system of cubes $\mathcal{D}(\omega)$ we have

$$f = \sum_{Q\in\mathcal{D}(\omega)} \Delta_Q f$$

with the convergence understood in the L^2 sense. It is crucial that $\sigma(\Sigma) = \infty$, so that $f + C \in L^2(\sigma)$ if and only if $C = 0$ (in the case $\sigma(\Sigma) < \infty$ one would have to subtract from the left-hand side above the average of f).

Note that $\Delta_Q f$ are mutually orthogonal in $L^2(\sigma)$, so that

$$\|f\|_{L^2(\sigma)}^2 = \sum_{Q\in\mathcal{D}(\omega)} \|\Delta_Q f\|_{L^2(\sigma)}^2. \tag{3.1}$$

Moreover, if $Q \in \mathcal{D}(\omega)$, then for σ-a.e. $x \in Q$

$$f(x) = \langle f\rangle_Q + \sum_{P\in\mathcal{D}(Q)} \Delta_P f(x). \tag{3.2}$$

Lemma 3.2. *Suppose $Q \in \mathcal{D}$, and let $R = R(Q) \in \mathcal{D}(\omega(Q))$ be as in Subsection 2.2. Then, for $f \in L^2(\sigma)$ we have*

$$\alpha_{f\sigma}(Q) \lesssim |\langle f\rangle_R|\alpha_\sigma(R) + \sum_{P\in\mathcal{D}(R)} \frac{\ell(P)^{1+d/2}}{\ell(Q)^{1+d}}\|\Delta_P f\|_2. \tag{3.3}$$

Proof: Let $\varphi \in \operatorname{Lip}_1(B_Q)$ and consider a candidate for $\mathcal{L}_Q^{f\sigma}$ of the form $\langle f\rangle_R\mathcal{L}_Q^\sigma$. For all $x \in B_Q \cap \operatorname{supp}\sigma$ we have $x \in R$, so that using (3.2),

$$\begin{aligned}
&\left|\int \varphi(x)f(x)\,d\sigma(x) - \langle f\rangle_R \int \varphi(x)\,d\mathcal{L}_Q^\sigma(x)\right| \\
&\quad= \left|\int \varphi(x)\langle f\rangle_R + \sum_{P\in\mathcal{D}(R)} \varphi(x)\Delta_P f(x)\,d\sigma(x) - \int \varphi(x)\langle f\rangle_R\,d\mathcal{L}_Q^\sigma(x)\right| \\
&\quad\le |\langle f\rangle_R|\left|\int\varphi(x)\,d\sigma(x) - \int\varphi(x)\,d\mathcal{L}_Q^\sigma(x)\right| + \sum_{P\in\mathcal{D}(R)}\left|\int\varphi(x)\Delta_P f(x)\,d\sigma(x)\right| \\
&\quad=: I_1 + I_2.
\end{aligned}$$

It is clear that

$$I_1 \le |\langle f\rangle_R|\alpha_\sigma(Q)\ell(Q)^{d+1} \overset{(2.2)}{\lesssim} |\langle f\rangle_R|\alpha_\sigma(R)\ell(Q)^{d+1},$$

which gives rise to the first term on the right-hand side of (3.3).

Concerning I_2, we use the zero mean property of martingale difference operators, and the fact that $\varphi \in \operatorname{Lip}_1(B_Q)$, to get

$$\begin{aligned}
I_2 &= \sum_{P\in\mathcal{D}(R)}\left|\int(\varphi(x) - \varphi(z_P))\Delta_P f(x)\,d\sigma(x)\right| \\
&\le \sum_{P\in\mathcal{D}(R)}\int|\varphi(x) - \varphi(z_P)||\Delta_P f(x)|\,d\sigma(x) \\
&\lesssim \sum_{P\in\mathcal{D}(R)}\ell(P)\|\Delta_P f\|_1 \overset{\text{Hölder}}{\lesssim} \sum_{P\in\mathcal{D}(R)}\ell(P)^{1+d/2}\|\Delta_P f\|_2.
\end{aligned}$$

Dividing by $\ell(Q)^{d+1}$ and taking the supremum over $\varphi \in \operatorname{Lip}_1(B_Q)$ yields (3.3). $\square$

Proof of Proposition 3.1: We begin by noting that, since σ is uniformly rectifiable, $\alpha_\sigma(Q)^2\ell(Q)^d$ is a Carleson measure by the results from [**19**]; see Theorem 1.1. Therefore, the estimate

$$\sum_{Q\in\mathcal{D}} |f|_{B_Q}^2\alpha_\sigma(Q)^2\ell(Q)^d \lesssim \|f\|_{L^2(\sigma)}^2$$

follows immediately from Carleson's embedding theorem (see e.g. [**21**, Theorem 5.8]), and we only need to estimate the sum involving $\alpha_{f\sigma}(Q)$.

Observe that for each $\omega \in \Omega$ and $R \in \mathcal{D}(\omega)$ there is at most a bounded number of cubes $Q \in \mathcal{D}$ such that $R(Q) = R$.

Fix some $\omega \in \Omega$. Recall that $\mathcal{G}(\omega)$ is the family of cubes $Q \in \mathcal{D}$ such that $\omega(Q) = \omega$. We apply (3.3) and the observation above to get

$$\begin{aligned}\sum_{Q\in\mathcal{G}(\omega)} \alpha_{f\sigma}(Q)^2\ell(Q)^d &\lesssim \sum_{R\in\mathcal{D}(\omega)} |\langle f\rangle_R|^2\alpha_\sigma(R)^2\ell(R)^d \\ &\quad + \sum_{R\in\mathcal{D}(\omega)}\Big(\sum_{P\in\mathcal{D}(R)} \frac{\ell(P)^{1+d/2}}{\ell(R)^{1+d/2}}\|\Delta_P f\|_2\Big)^2 \\ &=: S_1 + S_2.\end{aligned}$$

Concerning S_1, we may use Carleson's embedding theorem again to estimate $S_1 \lesssim \|f\|_2^2$.

Moving on to S_2, we apply the Cauchy–Schwarz inequality to get

$$S_2 \le \sum_{R\in\mathcal{D}(\omega)}\Big(\sum_{P\in\mathcal{D}(R)}\frac{\ell(P)}{\ell(R)}\|\Delta_P f\|_2^2\Big)\Big(\sum_{P\in\mathcal{D}(R)}\frac{\ell(P)^{d+1}}{\ell(R)^{d+1}}\Big).$$

It is easy to see that, due to the Ahlfors regularity of σ, $\sum_{P\in\mathcal{D}(R)}\frac{\ell(P)^{d+1}}{\ell(R)^{d+1}} \lesssim 1$. Thus,

$$\begin{aligned}S_2 &\le \sum_{R\in\mathcal{D}(\omega)}\sum_{P\in\mathcal{D}(R)}\frac{\ell(P)}{\ell(R)}\|\Delta_P f\|_2^2 \\ &= \sum_{P\in\mathcal{D}(\omega)}\|\Delta_P f\|_2^2 \sum_{\substack{R\in\mathcal{D}(\omega)\\ R\supset P}}\frac{\ell(P)}{\ell(R)} \lesssim \sum_{P\in\mathcal{D}(\omega)}\|\Delta_P f\|_2^2 \overset{(3.1)}{\lesssim} \|f\|_2^2.\end{aligned}$$

Putting the estimates above together we arrive at

$$\sum_{Q\in\mathcal{G}(\omega)} \alpha_{f\sigma}(Q)^2\ell(Q)^d \lesssim \|f\|_2^2.$$

Summing over all $\omega \in \Omega$ (recall that $\#\Omega$ is bounded) we get the desired estimate. □

4. The estimate $\|Jf\|_p \lesssim \|f\|_p$ for $1 < p < \infty$

In this section we use the estimate $||Jf||_2 \lesssim \|f\|_2$ to prove $||Jf||_p \lesssim \|f\|_p$ for general $1 < p < \infty$. More precisely, we will show a localized version of the estimate, which implies the global estimate via a limiting argument.

Fix an arbitrary $Q_0 \in \mathcal{D}$ and set

$$J_0 f(x) := \Big(\sum_{x\in Q\in\mathcal{D}(Q_0)} \alpha_{f\sigma}(Q)^2 + |f|_{B_Q}^2\alpha_\sigma(Q)^2\Big)^{1/2}.$$

Proposition 4.1. *Let $1 < p < \infty$ and $f \in L^p(\sigma)$. Then,*

$$\|J_0 f\|_{L^p(Q_0)} \lesssim_p \|f\|_{L^p(B_{Q_0})}.$$

The proposition follows easily from a good-lambda inequality as stated below. Let M denote the noncentered maximal Hardy–Littlewood operator with respect to σ, i.e.

$$Mf(x) = \sup\{|f|_B : x \in B,\ B \text{ is a ball}\}.$$

Since σ is Ahlfors regular, the operator M is bounded on $L^p(\sigma)$ for $p > 1$; see e.g. [**21**, Theorem 2.6, Remark 2.7].

Lemma 4.2. *Let $f \in L^1_{\mathrm{loc}}(\sigma)$. For any $\alpha > 1$ there exists $\varepsilon = \varepsilon(\alpha) > 0$ such that for all $\lambda > 0$*

$$\sigma(\{x \in Q_0 : J_0 f(x) > \alpha\lambda,\ Mf(x) \le \varepsilon\lambda\}) \le \frac{9}{10}\sigma(\{x \in Q_0 : J_0 f(x) > \lambda\}). \tag{4.1}$$

Let us show how to use the above to prove Proposition 4.1.

Proof of Proposition 4.1: Note that $J_0 f = J_0(f \mathbb{1}_{B_{Q_0}})$, so without loss of generality we may assume that $\operatorname{supp} f \subset B_{Q_0}$. Let $\alpha = \alpha(p) > 1$ be so close to 1 that $0.9\alpha^p < 0.95$, and let $\varepsilon = \varepsilon(\alpha)$ be as in Lemma 4.2. We use the layer cake representation to get

$$\begin{aligned}
\int_{Q_0} J_0 f(x)^p \, d\sigma(x) &= p \int_0^\infty \lambda^{p-1} \sigma(\{x \in Q_0 : J_0 f(x) > \lambda\}) \, d\lambda \\
&= p\alpha^p \int_0^\infty \lambda^{p-1} \sigma(\{x \in Q_0 : J_0 f(x) > \alpha\lambda\}) \, d\lambda \\
&\le p\alpha^p \int_0^\infty \lambda^{p-1} \sigma(\{x \in Q_0 : J_0 f(x) > \alpha\lambda,\ Mf(x) \le \varepsilon\lambda\}) \, d\lambda \\
&\quad + p\alpha^p \int_0^\infty \lambda^{p-1} \sigma(\{x \in Q_0 : Mf(x) > \varepsilon\lambda\}) \, d\lambda \\
&\overset{(4.1)}{\le} \frac{9}{10} p\alpha^p \int_0^\infty \lambda^{p-1} \sigma(\{x \in Q_0 : J_0 f(x) > \lambda\}) \, d\lambda \\
&\quad + \alpha^p \varepsilon^{-p} \int_{Q_0} Mf(x)^p \, d\sigma(x) \\
&\le \frac{19}{20} p \int_0^\infty \lambda^{p-1} \sigma(\{x \in Q_0 : J_0 f(x) > \lambda\}) \, d\lambda \\
&\quad + \alpha^p \varepsilon^{-p} \int_{Q_0} Mf(x)^p \, d\sigma(x) \\
&= \frac{19}{20} \int_{Q_0} J_0 f(x)^p \, d\sigma(x) + \alpha^p \varepsilon^{-p} \int_{Q_0} Mf(x)^p \, d\sigma(x).
\end{aligned}$$

Absorbing the first term from the right-hand side into the left-hand side, we arrive at

$$\int_{Q_0} J_0 f(x)^p \, d\sigma(x) \le 20\alpha^p \varepsilon^{-p} \int Mf(x)^p \, d\sigma(x).$$

We use the L^p boundedness of M and the assumption $\operatorname{supp} f \subset B_{Q_0}$ to conclude

$$\int_{Q_0} J_0 f(x)^p \, d\sigma(x) \lesssim_{\alpha,\varepsilon} \int_{B_{Q_0}} f(x)^p \, d\sigma(x). \qquad \square$$

The remainder of this section is dedicated to proving Lemma 4.2.

4.1. Preliminaries. Fix $\alpha > 1$ and $\lambda > 0$. First, we set

$$E_\lambda = \{x \in Q_0 : J_0 f(x) > \lambda\}.$$

Consider the covering of E_λ with a family of cubes $\mathscr{C}_\lambda \subset \mathcal{D}(Q_0)$ such that for every $S \in \mathscr{C}_\lambda$ we have

$$\sigma(S \cap E_\lambda) \ge 0.99\sigma(S)$$

and S is the maximal cube with this property. Since the cubes from $\mathscr{C}_\lambda$ are pairwise disjoint, to get (4.1) it is enough to find $\varepsilon = \varepsilon(\alpha)$ such that for each $S \in \mathscr{C}_\lambda$ we have

$$\sigma(\{x \in S : J_0 f(x) > \alpha\lambda,\, Mf(x) \le \varepsilon\lambda\}) \le \frac{8}{10}\sigma(S). \tag{4.2}$$

Fix $S \in \mathscr{C}_\lambda$. Without loss of generality assume that

$$\sigma(\{x \in S : Mf(x) \le \varepsilon\lambda\}) > \frac{8}{10}\sigma(S), \tag{4.3}$$

otherwise there is nothing to prove.

Given $x \in S$, we split the sum from the definition of $J_0 f(x)$ into two parts:

$$\begin{aligned} J_0 f(x)^2 &= \sum_{x \in Q \in \mathcal{D}(S)} (\alpha_{f\sigma}(Q)^2 + |f|_{B_Q}^2 \alpha_\sigma(Q)^2) \\ &\quad + \sum_{S \subsetneq Q \in \mathcal{D}(Q_0)} (\alpha_{f\sigma}(Q)^2 + |f|_{B_Q}^2 \alpha_\sigma(Q)^2) \\ &=: J_1 f(x)^2 + J_2 f(x)^2. \end{aligned} \tag{4.4}$$

Clearly, $J_2 f(x) \equiv J_2 f$ is just a constant. We claim that $J_2 f \le \lambda$. There are two cases: either $S = Q_0$ (in which case $J_2 f = 0$), or $S \subsetneq \hat{S} \subset Q_0$, where $\hat{S}$ is the parent of S. If the latter is true, then by the definition of $\mathscr{C}_\lambda$ there exists $y \in \hat{S}$ such that $y \notin E_\lambda$. Hence, by the definition of E_λ, we get that

$$J_2 f \le J_0 f(y) \le \lambda.$$

We will show the following.

Lemma 4.3. *There exists a set $S_1 \subset S$ such that $\sigma(S_1) \geq 0.5\sigma(S)$ and*

$$\int_{S_1} J_1 f(x)^2 \, d\sigma(x) \lesssim \varepsilon^2 \lambda^2 \sigma(S_1).$$

The estimate (4.2) follows from the above easily. Indeed, using Chebyshev, we can find $S_2 \subset S_1$ such that for all $x \in S_2$ we have $J_1 f(x) \lesssim \varepsilon\lambda$ and $\sigma(S_2) \geq 0.5\sigma(S_1) \geq 0.2\sigma(S)$. Then, choosing $\varepsilon = \varepsilon(\alpha)$ small enough, (4.4) gives $J_0 f(x)^2 \leq \lambda^2 + C\varepsilon^2\lambda^2 \leq \alpha^2\lambda^2$ on S_2, so that

$$\sigma(\{x \in S : J_0 f(x) > \alpha\lambda,\, Mf(x) \leq \varepsilon\lambda\}) \leq \sigma(S \setminus S_2) \leq \frac{8}{10}\sigma(S).$$

So our goal is to prove Lemma 4.3.

4.2. Calderón–Zygmund decomposition. Let $R = R(S)$ be as in Subsection 2.2, so that $B_S \cap \operatorname{supp} \sigma \subset R$. We consider a variant of the Calderón–Zygmund decomposition of $f\mathbb{1}_R$ with respect to $\mathcal{D}(R)$ at the level $2\varepsilon\lambda$.

First, let $\{Q_j\}_j \subset \mathcal{D}(R)$ be maximal cubes satisfying $|f|_{B_{Q_j}} \geq 2\varepsilon\lambda$. Note that for all $x \in Q_j$ (and recalling that M is the noncentered maximal function) we have

$$Mf(x) \geq |f|_{B_{Q_j}} \geq 2\varepsilon\lambda.$$

Hence, $\bigcup_j Q_j \subset \{x \in R : Mf(x) \geq 2\varepsilon\lambda\}$, and so

$$\begin{aligned} \sigma\Big(R \setminus \bigcup_j Q_j\Big) &\geq \sigma\Big(S \setminus \bigcup_j Q_j\Big) \geq \sigma(\{x \in S : Mf(x) \leq \varepsilon\lambda\}) \\ &\overset{(4.3)}{\geq} \frac{8}{10}\sigma(S) \sim \ell(S)^d \sim \ell(R)^d. \end{aligned} \tag{4.5}$$

In particular, $Q_j \neq R$ for all j. Thus, by the maximality of Q_j we get easily

$$|f|_{B_{Q_j}} \sim \varepsilon\lambda. \tag{4.6}$$

We define $g \in L^\infty(\sigma)$ by

$$g(x) = f(x)\mathbb{1}_{R\setminus\bigcup_j Q_j}(x) + \sum_j \langle f\rangle_{Q_j} \mathbb{1}_{Q_j}(x).$$

From the definition of Q_j and (4.6) it follows that $\|g\|_\infty \lesssim \varepsilon\lambda$. We define also $b \in L^1(\sigma)$ as

$$b(x) = \sum_j (f(x) - \langle f\rangle_{Q_j})\mathbb{1}_{Q_j}(x) =: \sum_j b_j(x).$$

Note that $f = g + b$ and for all j we have $\int b_j \, d\sigma = 0$.

4.3. Definition of S_1. We set $S_1 = S \setminus N_\eta$, where N_η is some small neighborhood of $\bigcup_j Q_j$. To make this more precise, given a small $\eta > 0$ we define $N_\eta = \bigcup_j N_{\eta,j}$, where

$$N_{\eta,j} = \{x \in \operatorname{supp}\sigma : \operatorname{dist}(x, Q_j) < \eta\ell(Q_j)\}.$$

The thin boundaries property of $\mathcal{D}$ (2.1) gives

$$\sigma(N_{\eta,j} \setminus Q_j) \leq \eta^\gamma \sigma(Q_j)$$

for some $\gamma \in (0,1)$. From (4.5) and the fact that $\sigma(S) \sim \sigma(R)$ we get

$$\begin{aligned}\sigma(S \setminus N_\eta) &\geq \sigma\Big(S \setminus \bigcup_j Q_j\Big) - \sum_j \sigma(N_{\eta,j} \setminus Q_j) \overset{(4.5)}{\geq} \frac{8}{10}\sigma(S) - \sum_j \eta^\gamma \sigma(Q_j) \\ &\geq \frac{8}{10}\sigma(S) - \eta^\gamma\sigma(R) \geq \frac{8}{10}\sigma(S) - C\eta^\gamma\sigma(S) = \Big(\frac{8}{10} - C\eta^\gamma\Big)\sigma(S).\end{aligned}$$

Here C depends only on the implicit constant in $\sigma(S) \sim \sigma(R)$, which in turn depends on the Ahlfors regularity constant of σ and on the parameters from the definition of the system $\mathcal{D}$.

Choosing η so small that $C\eta^\gamma < 0.1$, we get that $S_1 = S \setminus N_\eta$ satisfies

$$\sigma(S_1) \geq \frac{7}{10}\sigma(S).$$

4.4. Estimating $J_1 f$. Now, we will show that

$$\int_{S_1} J_1 f(x)^2 \, d\sigma(x) \lesssim \varepsilon^2\lambda^2\sigma(S_1).$$

Recall that

$$J_1 f(x)^2 = \sum_{x \in Q \in \mathcal{D}(S)} \alpha_{f\sigma}(Q)^2 + \sum_{x \in Q \in \mathcal{D}(S)} |f|_{B_Q}^2 \alpha_\sigma(Q)^2 =: J_1' f(x)^2 + J_1'' f(x)^2.$$

First we deal with $J_1'' f$. Observe that for all $Q \in \mathcal{D}(S)$ intersecting S_1 we have

$$|f|_{B_Q} \lesssim \varepsilon\lambda. \tag{4.7}$$

Indeed, let $y \in Q \cap S_1$, and let $P \in \mathcal{D}(R)$ be such that $y \in P$, $\ell(Q) \sim \ell(P)$, and $B_Q \subset B_P$. By the maximality of Q_j and the fact that $P \setminus \bigcup_j Q_j \neq \varnothing$ we get $|f|_{B_P} \leq 2\varepsilon\lambda$. Estimate (4.7) follows from the inclusion $B_Q \subset B_P$.

Using (4.7) as well as Theorem 1.1 we get

$$\begin{aligned}\int_{S_1}\sum_{x\in Q\in\mathcal{D}(S)}|f|^2_{B_Q}\alpha_\sigma(Q)^2\,d\sigma(x) &\lesssim \varepsilon^2\lambda^2\sum_{Q\in\mathcal{D}(S)}\alpha_\sigma(Q)^2\sigma(Q\cap S_1)\\ &\lesssim \varepsilon^2\lambda^2\sum_{Q\in\mathcal{D}(S)}\alpha_\sigma(Q)^2\sigma(Q)\\ &\lesssim \varepsilon^2\lambda^2\sigma(S)\sim\varepsilon^2\lambda^2\sigma(S_1).\end{aligned}$$

Thus, we are only left with showing

$$\int_{S_1} J_1' f(x)^2\,d\sigma(x)=\int_{S_1}\sum_{x\in Q\in\mathcal{D}(S)}\alpha_{f\sigma}(Q)^2\,d\sigma(x)\lesssim\varepsilon^2\lambda^2\sigma(S_1). \tag{4.8}$$

Lemma 4.4. *For $Q\in\mathcal{D}(S)$ we have*

$$\alpha_{f\sigma}(Q)\lesssim\alpha_{g\sigma}(Q)+\varepsilon\lambda\sum_{j:Q_j\cap B_Q\neq\varnothing}\frac{\ell(Q_j)^{d+1}}{\ell(Q)^{d+1}}.$$

Proof: Let $\varphi\in\operatorname{Lip}_1(B_Q)$. Then, using the decomposition $f(y)=g(y)+b(y)$ valid for all $y\in R\supset B_S\cap\operatorname{supp}\sigma\supset B_Q\cap\operatorname{supp}\sigma$,

$$\begin{aligned}&\left|\int\varphi(y)f(y)\,d\sigma(y)-\int\varphi(y)\,d\mathcal{L}_Q^{g\sigma}(y)\right|\\ &\quad\le\left|\int\varphi(y)g(y)\,d\sigma(y)-\int\varphi(y)\,d\mathcal{L}_Q^{g\sigma}(y)\right|+\left|\int\varphi(y)b(y)\,d\sigma(y)\right|\\ &\quad\lesssim\ell(Q)^{d+1}\alpha_{g\sigma}(Q)+\sum_j\left|\int\varphi(y)b_j(y)\,d\sigma(y)\right|.\end{aligned}$$

Concerning the second term on the right-hand side, recall that $\int b_j\,d\sigma=0$ and that $\operatorname{supp} b_j\subset Q_j$. Keeping that in mind, denoting by x_j the center of Q_j, we estimate in the following way:

$$\begin{aligned}&\sum_j\left|\int\varphi(y)b_j(y)\,d\sigma(y)\right|=\sum_j\left|\int(\varphi(y)-\varphi(x_j))b_j(y)\,d\sigma(y)\right|\\ &\quad\le\sum_j\int|(\varphi(y)-\varphi(x_j))b_j(y)|\,d\sigma(y)\lesssim\sum_{j:Q_j\cap B_Q\neq\varnothing}\ell(Q_j)\int|b_j(y)|\,d\sigma(y)\\ &\quad=\sum_{j:Q_j\cap B_Q\neq\varnothing}\ell(Q_j)\int_{Q_j}|f(y)-\langle f\rangle_{Q_j}|\,d\sigma(y)\lesssim\sum_{j:Q_j\cap B_Q\neq\varnothing}\ell(Q_j)^{d+1}\langle|f|\rangle_{Q_j}\\ &\quad\overset{(4.6)}{\lesssim}\varepsilon\lambda\sum_{j:Q_j\cap B_Q\neq\varnothing}\ell(Q_j)^{d+1}.\end{aligned}$$

Together with the previous string of estimates, taking the supremum over all $\varphi \in \mathrm{Lip}_1(B_Q)$, we get

$$\alpha_{f\sigma}(Q) \lesssim \alpha_{g\sigma}(Q) + \varepsilon\lambda \sum_{j:Q_j\cap B_Q\neq\varnothing} \frac{\ell(Q_j)^{d+1}}{\ell(Q)^{d+1}}. \qquad \square$$

An immediate consequence of Lemma 4.4 is the estimate

$$\begin{aligned}\int_{S_1} J_1' f(x)^2\, d\sigma(x) \lesssim& \int_{S_1} J_1' g(x)^2\, d\sigma(x)\\ &+\varepsilon^2\lambda^2 \int_{S_1} \sum_{x\in Q\in\mathcal{D}(S)} \Bigg(\sum_{j:Q_j\cap B_Q\neq\varnothing} \frac{\ell(Q_j)^{d+1}}{\ell(Q)^{d+1}}\Bigg)^2 d\sigma(x).\end{aligned} \tag{4.9}$$

Using Proposition 3.1 and the fact that $\|g\|_\infty \lesssim \varepsilon\lambda$, $\operatorname{supp} g \subset R$, we get

$$\begin{aligned}\int_{S_1} J_1' g(x)^2\, d\sigma(x) &\le \|Jg\|_2^2 \lesssim \|g\|_2^2\\ &\le \|g\|_\infty^2 \sigma(R) \lesssim \varepsilon^2\lambda^2\sigma(R) \sim \varepsilon^2\lambda^2\sigma(S_1).\end{aligned} \tag{4.10}$$

Moving on to the second term from the right-hand side of (4.9), denote by $\mathsf{Tree} \subset \mathcal{D}(S)$ the family of cubes contained in S that intersect S_1. We have

$$\begin{aligned}&\int_{S_1} \sum_{x\in Q\in\mathcal{D}(S)} \Bigg(\sum_{j:Q_j\cap B_Q\neq\varnothing} \frac{\ell(Q_j)^{d+1}}{\ell(Q)^{d+1}}\Bigg)^2 d\sigma(x)\\ &\qquad\le \sum_{Q\in\mathsf{Tree}} \sigma(Q) \Bigg(\sum_{j:Q_j\cap B_Q\neq\varnothing} \frac{\ell(Q_j)^{d+1}}{\ell(Q)^{d+1}}\Bigg)^2\\ &\qquad\overset{\text{Cauchy–Schwarz}}{\lesssim} \sum_{Q\in\mathsf{Tree}} \ell(Q)^{-d-2} \Bigg(\sum_{j:Q_j\cap B_Q\neq\varnothing} \ell(Q_j)^{d+2}\Bigg)\\ &\qquad\qquad\times \Bigg(\sum_{j:Q_j\cap B_Q\neq\varnothing} \ell(Q_j)^{d}\Bigg).\end{aligned} \tag{4.11}$$

Note that since $Q \in \mathsf{Tree}$ we have $Q \cap S_1 \neq \varnothing$. By the definition of S_1, this implies that for all j such that $Q_j \cap B_Q \neq \varnothing$ we have $\ell(Q) \gtrsim_\eta \ell(Q_j)$. Indeed, if $\ell(Q) \ll \eta\ell(Q_j)$, then $B_Q \cap Q_j \neq \varnothing$ implies $Q \subset N_{\eta,j}$, which would contradict $Q \cap S_1 \neq \varnothing$.

By the observation above, we have some $C = C(\eta)$ such that if $B_Q \cap Q_j \neq \varnothing$, then $Q_j \subset CB_Q$. Consequently,

$$\sum_{j:Q_j\cap B_Q\neq\varnothing} \ell(Q_j)^d \lesssim \sum_{j:Q_j\subset CB_Q} \sigma(Q_j) \leq \sigma(CB_Q) \sim_\eta \ell(Q)^d.$$

Thus, the right-hand side of (4.11) can be estimated by

$$\sum_{Q\in\mathsf{Tree}} \ell(Q)^{-2} \sum_{j:Q_j\cap B_Q\neq\varnothing} \ell(Q_j)^{d+2} = \sum_j \ell(Q_j)^{d+2} \sum_{Q\in\mathsf{Tree}:Q_j\cap B_Q\neq\varnothing} \ell(Q)^{-2}. \tag{4.12}$$

As noted above, $Q_j \cap B_Q \neq \varnothing$ implies $\ell(Q) \gtrsim_\eta \ell(Q_j)$. Hence,

$$\sum_{Q\in\mathsf{Tree}:Q_j\cap B_Q\neq\varnothing} \ell(Q)^{-2} \lesssim_\eta \ell(Q_j)^{-2},$$

where we use the fact that the sum above is essentially a geometric series. Putting this together with (4.12) and (4.11), we get

$$\int_{S_1} \sum_{x\in Q\in\mathcal{D}(S)} \Bigg(\sum_{j:Q_j\cap B_Q\neq\varnothing} \frac{\ell(Q_j)^{d+1}}{\ell(Q)^{d+1}} \Bigg)^2 d\sigma(x) \lesssim_\eta \sum_j \ell(Q_j)^d \lesssim \ell(R)^d \sim \sigma(S_1).$$

Together with (4.9) and (4.10) this gives the desired estimate (4.8):

$$\int_{S_1} J_1' f(x)^2\, d\sigma(x) \lesssim_\eta \varepsilon^2 \lambda^2 \sigma(S_1).$$

This finishes the proof of Lemma 4.3.

5. The estimate $\|f\|_p \lesssim \|Jf\|_p$ for $1 < p < \infty$

In this section we show the second inequality of Theorem 2.4.

Proposition 5.1. *Let $f \in L^p(\sigma)$ for some $1 < p < \infty$. Then*

$$\|f\|_{L^p(\sigma)} \lesssim \|Jf\|_{L^p(\sigma)}. \tag{5.1}$$

5.1. Littlewood–Paley theory. Our main tool will be the Littlewood–Paley theory for spaces of homogeneous type developed by David, Journé, and Semmes in [**9**]. We follow the way it was paraphrased (in English) in [**22**, Section 15].

For $r > 0$, $x \in \Sigma$, and $g \in L^1_{\mathrm{loc}}(\sigma)$, let

$$D_r g(x) = \frac{\phi_r * (g\sigma)(x)}{\phi_r * \sigma(x)} - \frac{\phi_{2r} * (g\sigma)(x)}{\phi_{2r} * \sigma(x)},$$

where $\phi_r(y) = r^{-d}\phi(y/r)$ and ϕ is a radially symmetric smooth nonnegative function supported in $B(0,1)$ with $\int_{\mathbb{R}^n} \phi = 1$.

For a function $g \in L^1_{\mathrm{loc}}(\sigma)$ and $r > 0$, we denote

$$S_r g(x) = \frac{\phi_r * (g\sigma)(x)}{\phi_r * \sigma(x)},$$

so that

$$D_r g = S_r g - S_{2r} g.$$

Let W_r be the operator of multiplication by $1/S_r^* 1$. We consider the operators

$$\tilde{S}_r = S_r\, W_r\, S_r^* \quad \text{and} \quad \tilde{D}_r = \tilde{S}_r - \tilde{S}_{2r}.$$

Note that $\tilde{S}_r$, and thus $\tilde{D}_r$, are self-adjoint and $\tilde{S}_r 1 \equiv 1$, so that

$$\tilde{D}_r 1 = \tilde{D}_r^* 1 = 0. \tag{5.2}$$

Let $s_r(x,y)$ be the kernel of S_r with respect to σ, that is, so we can write

$$S_r g(x) = \int s_r(x,y)\, g(y)\, d\sigma(y).$$

Observe that

$$s_r(x,y) = \frac{1}{\phi_r * \sigma(x)}\, \phi_r(x-y)$$

and the kernel of $\tilde{S}_r$ is

$$\tilde{s}_r(x,y) = \int s_r(x,z) \frac{1}{S_r^* 1(z)} s_r(y,z)\, d\sigma(z).$$

We claim that the kernel $\tilde{d}_r(x,\cdot)$ for the operator $\tilde{D}_r$ is supported in $B(x,4r)$ and satisfies the Lipschitz bounds

$$|\tilde{d}_r(x,y) - \tilde{d}_r(x,z)| \lesssim |y-z| r^{-d-1}. \tag{5.3}$$

Indeed, let $x, x' \in \operatorname{supp} \sigma$. Since ϕ_r is Cr^{-d-1}-Lipschitz and σ is Ahlfors regular,

$$\begin{aligned}
|\phi_r * \sigma(x) - \phi_r * \sigma(x')| &= \left| \int (\phi_r(x-y) - \phi_r(x'-y))\, d\sigma(y) \right| \\
&\lesssim \frac{|x-x'|}{r^{d+1}} \sigma(B(x,r) \cup B(x',r)) \\
&\lesssim \frac{|x-x'|}{r}.
\end{aligned}$$

Thus, for $y \in \operatorname{supp} \sigma$,

$$\begin{aligned}|s_r(x,y) - s_r(x',y)| &\leq \frac{|\phi_r(x-y) - \phi_r(x'-y)|}{\phi_r * \sigma(x)} \\ &\quad + \phi_r(x'-y)\left|\frac{1}{\phi_r * \sigma(x)} - \frac{1}{\phi_r * \sigma(x')}\right| \\ &\lesssim \frac{|x-x'|}{r^{d+1}} + r^{-d}\frac{|\phi_r * \sigma(x) - \phi_r * \sigma(x')|}{\phi_r * \sigma(x)^2} \sim \frac{|x-x'|}{r^{d+1}}.\end{aligned}$$

Hence,

$$\begin{aligned}|\tilde{s}_r(x,y) - \tilde{s}_r(x',y) &= \left|\int (s_r(x,z) - s_r(x',z))\frac{1}{S_r^*1(z)}\, s_r(y,z)\, d\sigma(z)\right| \\ &\lesssim \frac{|x-x'|}{r^{d+1}}\left|\int \frac{1}{S_r^*1(z)}\, s_r(y,z)\, d\sigma(z)\right| \lesssim \frac{|x-x'|}{r^{d+1}},\end{aligned}$$

where in the last line we use the fact that $\int s_r(y,z)\, d\sigma(z) = 1$ and

$$S_r^*1(z) = \int \frac{\phi_r(x-z)}{\phi_r * \sigma(x)}\, d\sigma(x) \geq \int_{B(z,r/2)} \frac{r^{-d}}{\phi_r * \sigma(x)}\, d\sigma(x) \sim 1.$$

Since $\tilde{d}_r = \tilde{s}_r - \tilde{s}_{2r}$ and is symmetric, this proves (5.3). Moreover, notice that if $x \in \operatorname{supp} \sigma$, $\operatorname{supp} s_r(x,\cdot) \subseteq B(x,r)$, and so the integrand of $\tilde{s}_r$ is nonzero only when $z \in B(x,r) \cap B(y,r)$, meaning $|x-y| \leq 2r$, and so $\operatorname{supp} \tilde{s}_r \subseteq B(x,2r)$, hence $\operatorname{supp} \tilde{d}_r \subseteq B(x,4r)$, which proves our claim.

Theorem 5.2. **[9]** *Let $r_k = 2^{-k}$, and $g \in L^p(\sigma)$, $1 < p < \infty$. We have*

$$\|g\|_{L^p(\sigma)} \sim \left\|\left(\sum_{k\in\mathbb{Z}} |\tilde{D}_{r_k} g|^2\right)^{\frac{1}{2}}\right\|_{L^p(\sigma)}. \tag{5.4}$$

The original result is stated for $p = 2$, but this case implies the other cases (see for example the proof of [**17**, Corollary 6.1]).

Let $\tilde{D}_k := \tilde{D}_{r_k}$, $\tilde{d}_k := \tilde{d}_{r_k}$. By (5.4), it is clear that to prove (5.1) it suffices to show that

$$\left\|\left(\sum_{k\in\mathbb{Z}} |\tilde{D}_k f|^2\right)^{\frac{1}{2}}\right\|_{L^p(\sigma)} \lesssim \|Jf_{L^p(\sigma)}.$$

In fact, we will show a stronger, pointwise inequality which immediately implies the one above.

Lemma 5.3. *Let $x \in \Sigma$, $k \in \mathbb{Z}$, and let $Q \in \mathcal{D}$ be the smallest cube containing x and such that $B(x, 4r_k) \subset 0.5B_Q$. Then, $\ell(Q) \sim r_k$ and*

$$|\tilde{D}_k f(x)| \lesssim \alpha_{f\sigma}(Q) + |f|_{B_Q} \alpha_\sigma(Q). \tag{5.5}$$

The remainder of this section is devoted to the proof of this lemma.

5.2. Preliminaries. Fix $x \in \Sigma$, $k \in \mathbb{Z}$, and let Q be as above. The estimate $\ell(Q) \sim r_k$ follows immediately by the definition of Q. As noted just above (5.3), we have $\operatorname{supp} \tilde{d}_k(x, \cdot) \subset B(x, 4r_k) \subset 0.5B_Q$.

We make a few simple reductions.

Remark 5.4. Without loss of generality we may assume that $\alpha_\sigma(Q) \le \varepsilon$ for some small ε. Indeed, if we had $\alpha_\sigma(Q) \ge \varepsilon$, then using (5.3) and the fact that $\operatorname{supp} \tilde{d}_k(x, \cdot) \subset B_Q$

$$\begin{aligned} |\tilde{D}_k f(x)| &= \left| \int \tilde{d}_k(x, y) f(y) \, d\sigma(y) \right| \le \|\tilde{d}_k(x, \cdot)\|_\infty \int_{B_Q} |f(y)| \, d\sigma(y) \\ &\lesssim \ell(Q)^{-d} \int_{B_Q} |f(y)| \, d\sigma(y) \sim |f|_{B_Q} \lesssim_\varepsilon |f|_{B_Q} \alpha_\sigma(Q), \end{aligned}$$

and so in this case (5.5) holds. From now on we assume $\alpha_\sigma(Q) \le \varepsilon$.

Remark 5.5. Similarly, without loss of generality we may assume that $L_Q^{f\sigma} \cap 0.5B_Q \neq \varnothing$. If we had $L_Q^{f\sigma} \cap 0.5B_Q = \varnothing$, then $L_Q^{f\sigma} \cap \operatorname{supp} \tilde{d}_r(x, \cdot) = \varnothing$ so that

$$\int \tilde{d}_k(x, y) \, d\mathcal{L}_Q^{f\sigma}(y) = 0.$$

This implies

$$|\tilde{D}_k f(x)| = \left| \int \tilde{d}_k(x, y) f(y) \, d\sigma(y) \right| \lesssim \alpha_{f\sigma}(Q),$$

and so (5.5) is true also in this case.

Recall that $c_Q^{f\sigma}$, c_Q^σ are the constants minimizing $\alpha_{f\sigma}(Q)$, $\alpha_\sigma(Q)$, respectively. Since σ is Ahlfors regular and $\alpha_\sigma(Q) \le \varepsilon$, choosing $\varepsilon > 0$ small enough (depending on the Ahlfors regularity constants of σ) we get by [**3**, Lemma 3.3]

$$c_Q^\sigma \sim 1. \tag{5.6}$$

To show (5.5) we begin by using (5.2) and the triangle inequality:

$$
\begin{aligned}
|\tilde{D}_k f(x)| &= \left| \int_\Sigma \tilde{d}_k(x,y) f(y)\, d\sigma(y) \right| \\
&\overset{(5.2)}{=} \left| \int_\Sigma \tilde{d}_k(x,y) f(y)\, d\sigma(y) - \frac{c_Q^{f\sigma}}{c_Q^\sigma} \int_\Sigma \tilde{d}_k(x,y)\, d\sigma(y) \right| \\
&\leq \left| \int_\Sigma \tilde{d}_k(x,y) f(y)\, d\sigma(y) - \int_{L_Q^{f\sigma}} \tilde{d}_k(x,y)\, d\mathcal{L}_Q^{f\sigma}(y) \right| \\
&\quad + \left| \int_{L_Q^{f\sigma}} \tilde{d}_k(x,y)\, d\mathcal{L}_Q^{f\sigma}(y) - \frac{c_Q^{f\sigma}}{c_Q^\sigma} \int_{L_Q^\sigma} \tilde{d}_k(x,y)\, d\mathcal{L}_Q^\sigma(y) \right| \\
&\quad + \left| \frac{c_Q^{f\sigma}}{c_Q^\sigma} \right| \left| \int_{L_Q^\sigma} \tilde{d}_k(x,y)\, d\mathcal{L}_Q^\sigma(y) - \int_\Sigma \tilde{d}_k(x,y)\, d\sigma(y) \right| \\
&=: (I) + (II) + (III).
\end{aligned} \tag{5.7}
$$

Using the Lipschitz property of $\tilde{d_k}$ (5.3) we immediately get that $(I) \lesssim \alpha_{f\sigma}(Q)$, and that

$$
(III) \lesssim \left| \frac{c_Q^{f\sigma}}{c_Q^\sigma} \right| \alpha_\sigma(Q) \overset{(5.6)}{\sim} |c_Q^{f\sigma}| \alpha_\sigma(Q). \tag{5.8}
$$

Lemma 5.6. *We have* $|c_Q^{f\sigma}| \lesssim |f|_{B_Q}$.

Proof: Indeed, if we had $|c_Q^{f\sigma}| \geq \Lambda |f|_{B_Q}$ for some big $\Lambda > 10$, then $\tilde{c}_Q^{f\sigma} = 0$ would be a better competitor for a constant minimizing $\alpha_{f\sigma}(Q)$. To see that, note that for any $\varphi \in \mathrm{Lip}_1(B_Q)$

$$
\left| \int \varphi f\, d\sigma - 0 \right| \leq C \ell(Q)^{d+1} |f|_{B_Q}.
$$

That is, $F_{B_Q}(f\sigma, 0) \leq C\ell(Q)^{d+1}|f|_{B_Q}$. On the other hand, taking a positive $\psi \in \mathrm{Lip}_1(B_Q)$ such that $\psi(x) = \ell(Q)$ for $x \in 0.7B_Q$ and using the assumption $L_Q^{f\sigma} \cap 0.5B_Q \neq \varnothing$ we get

$$
\begin{aligned}
\alpha_{f\sigma}(Q)\ell(Q)^{d+1} &\gtrsim \left| \int \psi f\, d\sigma - c_Q^{f\sigma} \int_{L_Q^{f\sigma}} \psi\, d\mathcal{H}^d \right| \\
&\geq |c_Q^{f\sigma}| \ell(Q) \mathcal{H}^d(0.7B_Q \cap L_Q^{f\sigma}) - \left| \int \psi f\, d\sigma \right| \\
&\geq \tilde{C}\Lambda |f|_{B_Q} \ell(Q)^{d+1} - C\ell(Q)^{d+1}|f|_{B_Q} \\
&\geq \frac{\tilde{C}\Lambda}{2} |f|_{B_Q} \ell(Q)^{d+1} > F_{B_Q}(f\sigma, 0),
\end{aligned}
$$

assuming Λ big enough. This contradicts the optimality of $c_Q^{f\sigma}$. $\square$

Using the lemma above and (5.8) we get

$$(III) \lesssim |f|_{B_Q} \alpha_\sigma(Q).$$

Hence, by (5.7), to finish the proof of (5.5) it remains to show that

$$\begin{aligned}(II) = |c_Q^{f\sigma}| \left| \int_{L_Q^{f\sigma}} \tilde{d}_k(x,y)\, d\mathcal{H}^d(y) - \int_{L_Q^\sigma} \tilde{d}_k(x,y)\, d\mathcal{H}^d(y) \right| \\ \lesssim \alpha_{f\sigma}(Q) + |f|_{B_Q} \alpha_\sigma(Q).\end{aligned}$$

This can be seen as an estimate of how far from each other the planes $L_Q^{f\sigma}$ and L_Q^σ are.

The inequality above follows immediately from Proposition 5.7 proved in the next subsection, together with the already established estimate $|c_Q^{f\sigma}| \lesssim |f|_{B_Q}$.

5.3. Angles between planes approximating $\boldsymbol{f\sigma}$ and $\boldsymbol{\sigma}$. In the following proposition we do not use uniform rectifiability in any way, and so we state it for a general Ahlfors regular measure μ. Recall that given a ball B we defined $\mathcal{P}_B^\mu = \mathcal{H}^d \llcorner L_B^\mu$.

Proposition 5.7. *Let μ be an Ahlfors d-regular measure on $\mathbb{R}^n$, and let $f \in L^1_{\mathrm{loc}}(\mu)$. Let $x \in \operatorname{supp}\mu$, $r > 0$, $B = B(x,r)$, and suppose that $L_B^{f\mu} \cap 0.5B \neq \varnothing$. Then,*

$$|c_B^{f\mu}| \frac{1}{r^{d+1}} F_B(\mathcal{P}_B^\mu, \mathcal{P}_B^{f\mu}) \lesssim \alpha_{f\mu}(B) + |c_B^{f\mu}| \alpha_\mu(B). \tag{5.9}$$

In the proof of Proposition 5.7 we will use the following lemma.

Lemma 5.8. *Let $B = B(x,r)$ and let L_1, L_2 be two d-planes intersecting $0.5B$. Set $\mathcal{P}_1 = \mathcal{H}^d \llcorner L_1$, $\mathcal{P}_2 = \mathcal{H}^d \llcorner L_2$. Then,*

$$\frac{1}{r^d} F_B(\mathcal{P}_1, \mathcal{P}_2) \lesssim \operatorname{dist}_H(L_1 \cap B, L_2 \cap B). \tag{5.10}$$

Proof: First, set

$$D = \frac{\operatorname{dist}_H(L_1 \cap B, L_2 \cap B)}{r}.$$

Note that we always have $F_B(\mathcal{P}_1, \mathcal{P}_2) \lesssim r^{d+1}$ so that if $D \gtrsim 1$, then (5.10) follows trivially. Hence, without loss of generality we may assume that $D \le \varepsilon$ for some $\varepsilon > 0$ to be fixed later.

We claim that if ε is chosen small enough (depending only on n, d), then there exists an isometry $A\colon L_1 \to L_2$ such that for $y \in B \cap L_1$ we have $|y - A(y)| \lesssim Dr$. To see that, let $y_1 \in L_1 \cap B$ be arbitrary. Set $y_2 = \pi_{L_2}(y_1)$. Clearly,

$$|y_1 - y_2| \le Dr \le \varepsilon r.$$

Let $v_1, \dots, v_d$ be an orthonormal basis of the linear plane $L_1' := L_1 - y_1$. For $i = 1, \dots, d$ define

$$w_i := \pi_{L_2}(y_1 + v_i) - y_2 \in L_2 - y_2 =: L_2'.$$

In fact, since $y_2 = \pi_{L_2}(y_1)$, we have $w_i = \pi_{L_2'}(v_i)$. It is easy to see that for all $v \in L_1'$ we have

$$|\pi_{L_2'}(v) - v| \lesssim D|v|.$$

Hence, $|w_i - v_i| \lesssim D \le \varepsilon$ and for $i \neq j$

$$|w_i \cdot w_j| = |(w_i - v_i) \cdot (w_j - v_j) + (w_i - v_i) \cdot v_j + v_i \cdot (w_j - v_j)| \lesssim D \le \varepsilon.$$

Choosing ε small enough (depending only on dimensions), we get easily that $\{w_i\}$ is a basis of L_2'. Moreover, if $\{\hat{w}_i\}$ is the orthonormal basis of L_2' constructed from $\{w_i\}$ using the Gram–Schmidt process, then it follows from the estimates above that for all $i = 1, \dots, d$

$$|\hat{w}_i - v_i| \lesssim D.$$

We define the map $A\colon L_1 \to L_2$ as the unique isometry such that $A(y_1) = y_2$ and $A(y_1 + v_i) = y_2 + \hat{w}_i$. It follows immediately from basic linear algebra that for $y \in L_1 \cap B$ we have $|y - A(y)| \lesssim Dr$.

Now, let $\varphi \in \operatorname{Lip}_1(B)$. We have

$$\begin{aligned} &\left| \int_{L_1} \varphi(y)\, d\mathcal{H}^d(y) - \int_{L_2} \varphi(y)\, d\mathcal{H}^d(y) \right| \\ &\qquad = \left| \int_{L_1} \varphi(y)\, d\mathcal{H}^d(y) - \int_{L_1} \varphi(A(y))\, d\mathcal{H}^d(y) \right| \\ &\qquad \le \int_{L_1} |\varphi(y) - \varphi(A(y))|\, d\mathcal{H}^d(y) \lesssim \int_{L_1 \cap B} Dr\, d\mathcal{H}^d(y) \lesssim Dr^{d+1}. \end{aligned}$$

Taking the supremum over $\varphi \in \operatorname{Lip}_1(B)$ finishes the proof. □

Proof of Proposition 5.7: For simplicity of notation we will usually omit the subscript B, i.e. we will write $L^\mu := L^\mu_B$, $c^{f\mu} := c^{f\mu}_B$, and so on.

Without loss of generality we can assume that $c^{f\mu} \ge 0$. Indeed, if that were not the case, we could consider $g = -f$. Then the plane $L^{g\mu} = L^{f\mu}$ and the constant $c^{g\mu} = -c^{f\mu} \ge 0$ are minimizing for $\alpha_{g\mu}(B)$, and we have $\alpha_{g\mu}(B) = \alpha_{f\mu}(B)$. Thus, proving (5.9) for g is equivalent to proving it for f, and $c^{g\mu} \ge 0$.

Note that we always have $F_B(\mathcal{P}^\mu, \mathcal{P}^{f\mu}) \lesssim r^{d+1}$ so that if $\alpha_\mu(B) \gtrsim 1$, then (5.9) is trivial. Assume that $\alpha_\mu(B) \le \varepsilon$ for some small $\varepsilon > 0$ (depending on dimensions and Ahlfors regularity constants), to be fixed later.

Note that if ε is small enough, then one can use the Ahlfors regularity of μ to conclude that $L^\mu \cap 0.5B \neq \varnothing$ (see for example [**19**, Lemma 3.1]). We use this observation, the assumption $L^{f\mu} \cap 0.5B \neq \varnothing$ and (5.10) to estimate

$$c^{f\mu}\frac{1}{r^{d+1}}F_B(\mathcal{P}^\mu, \mathcal{P}^{f\mu}) \lesssim c^{f\mu}\frac{\operatorname{dist}_H(L^\mu \cap B, L^{f\mu} \cap B)}{r} =: c^{f\mu}D.$$

Our aim is to show that

$$c^{f\mu}D \lesssim c^{f\mu}\alpha_\mu(B) + \alpha_{f\mu}(B). \tag{5.11}$$

Let $0 < \eta < 0.01$ be some dimensional constant. Note that, since $L^{f\mu} \cap 0.5B \neq \varnothing$, the set $L^{f\mu} \cap 0.9B$ is a d-dimensional ball with $\mathcal{H}^d(L^{f\mu} \cap 0.9B) \sim r^d$. We claim that we can find a d-dimensional ball B_0 contained in $L^{f\mu} \cap 0.9B$, of radius ηr (in particular $r_{\mathsf{B}_0} \sim_\eta r_B$), and such that

$$\operatorname{dist}(z, L^\mu) \geq 10\eta D r \quad \text{for all } z \in \mathsf{B}_0. \tag{5.12}$$

Indeed, if there were no such ball, i.e. if for all d-dimensional balls $\mathsf{B}_0 \subset L^{f\mu} \cap 0.9B$ of radius ηr there were some $z \in \mathsf{B}_0$ with $\operatorname{dist}(z, L^\mu) \leq 10\eta D r$, then it would follow easily from the definition of the Hausdorff distance, and from the fact that L^μ and $L^{f\mu}$ are d-planes intersecting $0.5B$, that

$$\operatorname{dist}_H(L^\mu \cap B, L^{f\mu} \cap B) \lesssim \eta D r = \eta \operatorname{dist}_H(L^\mu \cap B, L^{f\mu} \cap B).$$

For η small enough, this is a contradiction. We omit the details, which can be readily filled in e.g. using [**2**, Lemma 6.4] (with $\varepsilon \sim \eta D$ and X an appropriate subset of $0.9B \cap L^{f\mu}$ spanning $L^{f\mu}$).

Consider an open neighborhood of B_0 given by

$$U := \{y \in \mathbb{R}^n : \operatorname{dist}(y, \mathsf{B}_0) < \eta D r\},$$

and also for $\lambda > 0$ set

$$\lambda U := \{y \in \mathbb{R}^n : \operatorname{dist}(y, \mathsf{B}_0) < \lambda \eta D r\}.$$

Since $D \leq 1$, one should think of U as an n-dimensional pancake around B_0 of thickness $\eta D r$, so that the smaller D, the flatter the pancake. Note that by (5.12) for all $0 < \lambda < 10$ we have $\lambda U \cap L^\mu = \varnothing$, and also $\lambda U \subset B$ because $\mathsf{B}_0 \subset 0.9B$.

Let $\varphi\colon \mathbb{R}^n \to [0, \eta D r]$ be a function satisfying $\varphi \equiv \eta D r$ in U, $\operatorname{supp}\varphi \subset 2U$, and $\operatorname{Lip}(\varphi) \leq 1$. Clearly, $\varphi \in \operatorname{Lip}_1(B)$, and so

$$\left| \int \varphi f \, d\mu - \int \varphi \, d\mathcal{L}^{f\mu} \right| \leq \alpha_{f\mu}(B) r^{d+1}. \tag{5.13}$$

Furthermore, note that $\varphi \equiv \eta D r$ on B_0, so that

$$\int \varphi \, d\mathcal{L}^{f\mu} = c^{f\mu} \int_{L^{f\mu}} \varphi \, d\mathcal{H}^d \geq c^{f\mu} \eta D r \mathcal{H}^d(\mathsf{B}_0) = C(d) c^{f\mu} D \eta^{d+1} r^{d+1}.$$

Together with (5.13) this implies

$$\int \varphi f \, d\mu \geq C(\eta, d) c^{f\mu} D r^{d+1} - \alpha_{f\mu}(B) r^{d+1}. \tag{5.14}$$

Recall that we are trying to prove $c^{f\mu} D \lesssim c^{f\mu} \alpha_\mu(B) + \alpha_{f\mu}(B)$. If we had $c^{f\mu} D \leq \Lambda \alpha_{f\mu}(B)$ for some $\Lambda = \Lambda(\eta, d) > 100$, then there would be nothing to prove. So without loss of generality assume that $c^{f\mu} D \geq \Lambda \alpha_{f\mu}(B)$. In that case (5.14) gives

$$\int \varphi f \, d\mu \gtrsim_\eta c^{f\mu} D r^{d+1}. \tag{5.15}$$

Now we define a modified version of φ. Recall that $\operatorname{supp} \varphi \subset 2U$. For all $y \in \operatorname{supp} \mu \cap 2U$ let $B_y = B(y, \eta D r/5)$. We use the $5r$ covering theorem to extract from $\{B_y\}_{y \in \operatorname{supp} \mu \cap 2U}$ a subfamily of pairwise disjoint balls $\{B_i\}_{i \in I}$ such that $\operatorname{supp} \mu \cap 2U \subset \bigcup_i 5B_i$. Note that $\bigcup_i 10 B_i \subset 4U$, and in particular, $\bigcup_i 10 B_i \cap L^\mu = \varnothing$. Moreover, the balls $10B_i$ have bounded intersection. Thus, we may consider a partition of unity

$$\Psi = \sum_{i \in I} \psi_i,$$

such that $\operatorname{supp} \psi_i \subset 10 B_i$ for each $i \in I$, $\Psi \equiv 1$ on $\bigcup_i 5B_i$, and $\operatorname{Lip} \Psi \lesssim (\eta D r)^{-1}$.

Consider $\Phi = \varphi \Psi$. We have

$$\|\nabla \Phi\|_\infty \leq \|\nabla \varphi\|_\infty \|\Psi\|_\infty + \|\varphi\|_\infty \|\nabla \Psi\|_\infty \lesssim 1 + \eta D r (\eta D r)^{-1} = 1.$$

Hence, $C\Phi \in \operatorname{Lip}_1(B)$ for some $C \sim 1$, so that

$$\left| \int \Phi f \, d\mu - \int \Phi \, d\mathcal{L}^{f\mu} \right| \leq C^{-1} \alpha_{f\mu}(B) r^{d+1}. \tag{5.16}$$

On the other hand, observe that $\Psi \equiv 1$ on $\operatorname{supp} \varphi \cap \operatorname{supp} \mu$. By (5.15)

$$\int \Phi f \, d\mu = \int \varphi f \, d\mu \gtrsim_\eta c^{f\mu} D r^{d+1}.$$

Together with (5.16) this gives

$$\int \Phi \, d\mathcal{L}^{f\mu} \geq C(\eta) c^{f\mu} D r^{d+1} - C^{-1} \alpha_{f\mu}(B) r^{d+1} \gtrsim_\eta c^{f\mu} D r^{d+1}, \tag{5.17}$$

where we use once again the additional assumption $c^{f\mu} D \geq \Lambda \alpha_{f\mu}(B)$ we made along the way (and choosing Λ large).

Now we will show that

$$\int_{L^{f_\mu}} \Phi \, d\mathcal{H}^d \lesssim_\eta \alpha_\mu(B) r^{d+1}. \tag{5.18}$$

Since $\mathcal{L}^{f_\mu} = c^{f_\mu}\mathcal{H}^d \llcorner L^{f_\mu}$, together with (5.17) this will give $c^{f_\mu} D \lesssim_\eta c^{f_\mu}\alpha_\mu(B)$, and so the proof of (5.11) will be finished.

Recall that $\operatorname{supp}\Phi \subset \operatorname{supp}\Psi \subset \bigcup_i 10B_i$, and that $\|\Phi\|_\infty \leq \|\varphi\|_\infty = \eta D r$. Hence,

$$\int_{L^{f_\mu}} \Phi \, d\mathcal{H}^d \lesssim_\eta Dr \sum_{i\in I} \mathcal{H}^d(L^{f_\mu} \cap 10B_i) \lesssim_\eta \#I(Dr)^{d+1}.$$

To estimate $\#I$ we will use the Ahlfors regularity of μ. Recall that $\{B_i\}_{i\in I}$ are pairwise disjoint, they are centered at points from $\operatorname{supp}\mu \cap 2U$, and $r(B_i) = \eta r D/5$. Thus,

$$\#I(rD)^d \sim_\eta \sum_{i\in I} \mu(B_i) = \mu\Bigl(\bigcup_{i\in I} B_i\Bigr).$$

On the other hand, since the balls $\{B_i\}$ are centered at points from $2U$, we have $\bigcup_{i\in I} B_i \subset 3U$ and

$$\mu\Bigl(\bigcup_{i\in I} B_i\Bigr) \leq \mu(3U).$$

To bound $\mu(3U)$ consider $\tilde{\varphi} \in \operatorname{Lip}_1(B)$ such that $\tilde{\varphi} \geq 0$, $\tilde{\varphi} \equiv \eta r D$ on $3U$ and $\operatorname{supp}\tilde{\varphi} \subset 4U$. Recalling that $4U \cap L^\mu = \varnothing$, we arrive at

$$rD\mu(3U) \lesssim_\eta \int \tilde{\varphi}\, d\mu = \left| \int \tilde{\varphi}\, d\mu - \int \tilde{\varphi}\, d\mathcal{L}^\mu \right| \leq \alpha_\mu(B) r^{d+1}.$$

Putting all the estimates above together we get (5.18):

$$\int_{L^{f_\mu}} \Phi \, d\mathcal{H}^d \lesssim_\eta \#I(Dr)^{d+1} \lesssim_\eta rD\mu(3U) \lesssim_\eta \alpha_\mu(B) r^{d+1}. \qquad \square$$

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