Publ. Mat. **68** (2024), 103–109
DOI: 10.5565/PUBLMAT6812405

A REMARK ON FIRST INTEGRALS OF VECTOR FIELDS

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Abstract: We provide examples of vector fields on $(\mathbb{C}^3, 0)$ admitting a formal first integral but no holomorphic first integral. These examples are related to a question raised by D. Cerveau and motivated by the celebrated theorems of Malgrange and Mattei–Moussu.

2020 Mathematics Subject Classification: Primary: 32S65; Secondary: 32M25.

Key words: holomorphic vector field, first integral, formal power series, Stokes phenomena.

1. Introduction

A celebrated theorem due to Mattei and Moussu ([**10**]) states that a holomorphic codimension 1 foliation admitting a formal first integral necessarily possesses a holomorphic first integral as well. The theorem and its proof completely clarify the relationship between formal and holomorphic first integrals for codimension 1 foliations. The general investigation of the existence of first integrals also includes an influential work of Malgrange [**8**]. However, for higher codimension foliations the relationship between formal and holomorphic first integrals remains quite mysterious. In this context, D. Cerveau naturally asked whether a holomorphic vector field X defined on a neighborhood of the origin of $\mathbb{C}^3$ and admitting one—or two independent—formal first integrals must possess holomorphic first integrals as well. The goal of this paper is to show that the existence of a single formal first integral is not enough to guarantee the existence of a holomorphic one. This is done by means of the following theorem:

Theorem 1.1. *Consider the family $X_{a,b,c}$ of vector fields on $\mathbb{C}^3$ defined by*

$$X_{a,b,c} = x^2 \frac{\partial}{\partial x} + (1+ax)\left(y_1 \frac{\partial}{\partial y_1} - y_2 \frac{\partial}{\partial y_2}\right) + bx\, y_2 \frac{\partial}{\partial y_1} + cx\, y_1 \frac{\partial}{\partial y_2}, \tag{1}$$

where a, b, and c are complex parameters. Assume that the parameters are such that

$$\cos(2\pi a) \neq \cos(2\pi\sqrt{a^2+bc}).$$

Then the vector field $X_{a,b,c}$ does not possess any (non-constant) holomorphic first integral, albeit it possesses formal first integrals.

In particular, the vector field $X_{1,1,1}$ obtained by setting $a = b = c = 1$ admits a formal first integral but no holomorphic one. The existence of these examples was certainly expected, given the transcendental behavior of singular foliations, but we highlight their simplicity which suggests that this might be a fairly common phenomenon in applications. The issue is also related to Malgrange's theorem in [**9**] in that it confirms that some (strong) additional assumptions are in fact needed (see below for further information). As a side note, the simple nature of the example provided by Theorem 1.1 also bears some similarities with certain results quoted in the survey article [**12**] of Stolovitch: many normalization results for "simple" vector fields having formulas not too different from $X_{a,b,c}$ are presented under some additional geometric conditions (for example "volume-preserving" or "Hamiltonian"). We might wonder what influence these conditions have on the problem discussed here. Conversely, it is also fair to wonder if the (potential) ability to turn formal first integrals into holomorphic ones may shed new light on more general normal form problems.

Let us remark that in the real domain there are well-known examples of real analytic vector fields which are $\mathcal{C}^\infty$-integrable but not analytically, nor formally. The most basic situation would be any saddle-node singularity in $\mathbb{R}^2$, e.g., $x^2\frac{\partial}{\partial x} + (1+ax)y\frac{\partial}{\partial y}$, $a \in \mathbb{R}$. In a Hamiltonian setting, a $\mathcal{C}^\infty$-Liouville integrable but formally non-integrable vector field in $\mathbb{R}^4$ was provided in [**5**]. Likewise, when $a, b, c \in \mathbb{R}$, then the restriction of $X_{a,b,c}$ to $\mathbb{R}^3$ possesses $\mathcal{C}^\infty$-first integrals (see Remark 2.5).

Our observation of the vector field (1) is a by-product of our study of the global dynamics of the Airy and Painlevé I and II equations [**3**]. Our original motivation was the local analysis of the saddle-node singularity associated with the vector field

$$Y_A = -\tfrac{1}{2}x^4\frac{\partial}{\partial x} + \left(z - \tfrac{1}{2}x^3 y\right)\frac{\partial}{\partial y} + (y - x^3 z)\frac{\partial}{\partial x},$$

which appears in a convenient birational model for the compactified Airy equation. The formal normal form of Y_A as well as the corresponding Stokes phenomenon can be accurately computed with the same technique as is detailed in Section 2 for the vector field $X_{a,b,c}$. In doing so, it follows that Y_A admits a first integral in the field of fractions of formal power series, i.e., there is a formal first integral of the form F/G with $F, G \in \mathbb{C}[[x, y, z]]$. Yet Y_A has no holomorphic or meromorphic first integral. Basically, the difference between the example provided by Y_A and Cerveau's general questions lies in the fact that the "formal first integral" of the vector field Y_A has a "meromorphic" nature rather than a more standard power series representation without negative terms. In turn, there are deep differences between first integrals of "holomorphic" and of "meromorphic" natures as already underlined in the topological context. In fact, in codimension 1, the Mattei–Moussu theorem ([**10**]) asserts that first integrals are topological invariants and the existence of formal first integrals implies the existence of holomorphic ones. On the other hand, the existence of meromorphic first integrals is not a topological invariant already in the two-dimensional ambient case; cf. [**4, 7, 11**]. Similarly, in codimension 2 complete integrability in the holomorphic sense is not a topological invariant either [**11**]. From this point of view, the vector field Y_A falls genuinely short of shedding light on Cerveau's questions due to the nature of its formal first integral.

It is now interesting to investigate whether a holomorphic vector field X defined on a neighborhood of the origin of $\mathbb{C}^3$ and admitting two formal first integrals F_1, F_2 such that $\mathrm{d}F_1 \wedge \mathrm{d}F_2 \not\equiv 0$ necessarily admits at least one holomorphic first integral. The best result in this direction, as far as we are aware of, remains the previously mentioned theorem of Malgrange [**9**] concerning Pfaffian systems in arbitrary dimensions. More precisely, given a codimension r foliation defined in some open set of $\mathbb{C}^n$ and generated by r one-forms $\Omega = \{\omega_1, \ldots, \omega_r\}$, denote by $S(\Omega)$ the singular locus of Ω, that is, the set of points where the r-form $\omega_1 \wedge \cdots \wedge \omega_r$ is identically zero. We say that Ω is integrable (respectively formally integrable) at $x \in \mathbb{C}^n$ if there exist r holomorphic function germs $f_1, \ldots, f_r \in \mathcal{O}_x$ (respectively r formal power series in $\widehat{\mathcal{O}}_x$) such that the module generated by $\{\mathrm{d}f_1, \ldots, \mathrm{d}f_r\}$ coincides with $\Omega \cdot \mathcal{O}_x$ (respectively with $\Omega \cdot \widehat{\mathcal{O}}_x$). In [**9**], Malgrange shows that if $S(\Omega)$ has codimension 3, or if Ω is formally integrable and $S(\Omega)$ has codimension 2, then Ω is integrable. As mentioned, these hypotheses are generally quite strong when we consider a Pfaffian system obtained as the dual of a vector field.

The proof of Theorem 1.1 relies on the standard theory of linear systems (normal forms and Stokes phenomena, among others). We refer the reader to [**6**, §16 and 20] and references therein (or [**1, 13**]) for an introduction to the methods used in this work.

Acknowledgments. H. Reis was partially supported by CMUP, which is financed by national funds through FCT – Fundação para a Ciência e Tecnologia, I. P., under the project with reference UIDB/00144/2020. J. Rebelo and H. Reis are also partially supported by CIMI through the project "Complex dynamics of group actions, Halphen and Painlevé systems".

2. Proof of Theorem 1.1

Let us begin our approach to Theorem 1.1 by noticing that the vector field $X_{a,b,c}$ is associated with the time-dependent linear differential system

$$x^2\frac{\mathrm{d}y}{\mathrm{d}x} = \begin{bmatrix} 1+ax & bx \\ cx & -1-ax \end{bmatrix} y, \quad y = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}. \tag{2}$$

Following classical terminology of linear systems, the system above has a non-resonant irregular singular point of Poincaré rank 1 at $x = 0$; see for example [**6**, §20]. Note that this differential system is in the so-called Birkhoff normal form: the system is well defined for all $x \in \mathbb{CP}^1$ and has only two singular points, namely $x = 0$ and $x = \infty$; see e.g. [**6**, §20B]. In turn, the singularity at $x = \infty$ is a *Fuchsian one*. In other words, the system has a simple pole at $x = \infty$; see e.g. [**6**, Definition 16.9]. In addition, since the linear system (2) is non-resonant, it can formally be transformed into a diagonal linear system by means of the standard Poincaré–Dulac method [**6**, Theorem 20.7]. Whereas the resulting (formal) power series is divergent, Sibuya's theorem asserts that it is Borel 1-summable in all directions $x \in \mathrm{e}^{\mathrm{i}\alpha}\mathbb{R}_{>0}$ with the exception of the singular directions, namely the directions corresponding to $\alpha \in \pi\mathbb{Z}$. The preceding is made accurate by the lemma below:

Lemma 2.1. *There exists a formal linear change of coordinates having the form $y = \hat{T}(x)u$, with $\hat{T}(0) = I$, which conjugates system (2) to the (diagonal) linear system*

$$x^2\frac{\mathrm{d}u}{\mathrm{d}x} = \begin{bmatrix} 1+ax & 0 \\ 0 & -1-ax \end{bmatrix} u, \quad u = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}. \tag{3}$$

Moreover, for every $\alpha \in {]0,\pi[} \cup {]\pi, 2\pi[}$, there exists a holomorphic transformation $y = T_\alpha(x)u$ conjugating systems (2) and (3) and satisfying the following conditions:

(i) *$T_\alpha(x)$ is analytic on the open sector of opening angle π bisected by the half-line $\mathrm{e}^{\mathrm{i}\alpha}\mathbb{R}_{>0}$.*
(ii) *$T_\alpha(x)$ and $T_\beta(x)$, with $\alpha < \beta$, coincide on the intersection of the corresponding half-planes provided that the interval $]\alpha, \beta[$ does not contains an integral multiple of π.*
(iii) *$T_\alpha(x)$ is asymptotic to $\hat{T}(x)$.*

Proof: As previously stated, the existence of a formal change of variables conjugating systems (2) and (3), as well as its analytic nature on the indicated sectors, goes back to classical results by Birkhoff and Malmquist (or more general versions by Hukuhara, Turittin, and Sibuya; see [**6**, Theorems 20.7 and 20.16]). Therefore it only remains to check that the diagonal matrix appearing in (3) has the indicated form. To do this, note that the formal invariants of the initial system (2) can be read off a suitable finite jet of the eigenvalue functions associated with the matrix

$$\begin{bmatrix} 1+ax & bx \\ cx & -1-ax \end{bmatrix}.$$

Clearly these eigenvalue functions are equal to $\pm\sqrt{(1+ax)^2+bcx^2}$. Now, since the Poincaré rank of the singularity is 1, only the 1-jet of the eigenvalue function is a formal invariant; c.f. [**6**, Proposition 20.2]. Therefore $\pm(1+ax)$ are the only formal invariants of the system. $\square$

Next, note that system (3) clearly admits

$$U(x) = \begin{bmatrix} x^a \mathrm{e}^{-1/x} & 0 \\ 0 & x^{-a}\mathrm{e}^{1/x} \end{bmatrix} \tag{4}$$

as a fundamental matrix solution. Consider one of the two singular directions, namely $\beta = 0$ or $\beta = \pi$. Let $T_{\beta+}$ and $T_{\beta-}$ denote, respectively, the Borel sums on the "left" and on the "right" of the fixed singular direction β. Then, there is a constant matrix S_β satisfying

$$T_{\beta-}(x) = T_{\beta+}(x)U(x)S_\beta U(x)^{-1},$$

for $x \in \mathrm{e}^{\mathrm{i}\beta}\mathbb{R}_{>0}$. The matrices S_0 and S_π are called the *Stokes matrices* and they have the general forms

$$S_0 = \begin{bmatrix} 1 & s_0 \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad S_\pi = \begin{bmatrix} 1 & 0 \\ s_\pi & 1 \end{bmatrix},$$

for suitable constants $s_0, s_\pi \in \mathbb{C}$; see e.g. [**6**, §20G]. In the particular case in question, explicit formulas for s_0 and s_π are known; see [**2**, pp. 86 and 87]. However, for our purposes, it suffices to prove that:

Lemma 2.2. $s_0 s_\pi \neq 0$ *if and only if* $\cos(2\pi a) \neq \cos(2\pi\sqrt{a^2+bc})$.

Proof: The lemma will be proved by explicitly computing the monodromy matrix M associated with system (2) around $x = 0$ in two different ways: first we will compute the matrix directly around $x = 0$ by using the Stokes matrices and then we will compute the monodromy (holonomy) around $x = \infty$ which is a Fuchsian singular point. The monodromy around $x = \infty$ is the inverse of the monodromy matrix M since the system in question has only two singular points (corresponding to $x = 0$ and to $x = \infty$). The result will then easily follow by computing the trace of M in each situation.

Let us compute the monodromy matrix around the origin with respect to the fundamental matrix solution $T_{0+}(x)U(x)$. By Lemma 2.1 $T_{0+}(x) = T_{\pi-}(x)$ and $T_{\pi+}(x) = T_{2\pi-}(x)$, and when considered on the Riemann surface of the logarithm then $T_\alpha(x) = T_{\alpha+2\pi}(\mathrm{e}^{2\pi\mathrm{i}}x)$ for any non-singular direction α. Hence

$$\begin{aligned} T_{0+}(\mathrm{e}^{2\pi\mathrm{i}}x)U(\mathrm{e}^{2\pi\mathrm{i}}x) &= T_{\pi-}(\mathrm{e}^{2\pi\mathrm{i}}x)U(\mathrm{e}^{2\pi\mathrm{i}}x) = T_{\pi+}(\mathrm{e}^{2\pi\mathrm{i}}x)U(\mathrm{e}^{2\pi\mathrm{i}}x)S_\pi \\ &= T_{2\pi-}(\mathrm{e}^{2\pi\mathrm{i}}x)U(\mathrm{e}^{2\pi\mathrm{i}}x)S_\pi = T_{0-}(x)U(x)NS_\pi \\ &= T_{0+}(x)U(x)S_0NS_\pi, \end{aligned}$$

where

$$N = \begin{bmatrix} \mathrm{e}^{2\pi\mathrm{i}a} & 0 \\ 0 & \mathrm{e}^{-2\pi\mathrm{i}a} \end{bmatrix}$$

is the "formal monodromy" of the fundamental matrix solution $U(x)$ (4). Therefore $M = S_0NS_\pi$ is the monodromy matrix, and it follows that

$$\operatorname{tr} M = 2\cos(2\pi a) + \mathrm{e}^{-2\pi\mathrm{i}a}s_0 s_\pi.$$

Let us now compute $\operatorname{tr} M$ by looking at the singular point $x = \infty$. Let $v = 1/x$ so that system (2) becomes

$$v\frac{\mathrm{d}y}{\mathrm{d}v} = \begin{bmatrix} -a-v & -b \\ -c & a+v \end{bmatrix} y = A(v)y, \tag{5}$$

and note that $v = 0$ corresponds to $x = \infty$. Denote by λ_1 and λ_2 the eigenvalues of the matrix $A(0)$. Naturally the matrix $A(0)$ is the so-called *residue matrix* of system (5). Clearly these two eigenvalues are symmetric and, up to relabeling, we set $\lambda_1 = \lambda$ and $\lambda_2 = -\lambda$, where $\lambda = \sqrt{a^2 + bc}$. In this case, the system is non-resonant if $2\lambda \notin \mathbb{Z}$; see e.g. [**6**, Definition 16.12]. In turn, provided that there is no resonance, the system is locally holomorphically equivalent to the Euler system $tv' = A(0)v$; see e.g. [**6**, Theorem 16.16]. In turn, the monodromy matrix around $v = 0$ is conjugate to the exponential of $2\pi \mathrm{i} A(0)$ and the latter matrix is conjugate to the inverse of the initial monodromy matrix M. Since traces of matrices remain invariant under conjugations, the preceding finally yields

$$\operatorname{tr} M = 2\cos(2\pi\lambda).$$

In fact, this last formula holds whether or not system (5) is resonant, as it immediately follows from the continuity of $\operatorname{tr} M$ with respect to the parameters a, b, and c (the set of non-resonant systems is open and dense). Lemma 2.2 promptly follows. $\square$

Next, note that the diagonal differential system (3) is naturally equivalent to the following family of vector fields on $\mathbb{C}^3$:

$$X_a = x^2 \tfrac{\partial}{\partial x} + (1 + ax)\left(u_1 \tfrac{\partial}{\partial u_1} - u_2 \tfrac{\partial}{\partial u_2}\right),$$

where $a \in \mathbb{C}$. Clearly vector fields in the family X_a admit the function $h(u) = u_1 u_2$ as a holomorphic first integral. Furthermore, we have:

Lemma 2.3. *The function $h(u) = u_1 u_2$ is a primitive first integral of X_a in the following sense: if $\hat{F} = \hat{F}(x, u) \in \mathbb{C}[[x, u]]$ is a formal first integral of X_a, then there exists a formal power series $\hat{G} \in \mathbb{C}[[z]]$ such that $\hat{F} = \hat{G} \circ h$.*

Proof: Assume that $\hat{F}(x, u)$ is a formal first integral of X_a and consider a Taylor expansion of the form:

$$\hat{F}(x, u) = \sum_{j=0}^{\infty} x^j \hat{f}_j(u) = \sum_{j=0}^{\infty} x^j \sum_{k \in \mathbb{Z}} u_1^k \hat{f}_{j,k}(u_1 u_2). \tag{6}$$

Let us show that in fact $\hat{F}(x, u) = \hat{f}_{0,0}(u_1 u_2)$, which proves the lemma.

We argue by induction. Assume $\hat{F}(x, u) = \hat{f}_{0,0}(u_1 u_2) + O(x^n)$ for some $n \geq 0$. Since $\hat{F}$ is a formal first integral of X_a, a direct computation yields

$$0 = X_a.\hat{F} = x^n \left(\sum_{k \in \mathbb{Z}} k u_1^k \hat{f}_{n,k}(u_1 u_2) \right) + O(x^{n+1}).$$

By comparing monomial degrees, it follows that all the functions $\hat{f}_{n,k}(\cdot)$ must vanish identically provided that $k \neq 0$. Thus the power series expansion (6) of $\hat{F}$ takes on the form

$$\hat{F} = \hat{f}_{0,0}(u_1 u_2) + x^n \hat{f}_{n,0}(u_1 u_2) + x^{n+1} \sum_{k \in \mathbb{Z}} u_1^k \hat{f}_{n+1,k}(u_1 u_2) + O(x^{n+2}).$$

In turn, this refined formula for $\hat{F}$ yields

$$0 = X_a.\hat{F} = x^{n+1} \left(n \hat{f}_{n,0}(u_1 u_2) + \sum_{k \in \mathbb{Z}} k u_1^k \hat{f}_{n+1,k}(u_1 u_2) \right) + O(x^{n+2}).$$

Therefore also $\hat{f}_{n,0}(\cdot)$ must vanish identically unless $n = 0$. Hence $\hat{F}(x, u) = \hat{f}_{0,0}(u_1 u_2) + O(x^{n+1})$, which establishes the induction step. $\square$

Remark 2.4. The computation carried out in the proof of Lemma 2.3 is related to a qualitative issue that is worth pointing out. For this, note first that the general solution of the diagonal system (3) has the form

$$\begin{cases} u_1(x) = c_1 \mathrm{e}^{-1/x} x^a, \\ u_2(x) = c_2 \mathrm{e}^{1/x} x^{-a} \end{cases}$$

for suitable constants $c_1, c_2 \in \mathbb{C}$. It follows that for any formal first integral $\hat{F} = \hat{F}(x, u_1, u_2)$ of X_a, the composition $\hat{F}(x, c_1 \mathrm{e}^{-\frac{1}{x}} x^a, c_2 \mathrm{e}^{\frac{1}{x}} x^{-a})$ must be a constant, and hence must factor through h due to the presence of the essential singularity arising from $\mathrm{e}^{\frac{1}{x}}$.

We are now able to provide the proof of Theorem 1.1.

Proof of Theorem 1.1: Owing to Lemma 2.1, the vector field $X_{a,b,c}$ has a *formal first integral* $\hat{f} = \hat{f}(x, y)$ which is obtained out of the first integral $h(u) = u_1 u_2$ of X_a by means of the equation

$$\hat{f}(x, \hat{T}(x) u) = h(u). \tag{7}$$

Furthermore, according to Lemma 2.3, every other formal first integral of $X_{a,b,c}$ must formally factor through $\hat{f}$. The proof of Theorem 1.1 is then reduced to showing that if a (non-constant) first integral of $X_{a,b,c}$ is holomorphic, then we necessarily have $\cos(2\pi a) = \cos(2\pi\sqrt{a^2 + bc})$.

Let us then assume there is a (non-constant) holomorphic first integral f for the vector field $X_{a,b,c}$ defined as in (1). It follows from Lemmas 2.1 and 2.3 that there exists a formal series $\hat{G} \in \mathbb{C}[[z]]$ such that $f(x, \hat{T}(x)u) = \hat{G} \circ h(u)$. The formal series on the right-hand side is independent of x; therefore, specializing the left-hand side to $x = 0$ (where $\hat{T}(0) = I$) implies that $f(0, u) = \hat{G} \circ h(u)$ is an analytic function of u, hence $\hat{G} = G$ is analytic. Since the Borel summation preserves analytic relations, it follows that $T_\alpha(x)$ also satisfies $f(x, T_\alpha(x)u) = G \circ h(u)$. For each of the singular directions $\beta = 0$ and $\beta = \pi$ we have

$$\begin{aligned} G \circ h(u) &= f(x, T_{\beta-}(x)u) = f(x, T_{\beta+}(x) U(x) S_\beta U(x)^{-1} u) \\ &= G \circ h(U(x) S_\beta U(x)^{-1} u). \end{aligned}$$

In other words, the function $h(u) = u_1 u_2$ must be invariant by both Stokes operators

$$u \mapsto U S_0 U^{-1} u = \begin{bmatrix} 1 & s_0 x^{2a} \mathrm{e}^{-\frac{2}{x}} \\ 0 & 1 \end{bmatrix} u, \quad u \mapsto U S_\pi U^{-1} u = \begin{bmatrix} 1 & 0 \\ s_\pi x^{-2a} \mathrm{e}^{\frac{2}{x}} & 1 \end{bmatrix} u,$$

which means that both $s_0 = s_\pi = 0$. In particular, $s_0 s_\pi = 0$, which is by Lemma 2.2 equivalent to $\cos(2\pi a) = \cos(2\pi\sqrt{a^2 + bc})$. This ends the proof of Theorem 1.1. $\square$

Remark 2.5. In the case when $a, b, c \in \mathbb{R}$, the sectorial normalizing transformations of Lemma 2.1 satisfy $\overline{T_\alpha(x)} = T_{-\alpha}(\overline{x})$, which means that

$$y = T_{\mathbb{R}}(x) u, \quad T_{\mathbb{R}}(x) = \tfrac{1}{2}(T_\alpha(x) + T_{-\alpha}(x)), \quad x \in \mathbb{R},$$

is a $\mathcal{C}^\infty$-normalizing transformation between the restrictions of system (2) and its formal normal form (3) to $x \in \mathbb{R}$. Consequently, each of the following three functions $f(x, y)$ defined by

$$f(x, T_{\mathbb{R}}(x) u) = \begin{cases} \text{(i)} & u_1 u_2, \\ \text{(ii)} & \mathbb{1}_{\mathbb{R}_{<0}}(x) \cdot x^{-a} \mathrm{e}^{1/x} u_1, \\ \text{(iii)} & \mathbb{1}_{\mathbb{R}_{>0}}(x) \cdot x^{a} \mathrm{e}^{-1/x} u_2, \end{cases}$$

where $\mathbb{1}_{\mathbb{R}_{\leqslant 0}}(x)$ is the characteristic function, are $\mathcal{C}^\infty$-first integrals for the restriction of $X_{a,b,c}$ to $\mathbb{R}^3$. In the case (i) this $\mathcal{C}^\infty$-first integral is asymptotic to the formal one defined by (7).

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Received on October 29, 2021.

Accepted on June 15, 2022.