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CYCLIC COVERINGS OF RATIONAL NORMAL SURFACES WHICH ARE QUOTIENTS OF A PRODUCT OF CURVES

Enrique Artal Bartolo, José Ignacio Cogolludo-Agustín, and Jorge Martín-Morales

Abstract: This paper deals with cyclic covers of a large family of rational normal surfaces that can also be described as quotients of a product, where the factors are cyclic covers of algebraic curves. We use a generalization of the Esnault–Viehweg method to show that the action of the monodromy on the first Betti group of the covering (and its Hodge structure) splits as a direct sum of the same data for some specific cyclic covers over $\mathbb{P}^1$.

This has applications to the study of Lê–Yomdin surface singularities, in particular to the action of the monodromy on the mixed Hodge structure, as well as to isotrivial fibered surfaces.

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Introduction

The general framework of this paper is the study of the cohomology of cyclic covers of normal projective surfaces ramified along a curve, i.e., a Weil divisor.

In the smooth case, for instance the projective plane, cyclic coverings of $\mathbb{P}^2$ ramified along a curve have been intensively studied since Zariski [**33, 34**]. The first approaches used the degree of the Alexander polynomial of the complement of the ramification locus to calculate the irregularity of the covering ramified along the curve with order the degree of the curve. In the 1980's a series of papers ([**18, 17, 21, 29, 22, 3**]) allowed for a computation of the irregularity of the covering that was independent of its fundamental group.

The problem of computing the irregularity and structure of cyclic (or more generally, abelian) covers is relevant in its own right, e.g. in [**1, 28**] for the non-cyclic case, with a focus on their global structure (including their singularities, whether the base is smooth or singular). Note that the first Betti numbers (i.e., twice the irregularity) of arbitrary finite abelian coverings can be retrieved from those of finite cyclic covers.

Our main motivation for this work, however, stems from the study of surface singularities. An important invariant of a singular surface in $\mathbb{C}^3$ is the mixed Hodge structure of the cohomology of its Milnor fiber [**14, 15, 16**]. In the isolated case this structure can be described by using the Steenbrink spectral sequence [**31**] associated with the semistable reduction of a resolution of the singularity [**27**]. A crucial ingredient to understand this spectral sequence is the fact that the restriction of the semistable reduction to each of the exceptional divisors of the resolution is a cyclic

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branched covering ramified on a normal crossing divisor on a smooth surface. Esnault and Viehweg's theory ([**18**]) can be used to compute the equivariant first Betti numbers of these coverings (by the Hodge symmetries only these are needed).

In practice, however, either the embedded resolutions are too difficult to compute or their structure is too complicated, and only a few explicit examples are known in the literature.

The first named author computed in [**2**] the Hodge structure of superisolated surface singularities [**23**], providing a counterexample to Yau's conjecture [**32**].

In order to simplify this combinatorial problem, the third named author used in [**25**] embedded $\mathbb{Q}$-resolutions instead, which are less complicated. For these resolutions, one needs to deal with cyclic branched coverings of surfaces with quotient singularities. This motivated us to develop a generalization of Esnault and Viehweg's theory to this setting [**6**]. An explicit embedded $\mathbb{Q}$-resolution is constructed for Lê–Yomdin surface singularities in [**24**]. In [**26**], it is shown that only cyclic covers ramified along $\mathbb{Q}$-normal crossing divisors are required both for weighted projective planes and for the type of surfaces studied in this paper.

The construction of the surfaces studied in this work involves three cyclic branched covers:

$$\begin{array}{ccc} \mathbb{P}^1 & \xrightarrow{m_\kappa} & \mathbb{P}^1, \\ z & \longmapsto & z^\kappa \end{array} \qquad G \xrightarrow[\kappa:1]{\tau} \mathbb{P}^1, \qquad F_{(d)} \xrightarrow[d:1]{\pi_F} \mathbb{P}^1.$$

The cover τ will be interpreted as an orbifold map $\tau\colon G \to \mathcal{O}$, where $\mathcal{O}$ is an orbifold whose underlying manifold is $\mathbb{P}^1$ and has r orbifold points (the images of the branching points); in the same way m_κ induces an orbifold structure $\mathbb{P}^1_{d,d}$ with two orbifold points of order d at 0, ∞. We consider a surface S as a diagonal quotient of $G \times \mathbb{P}^1$ by the action of $\mathbb{Z}/\kappa$; see (2.3) for details. The cover π_F appears as the restriction to the second factor in the vertical part of the pull-back of π and τ_2:

$$\begin{tikzcd} G \times F_{(d)} \arrow[r] \arrow[d, "\mathbf{1}_G \times \pi_F"'] & S_d \arrow[d, "\pi"] \\ G \times \mathbb{P}^1 \arrow[r, "\tau_2"] \arrow[dr, "\tau \times m_\kappa"'] & S \arrow[d, "\tau_3"] \\ & \mathbb{P}^1 \times \mathbb{P}^1 \end{tikzcd}$$

The normal surface S has $2r$ cyclic quotient points. There are several isotrivial fibrations hidden in the diagram above. The composition of τ_3 with the first projection can be seen as a ruled surface $S \to \mathcal{O}$; the composition of τ_3 with the second projection is an isotrivial fibration $S \to \mathbb{P}^1_{d,d}$.

The surface S_d inherits an $F_{(d)}$-isotrivial fibration structure $S_d \to \mathcal{O}$ and the main goal of the paper is to compute its cohomology of degree 1, more precisely its eigen-decomposition by the monodromy of π. The surface S_d is also the finite quotient of $G \times F_{(d)}$ by a non-free action of $\mathbb{Z}/\kappa$.

The case $r = 2$ was studied in [**4**] and some results will be used here. The orbifolds in [**4**] are rational, while in this work they are arbitrary orbifolds supported in $\mathbb{P}^1$.

This family can also be constructed by a series of weighted blow-ups and blow-downs of the Hirzebruch surface Σ_α. This fact allows us to determine the class groups of those surfaces S starting from particular presentations of Picard groups of Hirzebruch surfaces.

Our strategy to describe the cyclic coverings is to use a generalization of the Esnault and Viehweg's theory for cyclic coverings of smooth surfaces, developed in [**18**], to normal surfaces with quotient singularities (see [**6**]).

From an algebraic point of view, a d-cyclic covering $\tilde{X} \to X$ of a projective normal surface X with at most quotient singularities ramified along a Weil divisor D is determined by the choice (and existence) of a divisor class $H \in \operatorname{Cl}(X)$ satisfying $D \sim dH$. As long as $\operatorname{Cl}(X)$ has no torsion, the mere existence of H is enough. Otherwise, the choice of the particular $H \in \operatorname{Cl}(X)$ is also necessary (see Example 3.10). In this context, if D is a $\mathbb{Q}$-normal crossing divisor, the decomposition of $H^1(\tilde{X}, \mathbb{C})$ into invariant subspaces with respect to the action of the monodromy of the cover can be retrieved from the Hodge decomposition of $H^1(\tilde{X}, \mathcal{O}_{\tilde{X}}) \oplus H^0(\tilde{X}, \Omega^1_{\tilde{X}})$, where the invariant subspaces of the first term are naturally isomorphic to $H^1(X, \mathcal{O}_X(L^{(l)}))$ for certain divisors $L^{(l)}$, $l = 0, \dots, d-1$, and $H^0(\tilde{X}, \Omega^1_{\tilde{X}}) \cong \overline{H^1(\tilde{X}, \mathcal{O}_{\tilde{X}})}$. In case $X = \mathbb{P}^1 \times \mathbb{P}^1$ it is well known that this Hodge decomposition by the monodromy splits into two factors $\mathbb{H}_h \oplus \mathbb{H}_v$, that correspond to the restriction of the covering π to a vertical and a horizontal fiber of the birule of X; see Subsection 1.3.

The family of surfaces S presented here has a similar splitting that is described in detail in this paper. In order to do that, the concept of the *greatest common vertical covering* for a family of coverings will be introduced. For an explicit solution of the original problem, a description of the cohomology $H^*(S, \mathcal{O}_S(D))$ of a Weil divisor D is given. Such cohomology is often concentrated in a single degree. Concrete formulas are given in Section 3. The main result of this paper is proved in Section 4 (Theorem 4.12) and it refers to the H^1-eigenspace decomposition by the monodromy.

Theorem A. *Let S_d be the cyclic covering of S associated with (d, D, H), $D \in \operatorname{Div}(S)$, $H \in \operatorname{Cl}(S)$, $D \sim dH$, where D has $\mathbb{Q}$-normal crossings. Then*

$$H^1(S_d, \mathcal{O}_{S_d}) \cong \mathbb{H}_h \oplus \mathbb{H}_v,$$

where

- $\mathbb{H}_v$ *is the* 1*-cohomology of the structure sheaf of the restriction of an intermediate cover of π to a rational horizontal* fiber *and*
- $\mathbb{H}_h$ *is the* 1*-cohomology of the structure sheaf of the greatest common vertical cover of an intermediate cover.*

In particular, $H^1(S_d, \mathcal{O}_{S_d})$ splits as a direct sum of the cohomology of two cyclic covers of $\mathbb{P}^1$ and the splitting respects the eigenspaces of the monodromy and the Hodge structure.

The notions of the rational horizontal *fiber* and the *greatest common vertical cover* will be explained in the work; the degrees of intermediate covers will be made explicit in the text.

As an outline of the paper, in Section 1, the general theory of Esnault and Viehweg is reviewed for the sake of completeness. Section 2 is devoted to the description of the family of surfaces S and their divisor class group. In Section 3 we give explicit formulas for the cohomology of Weil divisors on S, proving when this cohomology is concentrated in a single degree. In Section 4, the main results are stated and proved. Some relevant examples are given in Section 5, including isotrivial fibered surfaces. The paper ends with cyclic covers appearing in weighted Lê–Yomdin singularities; see Section 6.

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1. Esnault and Viehweg's method

For notation, let $\zeta_m := \exp \frac{2\pi\sqrt{-1}}{m}$.

1.1. Cyclic covers of abelian quotient singular surfaces. In [**18**] the authors set the theory of ramified cyclic covers of projective smooth varieties. In this paper we are going to use a generalization of this theory for projective normal surfaces having cyclic quotient singularities [**6**]. Let X be either a projective smooth variety or a projective surface with cyclic quotient singularities. A cyclic cover $\pi\colon \tilde{X} \to X$ is algebraically determined by three data (d, D, H) where D is linearly equivalent to dH. The number of sheets of π is d and the ramification locus is a divisor D. We emphasize that $H \in \operatorname{Cl}(X)$ while D is a true Weil divisor in $\operatorname{Div}(X)$ and not just a linear equivalence class in $\operatorname{Cl}(X)$. Let

$$D := \sum_{i=0}^{r} n_i D_i$$

be the decomposition of D into irreducible divisors. Such a cyclic cover is defined topologically by a morphism $\rho\colon H_1(X^{\mathrm{reg}} \setminus D, \mathbb{Z}) \to \mathbb{Z}/d$. If μ_i is a meridian of D_i, then $\sigma(\mu_i) \equiv n_i \bmod d$. Usually one thinks of D as an effective divisor with $0 < n_i < d$ but actually the numbers n_i are only defined mod d and the divisor does not need to be effective. If the meridians of D generate $H_1(X^{\mathrm{reg}} \setminus D, \mathbb{Z})$, no more data are needed, but if not, different covers may have the same ramification divisor. To determine the cover one needs another Weil divisor H such that $D \sim dH$; to be more precise, only the class of H matters. The cover is the normalization of the zero locus of a multisection of the fiber bundle $\mathcal{O}_X(H)$ associated with the isomorphism $\mathcal{O}_X(H)^{\otimes d} \cong \mathcal{O}_X(D)$. The datum d is fixed; the divisor H can be replaced by any other linearly equivalent divisor. Moreover, given any divisor A we can replace D by $D + dA$ and H by $H + A$. In particular, the data (d, D, H) can be replaced by $(d, \tilde{D}, 0)$, where $\tilde{D} := D - dH$.

Note finally that the number of connected components of $\tilde{X}$ is the index of $\rho(H_1(X^{\mathrm{reg}} \setminus D, \mathbb{Z}))$ in $\mathbb{Z}/d$. If X is smooth and simply connected, then this number coincides with $\gcd(d, n_1, \ldots, n_r)$. We will state later what happens in the cyclic quotient case.

It is possible to track algebraically the action of the 1-cohomology of the monodromy $\sigma\colon \tilde{X} \to \tilde{X}$ of the covering. In fact, it is possible to get this action on $H^1(X, \mathcal{O}_X)$ and then derive the action on $H^1(X, \mathbb{C})$ via the Hodge decomposition. The theorem below has been proved in [**18**] in the smooth case and in [**6**, Theorem 2.3] in the cyclic quotient case (we restrict our attention to H^1).

Theorem 1.1. *With the previous notations, if D is a divisor with simple $\mathbb{Q}$-normal crossings, then*

$$H^1(\tilde{X}, \mathcal{O}_{\tilde{X}}) = \bigoplus_{l=0}^{d-1} H^1(X, \mathcal{O}_X(L^{(l)})), \quad L^{(l)} = -lH + \sum_{i=1}^{r} \left\lfloor \frac{l n_i}{d} \right\rfloor D_i,$$

where the monodromy of the cyclic covering acts on $H^1(X, \mathcal{O}_X(L^{(l)}))$ by multiplication by ζ_d^l.

Remark 1.2. The above divisors have nicer expressions if we apply them to $(d, \tilde{D}, 0)$. If we assume that

$$\tilde{D} = D - dH = \sum_{i=1}^{n} m_i D_i, \tag{1.1}$$

then

$$L^{(l)} = \sum_{i=1}^{n} \left\lfloor \frac{l m_i}{d} \right\rfloor D_i.$$

Note also that the condition on $\mathbb{Q}$-normal crossings is applied only to the reduction of $\tilde{D}$ mod d.

Let B be any Weil divisor in X. Then, since $B \cdot \tilde{D} = 0$, we have:

$$L^{(l)} \cdot B = \sum_{i=1}^{n} \left\lfloor \frac{lm_i}{d} \right\rfloor D_i \cdot B - \sum_{i=1}^{n} \frac{lm_i}{d} D_i \cdot B = - \sum_{i=1}^{n} \left\{ \frac{lm_i}{d} \right\} D_i \cdot B. \tag{1.2}$$

In particular, if B is effective and it does not have common components with $\tilde{D}$, then $L^{(l)} \cdot B \leq 0$. Moreover, $L^{(l)} \cdot B = 0$ if and only if l is a multiple of $\frac{d}{\gcd(d,m)}$ where $m = \gcd\{m_i \mid D_i \cdot B \neq 0\}$.

If we perform a (weighted) blow-up of $\hat{\pi} \colon \hat{X} \to X$, then we obtain a new cyclic cover ϖ by pulling back our original cover π. If π is defined by $(d, \tilde{D}, 0)$, then ϖ is defined by $(d, \hat{\pi}^* \tilde{D}, 0)$. Note that as $\tilde{D} \sim 0$, it is an integral Cartier divisor and then in particular $\hat{\pi}^* \tilde{D}$ is also an integral Cartier divisor. It is determined by D and the multiplicity in $\hat{\pi}^* \tilde{D}$ of the exceptional divisor of $\hat{\pi}$.

Example 1.3. Let us suppose that $(X, P) \cong \frac{1}{n}(a, b)$ and we perform a (p, q)-weighted blow-up $\hat{\pi} \colon \hat{X} \to X$; for simplicity, assume n, a, b are pairwise coprime and $\gcd(p, q) = 1$. Let $e := \gcd(n, pb - qa)$. Assume that $\tilde{D}$ is as in (1.1), and let ν_i be the (p, q)-multiplicity of D_i (it vanishes if $P \notin D_i$). Let E be the exceptional component of $\hat{\pi}$ and let us denote by D_i the strict transform of D_i in $\hat{X}$ (the context will indicate which divisor we are referring to). Then,

$$\hat{\pi}^* \tilde{D} = \frac{1}{e} \left(\sum_{i=1}^{n} \nu_i m_i \right) E + \sum_{i=1}^{n} m_i D_i.$$

If $\tilde{D}$ is a simple $\mathbb{Q}$-normal crossing divisor mod d, then it is also the case for $\hat{\pi}^* \tilde{D}$.

Definition 1.4. Let (W, P) be a germ of type $\frac{1}{n}(a, b)$, where n, a, b are pairwise coprime. This space admits two special expressions, namely $\frac{1}{n}(1, b')$ or $\frac{1}{n}(a', 1)$, where a', b' are well defined mod n. A weighted blow-up of W is said to be *special* if it is of weight $(a', 1)$ or $(1, b')$.

These special blow-ups have nice properties. The curvettes of the exceptional components are the extremal smooth curve germs of W. For example, if a', b' are reduced mod n, then they are the first step of the Jung–Hirzebruch resolutions.

Lemma 1.5. *Let (W, P) be a germ of type $\frac{1}{n}(a', 1) \cong \frac{1}{n}(1, b')$, $\gcd(a', n) = \gcd(b', n) = 1$. Let π_x, π_y be the special weighted blow-ups of weights $(a', 1)$, $(1, b')$, respectively, with exceptional components E_x, E_y. We denote by μ_x, μ_y, μ_x^e, μ_y^e suitable meridians of $\{x = 0\}$, $\{y = 0\}$, E_x, E_y, in $W \setminus \{xy = 0\}$.*

(i) *The local fundamental group of $W \setminus \{P\}$ is cyclic of order n; μ_x^e and μ_y^e are (separately) generators of this group.*
(ii) *The local fundamental group of $W \setminus \{y = 0\}$ is $\mathbb{Z}$. The sets $\{\mu_x^e\}$ and $\{\mu_y^e, \mu_y\}$ generate this group (separately). A similar statement holds for $W \setminus \{x = 0\}$.*
(iii) *The local fundamental group of $W \setminus \{xy = 0\}$ is $\mathbb{Z}^2$. The sets $\{\mu_x^e, \mu_x\}$ and $\{\mu_y^e, \mu_y\}$ generate this group (separately).*

Proof: Iterating the special blow-ups one obtains the Jung–Hirzebruch resolution of W. Using Mumford's method [**27**] a presentation of the distinct fundamental groups (generated by the meridians of all divisors) is given and the result follows. $\square$

Note that we are not asking the coverings π, ϖ above to be connected. In the classical case (X smooth and simply connected) it is easy to relate the number of connected components and the arithmetic of the coefficients of D. If we drop the smoothness, more conditions are needed.

Proposition 1.6. *Let X be a simply connected projective surface with normal cyclic quotient singularities. Let $\pi\colon \tilde{X} \to X$ be a cyclic branched cover associated with (d, D, H), where D has* smooth *components and $\mathbb{Q}$-normal crossings* mod d. *Let $\hat{\sigma}\colon \hat{X} \to X$ be the composition of* one *special blow-up for each singular point of X.*

Let m be the greatest common divisor of d and the coefficients of the divisor $\hat{\sigma}^*(D-dH)$. Then $\tilde{X}$ has m connected components.

Proof: Let $\sigma_Y\colon Y \to X$ be the minimal resolution of the singularities of X. Let

$$D_Y := d\left\{\frac{\sigma_Y^*(D-dH)}{d}\right\}, \quad H_Y := -\left\lfloor\frac{\sigma_Y^*(D-dH)}{d}\right\rfloor.$$

Let us denote also

$$\hat{D} := d\left\{\frac{\hat{\sigma}^*(D-dH)}{d}\right\}, \quad \hat{H} := -\left\lfloor\frac{\hat{\sigma}^*(D-dH)}{d}\right\rfloor.$$

Let E be an exceptional component of $\hat{\sigma}$; its multiplicity in $\hat{D}$ coincides with the multiplicity of its strict transform in D_Y. The same applies for the irreducible components of D.

Since X is simply connected, it is the case for Y. Then $H_1(Y \setminus D_Y, \mathbb{Z})$ is generated by the meridians of the irreducible components of D_Y. Let C be an irreducible component of D_Y and μ_C the class of its meridians. Let $\rho\colon H_1(Y \setminus D_Y, \mathbb{Z}) \to \mathbb{Z}/d$ be the morphism determining the covering over Y (which has the same number m of connected components as $\tilde{X}$). Recall that $\rho(\mu_C)$ is the coefficient of C in D_Y (mod d). Then, m equals the greatest common divisor of d and the coefficients of the divisor D_Y.

This comes from the fact that the whole set of meridians generate $H_1(Y \setminus D_Y, \mathbb{Z})$. But Lemma 1.5 implies that only the strict transforms of the irreducible components of $\hat{D}$ suffice and the result follows. $\square$

Let π be a cyclic cover of a surface X with cyclic quotient singular points, associated with (d, D, H), where D is a simple $\mathbb{Q}$-normal crossing divisor. Let $C \subset X$ be an irreducible curve (with only unibranch points) such that the union of C and the support of D is a $\mathbb{Q}$-normal crossing divisor. Then, $\pi_C := \pi_|\colon \pi^{-1}(C) \to C$ is a (maybe non-connected) cyclic cover of the curve C. In the smooth case it is easy to obtain the divisors defining this cover. In the cyclic quotient case some work has to be done.

Let $P_1, \dots, P_s \in C \cap \operatorname{Sing} X$ and let $\hat{\sigma}\colon \hat{X} \to X$ be a composition of weighted blow-ups at $P_1, \dots, P_s$ such that the strict transform of C (still denoted by C) is contained in the regular part of $\hat{X}$, which exists because of the $\mathbb{Q}$-normal crossing condition. Let $\hat{\pi}$ be the pull-back of π by $\hat{\sigma}$. Note that π_C and $\hat{\pi}_C$ can be identified.

The covering $\hat{\pi}$ is associated with a triple $(d, \hat{D}, \hat{H})$ obtained as follows. Consider $\tilde{D} = D - dH$; then $\hat{\sigma}^*\tilde{D} = \hat{D} - d\hat{H}$, where the support of $\hat{D}$ is contained in the support of the $\mathbb{Q}$-divisor $\pi^*(D)$.

Proposition 1.7. *Let $C \subset X$ be an irreducible curve (with only unibranch points) such that the union of the support of D with C has $\mathbb{Q}$-normal crossings.*

Then, the divisor D_C of C defining π_C has support at $D \cap C \equiv \hat{D} \cap C$, and the multiplicity of each point $P \in \hat{D} \cap C$ is the coefficient of the irreducible component D_i of $\hat{D}$ containing P.

Proof: We add some blow-ups of X such that $\hat{\sigma}$ is as in Proposition 1.6. The extra blow-ups do not affect the multiplicities of the components intersecting C and then the result follows. $\square$

Note that, since the $\mathrm{Cl}(\mathbb{P}^1)$ is completely determined by the degree, we deduce that d divides $\deg D_C$ and H_C is a divisor of degree $\frac{\deg D_C}{d}$.

1.2. Application to covers of $\mathbb{P}^1$. We are going to study the characteristic polynomial and the H^1-eigenspace decomposition of a d-cyclic covering of $\mathbb{P}^1$ associated with a divisor

$$D = \sum_{j=1}^{s} m_j \langle p_j \rangle, \quad \sum_{j=1}^{s} m_j = dh, \quad h \in \mathbb{Z},$$

i.e., $\deg H = h$; no coprimality conditions are assumed. Then we have

$$L^{(l)} \sim -lH + \sum_{j=1}^{s} \left\lfloor \frac{lm_j}{d} \right\rfloor \langle p_j \rangle \Longrightarrow \deg L^{(l)} = -\sum_{j=1}^{s} \left\{ \frac{lm_j}{d} \right\}.$$

Note that $L^{(l)}$ depends only on $l \bmod d$ and $L^{(0)} = 0$. Let $n := \gcd(d, m_1, \dots, m_r)$ and $\hat{d} = \frac{d}{n}$, $\hat{m}_j := \frac{m_j}{n}$. Then

$$\deg L^{(l)} = -\sum_{j=1}^{s} \left\{ \frac{l\hat{m}_j}{\hat{d}} \right\}$$

and actually $L^{(l)}$ depends only on $l \bmod \hat{d}$. We conclude that for $l \in \{0, 1, \dots, d-1\}$:

$$h^1(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(L^{(l)})) = \begin{cases} 0 & \text{if } l \equiv 0 \bmod \hat{d}, \\ -1 + \displaystyle\sum_{j=1}^{s} \left\{ \frac{l\hat{m}_j}{\hat{d}} \right\} & \text{otherwise.} \end{cases} \tag{1.3}$$

Geometrically the d-cyclic cover of $\mathbb{P}^1$ associated with D is the disjoint union of n copies of a $\hat{d}$-cyclic cover of $\mathbb{P}^1$ associated with $\frac{1}{n}D \in \mathrm{Div}(\mathbb{P}^1)$, where the monodromy exchanges cyclically these copies. Applying Lemma B.1 the characteristic polynomial of the monodromy is

$$\Delta(t) = \frac{(t^n - 1)^2 (t^d - 1)^{s-2}}{\prod_{j=1}^{s} (t^{\gcd(d, m_j)} - 1)}. \tag{1.4}$$

Note that we have obtained more than that since we have the monodromy action on the Hodge structure of the covering space.

1.3. Normal crossing covers of $\mathbb{P}^1 \times \mathbb{P}^1$. We illustrate the coverings ramified along normal crossing divisors on surfaces, studying $\mathbb{P}^1 \times \mathbb{P}^1$. Let $\pi \colon \tilde{X} \to \mathbb{P}^1 \times \mathbb{P}^1$ be a cover associated with (d, D, H), where D is a normal crossing divisor. The condition $D \sim dH$ completely determines H using bidegrees (this is the first difference with *reducible normal fake quadrics*, see Definition 2.3, since their class group may not be torsion-free).

The cohomology of $\tilde{X}$ can be studied using Theorem 1.1 (or more precisely the original theorem of Esnault and Viehweg), for which we need to know the 1-cohomology of some divisors. The following result is well known.

Proposition 1.8. *Let S be a section and let F be a fiber of $\mathbb{P}^1 \times \mathbb{P}^1$. Then:*

(i) $\dim H^0(\mathbb{P}^1 \times \mathbb{P}^1; \mathcal{O}(aS+bF)) = \begin{cases} (a+1)(b+1) & \textit{if } a,b \geq 0, \\ 0 & \textit{otherwise.} \end{cases}$

(ii) $\dim H^1(\mathbb{P}^1 \times \mathbb{P}^1; \mathcal{O}(aS+bF)) = \begin{cases} (a+1)(b+1) & \textit{if } a,b \leq -2, \\ 0 & \textit{otherwise.} \end{cases}$

(iii) $\dim H^2(\mathbb{P}^1 \times \mathbb{P}^1; \mathcal{O}(aS+bF)) = \begin{cases} -(a+1)(b+1) & \textit{if } (a+2)(b+2) < 0, \\ 0 & \textit{otherwise.} \end{cases}$

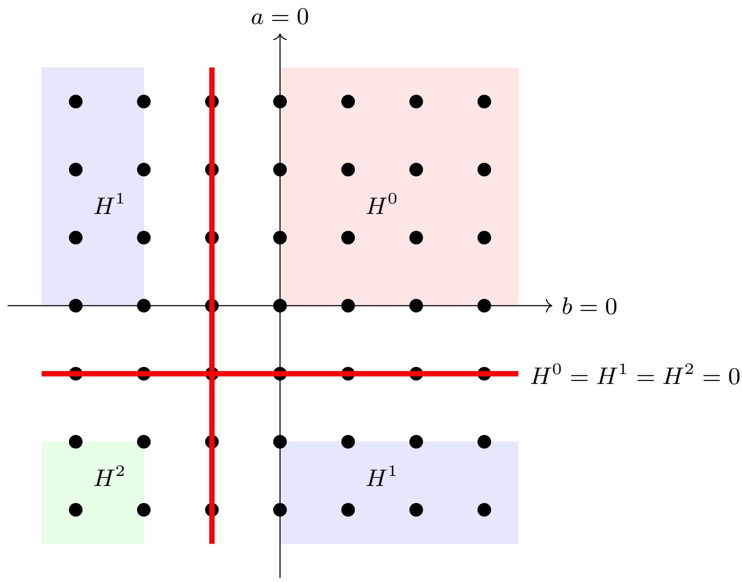

FIGURE 1. Map of the cohomology of $H^*(\mathbb{P}^1 \times \mathbb{P}^1; \mathcal{O}(aS+bF))$, where S is a section and F is a fiber.

This result is a direct consequence of the following:

($\mathcal{H}1$) The space $\dim H^0(\mathbb{P}^1 \times \mathbb{P}^1; \mathcal{O}(aS+bF))$ is isomorphic to the space of polynomials of bidegree (a,b) and (i) follows.
($\mathcal{H}2$) Using Serre duality (ii) follows.
($\mathcal{H}3$) Using Riemann–Roch and combining the previous results, (iii) follows.

Actually we are only interested in H^1 for $a,b \leq 0$ and only for $(a,0)$, $(0,a)$, $a<0$, the contribution is positive. Let us decompose $D = \nu_v D_v + \nu_h D_h + \nu_m D_m$, where D_v, D_h, D_m are primitive (their multiplicities are coprime), and all the components of D_v have bidegree of type $(0,b)$, all the components of D_h have bidegree of type $(a,0)$, and all the components of D_h have bidegree of type (a,b), $a,b>0$. The following result is well known.

Theorem 1.9. *The space $H^1(\tilde{X}; \mathcal{O}_{\tilde{X}})$ is decomposed as a direct sum $H^1(\tilde{X}_v; \mathcal{O}_{\tilde{X}_v}) \oplus H^1(\tilde{X}_h; \mathcal{O}_{\tilde{X}_h})$ where $\pi_v \colon \tilde{X}_v \to F$ is the restriction to $F \cong \mathbb{P}^1$ of the intermediate cover of degree $\gcd(d,\nu_h,\nu_m)$ and $\pi_h \colon \tilde{X}_h \to S$ is the restriction to $S \cong \mathbb{P}^1$ of the intermediate cover of degree $\gcd(d,\nu_v,\nu_m)$.*

2. Reducible normal fake quadrics

2.1. A ramified covering of the projective line. For the sequel we need to define an orbifold $\mathcal{O}$, with the following data. The orbifold is supported by $\mathbb{P}^1 = \mathbb{C} \cup \{\infty\}$ and the orbifold points are $\gamma_1, \dots, \gamma_r \in \mathbb{C} \subset \mathbb{P}^1$, $r \geq 0$, of orders $d_1, \dots, d_r \in \mathbb{Z}_{>1}$. Let us consider also $q_i \in \{1, \dots, d_i - 1\}$, $\gcd(d_i, q_i) = 1$, $i = 1, \dots, r$, such that

$$\alpha = \sum_{i=1}^{r} \frac{q_i}{d_i} \in \mathbb{Z}. \tag{2.1}$$

This imposes strong conditions on $d_1, \dots, d_r$. For instance,

$$d_r \text{ divides } \operatorname{lcm}(d_1, \dots, d_{r-1}).$$

In particular if $\kappa := \operatorname{lcm}(d_1, \dots, d_r)$, then κ^2 divides $d_1 \cdot \ldots \cdot d_r$ (see Remark 2.12).

There is an orbifold κ-cyclic covering $\tau\colon G \to \mathcal{O}$ associated with the epimorphism (well defined from (2.1))

$$\begin{aligned} \pi_1^{\mathrm{orb}}(\mathcal{O}) = \langle \mu_1, \dots, \mu_r \mid \mu_1 \cdot \ldots \cdot \mu_r = 1, \mu_1^{d_1} = \cdots = \mu_r^{d_r} = 1 \rangle &\longrightarrow \mathbb{Z}/\kappa, \\ \mu_i &\longmapsto q_i \frac{\kappa}{d_i} \mod \kappa; \end{aligned}$$

the covering τ was mentioned in the introduction. The position of the orbifold points has an influence on the analytic type of G but not on its topological type. The following result is a direct consequence of the definition of a cover associated with an epimorphism onto a cyclic group.

Lemma 2.1. *There exists a unique generator $\eta\colon G \to G$ of the monodromy of τ such that for any $i \in \{1, \dots, r\}$ and $p \in \tau^{-1}(\gamma_i)$ there exists a local coordinate y of G centered at p such that*

$$\eta^{\frac{\kappa}{d_i}}(y) = \zeta_{d_i}^{q_i} y.$$

Using the Riemann–Hurwitz method, one can compute the genus of G:

$$2 - 2g(G) = \chi(G) = \kappa(2 - r) + \sum_{i=1}^{r} \frac{\kappa}{d_i} = \kappa \chi^{\mathrm{orb}},$$

where

$$\chi^{\mathrm{orb}} := \chi^{\mathrm{orb}}(\mathcal{O}) = 2 - \sum_{i=1}^{r} \left(1 - \frac{1}{d_i}\right) \in \mathbb{Z}\frac{1}{\kappa}. \tag{2.2}$$

Lemma 2.2. *With the previous notations, if $r > 2$, then $g(G) > 0$ and $\chi^{\mathrm{orb}} \leq 0$.*

Proof: It is enough to prove that $\chi^{\mathrm{orb}} \leq 0$. Since $r > 2$ and

$$\chi^{\mathrm{orb}} \leq \frac{r}{2} + 2 - r = \frac{4 - r}{2},$$

it is enough to rule out the case $r = 3$.

In that case χ^{orb} can be positive only if (d_1, d_2, d_3) is one of the following: $(2, 2, n)$, $(2, 3, 3)$, $(2, 3, 4)$, or $(2, 3, 5)$, but none of them satisfy (2.1). $\square$

2.2. Definition and description of a reducible normal fake quadric. Recall the classical definition of a *fake quadric* as a smooth projective surface with the same rational cohomology as, but not biholomorphic to, the quadric surface $\mathbb{P}^1 \times \mathbb{P}^1$. Examples of fake quadrics are quotients of a product of two smooth projective curves by a free diagonal action. Such surfaces are called *reducible* (also called isogenous to a product of curves of an *unmixed* type; see [**12, 10**]). We extend this concept to *reducible normal fake quadrics*, where the freeness condition is dropped.

Let us consider the following $(\mathbb{Z}/\kappa)^2$-Galois and $\mathbb{Z}/\kappa$-Galois covers:

$$\begin{tikzcd}
 & S := (G \times \mathbb{P}^1)/(\mathbb{Z}/\kappa) \arrow[dr] & \\
G \times \mathbb{P}^1 \arrow[ur, "\tau_2"] \arrow[rr] & & \mathbb{P}^1 \times \mathbb{P}^1 \\
(p, z) \arrow[rr, mapsto] & & (\tau(p), z^\kappa)
\end{tikzcd}$$

The action defining S is given by a diagonal action

$$(1 \bmod \kappa, (p, z)) \longmapsto (\eta(p), \zeta_\kappa^{-1} z). \tag{2.3}$$

Definition 2.3. The surface $S := (G \times \mathbb{P}^1)/(\mathbb{Z}/\kappa)$ with the action (2.3) is called the *reducible normal fake quadric* associated with τ.

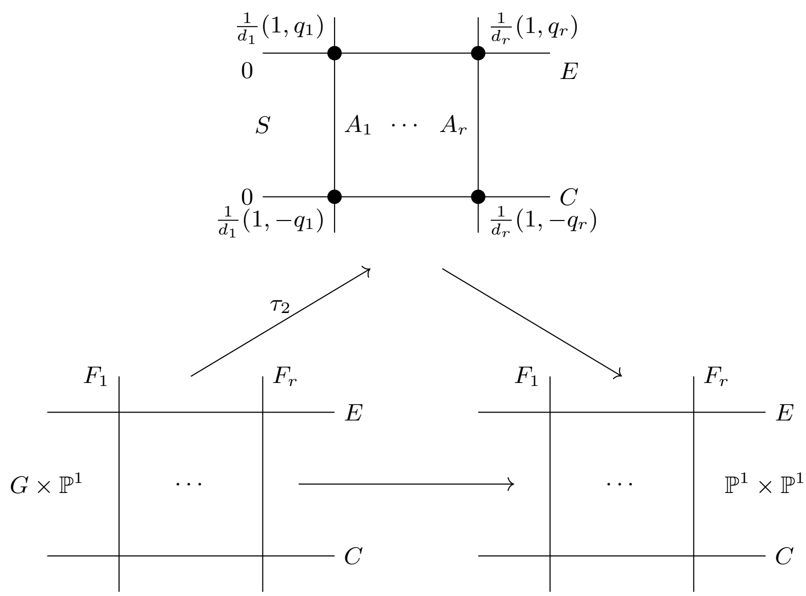

FIGURE 2. Covering construction of S.

The reason for the name *reducible normal fake quadric* will be clarified by the following description.

Lemma 2.4. *Let S be the reducible normal fake quadric associated with τ. It is a normal ruled surface $\pi_S \colon S \to \mathbb{P}^1$, $[(p, z)] \mapsto \tau(p)$ with two sets* $\operatorname{Sing}(C) := \{P_1, \dots, P_r\}$ *and* $\operatorname{Sing}(E) := \{Q_1, \dots, Q_r\}$ *of singular points (of cyclic quotient type). The following holds:*

(S1) *The curves $C := \tau_2(G \times \{0\})$ and $E := \tau_2(G \times \{\infty\})$ are* sections *of π_S with self-intersection* 0.
(S2) *There are* fibers *A_i of π_S such that $\{P_i\} = A_i \cap C$ and $\{Q_i\} = A_i \cap E$.*
(S3) *The type of P_i is $\frac{1}{d_i}(1, -q_i)$ and the type of Q_i is $\frac{1}{d_i}(1, q_i)$.*

Proof: Figure 2 describes S as a middle cover. We can identify C and E with $\mathbb{C} \cup \{\infty\} \equiv \mathbb{P}^1$, where P_i and Q_i become γ_i. Let $p \in G$ such that $P_i = [(p, 0)]$. A neighborhood of P_i in S is isomorphic to a neighborhood of the origin in $\mathbb{C}^2/\mu_{d_i}$, where $\mu_{d_i} = \langle \zeta \rangle$ is the cyclic group of d_i-roots of unity in $\mathbb{C}^*$ and the action is defined by

$$\zeta \cdot (y, z) = (\zeta^{q_i} y, \zeta^{-1} z),$$

and thus the type of P_i as a quotient singular point is calculated. Since z^{-1} is a local coordinate at ∞, the type of Q_i is computed in the same way. The self-intersection computation is straightforward. □

Remark 2.5. The surface S does not determine the original data given by $(d_1, \ldots, d_r)$ and $(q_1, \ldots, q_r)$. For instance, interchanging 0 and ∞ in $\mathbb{P}^1$ and choosing η^{-1} as a generator of the monodromy of τ results in the same surface S, which is associated with the data $(d_1, \ldots, d_r)$, $q'_i := d_i - q_i$, and $\alpha' := r - \alpha$. However, note that in general replacing η by η^ℓ with $\gcd(\ell, \kappa) = 1$ does not result in a surface isomorphic to S.

2.3. An alternative construction. There is an alternative description of reducible normal fake quadrics in terms of generalized Nagata operations of ruled surfaces. Consider a reducible normal fake quadric S as above and (d_i, q_i), $i = 1, \ldots, r$, such that $\alpha := \sum_{i=1}^r \frac{q_i}{d_i} \in \mathbb{Z}$ as in Lemma 2.4.

Lemma 2.6. *The surface S and the smooth ruled surface Σ_α both have a common weighted blown-up space obtained as follows.*

(i) *From S: composition of $(1, q_i)$-weighted blow-ups of Q_i.*
(ii) *From Σ_α: composition of (d_i, q_i)-weighted blow-ups at points in a section with self-intersection α.*

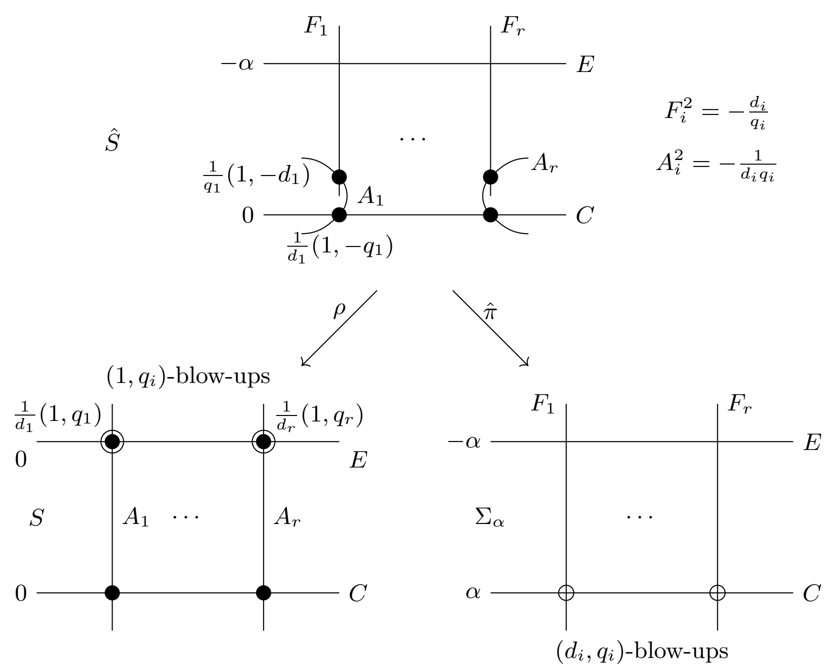

Figure 3. Blow-up construction of S.

Proof: The proof is depicted in Figure 3. Let us start from S. We perform the composition of the $(1,q_i)$-weighted blow-ups at Q_i, $i=1,\dots,r$; if (u_i,v_i) are the local variables, then E (resp. A_i) is given by $u_i=0$ (resp. $v_i=0$). Let $F_1,\dots,F_r$ be the exceptional components. From [**9**, Theorem 4.3] we obtain that $(F_i^2)_{\hat{S}}=-\frac{d_i}{q_i}$. For the strict transforms we have

$$(A_i^2)_{\hat{S}}=(A_i^2)_S-\frac{1^2}{d_iq_i}=-\frac{1}{d_iq_i},\quad (E^2)_{\hat{S}}=(E^2)_S-\sum_{i=1}^r\frac{q_i^2}{q_id_i}=-\sum_{i=1}^r\frac{q_i^2}{q_id_i}=-\alpha.$$

Since the centers of the blow-ups are disjoint to C, we still have $(C^2)_{\hat{S}}=0$. Moreover, the surface $\hat{S}$ is smooth along E, and $\hat{S}$ has cyclic quotient singular points of type $\frac{1}{q_i}(1,-d_i)$ at $F_i\cap A_i$.

Note that:

$$\hat{\pi}^*(F_i)=F_i+d_iA_i,\quad \hat{\pi}^*(C)=C+\sum_{i=1}^r q_iA_i,$$

$$\rho^*(A_i)=A_i+\frac{1}{d_i}F_i,\quad \rho^*(E)=E+\sum_{i=1}^r\frac{q_i}{d_i}F_i.$$

The surface $\hat{S}$ along A_i looks like the exceptional component of a weighted blow-up of type (d_i,q_i) at a smooth point. This shows that the stated weighted blow-ups of Σ_α also yield $\hat{S}$. $\square$

Remark-Definition 2.7. As a consequence of Lemmas 2.4 and 2.6, associated with any (d_i,q_i), $i=1,\dots,r$, such that $\gcd(d_i,q_i)=1$ and $\alpha:=\sum_{i=1}^r\frac{q_i}{d_i}\in\mathbb{Z}$, there is a reducible normal fake quadric, say S. According to Remark 2.5, this correspondence is not one to one, but we can still refer to S as *the reducible normal fake quadric associated with* (d_i,q_i), $i=1,\dots,r$.

Summarizing, the following describes the numerical relationship between the more relevant divisors on S:

$$A_i^2=E^2=C^2=F^2=0,\quad F\cdot C=F\cdot E=1,\quad C\cdot A_i=E\cdot A_i=\frac{1}{d_i},$$

where F is a generic fiber of π_S.

Remark 2.8. The surface S admits another map $\pi_G\colon S\to\mathbb{P}^1$ corresponding to the map $[(p,z)]\mapsto z^\kappa$. The fiber corresponding to 0 is κC, the fiber corresponding to ∞ is κE, and the other fibers are isomorphic to G (and they will again be denoted by G). These curves admit a simple characterization.

Lemma 2.9. *Let $D\subset S$ be an irreducible curve.*

(i) *If $D\cdot F=0$, then D is equal to one of these curves: $A_1,\dots,A_r$ or a generic fiber F of π_S. A linear combination of such divisors will be called a* vertical divisor.

(ii) *If $D\cdot C=0$, then D is equal to one of these curves: C, E, or a curve G. A linear combination of such divisors will be called a* horizontal divisor.

A linear combination of irreducible divisors which are neither vertical nor horizontal will be called a slanted divisor.

Proof: Let us start with (i). Let $p\in D$ and $w:=\pi_S(p)\in\mathbb{P}^1$. Note that $\pi_S^{-1}(w)$ is either $A_1,\dots,A_r$ or a fiber F; then $D\cdot\pi_S^{-1}(w)=0$ and $D\cap\pi_S^{-1}(w)\neq\emptyset$. Since D and $\pi_S^{-1}(w)$ are irreducible the only option is $D=\pi_S^{-1}(w)$. For (ii) we follow the same ideas using the map π_G. $\square$

2.4. Weil divisor class group. Consider S the reducible normal fake quadric associated with (d_i, q_i), $i = 1, \ldots, r$, such that $\gcd(d_i, q_i) = 1$ and $\alpha := \sum_{i=1}^{r} \frac{q_i}{d_i} \in \mathbb{Z}$ as defined in Remark-Definition 2.7. The descriptions of S given in Section 2 together with [**4**, §2.3] are the main ingredients for the computation of the Weil divisor class group $\mathrm{Cl}(S)$. Despite Σ_α having a simple class group isomorphic to $\mathbb{Z}^2$, note the following description of this group in terms of the divisor classes involved in the construction of Σ_α:

$$\mathrm{Cl}(\Sigma_\alpha) = \langle C, E, F, F_1, \ldots, F_r \mid E \sim C - \alpha F,\ F \sim F_1 \sim \cdots \sim F_r \rangle,$$

where F is a generic fiber. In order to obtain $\mathrm{Cl}(\hat{S})$, see Figure 3, the previous generators need to be replaced by their strict transforms, the classes of the exceptional components added, and the linear equivalence relations rewritten in terms of the new generators (see [**4**, Proposition 2.10]):

$$\mathrm{Cl}(\hat{S}) = \left\langle C, E, F, F_1, \ldots, F_r, A_1, \ldots, A_r \mid E \sim C + \sum_{i=1}^{r} q_i A_i - \alpha F,\ F \sim F_i + d_i A_i \right\rangle. \tag{2.4}$$

As in [**4**, Proposition 2.12], a presentation of the class group for a blow-down can easily be obtained if the exceptional components are part of the presentation of the class group of the source, and hence presentation (2.4) comes in handy. In this situation, it is enough to “forget” those exceptional components, that is,

$$\mathrm{Cl}(S) = \left\langle C, E, F, A_1, \ldots, A_r \mid E \sim C + \sum_{i=1}^{r} q_i A_i - \alpha F,\ F \sim d_i A_i \right\rangle. \tag{2.5}$$

Proposition 2.10. *The class group* $\mathrm{Cl}(S)$ *has the following structure as an abelian group:*

$$\mathrm{Cl}(S) \cong \mathbb{Z}^2 \oplus \bigoplus_{i=1}^{r-1} \mathbb{Z}/m_i, \tag{2.6}$$

where $m_i := \frac{\hat{d}_i}{\hat{d}_{i-1}}$, $\hat{d}_0 = 1$, *and* $\hat{d}_i = \gcd\left(\{\prod_{j\in I} d_j\}_{I\subset\{1,\ldots,r\},\,|I|=i}\right)$.

Moreover, the following holds:

(Cl1) *The free part is generated by the class of C and the class of a suitable linear combination of $A_1, \ldots, A_r$ (which is a rational multiple of F).*
(Cl2) *The torsion part has order* $m_1 \cdot \ldots \cdot m_{r-1} = \frac{d_1 \cdot \ldots \cdot d_r}{\kappa}$.
(Cl3) *The element* $T := E - C \in \operatorname{Tor}\mathrm{Cl}(S)$ *has maximal order* $\kappa = m_{r-1}$ *in* $\operatorname{Tor}\mathrm{Cl}(S)$.

Remark 2.11. Note that there might be more than one subgroup of order κ in $\operatorname{Tor}\mathrm{Cl}(S)$, but the one generated by T will be specially useful for our purposes. Note that T is *horizontal* since it is the difference of two sections, but it is also *vertical* as it is linearly equivalent to $\sum_{i=1}^{r} q_i A_i - \alpha F$.

Proof: From the presentation matrix one can easily see that $\mathrm{Cl}(S)$ is the direct sum of the free subgroup $\mathrm{Cl}_C(S) := \mathbb{Z}\langle C\rangle$ and $\mathrm{Cl}_F(S) := \mathbb{Z}\langle A_1, \ldots, A_r\rangle$. Note that $\mathrm{Cl}_F(S) \otimes_{\mathbb{Z}} \mathbb{Q} = \mathbb{Q}\langle F\rangle$ (of dimension 1) and that $\operatorname{Tor}\mathrm{Cl}(S) \subset \mathrm{Cl}_F(S)$. This shows part (Cl1).

The presentation matrix for $\mathrm{Cl}_F(S)$ is given as

$$\begin{pmatrix} d_1 & \ldots & 0 & -d_r \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \ldots & d_{r-1} & -d_r \end{pmatrix}.$$

Its Fitting ideals are $(\hat{d}_i)$, $i = 1, \ldots, r-1$, and hence its invariant factors are $m_i := \frac{\hat{d}_i}{\hat{d}_{i-1}}$, which ends the structure shown in (2.6).

Part (Cl2) follows from the formula $\hat{d}_{r-1} := \gcd\left(\frac{d_1 \cdot \ldots \cdot d_r}{d_1}, \ldots, \frac{d_1 \cdot \ldots \cdot d_r}{d_r}\right) = \frac{d_1 \cdot \ldots \cdot d_r}{\operatorname{lcm}(d_1, \ldots, d_r)}$, see (A.4) in the appendix, and the definition of $\kappa = \operatorname{lcm}(d_1, \ldots, d_r)$.

For part (Cl3), note that from the presentation matrix it follows that the maximal order of $\operatorname{Tor}\operatorname{Cl}(S)$ is κ. Hence, it remains to verify that the order of T is κ,

$$\kappa T \sim \kappa(E - C) \sim \kappa \sum_{i=1}^{r} q_i A_i - \kappa\alpha F \sim \left(\sum_{i=1}^{r} \frac{\kappa}{d_i} q_i - \kappa\alpha\right) F = 0.$$

Then $\kappa T \sim 0$. To check that κ is exactly the order of T, we need Lemma 2.13 below. Assume that another integer κ_1 satisfies $\kappa_1 T \sim 0$. Then, d_i divides $\kappa_1 q_i$. Since q_i and d_i are coprime, κ_1 is a multiple of d_i, and hence it is a multiple of κ. The fact that κ is precisely m_{r-1} is a consequence of another arithmetic property; see (A.5) in the appendix. $\square$

Remark 2.12. Note that only part (Cl3) depends on the condition (2.1). Also, as a consequence of parts (Cl2) and (Cl3), note that κ divides $\frac{d_1 \cdot \ldots \cdot d_r}{\kappa}$ and thus κ^2 divides $d_1 \cdot \ldots \cdot d_r$.

Lemma 2.13. *Let $D \in \operatorname{Cl}_F(S)$. Then, there are unique $f, a_i \in \mathbb{Z}$, $0 \leq a_i < d_i$, $i = 1, \ldots, r$, such that*

$$D \sim \sum_{i=1}^{r} a_i A_i + fF.$$

Proof: If D had two representations as in the statement, then the difference would represent 0 as a combination $0 = \sum_{i=1}^{r} a_i A_i + fF \in \operatorname{Cl}_F(S)$, where $-d_i < a_i < d_i$. Then, this expression would be a linear combination of $F - d_i A_i$ and hence d_i would divide a_i, which can only happen if $a_i = 0$. This implies $f = 0$. $\square$

Note the following additional linear equivalences $G \sim \kappa C \sim \kappa E$ given by the projection π_G.

Remark 2.14. There are canonical ways to represent a divisor class in S up to linear equivalence, but for technical reasons we will often use non-canonical expressions. However, one can apply Lemma 2.13 to find a unique representative for a divisor class. Note that any divisor D is linearly equivalent to a non-unique expression of the form

$$cC + eE + \sum_{i=1}^{r} a_i A_i + fF.$$

The term $c_D := F \cdot D = c + e \in \mathbb{Z}$ is intrinsic to D. Hence, $D \sim c_D C + \hat{D}$, where

$$\hat{D} = \sum_{i=1}^{r} (a_i + eq_i) A_i + (f - e\alpha) F \in \operatorname{Cl}_F(S).$$

Using Lemma 2.13 on $\hat{D}$ one obtains the *canonical form*

$$D \sim c_D C + \sum_{i=1}^{r} \hat{a}_i A_i + \hat{f} F, \tag{2.7}$$

where

(D1) $c_D = F \cdot D \in \mathbb{Z}$,
(D2) $\hat{a}_i \equiv (a_i + eq_i) \bmod d_i$ are integers in $[0, d_i)$, and
(D3) $\hat{f} = f + a - \hat{a} \in \mathbb{Z}$, for $\hat{a} := \sum_{i=1}^{r} \frac{\hat{a}_i}{d_i} \in \mathbb{Z}\frac{1}{\kappa}$ and $a := \sum_{i=1}^{r} \frac{a_i}{d_i} \in \mathbb{Z}\frac{1}{\kappa}$.

By Lemma 2.13, the triple $(c_D,(\hat{a}_i)_{i=1,\dots,r},\hat{f})$ characterizes the linear equivalence class of D. Also, note that

$$\varphi_D := C\cdot D = \hat{f}+\hat{a} = f+a \in \mathbb{Z}\frac{1}{\kappa},$$

and, moreover, the pair $(C\cdot D, F\cdot D) = (\varphi_D, c_D)\in \mathbb{Z}\frac{1}{\kappa}\times\mathbb{Z}$ determines the linear class of D up to torsion.

In a natural way, we have the following exact sequence involving the *horizontal* part $\mathrm{Cl}_H(S) := \mathbb{Z}\langle C,E\rangle$ of $\mathrm{Cl}(S)$:

$$0 \longrightarrow \mathbb{Z}\oplus\mathbb{Z}/\kappa \cong \mathrm{Cl}_H(S) \hookrightarrow \mathrm{Cl}(S) \longrightarrow \mathbb{Z}\oplus\bigoplus_{i=1}^{r-2}\mathbb{Z}/m_i \longrightarrow 0. \tag{2.8}$$

Remark 2.15. By (2.7), the condition for D to be in $\mathrm{Cl}_H(S)$ is equivalent to $\varphi_D=0$ and $\hat{a}_i\equiv 0 \bmod d_i$. By (D2), the latter is equivalent to the existence of a solution of

$$x\equiv a_i q_i^{-1} \mod d_i, \quad \forall i=1,\dots,r. \tag{2.9}$$

2.5. Canonical divisor. In the forthcoming calculations, the role of the class of the canonical divisor on S will be essential. The following result describes it.

Proposition 2.16. *The divisor*

$$K_S := -(C+E)+(r-2)F-\sum_{i=1}^{r} A_i$$

is a canonical divisor of S.

Proof: Recall the blow-up-down construction starting from Σ_α described in Figure 3. A canonical divisor of Σ_α is $-(C+E+F+F')$, where F and F' are two fibers. It is more convenient to consider the following linearly equivalent divisor:

$$K_{\Sigma_\alpha} := -(C+E)+(r-2)F-\sum_{i=1}^{r} F_i.$$

Recall $K_{\hat{S}} = \hat{\pi}^* K_{\Sigma_\alpha} + K_{\hat{\pi}}$, where $K_{\hat{\pi}}$ is the relative canonical divisor of $\hat{\pi}\colon \hat{S}\to\Sigma_\alpha$. Since the divisor K_{Σ_α} is *logarithmic* at the centers of the blow-ups, then

$$K_{\hat{S}} := -(C+E)+(r-2)F-\sum_{i=1}^{r}(A_i+F_i).$$

The direct image under ρ gives the result. □

Remark 2.17. Note that $K_S\cdot F=-2$ and $K_S\cdot C = -\chi^{\mathrm{orb}}$.

3. Cohomology of line bundles

Let D be a Weil divisor of S. The main goal of this section is to compute the cohomology groups $H^i(S,\mathcal{O}_S(D))$ for $i=0,1,2$. The key point in these calculations relies on the interpretation of the global sections of $\mathcal{O}_S(D)$ as global sections of a line bundle on a weighted projective plane, that is, the vector space of quasihomogeneous polynomials of a fixed degree satisfying certain vanishing conditions. Then, Serre's duality and the Riemann–Roch formula for normal surfaces is applied to obtain the second cohomology group and the Euler characteristic, respectively. Finally, the first cohomology group is obtained as a side product. For this reason, we have organized this section in four parts, where the different objects are studied, namely Subsection 3.1, global sections; Subsection 3.2, Euler characteristics; Subsection 3.3, general vanishing results; and Subsection 3.4, special cases (the last two both serve us to understand the first cohomology group).

3.1. Global sections. Consider D a Weil divisor in S. By (2.5), its class in $\mathrm{Cl}(S)$ can be written as $cC + eE + \sum_{i=1}^r a_i A_i + fF$, where $c, e, a_i \in \mathbb{Z}$, $i = 1, \dots, r$, that is,

$$D \sim cC + eE + \sum_{i=1}^r a_i A_i + fF \sim c_D\, C + \sum_{i=1}^r \hat{a}_i A_i + \hat{f} F,$$

where the right-most expression is unique, as described in Remark 2.14. Note, however, that c, e, a_i are not uniquely determined by D, since the group $\mathrm{Cl}(S)$ is not torsion-free and C, E, A_i are not linearly independent.

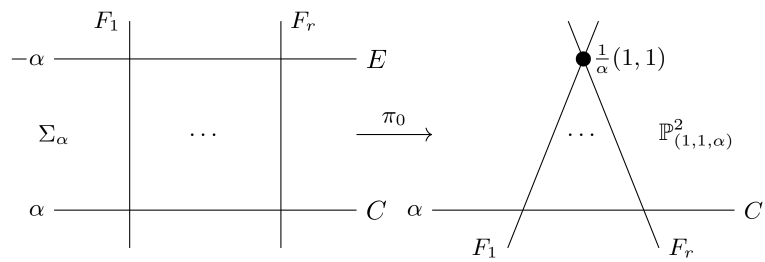

FIGURE 4. Birational transformation to $\mathbb{P}^2_{(1,1,\alpha)}$.

Recall that S is birationally equivalent to $\mathbb{P}^2_{(1,1,\alpha)}$ following the diagram

$$\begin{tikzcd}
\mathbb{P}^2_{(1,1,\alpha)} & \Sigma_\alpha \arrow[l, "\pi_0"'] & \widehat{S} \arrow[l, "\hat{\pi}:=\pi_1\circ\cdots\circ\pi_r"'] \arrow[ll, bend left, "\pi"] \arrow[r, "\rho"] & S,
\end{tikzcd}$$

where

$$\rho^*(A_i) = A_i + \frac{1}{d_i} F_i, \qquad \rho^*(E) = E + \sum_{i=1}^r \frac{q_i}{d_i} F_i, \qquad \rho^*(C) = C,$$

$$\pi^*(F_i) = F_i + \frac{1}{\alpha} E + d_i A_i, \qquad \pi^*(C) = C + \sum_{i=1}^r q_i A_i, \qquad \pi^*(F) = F + \frac{1}{\alpha} E.$$

By the projection formula for normal surfaces (see [**30**, Theorem 2.1])

$$H^0(S, \mathcal{O}_S(D)) \simeq H^0(\widehat{S}, \mathcal{O}_{\widehat{S}}(D')),$$

where

$$D' := \lfloor \rho^*(D) \rfloor = cC + eE + \sum_{i=1}^r a_i A_i + fF + \sum_{i=1}^r \left\lfloor \frac{eq_i + a_i}{d_i} \right\rfloor F_i.$$

Using the morphism π, there is a natural identification of the global sections of $\mathcal{O}_{\widehat{S}}(D')$ with those of $\mathcal{O}_{\mathbb{P}^2_{(1,1,\alpha)}}(\pi_*(D'))$. More precisely, according to [**6**, Proposition 4.2(2)],

$$H^0(\widehat{S}, \mathcal{O}_{\widehat{S}}(D')) \simeq \left\{ H \in \mathbb{C}[x,y,z]_{(1,1,\alpha),d} \;\middle|\; \begin{array}{l} \mathrm{mult}_{E'}(\pi^*(H)) \geq \mathrm{mult}_{E'}(\pi^*(\pi_*(D')) - D') \\ \forall E' \in \mathrm{Exc}(\pi) = \{E, A_1, \dots, A_r\} \end{array} \right\},$$

where $\mathbb{C}[x,y,z]_{(1,1,\alpha),d}$ denotes the $(1,1,\alpha)$-quasihomogeneous polynomials in x, y, z of degree $d := \deg_{(1,1,\alpha)}(\pi_*(D'))$.

Note that

$$\pi_*(D') = c\,C + fF + \sum_{i=1}^{r} \left\lfloor \frac{eq_i + a_i}{d_i} \right\rfloor F_i,$$

which has degree $d = \alpha c + f + \sum_{i=1}^r \lfloor \frac{eq_i+a_i}{d_i} \rfloor = \alpha c_D + \hat{f}$, and thus

$$\begin{aligned}\pi^*(\pi_*(D')) - D' = &\frac{1}{\alpha}\left(\sum_{i=1}^{r} \left\lfloor \frac{eq_i + a_i}{d_i} \right\rfloor - e\alpha + f\right)E \\ &+ \sum_{i=1}^{r}\left(cq_i + \left\lfloor \frac{eq_i + a_i}{d_i} \right\rfloor d_i - a_i\right)A_i.\end{aligned} \tag{3.1}$$

Using that $(c+e) = D\cdot F = c_D$ (see (D1)) and $eq_i+a_i - \lfloor \frac{eq_i+a_i}{d_i} \rfloor d_i \equiv (eq_i+a_i) \equiv \hat{a}_i \mod d_i$ (see (D2)), the coefficient of A_i in (3.1) can be rewritten as $c_D q_i - \hat{a}_i$ and that of E as $\frac{\hat{f}}{\alpha}$.

Assume, without loss of generality, that $\pi_0\colon \Sigma_\alpha \to \mathbb{P}^2_{(1,1,\alpha)}$ is the $(1,1)$-blow-up at the point $[0:0:1] \in \mathbb{P}^2_{(1,1,\alpha)}$. Then,

$$\operatorname{mult}_E(\pi^*(H)) = \frac{1}{\alpha}\operatorname{ord}(H(x,y,1)).$$

Also assume that F_i is the line in $\mathbb{P}^2_{(1,1,\alpha)}$ given by $x - \gamma_i y = 0$, $i = 1,\ldots,r$ ($\gamma_i \neq \gamma_j$, if $i \neq j$), and $C = \{z = 0\}$ so that $F_i \cap C = \{[\gamma_i : 1 : 0]\}$. Then π_i is the (d_i, q_i)-blow-up at a smooth point $(\gamma_i, 0) \in \mathbb{C}^2$ with local coordinates (x,z). Hence

$$\operatorname{mult}_{A_i}(\pi^*(H)) = \operatorname{ord}(H(x^{d_i} + \gamma_i, 1, z^{q_i})).$$

Summarizing, $H^0(S, \mathcal{O}_S(D))$ can be identified via π and ρ with the vector space of $(1,1,\alpha)$-quasihomogeneous polynomials $H(x,y,z)$ in x, y, z satisfying

$$\begin{cases}\deg(H(x,y,z)) = c_D\alpha + \hat{f} = c_D\alpha + f + a - \hat{a} = d,\\ \operatorname{ord}(H(x,y,1)) \geq \hat{f} = f + a - \hat{a},\\ \operatorname{ord}(H(x^{d_i} + \gamma_i, 1, z^{q_i})) \geq c_D q_i - \hat{a}_i, \quad \forall i = 1,\ldots,r,\end{cases} \tag{3.2}$$

where $a = \sum_{i=1}^r \frac{a_i}{d_i}$ and $\hat{a} = \sum_{i=1}^r \frac{\hat{a}_i}{d_i}$, as described in Remark 2.14.

To describe the contribution of the different cohomology spaces $H^i(S, \mathcal{O}_S(D))$ it is very convenient to construct a lattice that will encode relevant properties of divisor classes.

Definition 3.1. We shall define the *divisor lattice* $L := \mathbb{Z}\frac{1}{\kappa} \times \mathbb{Z} \subset \mathbb{Q}^2$ and the map $\operatorname{Cl}(S) \to L$ given by $D \mapsto \ell_D := (\varphi_D, c_D) = (D\cdot C, D\cdot F) \in L$. By the discussion in Remark 2.14, this map is onto. Moreover, its kernel is given by the torsion part of $\operatorname{Cl}(S)$. In particular, given a lattice point $\ell \in L$, there are exactly $\frac{d_1\cdot\ldots\cdot d_r}{\kappa}$ divisor classes in $\operatorname{Cl}(S)$ whose images coincide with $\ell \in L$.

For instance, the following proposition states that $H^0(S, \mathcal{O}_S(D)) \neq 0$ is only possible if ℓ_D sits on the first quadrant $L_{\geq 0} := L \cap \mathbb{Q}^2_{\geq 0}$ (lattice axes included) of L.

Proposition 3.2. *Using the previous notation,*

$$H^0(S, \mathcal{O}_S(D)) \neq 0 \Longrightarrow \ell_D \in L_{\geq 0}; \tag{3.3}$$

see the left-hand side of Figure 5.

Proof: We will use the intersection theory for weighted projective planes developed in [**9**, Proposition 5.2]. Choose $H(x,y,z)\in H^0(S,\mathcal{O}_S(D))$ different from zero.

Since F is generic, H and F do not have common components and the intersection $H\cap F$ consists of a finite number of points. Moreover,

$$\frac{d}{\alpha}=\frac{\deg(H)\cdot\deg(F)}{\alpha}=H\cdot F=\sum_{P\in\mathbb{P}^2_{(1,1,\alpha)}}(H\cdot F)_P\geq (H\cdot F)_{[0:0:1]}=\frac{\operatorname{ord}(H(x,y,1))}{\alpha}.$$

Therefore $d\geq\operatorname{ord}(H(x,y,1))$. Recall that $d=c_D\alpha+\hat{f}$ and, due to (3.2), $\operatorname{ord}(H(x,y,1))\geq \hat{f}=f+a-\hat{a}$. Hence $c_D\alpha\geq 0$ and $D\cdot F=c_D=c+e\geq 0$.

Let us now check that $D\cdot C\geq 0$. Assume $H(x,y,z)=z^mH'(x,y,z)$, $m\geq 0$, where H' and $C=\{z=0\}$ do not have common components. According to (3.2),

$$\begin{aligned}
&\deg(H'(x,y,z))=\alpha(c-m)+\sum_{i=1}^{r}\left\lfloor\frac{eq_i+a_i}{d_i}\right\rfloor=\hat{f}+(c_D-m)\alpha,\\
&\operatorname{ord}(H'(x,y,1))\geq\hat{f},\\
&\operatorname{ord}(H'(x^{d_i}+\gamma_i,1,z^{q_i}))\geq (c_D-m)q_i-\hat{a}_i,\quad \forall i=1,\ldots,r.
\end{aligned}$$

We apply Bézout's identity to H' and C and obtain

$$\begin{aligned}
\hat{f}+(c_D-m)\alpha&=\frac{\deg(H')\cdot\deg(C)}{\alpha}=H'\cdot C\\
&=\sum_{P\in\mathbb{P}^2_{(1,1,\alpha)}}(H'\cdot C)_P\geq\sum_{i=1}^{r}(H'\cdot C)_{F_i\cap C}.
\end{aligned}\tag{3.4}$$

It can be checked that $\operatorname{mult}_{A_i}(\pi^*(H'))=\operatorname{ord}(H'(x^{d_i}+\gamma_i,1,z^{q_i}))\leq d_i(H'\cdot C)_{F_i\cap C}$. Indeed, since this is a local problem, one can assume $\gamma_i=0$, that is, $F_i=\{x=0\}$ and $C=\{z=0\}$. If $(H'\cdot C)_{F_i\cap C}=n$, then x^n is a term of $H'(x,1,z)$ and thus $\operatorname{ord}_{(d_i,q_i)}(H'(x,1,z))\leq nd_i$. The multiplicity of $\pi^*(H')$ along A_i equals the (d_i,q_i)-order of $H'(x,1,z)$ because π_i is nothing but the (d_i,q_i)-blow-up at the point $F_i\cap C$. Then, using (2.1), one has

$$\sum_{i=1}^{r}(H'\cdot C)_{F_i\cap C}\geq\sum_{i=1}^{r}\frac{\operatorname{mult}_{A_i}(\pi^*(H'))}{d_i}\geq\sum_{i=1}^{r}\frac{(c_D-m)q_i-\hat{a}_i}{d_i}=(c_D-m)\alpha-\hat{a}.\tag{3.5}$$

Combining (3.4) and (3.5) gives $\hat{a}+\hat{f}=a+f\geq 0$, as desired. □

Theorem 3.3. *Let $D\sim cC+eE+\sum_{i=1}^r a_iA_i+fF$ be a Weil divisor on the normal surface S. The dimension of $H^0(S,\mathcal{O}_S(D))$ as a $\mathbb{C}$-vector space is*

$$h^0(S,\mathcal{O}_S(D))=\sum_{j=0}^{c_D}\max\{b_j(D),0\},$$

where

$$b_j(D):=1+\varphi_D-\sum_{i=1}^{r}\left\{\frac{a_i+(j-c)q_i}{d_i}\right\}\in\mathbb{Z}.$$

In particular, $h^0(S,\mathcal{O}_S(D))$ does not depend on the position of the singular points of S, but only on the singular types $\{(d_i;1,q_i)\}_{i=1}^r$ and the class of D in $\operatorname{Cl}(S)$.

Proof: We come back to the description given in (3.2) by considering $H(x,y,z)$ a generic $(1,1,\alpha)$-weighted homogeneous polynomial of degree d. Hence, let us write

$$H(x,y,z)=\sum_{j\geq 0} h_{d-\alpha j}(x,y)z^j, \tag{3.6}$$

where $h_{d-\alpha j}(x,y)$ is a homogeneous polynomial of degree $d-\alpha j$.

Let $j_{\max}=\left\lfloor \frac{d}{\alpha}\right\rfloor$ denote the maximum value of j such that $d-\alpha j\geq 0$. Then $H(x,y,1)=\sum_{j\geq 0} h_{d-\alpha j}(x,y)$ and its order is

$$\operatorname{ord}(H(x,y,1))=d-\alpha j_{\max}=(c_D-j_{\max})\alpha+\hat{f},$$

which is greater than or equal to $\hat{f}$ if and only if $j_{\max}\leq c_D$. Hence the sum in (3.6) runs from $j=0$ to $j=c_D$.

The condition $\operatorname{ord}(H(x^{d_i}+\gamma_i,1,z^{q_i}))\geq c_D q_i-\hat{a}_i$, $\forall i=1,\dots,r$, implies that $h_{d-\alpha j}(x,y)$ is of the form

$$h_{d-\alpha j}(x,y)=\prod_{i=1}^{r}(x-\gamma_i y)^{m_{ij}} g_j(x,y),$$

where $g_j(x,y)$ is a homogeneous polynomial of degree $d-\alpha j-\sum_{i=1}^r m_{ij}$, where

$$m_{ij}=\left\lceil \frac{(c_D-j)q_i-\hat{a}_i}{d_i}\right\rceil\geq 0.$$

Since the degrees of freedom of $g_j(x,y)$ is its degree plus 1 if the degree is non-negative, or zero otherwise, the required dimension is

$$\sum_{j=0}^{c_D}\max\left\{1+d-\alpha j-\sum_{i=1}^{r} m_{ij},0\right\}.$$

Note that

$$\begin{aligned}
1+d-\alpha j-\sum_{i=1}^{r} m_{ij} &= 1+\hat{f}+(c_D-j)\alpha-\sum_{i=1}^{r}\left\lceil \frac{(c_D-j)q_i-\hat{a}_i}{d_i}\right\rceil\\
&= 1+(\hat{f}+\hat{a})+\sum_{i=1}^{r}\left(\frac{(c_D-j)q_i-\hat{a}_i}{d_i}-\left\lceil\frac{(c_D-j)q_i-\hat{a}_i}{d_i}\right\rceil\right)\\
&= 1+\varphi_D-\sum_{i=1}^{r}\left\{\frac{\hat{a}_i+(j-c_D)q_i}{d_i}\right\}\\
&= 1+\varphi_D-\sum_{i=1}^{r}\left\{\frac{a_i+(j-c)q_i}{d_i}\right\}.
\end{aligned}$$

The last equality follows from the fact that

$$\frac{a_i+(j-c)q_i}{d_i}-\frac{\hat{a}_i+(j-c_D)q_i}{d_i}=\frac{a_i+eq_i-\hat{a}_i}{d_i}\in\mathbb{Z}.$$

Once a formula for computing $h^0(S,\mathcal{O}_S(D))$ has been found, the last part of the statement easily follows. $\square$

As a consequence one can determine the region of L where $H^2(S,\mathcal{O}_S(D))\neq 0$ is concentrated.

Corollary 3.4. *Using the previous notation,*

$$H^2(S,\mathcal{O}_S(D))\neq 0 \Longrightarrow -\ell_D\in(\chi^{\mathrm{orb}},2)+L_{\geq 0}; \tag{3.7}$$

see the middle part of Figure 5. In this case,

$$h^2(S,\mathcal{O}_S(D))=\sum_{j=0}^{-(2+c_D)}\max\{b_j(K_S-D),0\}$$

and

$$b_j(K_S-D)=1-(\varphi_D+\chi^{\mathrm{orb}})-\sum_{i=1}^{r}\left\{\frac{-1-a_i+(c+j+1)q_i}{d_i}\right\}.$$

Proof: By Serre's duality, $h^2(S,\mathcal{O}_S(D))=h^0(S,\mathcal{O}_S(K_S-D))$. Recall that the canonical divisor of S is $K_S=-C-E-\sum_{i=1}^r A_i+(r-2)F$. Then,

$$K_S-D=(-1-c)C+(-1-e)E+\sum_{i=1}^{r}(-1-a_i)A_i+(r-2-f)F.$$

From Proposition 3.2, $h^0(S,\mathcal{O}_S(K_S-D))\neq 0$ implies that $(K_S-D)\cdot F\geq 0$ and $(K_S-D)\cdot C\geq 0$, or equivalently, $D\cdot F\leq K_S\cdot F=-2$ and $D\cdot C\leq K_S\cdot C=-\sum_{i=1}^r\frac{1}{d_i}+r-2=-\chi^{\mathrm{orb}}$, as claimed.

The second part of the statement follows from Theorem 3.3 applied to K_S-D. ☐

3.2. Euler characteristic. The main purpose of this section is to compute the Euler characteristic of the sheaf $\mathcal{O}_S(D)$. As above, assume that

$$D\sim cC+eE+\sum_{i=1}^{r}a_iA_i+fF,$$

where $c,e,f,a_i\in\mathbb{Z}$, $i=1,\dots,r$, and recall that $K_S=-C-E-\sum_{i=1}^r A_i+(r-2)F$. In order to calculate $\chi(S,\mathcal{O}_S(D))$, we will use the Riemann–Roch formula on singular normal surfaces from [**11**, §1.2], see also [**13**], that is,

$$\chi(S,\mathcal{O}_S(D))=\chi(S,\mathcal{O}_S)+\frac{D\cdot(D-K_S)}{2}+R_S(D). \tag{3.8}$$

Recall that the correction term $R_S(D)$ is a sum of local invariants associated with each singular point P of S and the local class of D at P. In our case

$$R_S(D)=\sum_{i=1}^{r}R_{S_i^+}(D)+\sum_{i=1}^{r}R_{S_i^-}(D), \tag{3.9}$$

where $S_i^{\pm}$ denotes the local singularity type $\frac{1}{d_i}(1,\pm q_i)$.

First note that S is a rational surface (it is birationally equivalent to a weighted projective plane) and hence $\chi(S,\mathcal{O}_S)=1$. Also, using the fact that C and E are numerically equivalent to each other, D and K_S can numerically be described as

$$D\equiv c_D\,C+\varphi_DF,\quad K_S\equiv -2C-\chi^{\mathrm{orb}}F.$$

Then,

$$\begin{aligned}
&D^2=2c_D\,\varphi_D,\\
&D\cdot K_S=-c_D\,\chi^{\mathrm{orb}}-2\varphi_D,\\
&D^2-D\cdot K_S=2(c_D+1)\varphi_D+c_D\,\chi^{\mathrm{orb}},
\end{aligned}$$

and therefore

$$\chi(S,\mathcal{O}_S(D))=1+(c_D+1)\varphi_D+\frac{1}{2}c_D\,\chi^{\mathrm{orb}}+R_S(D). \tag{3.10}$$

In order to continue with this calculation, we need to understand the local contribution of each $R_{S_i^{\pm}}(D)$ to the correction term $R_S(D)$.

Consider $d, q \in \mathbb{Z}$ any three integers and denote by $S^{\pm}$ the cyclic singularity $\frac{1}{d}(1,\pm q)$. Since $R_{S^{\pm}}(D)$ only depends on the local class of the divisor D, we can consider it as a map $R_{S^{\pm}}\colon \operatorname{Weil}(S^{\pm})/\operatorname{Cart}(S^{\pm}) \cong \mathbb{Z}/d\mathbb{Z} \to \mathbb{Q}$. The following result gives a closed formula for the combined contribution $R_{S^+}(n) + R_{S^-}(n-mq)$ with the convention that $\sum_{j=k_1}^{k_2} f(j) = 0$ if $k_1 > k_2$.

Lemma 3.5. *Under the conditions above, let $n, m \in \mathbb{Z}$ be such that $m \geq -1$. Then,*

$$R_{S^+}(n) + R_{S^-}(n-mq) = -\sum_{j=0}^{m} \left\{ \frac{n+(j-m)q}{d} \right\} + m\frac{d-1}{2d}.$$

Proof: For $m \geq 0$, we proceed by induction on m. The first case, namely $R_{S^+}(n) + R_{S^-}(n) = -\left\{\frac{n}{d}\right\}$, was already proved in [**4**, Proposition 2.16(2)]. To be precise, in *loc. cit.* the result was stated in terms of the so-called Δ-invariant. In this context, the relationship between Δ and R is simply given by $R_{S^+}(n) = -\Delta_{S^+}(-n)$ (see [**13**, Introduction]). Similarly, $m = 1$ is a direct consequence of [**4**, Proposition 2.16(1)] and the case $m = 0$.

Assume the result is true for $m \geq 1$ and we will prove it for $m+1$. The cases $m = 1$, $m = 0$, and the induction hypothesis tell us respectively that

$$R_{S^+}(n) = -R_{S^-}(n-q) - \left\{ \frac{n-q}{d} \right\} - \left\{ \frac{n}{d} \right\} + \frac{d-1}{2d},$$

$$0 = R_{S^+}(n-q) + R_{S^-}(n-q) + \left\{ \frac{n-q}{d} \right\},$$

$$R_{S^-}(n-q-mq) = -R_{S^+}(n-q) - \sum_{j=0}^{m} \left\{ \frac{n-q+(j-m)q}{d} \right\} + m\frac{d-1}{2d}.$$

Adding up these three equations gives the result for $m+1$.

It remains to prove the case $m = -1$, that is, $R_{S^+}(n) + R_{S^-}(n+q) = -\frac{d-1}{2d}$, which is again a reformulation of [**4**, Proposition 2.16(1)]. $\square$

Theorem 3.6.

$$\chi(S, \mathcal{O}_S(D)) = \begin{cases} \displaystyle\sum_{j=0}^{c_D} b_j(D) & \text{if } c_D \geq 0, \\ 0 & \text{if } c_D = -1, \\ \displaystyle\sum_{j=0}^{-(c_D+2)} b_j(K_S - D) & \text{if } c_D \leq -2. \end{cases}$$

Proof: The result will follow after combining (3.8), (3.9), (3.10), and Lemma 3.5.

$$\begin{aligned} R_S(D) &= \sum_{i=1}^{r} (R_{S_i^+}(D) + R_{S_i^-}(D)) \\ &= \sum_{i=1}^{r} (R_{S_i^+}(eE + a_i A_i) + R_{S_i^-}(cC + a_i A_i)) \\ &= \sum_{i=1}^{r} (R_{S_i^+}(a_i + eq_i) + R_{S_i^-}(a_i - cq_i)). \end{aligned}$$

By Lemma 3.5, applied to $d = d_i$, $q = q_i$, $n = a_i + eq_i$, $m = c_D \geq -1$, each summand can be rewritten in terms of fractional parts so that

$$R_S(D) = \sum_{i=1}^{r} \left(-\sum_{j=0}^{c_D} \left\{ \frac{a_i + (j-c)q_i}{d_i} \right\} + c_D \frac{d_i - 1}{2d_i} \right).$$

Hence

$$\begin{aligned}\chi(S, \mathcal{O}_S(D)) &= 1 + (c_D+1)\varphi_D + \frac{1}{2} c_D\, \chi^{\text{orb}} - \sum_{i=1}^{r}\sum_{j=0}^{c_D} \left\{ \frac{a_i + (j-c)q_i}{d_i} \right\} + \frac{1}{2} c_D \sum_{i=1}^{r} \frac{d_i - 1}{d_i} \\ &= 1 + \frac{1}{2} c_D \left(\chi^{\text{orb}} + \sum_{i=1}^{r} \frac{d_i-1}{d_i} \right) + \sum_{j=0}^{c_D} \left(\varphi_D - \sum_{i=1}^{r} \left\{ \frac{a_i + (j-c)q_i}{d_i} \right\} \right) \\ &= \sum_{j=0}^{c_D} b_j(D),\end{aligned}$$

as claimed.

In particular, when $c_D = -1$, one has $\chi(S, \mathcal{O}_S(D)) = 0$. The last part of the statement follows from Serre's duality and the fact that $D \cdot F = c_D \leq -2$ implies that $(K_S - D) \cdot F = -(c_D + 2) \geq 0$. □

3.3. General vanishing results. According to Theorem 3.3, Corollary 3.4, and Theorem 3.6, the Euler characteristic of the sheaf $\mathcal{O}_S(D)$ coincides with the dimension of H^0 (resp. H^2) as long as $b_j(D) \geq 0$ (resp. $b_j(K_S - D) \geq 0$), $\forall j$. In this section we will investigate when each one of these conditions holds. This will affect the vanishing of the first cohomology group. As above, we assume that

$$D \sim cC + eE + \sum_{i=1}^{r} a_i A_i + fF \sim c_D\, C + \sum_{i=1}^{r} \hat{a}_i A_i + \hat{f} F, \tag{3.11}$$

where $(c_D, \{\hat{a}_i\}_{i=1,\ldots,r}, \hat{f})$ are uniquely determined by the canonical form of D (see Remark 2.14), and $\ell_D = (C \cdot D, F \cdot D) = (\varphi_D, c_D)$ (see Definition 3.1).

Figure 5 depicts a projection of $D \in \mathrm{Cl}(X)$ onto the divisor lattice $(\varphi_D, c_D) \in L$ (see Definition 3.1) and it describes regions where cohomological triviality is assured for all divisor classes corresponding to each value $(\varphi, c) \in L$. Combining these, one obtains in Figure 6 four cones where cohomology is concentrated in one degree.

The following result is obtained combining the previous results on global sections and Euler characteristics. It describes the general regions of L where the cohomology is concentrated in a single degree. The statement of the following theorem is summarized in the right-hand side of Figure 5.

Theorem 3.7. *Let D be as in* (3.11). *Then the following holds:*

(i) *If $\ell_D \in (-\chi^{\text{orb}}, -2) + L_{>0}$, then $h^*(S, \mathcal{O}_S(D))$ is concentrated in degree* 0.

(ii) *If $-\ell_D \in L_{>0}$, then $h^*(S, \mathcal{O}_S(D))$ is concentrated in degree* 2.

(iii) *If $\ell_D = (\varphi_D, c_D) \in L$ with either*

(a) *$\varphi_D < 0$ and $c_D \geq 0$, or*

(b) *$\varphi_D > -\chi^{\text{orb}}$ and $c_D \leq -2$,*

then $h^*(S, \mathcal{O}_S(D))$ is concentrated in degree 1.

(iv) *If $c_D = -1$, then $h^i(S, \mathcal{O}_S(D)) = 0$ for $i = 0, 1, 2$.*

Proof: To prove (i), let us assume $\varphi_D > -\chi^{\mathrm{orb}}$ and $c_D > -2$. Note that the maximum value of $\left\{\frac{n}{d_i}\right\}$ is reached when $n = d_i - 1$. Then,

$$b_j(D) = 1 + \varphi_D - \sum_{i=1}^{r} \left\{ \frac{a_i + (j-c)q_i}{d_i} \right\} \geq 1 + \varphi_D - \sum_{i=1}^{r} \frac{d_i - 1}{d_i} = \varphi_D + \chi^{\mathrm{orb}} - 1.$$

Under the current hypotheses this number is an integer strictly greater than -1 and hence $\max\{b_j(D), 0\} = b_j(D)$. By Theorems 3.3 and 3.6, $h^0(S, \mathcal{O}_S(D)) = \chi(S, \mathcal{O}_S(D))$. Corollary 3.4 provides the vanishing of $H^2(S, \mathcal{O}_S(D))$ and thus the vanishing of $H^1(S, \mathcal{O}_S(D))$.

Part (ii) is a consequence of Serre's duality.

Part (iii) follows from (3.3) in Proposition 3.2 and (3.7) in Corollary 3.4.

Finally, for part (iv) note that, if $c_D = -1$, then $h^0(S, \mathcal{O}_S(D)) = h^2(S, \mathcal{O}_S(D)) = 0$ from Proposition 3.2 and Corollary 3.4, respectively. Then $h^1(S, \mathcal{O}_S(D)) = -\chi(S, \mathcal{O}_S(D)) = 0$ from Theorem 3.6 and the result follows. $\square$

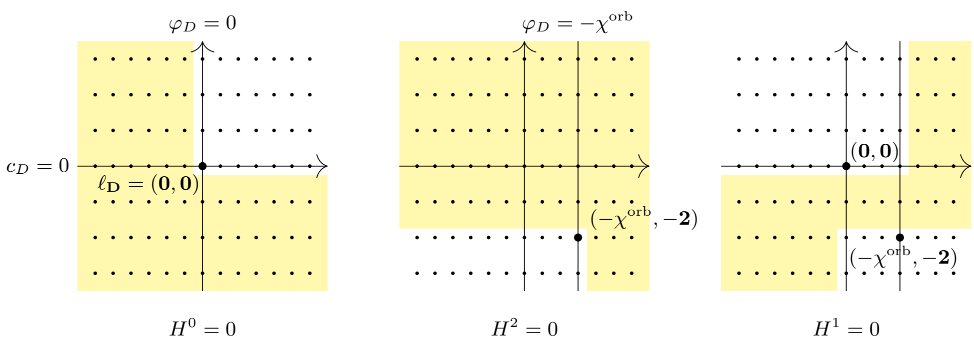

FIGURE 5. Vanishing of H^j.

3.4. Special cases. For our goal in Section 4 we are interested in finding explicit formulas for $h^1(S, \mathcal{O}_S(D))$ when $-\ell_D \in L_{\geq 0}$. By Theorem 3.7, one has $h^1(S, \mathcal{O}_S(D)) = 0$ whenever $-\ell_D \in L_{>0}$. The rest of this section will be devoted to the special case $-\ell_D \in L_{\geq 0}$ and either $\varphi_D = 0$ or $c_D = 0$.

The simplest case, which we want to exclude, is $D \sim 0$, for which $h^0(S, \mathcal{O}_S(D)) = 1$ and $h^1(S, \mathcal{O}_S(D)) = h^2(S, \mathcal{O}_S(D)) = 0$. Note that in this case $\varphi_D = c_D = 0$.

Let us first consider the case $c_D = 0$, $\varphi_D \leq 0$. Using the canonical form of D and using $c_D = 0$, one has

$$D \sim \sum_{i=1}^{r} \hat{a}_i A_i + \hat{f} F.$$

One has the following explicit formula for $h^1(S, \mathcal{O}_S(D))$.

Proposition 3.8. *Let D be such that $-\ell_D = (-\varphi_D, 0) \in L_{\geq 0}$ and $D \not\sim 0$. Then,*

$$h^0(S, \mathcal{O}_S(D)) = h^2(S, \mathcal{O}_S(D)) = 0$$

and

$$h^1(S, \mathcal{O}_S(D)) = -1 - \sum_{i=1}^{r} \left\lfloor \frac{\hat{a}_i}{d_i} \right\rfloor - \hat{f}. \tag{3.12}$$

In particular, if $\ell_D = (0,0)$, then

$$h^1(S, \mathcal{O}_S(D)) = -1 + \sum_{i=1}^{r} \left\{ \frac{\hat{a}_i}{d_i} \right\}. \tag{3.13}$$

Proof: By Corollary 3.4, $h^2(S,\mathcal{O}_S(D))=0$. On the other hand, $c_D=0$ together with Theorem 3.3 implies $h^0(S,\mathcal{O}_S(D))=\max\{b_0(D),0\}$, where

$$\mathbb{Z}\ni b_0(D)=1+\varphi_D-\sum_{i=1}^r\left\{\frac{\hat{a}_i}{d_i}\right\}=1+\hat{f}+\sum_{i=1}^r\left\lfloor\frac{\hat{a}_i}{d_i}\right\rfloor.$$

Since $\varphi_D\leq 0$, $D\not\sim 0$, and $b_0(D)\in\mathbb{Z}$, one deduces that $b_0(D)\leq 0$ and thus $h^0(S,\mathcal{O}_S(D))=0$.

Recall that $\chi(S,\mathcal{O}_S(D))=b_0(D)$ and the equality for $h^1(S,\mathcal{O}_S(D))$ holds. □

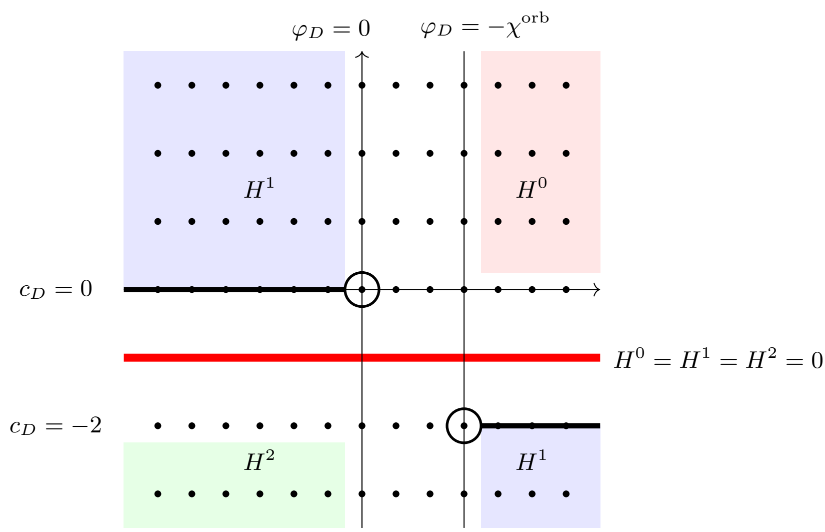

Figure 6. Cohomology concentrated in a single degree.

Now consider the case $D\not\sim 0$ and $-\ell_D=(0,-c_D)\in L_{\geq 0}$. Note that $h^0(S,\mathcal{O}_S(D))=0$ and, by Theorem 3.6 and Corollary 3.4,

$$h^1(S,\mathcal{O}_S(D))=\sum_{j=0}^{-(2+c_D)}\left(-b_j(K_S-D)+\max\{b_j(K_S-D),0\}\right),$$

where in this case

$$\begin{aligned}b_j(K_S-D)&=1-\chi^{\mathrm{orb}}-\sum_{i=1}^r\left\{\frac{-1-a_i+(c+j+1)q_i}{d_i}\right\}\\&\geq 1-\chi^{\mathrm{orb}}-\sum_{i=1}^r\left(\frac{d_i-1}{d_i}\right)=1-\chi^{\mathrm{orb}}-r+\sum_{i=1}^r\frac{1}{d_i}=-1.\end{aligned}\tag{3.14}$$

Hence

$$h^1(S,\mathcal{O}_S(D))=\#\{j\in\{0,1,\ldots,-(2+c_D)\}\mid b_j(K_S-D)=-1\}.\tag{3.15}$$

Proposition 3.9. *Let D be a divisor in S such that $\ell_D=(0,c_D)$ with $c_D<0$.*

(i) *If $D\notin\mathrm{Cl}_H(S):=\mathbb{Z}\langle C,E\rangle$, then $H^1(S,\mathcal{O}_S(D))=0$.*

(ii) *If $D\in\mathrm{Cl}_H(S)$, that is, $D\sim cC+eE+gG$ for some $c,e,g\in\mathbb{Z}$ ($c_D=c+e+\kappa g<0$), then*

$$h^1(S,\mathcal{O}_S(D))=-1-\left\lfloor\frac{c}{\kappa}\right\rfloor-\left\lfloor\frac{e}{\kappa}\right\rfloor-g.$$

Proof of Proposition 3.9: From Theorem 3.7, if $c_D = -1$, then $h^i(S, \mathcal{O}_S(D)) = 0$, $i = 0, 1, 2$, and the formula holds.

Assume $c_D \leq -2$, $\varphi_D = 0$, and $H^1(S, \mathcal{O}_S(D)) \neq 0$. According to (3.15), one has to study when $b_j(K_S - D) = -1$. This happens precisely when the inequality (3.14) becomes an equality

$$\sum_{i=1}^{r} \left\{ \frac{-1 - \hat{a}_i + (c_D + j + 1)q_i}{d_i} \right\} = \sum_{i=1}^{r} \frac{d_i - 1}{d_i}, \tag{3.16}$$

which happens if and only if

$$-1 - \hat{a}_i + (c_D + j + 1)q_i \equiv (d_i - 1) \bmod d_i,$$

i.e., there exists a solution for the system (2.9). Then, (i) has been proved.

Let us prove (ii). Since $G \sim \kappa C$, it is enough to show the result for $g = 0$. Then $D \sim cC + eE$ and the condition given in (3.16) becomes

$$\sum_{i=1}^{r} \left\{ \frac{-1 + (c + j + 1)q_i}{d_i} \right\} = \sum_{i=1}^{r} \frac{d_i - 1}{d_i},$$

which implies $j \equiv -(c+1) \bmod d_i$. In particular, there is a $0 \leq j_0 < \kappa$, such that $j_0 \equiv -(c+1) \bmod \kappa$. Moreover,

$$j_0 = -(c+1) - \kappa \left\lfloor \frac{-c-1}{\kappa} \right\rfloor.$$

The system (2.9) has $\ell := h^1(S, \mathcal{O}_S(D)) \geq 0$ solutions in $\{0, 1, \ldots, -(c_D+2)\}$, namely

$$j_0 + (\ell - 1)\kappa \leq -(c_D + 2) < j_0 + \ell\kappa.$$

This way $\ell - 1$ can be described as an integer satisfying

$$\frac{-(c_D+2) - j_0}{\kappa} - 1 < \ell - 1 \leq \frac{-(c_D+2) - j_0}{\kappa}.$$

In other words $\ell = 1 + \left\lfloor \frac{-(c_D+2)-j_0}{\kappa} \right\rfloor$ and then

$$\begin{aligned} h^1(S, \mathcal{O}_S(D)) = 1 + \left\lfloor \frac{-(c_D+2) - j_0}{\kappa} \right\rfloor &= 1 + \left\lfloor \frac{-e-1}{\kappa} + \left\lfloor \frac{-c-1}{\kappa} \right\rfloor \right\rfloor \\ &= 1 + \left\lfloor \frac{-e-1}{\kappa} \right\rfloor + \left\lfloor \frac{-c-1}{\kappa} \right\rfloor = -1 - \left\lfloor \frac{e}{\kappa} \right\rfloor - \left\lfloor \frac{c}{\kappa} \right\rfloor. \end{aligned}$$

The last equality follows from (A.2). $\square$

To summarize, Theorems 3.3 and 3.6 together with Serre's duality show that the Betti numbers $h^i(S, \mathcal{O}_S(D))$ are determined by the divisor class of D.

In addition, there are four translated cones in the divisor lattice L, as shown in Figure 6, where the cohomology is concentrated in a single degree. Moreover, the dimensions $h^i(S, \mathcal{O}_S(D))$ are in fact determined by the image ℓ_D in one of these translated cones.

The remaining threshold area is special in two directions. First, Betti numbers are not necessarily concentrated in a single degree anymore, and second, they need not be determined by ℓ_D.

To end this section, an example of the special behavior in the threshold area is provided.

Example 3.10. Let us consider the surface S associated with $(d_i) = (3,3,3,3)$ and $(q_i) = (1,2,1,2)$. In this case $\alpha = 2$, $\chi^{\text{orb}} = -\frac{2}{3}$, and $\kappa = 3$. According to Proposition 2.10, $\mathrm{Cl}(S) \simeq \mathbb{Z}^2 \times (\mathbb{Z}/3\mathbb{Z})^3$. The free part is generated by the classes of C and A_4 and the torsion part by the classes of $A_1 - A_4$, $A_2 - A_4$, $A_3 - A_4$. Let

$$D_1 = A_1, \quad D_2 = 2A_1 - A_2, \quad D_3 = 2A_1 - A_2 + A_3 - A_4.$$

Note that $\ell_{D_i} = \ell_{3C+D_i} = \left(\frac{1}{3}, 0\right)$ is in the threshold area since $0 < \varphi_{D_i} < -\chi^{\text{orb}} = \frac{2}{3}$. The following table can be obtained using Theorems 3.3 and 3.6 together with Serre's duality. It shows that Betti numbers of D are not determined by ℓ_D.

	D_1	D_2	D_3	$3C+D_1$	$3C+D_2$	$3C+D_3$
h^0	1	0	0	2	0	1
h^1	0	0	1	1	0	2
h^2	0	0	0	0	0	0

Table 1. Betti numbers for D_i and $3C + D_i$.

4. H^1-eigenspace decomposition and characteristic polynomial of the monodromy of a ramified cyclic cover

Let us consider a ramified d-cyclic covering of a curve or a normal surface and its monodromy σ. This monodromy acts on the first cohomology group of the cover and its characteristic polynomial is called the *characteristic polynomial* of the monodromy. For historical reasons we will also use the name *Alexander polynomial* of the covering for this characteristic polynomial.

Let $\pi\colon S_d \to S$ be the d-cyclic cover ramified along a divisor $D \sim dH$. Before we describe its characteristic polynomial, we will discuss two particular cases which will be essential for the general construction. These special covers are called *vertical* and *horizontal*.

4.1. Vertical coverings. Consider D a *vertical divisor*, that is,

$$D = \sum_{i=1}^{r} a_i A_i + \sum_{j=1}^{s} f_j F_j, \tag{4.1}$$

where F_j are generic fibers. Note that D is set as a Weil divisor, not only as a divisor class. Let H be a divisor, such that $D \sim dH$ for some $d \in \mathbb{Z}_{>0}$. The purpose of this section is to describe the eigenspace decomposition of $H^1(S_d, \mathbb{C})$ (also called the H^1-eigenspace decomposition of S_d) for S_d the cyclic cover $\pi\colon S_d \to S$ of S associated with (d, D, H). In particular, the characteristic polynomial of the monodromy of the d-cyclic covering of S coincides with that of $\pi_E\colon E_d := S_d|_{\pi^{-1}(E)} \to E = \mathbb{P}^1$, which is a d-cyclic covering of a rational curve as described in Proposition 1.7 and its preceding paragraph.

The following result will be proved.

Proposition 4.1. *Consider $\pi\colon S_d \to S$ the cyclic cover of S associated with (d, D, H) as described above. Then, the decomposition into invariant subspaces of the monodromy of the cover can be obtained by restricting the covering to a horizontal section of $\pi_S\colon S \to \mathbb{P}^1$ such as E (or C), that is, the decomposition of the cover $\pi_E\colon E_d \to E$ ($\pi_C\colon C_d \to C$) associated with (d, D_E, H_E).*

Proof: Let us break the proof into several steps.

Step 1. Reduction to $(d, D, 0)$.

By the discussion in Subsection 1.1 one can replace (d, D, H) by $(d, D - dH, 0)$. Also, using the canonical form for H given in (2.7), and since $dH \cdot F = F \cdot D = 0$, one has $H \sim \sum \hat{a}_i A_i + \hat{f} F$. In particular, $D' = D - dH$ is a vertical divisor and hence it is enough to study the case $(d, D', 0)$, where

$$D' = \sum_{i=1}^{r} a'_i A_i + \sum_{j=1}^{s} f'_j F_j.$$

Note that the condition $D' \sim 0$ is equivalent to

$$\varphi_D = \sum_{i=1}^{r} \frac{a'_i}{d_i} + \sum_{j=1}^{s} f'_j = 0 \quad \text{and} \tag{4.2a}$$

$$a'_i \equiv 0 \bmod d_i, \qquad i = 1, \ldots, r. \tag{4.2b}$$

Step 2. Calculation of $h^1(S, \mathcal{O}_S(L^{(l)}))$.

To apply Esnault and Viehweg's method we consider the following divisors for $l \in \mathbb{Z}$:

$$L^{(l)} = \sum_{i=1}^{r} \left\lfloor \frac{la'_i}{d} \right\rfloor A_i + \sum_{j=1}^{s} \left\lfloor \frac{lf'_j}{d} \right\rfloor F_j.$$

Note that $L^{(l)} \sim 0$ if and only if

$$\varphi_{L^{(l)}} = \sum_{i=1}^{r} \frac{1}{d_i} \left\lfloor \frac{la'_i}{d} \right\rfloor + \sum_{j=1}^{s} \left\lfloor \frac{lf'_j}{d} \right\rfloor = 0 \quad \text{and} \tag{4.3a}$$

$$\left\lfloor \frac{la'_i}{d} \right\rfloor \equiv 0 \bmod d_i, \qquad i = 1, \ldots, r.$$

Subtracting (4.3a) from $\frac{l}{d}$(4.2a) and using both $\{x\} = x - \lfloor x \rfloor$ and $\{x\} \geq 0$ one deduces

$$\sum_{i=1}^{r} \frac{1}{d_i} \left\{ \frac{la'_i}{d} \right\} + \sum_{j=1}^{s} \left\{ \frac{lf'_j}{d} \right\} = 0 \iff \left\{ \frac{la'_i}{d} \right\} = \left\{ \frac{lf'_j}{d} \right\} = 0 \iff \frac{la'_i}{d}, \frac{lf'_j}{d} \in \mathbb{Z}$$

for all $i = 1, \ldots, r$ and $j = 1, \ldots, s$. This is equivalent to $\frac{l}{d_n} \in \mathbb{Z}$, where $d_n = \frac{d}{n}$, for $n := \gcd(a'_1, \ldots, a'_r, f'_1, \ldots, f'_s, d)$. Summarizing, if l is a multiple of d_n, then $h^1(S, \mathcal{O}_S(L^{(l)})) = 0$. Otherwise, $l \not\equiv 0 \bmod d_n$, using (3.12):

$$h^1(S, \mathcal{O}_S(L^{(l)})) = -\sum_{i=1}^{r} \left\lfloor \frac{\lfloor \frac{la'_i}{d} \rfloor}{d_i} \right\rfloor - \sum_{j=1}^{s} \left\lfloor \frac{lf'_j}{d} \right\rfloor - 1 = -\sum_{i=1}^{r} \left\lfloor \frac{la'_i}{dd_i} \right\rfloor - \sum_{j=1}^{s} \left\lfloor \frac{lf'_j}{d} \right\rfloor - 1, \tag{4.4}$$

where the last equality follows from (A.1).

Step 3. Restriction to a 1-dimensional cover of $\mathbb{P}^1$.

Let us restrict this cover to E (for C it works the same way). We will be using Proposition 1.7 and the construction before it. Since E contains r singular points, say $\{\gamma_1,\dots,\gamma_r\}$ of the surface S, one needs to perform a $(1,q_i)$-weighted blow-up at each point γ_i with exceptional component E_i. The total transform of D is

$$\sum_{i=1}^{r} a_i' A_i + \sum_{i=1}^{r} \frac{a_i'}{d_i} E_i + \sum_{j=1}^{s} f_j' F_j$$

and hence $E_C = \sum_{i=1}^{r} \frac{a_i'}{d_i}\langle\gamma_i\rangle + \sum_{j=1}^{s} f_j\langle P_j\rangle$, where $P_j = E\cap F_j$ and $\frac{a_i'}{d_i}\in\mathbb{Z}$ by (4.2b).

The decomposition into invariant subspaces of the monodromy of this cover is calculated in Subsection 1.2 and it matches that obtained in (4.4). ☐

As a consequence of the proof and (1.4), one has the following specific formulas for the dimensions of the invariant subspaces and the Alexander polynomials of the monodromy.

Corollary 4.2. *Under the hypothesis of Proposition 4.1, after appropriate reduction to the case $(d,D',0)$ as in (4.1), the dimension of the invariant subspaces is given by*

$$h^1(S,\mathcal{O}_S(L^{(l)})) = -\sum_{i=1}^{r}\left\lfloor \frac{l a_i'}{d d_i}\right\rfloor - \sum_{j=1}^{s}\left\lfloor \frac{l f_j'}{d}\right\rfloor - 1.$$

Moreover, the characteristic polynomial of the monodromy is

$$\Delta_1(t) = \frac{(t^n-1)^2(t^d-1)^{r+s-2}}{\prod_{i=1}^{r}(t^{\gcd(d,\frac{a_i'}{d_i})}-1)\prod_{j=1}^{s}(t^{\gcd(d,f_j')}-1)}.$$

4.2. Horizontal coverings. The second type of special cyclic covers of S are those associated with (d,D,H), where D is a *horizontal divisor*, that is,

$$D = cC + eE + \sum_{j=1}^{s} g_j G_j \sim dH,$$

where $G_1,\dots,G_s$ are distinct fibers of π_G, all of them linearly equivalent to G; see Remark 2.8.

As was discussed in Subsection 4.1, it is enough to consider the case $(d, D-dH, 0)$. However, $D-dH$ is not necessarily a horizontal divisor, as described in (2.8). This subtlety makes this case a bit more involved than the vertical case.

The best reduction one can expect is given by the following result.

Lemma 4.3. *There exist $\gamma,\eta\in\mathbb{Z}$ and a divisor $T'\in\operatorname{Tor}\operatorname{Cl}(S)$ such that*

$$H\sim\gamma C+\eta E+T';$$

the order of T' is a divisor of $\gcd(d,\kappa)$ and the only common multiple of T' and T is 0 *(see Remark 2.11 for the definition of T).*

Moreover, there exist integers c', e' such that

$$e = d\eta+\kappa e',\quad c = d\gamma+\kappa c',\quad c'+e'+\sum_{j=1}^{s} g_j = 0.$$

Proof: Since $\gcd(q_i,d_i)=1$, we can assume that

$$H\sim\gamma' C+\eta' E+\sum_{i=1}^{r} q_i\alpha_i' A_i+\phi' F;$$

recall that C, E, F, A_i form a generator system of $\mathrm{Cl}(S)$. Moreover, since $d_iA_i \sim F$ and q_i, d_i are coprime, such an expression exists. The condition $D \sim dH$ is equivalent to

$$\sum_{i=1}^{r} \frac{q_i\alpha_i'}{d_i} + \phi' = 0, \quad c + e + \kappa\sum_{j=1}^{s} g_j = d(\gamma' + \eta'), \quad e \equiv d(\eta' + \alpha_i') \bmod d_i.$$

In particular, $\gcd(d, d_i)$ divides e, i.e.,

$$\operatorname{lcm}(\gcd(d, d_1), \ldots, \gcd(d, d_r)) = \gcd(d, \operatorname{lcm}(d_1, \ldots, d_r)) = \gcd(d, \kappa)$$

also divides e. Note that $\gcd(d, \kappa)$ divides c.

Let η_0 be a solution of $e \equiv d\eta' \bmod \kappa$; the solutions of this equation are $\eta_h := \eta_0 + h\kappa_1$, where $\kappa_1 := \frac{\kappa}{\gcd(d,\kappa)}$ and $h \in \mathbb{Z}$. Let

$$\gamma_h := \frac{c + e + \kappa\sum_{j=1}^{s} g_j}{d} - \eta_h \in \mathbb{Z}.$$

Let $H_h := \gamma_h C + \eta_h E = H_0 + h\kappa_1 T$. Note that

$$\begin{aligned} D - dH_h &\sim (c - d\gamma_h)C + (e - d\eta_h)E + \sum_{j=1}^{s} g_jG_j \\ &\sim \left(c - d\gamma_h + \kappa\sum_{j=1}^{s} g_j\right)C + (e - d\eta_h)E \sim (e - d\eta_h)T \sim 0. \end{aligned}$$

Then $T_h := H_h - H = T_0 + h\kappa_1 T$ defines a torsion class such that $dT_h \sim 0$. Since the maximal order of torsion classes is κ we deduce that $\gcd(\kappa, d)T_h = 0$.

Using the structure of $\mathrm{Cl}(S)$ given in Proposition 2.10, let us fix a direct-sum decomposition where the component of $\mathbb{Z}/m_{r-1} = \mathbb{Z}/\kappa$ is generated by T. The coordinate $\beta_h \bmod \kappa$ of T_h in this component must satisfy

$$\gcd(\kappa, d)\beta_h \equiv 0 \bmod \kappa \iff \beta_h \equiv 0 \bmod \kappa_1 \iff \beta_h = \hat{\beta}_h\kappa_1,$$

and note that $\hat{\beta}_h \equiv \hat{\beta}_0 + h \bmod \kappa_1$. Hence for a suitable h the coordinate $\hat{\beta}_h$ of H_h in T vanishes in $\mathbb{Z}/\kappa$.

Let us denote $\gamma = \gamma_h$, $\eta = \eta_h$, and $T' = T_h$. They are the values in the statement (in fact, those values may not be unique but we do not claim that).

The choice of H_0 is *well defined* up to congruence with $T_1 = \kappa_1 T$, with order exactly $\gcd(\kappa, d)$. The definition of γ, η as solutions of a congruence equation ends the proof. □

Reduction 4.4. As an immediate consequence of Lemma 4.3, considering $D' = D - d(\gamma C + \eta E)$, one can reduce the general case of horizontal coverings to those associated with (d, D', T'), where

$$D' = \kappa c' C + \kappa e' E + \sum_{j=1}^{s} g_jG_j \sim 0, \quad T' \sim \sum_{i=1}^{r} \alpha_i A_i. \tag{4.5}$$

The class T' is torsion of order d_τ and its only common multiple with T is 0. For convenience, we denote $\tau := \frac{d}{d_\tau} \in \mathbb{Z}$.

These integers c', e' are particularly important, since they provide an interesting feature of the restriction of the covering to the preimage $F_{(d)}$ of a generic fiber F.

Namely, this cover π_F ramifies at $\kappa s+2$ points with ramification indices κc (at $P_C = C|_F$), κe (at $P_E = C|_F$), and g_j, $j=1,\dots,s$ (at each of the κ points $G_j|_F = P_{j1}+\dots+P_{j\kappa}$).

$$\begin{array}{ccc} F_{(d)} & \longrightarrow & X \\ \downarrow{\scriptstyle \pi_F} & & \downarrow{\scriptstyle \pi_X} \\ F & \longrightarrow & \mathbb{P}^1 \\ z & \longmapsto & z^\kappa \end{array} \tag{4.6}$$

Actually this cover is the pull-back of a cover π_X with ramification indices c (at $z=0$), e (at $z=\infty$), and ramifications g_j at s points of $\mathbb{C}^*$.

Definition 4.5. The d-covering $\pi_X\colon X\to\mathbb{P}^1$ of (4.6) is called the *primitive vertical cover* of π.

The *vertical* coverings are the restrictions of the covering π to the preimage of the fibers. The above covering is called *primitive* because π_F can be retrieved from π_X due to (4.5) and how each G_j intersects the fibers at κ points.

This behavior is repeated for each special fiber A_i, replacing κ by $\frac{\kappa}{d_i}$ and taking into account that the factorization may not work for the restrictions $\pi_{A_i}\colon A_{i,(d)}\to A_i$ but only for intermediate covers $\pi_{A_i,d'}\colon A_{i,(d')}\to A_i$, where d' is a divisor of d. For each i, we set e_i as the maximal divisor of d such that the following diagram holds:

$$\begin{array}{ccc} A_{i,(e_i)} & \longrightarrow & X_{e_i} \\ \downarrow{\scriptstyle \pi_{A_i,e_i}} & & \downarrow{\scriptstyle \pi_{X,e_i}} \\ A_i & \longrightarrow & \mathbb{P}^1 \\ z & \longmapsto & z^{\frac{\kappa}{d_i}} \end{array}$$

Definition 4.6. Let $\hat{e}:=\gcd(e_1,\dots,e_r)$. The $\hat{e}$-covering $\pi_{X,\hat{e}}\colon X_{\hat{e}}\to\mathbb{P}^1$ is called the *greatest common vertical cover* of π.

In order to apply Esnault and Viehweg's method one has to consider the following divisors for $l=0,d-1$. The divisor for Esnault and Viehweg's method is

$$\begin{aligned} L^{(l)} &= \left\lfloor \frac{l\kappa c'}{d}\right\rfloor C + \left\lfloor \frac{l\kappa e'}{d}\right\rfloor E + \sum_{j=1}^{s}\left\lfloor \frac{lg_j}{d}\right\rfloor G_j - lT' \\ &\sim \left(\left\lfloor \frac{l\kappa c'}{d}\right\rfloor + \left\lfloor \frac{l\kappa e'}{d}\right\rfloor + \kappa\sum_{j=1}^{s}\left\lfloor \frac{lg_j}{d}\right\rfloor\right) C + \left\lfloor \frac{l\kappa e'}{d}\right\rfloor T - lT'. \end{aligned}$$

From Lemma 4.3, one has

$$l\kappa c' + l\kappa e' + \kappa\sum_{j=1}^{s} lg_j = l\kappa\left(c'+e'+\sum_{j=1}^{s} g_j\right) = 0$$

and hence

$$\left\{\frac{l\kappa c}{d}\right\} + \left\{\frac{l\kappa e}{d}\right\} + \kappa\sum_{j=1}^{s}\left\{\frac{lg_j}{d}\right\} \in \mathbb{Z}.$$

Let

$$\tilde{L}^{(l)} := -\left(\left\{\frac{l\kappa c'}{d}\right\} + \left\{\frac{l\kappa e'}{d}\right\} + \kappa\sum_{j=1}^{s}\left\{\frac{lg_j}{d}\right\}\right)C + \left\lfloor\frac{l\kappa e'}{d}\right\rfloor T - lT',$$

$$\begin{aligned} L^{(l)} - \tilde{L}^{(l)} &\sim \left(\left\lfloor\frac{l\kappa c'}{d}\right\rfloor + \left\{\frac{l\kappa c'}{d}\right\} + \left\lfloor\frac{l\kappa e'}{d}\right\rfloor + \left\{\frac{l\kappa e'}{d}\right\} + \kappa\sum_{j=1}^{s}\left(\left\lfloor\frac{lg_j}{d}\right\rfloor + \left\{\frac{lg_j}{d}\right\}\right)\right)C \\ &\sim \frac{l\kappa}{d}\left(c' + e' + \sum_{j=1}^{s} g_j\right)C = 0. \end{aligned} \tag{4.7}$$

Remark 4.7. The common divisor

$$n := \gcd(d, \kappa c, \kappa e, g_1, \ldots, g_s)$$

of the coefficients of D' and the degree of the cover will be useful as well as $d_n := \frac{d}{n}$. Note that n is also the greatest common divisor of d and the coefficients of D before the reduction.

As a first approach, we will consider the simpler case $(d, D', 0)$ for a horizontal divisor $D' \sim 0$, that is, $T' \sim 0$. In this case, one obtains the following result.

Proposition 4.8. *Let $\pi\colon S_d \to S$ be a horizontal cyclic cover of S associated with $(d, D', 0)$. Then its H^1-eigenspace decomposition can be described as a direct sum $\mathbb{H}_h \oplus \mathbb{H}_m$, where $\mathbb{H}_m$ comes from a vertical cover of type $(n, -\kappa eT, 0)$ and $\mathbb{H}_h$ comes from the* greatest common vertical cover *which is of degree d. Moreover, for any $l = 1, \ldots, d-1$,*

$$h^1(S, \mathcal{O}_S(L^{(l)})) = \begin{cases} -1 + \left\{\frac{lc}{d}\right\} + \left\{\frac{le}{d}\right\} + \sum_{j=1}^{s}\left\{\frac{lg_j}{d}\right\} & \text{if } d_n \nmid l, \\ -1 + \sum_{i=1}^{r}\left\{-\frac{l_2\kappa e q_i}{n d_i}\right\} & \text{if } l = l_2 d_n. \end{cases} \tag{4.8}$$

Proof: By (4.7), one obtains

$$c_{L^{(l)}} = L^{(l)} \cdot F = -\left(\left\{\frac{l\kappa c}{d}\right\} + \left\{\frac{l\kappa e}{d}\right\} + \kappa\sum_{j=1}^{s}\left\{\frac{lg_j}{d}\right\}\right) \leq 0. \tag{4.9}$$

Note that $c_{L^{(l)}} = 0$ exactly when $d_n | l$, say $l = l_2 d_n$ for some $l_2 \in \mathbb{Z}$, in which case $L^{(l)} \sim \frac{\kappa e}{n} l_2 T$. This falls in case (3.13) of Proposition 3.8 for $\hat{a}_i = \frac{\kappa e q_i}{n} l_2$ and one can check that the second part of formula (4.8) follows. For these terms one has

$$\sum_{l_2=0}^{n-1} h^1(S, \mathcal{O}_S(L^{(l_2 d_n)})) = h^1(S_n, \mathcal{O}_{S_n}),$$

where S_n is a vertical cover of S associated with $(n, 0, \frac{\kappa e}{n} T)$ or equivalently with $(n, -\kappa eT, 0)$. This is the *vertical* cover producing $\mathbb{H}_m$.

For the remaining terms $c_{L^{(l)}} < 0$ and hence, by the vanishing result in Proposition 3.9 and (A.1), one has

$$h^1(S, \mathcal{O}_S(L^{(l)})) = -1 - \left\lfloor\frac{lc}{d}\right\rfloor - \left\lfloor\frac{le}{d}\right\rfloor - \sum_{j=1}^{s}\left\lfloor\frac{lg_j}{d}\right\rfloor = -1 + \left\{\frac{lc}{d}\right\} + \left\{\frac{le}{d}\right\} + \sum_{j=1}^{s}\left\{\frac{lg_j}{d}\right\},$$

where the last equality follows from $\lfloor x\rfloor = x - \{x\}$ and $c + e + \sum g_j = 0$. This proves the first part of (4.8).

Now, let us describe the restrictions of the original d-covering to the curves A_i. Recall that the d-cover $\pi_F\colon F_{(d)} \to F \cong \mathbb{P}^1$ is the pull-back of a cover $\pi_X\colon X \to \mathbb{P}^1$ by the cyclic cover $z \mapsto z^\kappa$.

In order to describe the restriction to A_i one needs to perform a blow-up at the singular points of S on A_i as explained in Proposition 1.7 and the preceding paragraph. In particular, it is enough to perform a $(q_i', 1)$-weighted (resp. $(d_i - q_i', 1)$) blow-up at $A_i \cap E$ (resp. $A_i \cap C$), where $q_i q_i' \equiv 1 \bmod d_i$.

In addition, note that $D' \sim 0$ and horizontal implies that the multiplicity of the exceptional component E_i (resp. C_i) of the blow-up of $A_i \cap E$ (resp. $A_i \cap C$) is $\frac{\kappa e}{d_i}$ (resp. $\frac{\kappa c}{d_i}$), i.e., the following diagram holds:

$$\begin{array}{ccc}
A_{i,(d)} & \longrightarrow & X \\
\big\downarrow{\scriptstyle \pi_{A_i}} & & \big\downarrow{\scriptstyle \pi_X} \\
A_i & \longrightarrow & \mathbb{P}^1 \\
z & \longmapsto & z^{\frac{\kappa}{d_i}}
\end{array}$$

As a consequence, the covering π_X is the *greatest common vertical covering* of π (the divisor e of Definition 4.6 is exactly d).

Denote by $L_\kappa^{(l)}$ the Esnault–Viehweg divisors in $\mathbb{P}^1$ associated with π_X. Note that

$$L_\kappa^{(l)} = \left\lfloor \frac{lc}{d} \right\rfloor P_C + \left\lfloor \frac{le}{d} \right\rfloor P_E + \sum_{j=1}^{s} \left\lfloor \frac{lg_j}{d} \right\rfloor P_{G_j},$$

where

$$\deg L_\kappa^{(l_2 d_n)} = -\left\{\frac{l_2 c}{n}\right\} - \left\{\frac{l_2 e}{n}\right\} - \sum_{j=1}^{s} \left\{\frac{l_2 g_j}{n}\right\} = -\left\{\frac{l_2 c}{n}\right\} - \left\{\frac{l_2 e}{n}\right\} = \begin{cases} 0 & \text{if } \frac{l_2 c}{n} \in \mathbb{Z}, \\ -1 & \text{otherwise.} \end{cases}$$

The last equality follows since $\frac{l_2 c}{n} + \frac{l_2 e}{n} = -\sum_j \frac{l_2 g_j}{n} \in \mathbb{Z}$. Hence $h^1(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(L_\kappa^{(l_2 d_n)})) = 0$ either way (see (1.3)). This shows that

$$h^1(X, \mathcal{O}_X) = \sum_{l=0}^{d-1} h^1(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(L_\kappa^{(l)})) = \sum_{l=0,\, d_n \nmid l}^{d-1} h^1(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(L^{(l)})),$$

which ends the proof. □

According to Reduction 4.4, the general horizontal case can be given by (d, D', T'), where $T' \in \operatorname{Tor}\operatorname{Cl}(S)$ as in Lemma 4.3. The H^1-eigenspace decomposition in this case splits into a purely horizontal part and a vertical one as follows.

Proposition 4.9. *Let $\pi\colon S_d \to S$ be the horizontal cyclic cover of S associated with (d, D', T'). Then its H^1-eigenspace decomposition splits as $\mathbb{H}_h \oplus \mathbb{H}_v$, where $\mathbb{H}_v$ comes from the vertical cover associated with $(n, dT' - \kappa e T, 0)$ and $\mathbb{H}_h$ comes from the greatest common vertical cover of π which is of degree τ. In particular, it decomposes as a direct sum of the cohomology of two cyclic covers of $\mathbb{P}^1$ and the splitting respects the eigenspaces of the monodromy and the Hodge structure.*

Proof: The proof runs along the lines of that of Proposition 4.8. We will highlight the differences. Formula (4.9) holds and in this case $c_{L^{(l)}} = 0$ implies

$$L^{(l)} \sim l_2 \underbrace{\left(\frac{\kappa e}{n} T - d_n T'\right)}_{=:T''}, \quad L^{(l+d_n)} \sim L^{(l)} + T''.$$

(L1) The case $c_{L^{(l)}} = 0$ is equivalent to $l = l_2 d_n$, i.e., $L^{(l)} \sim l_2 T''$. The value of $h^1(S, \mathcal{O}_S(L^{(l)}))$ has been computed in Proposition 3.8. More precisely, this case corresponds to the vertical cover associated with $(n, 0, T'')$ (or equivalently $(n, dT' - \kappa e T, 0)$) considered in Subsection 4.1.

(L2) If $c_{L^{(l)}} < 0$, then by the vanishing result in Proposition 3.9, $h^1(S, \mathcal{O}_S(L^{(l)})) \neq 0$ only if $lT' \sim 0$. Hence, we assume $l = l_1 d_\tau$. By Proposition 3.9 and (A.1) one has

$$h^1(S, \mathcal{O}_S(L^{(l)})) = -1 + \left\{ \frac{l_1 c}{\tau} \right\} + \left\{ \frac{l_1 e}{\tau} \right\} + \sum_{j=1}^{s} \left\{ \frac{l_1 g_j}{\tau} \right\}.$$

Recall that the condition of Reduction 4.4 implies $0 \sim D' \sim dT' \sim \tau(d_\tau T')$ is a horizontal divisor.

In order to describe the restriction to A_i one needs to perform a blow-up at the singular points of S on A_i as explained in Proposition 1.7 and the preceding paragraph. In particular, it is enough to perform a $(q_i', 1)$-weighted (resp. $(d_i - q_i', 1)$) blow-up at $A_i \cap E$ (resp. $A_i \cap C$), where $q_i q_i' \equiv 1 \mod d_i$ and hence one can find $h_i \in \mathbb{Z}$ such that $q_i q_i' = 1 + h_i d_i$.

In addition, note that $D' \sim dT' \sim 0$ implies the existence of integers δ_i such that $d\alpha_i = d_i \delta_i$ (α_i is the coefficient of T' in A_i). Hence, the multiplicity of the exceptional component E_i of the blow-up of $A_i \cap E$ is

$$\frac{\kappa e + q_i' q_i d \alpha_i}{d_i} = e\frac{\kappa}{d_i} + d h_i \alpha_i + \frac{d\alpha_i}{d_i} = \frac{\kappa}{d_i} e + d h_i \alpha_i + \delta_i \equiv \frac{\kappa}{d_i} e + \delta_i \mod d,$$

since the multiplicities are only relevant mod d. Analogously, one can compute the multiplicity of the exceptional component C_i of the blow-up of $A_i \cap C$ as $\frac{\kappa}{d_i} c - \delta_i$.

Summarizing, the restriction of the original d-covering to the curve A_i is a d-cover of A_i ramified at $\frac{\kappa}{d_i} s + 2$ points with multiplicities $\frac{\kappa}{d_i} e + \delta_i$ (at $P_{E_i} = A_i \cap E_i$), $\frac{\kappa}{d_i} c - \delta_i$ (at $P_{C_i} = A_i \cap C_i$), and g_j, $j = 1, \ldots, s$ (at each of the $\frac{\kappa}{d_i}$ points of $G_j \cap A_i$).

Let us consider the intermediate τ-cover. Recall that $d_\tau T' \sim 0$. There exists β_i such that $d_\tau \alpha_i = d_i \beta_i$:

$$d_i \delta_i = d\alpha_i = \tau(d_\tau \alpha_i) = \tau(d_i \beta_i) \Rightarrow \delta_i \equiv 0 \mod \tau.$$

These congruences show that the degree of the greatest common vertical cover is τ. Its characteristic polynomial $\Delta_{2,h}(t)$ can be computed with the help of the divisors $L_\tau^{(l_1)}$ using again Lemma B.1. The same argument about $\deg L_\tau^{(l_1)}$ shows that the terms $\deg L_\tau^{(l_1)}$ for which $l = l_1 \tau$ is a multiple of d_n do not contribute to this horizontal part. This completes the proof. $\square$

Remark 4.10. The degree of the greatest common vertical cover can be obtained in two ways. From the above proposition, it can be obtained algebraically in terms of the torsion order of the divisor T'. And from the definition, it can be obtained topologically as the highest divisor of d for which the covers over A_i are pull-backs of the primitive vertical cover of π.

Corollary 4.11. *The characteristic polynomial $\Delta_2(t)$ of the monodromy of a horizontal cover of S associated with (d, D', T') as above factorizes as $\Delta_2(t) = \Delta_{2,h}(t)\Delta_{2,m}(t)$, where $\Delta_{2,h}(t)$ and $\Delta_{2,m}(t)$ are Alexander polynomials of coverings of $\mathbb{P}^1$.*

Proof: By Proposition 4.9, the H^1-eigenspace decomposition of the cover splits as a direct sum $\mathbb{H}_h \oplus \mathbb{H}_m$. By Proposition 4.1, the characteristic polynomial $\Delta_{2,m}(t)$ of the monodromy associated with $\mathbb{H}_m$ corresponds to that of the restriction of the vertical cover $(n, 0, T'')$ to E (or C), whereas $\Delta_{2,h}(t)$ corresponds to that of the horizontal τ-cover described in the proof. $\square$

4.3. General case. To end this section, we are in the position to describe the general case, that is, $\pi\colon S_d \to S$ is a covering associated with (d, D, H) such that $D \sim dH$ and

$$D = \sum_{j\in J} m_j D_j \in \operatorname{Div}(S), \quad H = \gamma C + \eta E + \sum_{i=1}^{r} \alpha_i A_i \in \operatorname{Cl}(S),$$

where $m_j \in \mathbb{Z}_{>0}$, D_j is an irreducible (effective) divisor, $j \in J$, $\gamma, \eta, \alpha_i \in \mathbb{Z}$, $i = 1, \ldots, r$, and the reduced support D_{red} of D is a $\mathbb{Q}$-simple normal crossing divisor. Following Lemma 2.9 we decompose D as

$$D = \nu_h D_h + \nu_v D_v + \nu_s D_s,$$

where D_h is horizontal, D_v is vertical, and D_s is slanted (see Lemma 2.9); all of them are primitive (i.e., the gcd of their multiplicities equals 1). We introduce the following notation:

$$d^h := \gcd(d, \nu_v, \nu_s), \quad d^v := \gcd(d, \nu_h, \nu_s).$$

The following result describes the H^1-eigenspace decomposition of a general cyclic cover of S. The notation used is defined in Subsections 4.1 (see Proposition 4.1) and 4.2 (see Proposition 4.9).

Theorem 4.12. *Consider $\pi\colon S_d \to S$ the cyclic cover of S associated with (d, D, H) as above. Then $H^1(S; \mathcal{O}_S) = \mathbb{H}_h \oplus \mathbb{H}_v$, $\mathbb{H}_v := H^1(E_{d^v}; \mathcal{O}_{E_{d^v}})$, and $\mathbb{H}_h := H^1(X_{e^h}; \mathcal{O}_{X_{e^h}})$, where $\pi_E\colon E_{d^v} \to E \cong \mathbb{P}^1$ is the restriction of the intermediate d^v-cover to E and $\pi_X\colon X_{e^h} \to \mathbb{P}^1$ is the greatest common vertical cover of the intermediate d^h-cover (e^h is a divisor of d^h as in Proposition 4.9). The eigenspace decomposition of $H^1(S; \mathcal{O}_S) = \mathbb{H}_h \oplus \mathbb{H}_v$ is the direct sum of the natural eigenspace decompositions of $\mathbb{H}_h$ and $\mathbb{H}_v$. In particular, each factor of the splitting is associated with a $\mathbb{P}^1$-cover.*

Proof: The divisor $L^{(l)}$ associated with Esnault and Viehweg's theory is

$$L^{(l)} = -lH + \sum_{j\in J} \left\lfloor \frac{lm_j}{d} \right\rfloor D_j.$$

Let $B \in \operatorname{Div}(S)$ be any divisor. Since $D \sim dH$, the equality

$$lH \cdot B = \frac{l}{d} \sum_{j\in J} m_j D_j \cdot B$$

holds. Taking into account (1.2) in Remark 1.2 one has

$$c_{L^{(l)}} = L^{(l)} \cdot F \le 0 \quad \text{and} \quad \varphi_{L^{(l)}} = L^{(l)} \cdot C \le 0.$$

According to Theorem 3.7, if they are both negative, then $h^1(S, \mathcal{O}_S(L^{(l)})) = 0$; the same happens if l is a multiple of d. The remaining cases for $l \in \mathbb{Z}$ are considered below.

(a) $c_{L^{(l)}} = 0$.

Let us denote $J_1 = \{j \in J \mid c_{D_j} \neq 0\}$, i.e., D_j is a term of either D_h or D_s. Then $c_{L^{(l)}} = 0$ if and only if d divides lm_j, $\forall j \in J_1$; see Remark 1.2. The latter is equivalent to asking $\frac{d}{d^v}$ to divide l. This way l can be written as $l = l_1 \frac{d}{d^v}$ for $l_1 = 0, 1, \ldots, d^v - 1$. In fact, these values measure the action of the monodromy of the intermediate d^v-cover.

Consider the cover of S associated with $(d^v, \nu_v D_v, H_v)$, where

$$H_v \sim \frac{d}{d^v} H - \frac{\nu_h}{d^v} D_h - \frac{\nu_s}{d^v} D_s.$$

This corresponds to a vertical cover. Let us denote by $L_v^{(l_1)}$ the Esnault–Viehweg divisors associated with this vertical cover. A simple check shows that $L_v^{(l_1)} = L^{(l_1 \frac{d}{d^v})}$. Hence, this invariant part of the cohomology is described by a vertical cover of d^v sheets in the sense of Subsection 4.1, which can be decomposed as that of a cover of $\mathbb{P}^1$ and has an associated characteristic polynomial $\Delta_1(t)$.

(b) $\varphi_{L^{(l)}} = 0$.
Analogously to the previous case, one can define $J_2 = \{j \in J \mid \varphi_{D_j} \neq 0\}$, i.e., D_j is a term of either D_v or D_s. Then $\varphi_{L^{(l)}} = 0$ if and only if l can be written as $l = l_2 \frac{d}{d^h}$ for $l_2 = 0, 1, \ldots, d^h - 1$. These values determine the action of the monodromy of the intermediate d^h-cover. Now consider the cover of S associated with $(d^h, \nu_h D_h, H_h)$, where

$$H_v \sim \frac{d}{d^h} H - \frac{\nu_v}{d^h} D_v - \frac{\nu_s}{d^h} D_s.$$

This is a horizontal cover. Analogously to the previous case, one can easily check that $L_h^{(l_2)} = L^{(l_2 \frac{d}{d^h})}$. Hence, this invariant part of the cohomology is described by a horizontal cover of d^h sheets in the sense of Subsection 4.2, which can be decomposed as a direct sum of a horizontal cover (greatest common vertical cover) and a vertical cover, an intermediate cover of order $\mu_0 := \gcd(d, \nu_h, \nu_v, \nu_s)$.

Let us denote by $\Delta_2(t)$ the characteristic polynomial of the cover of S associated with $(d^h, \nu_h D_h, H_h)$, and by $\Delta_{2,h}(t), \Delta_{2,m}(t)$ the characteristic polynomials of the horizontal and vertical covers of the preceding paragraph; see Corollary 4.11. Note that $\Delta_2(t) = \Delta_{2,m}(t)\Delta_{2,h}(t)$.

As a word of caution, note that the previous two cases (a) and (b) are not disjoint. That is why we need to consider a third case that accounts for repetitions.

(c) $\ell_{L^{(l)}} = (0, 0)$, that is, $c_{L^{(l)}} = \varphi_{L^{(l)}} = 0$.
Combining (a) and (b), this case occurs whenever $l = l_0 \frac{d}{\mu_0}$. Consider the vertical cover of S associated with $(\mu_0, 0, H_m)$, where

$$D_m = 0, \quad H_m \sim \frac{d}{\mu_0} H - \frac{1}{\mu_0} D. \tag{4.10}$$

Note that H_m is a torsion class. This case matches (L1) in Proposition 4.9, which accounts for the vertical part $\mathbb{H}_v$ in the decomposition of the horizontal cover associated with part (b). In order to see this, it is enough to check that the divisors $L_m^{(l_0)} := L^{(l_0 \frac{d}{\mu_0})}$ and $L_h^{(l_0 \frac{d^h}{\mu_0})}$ are related as follows:

$$L_m^{(l_0)} := L^{(l_0 \frac{d}{\mu_0})} = L^{(l_0 \frac{d^h}{\mu_0} \frac{d}{d^h})} = L_h^{(l_0 \frac{d^h}{\mu_0})}.$$

In particular, $\Delta_0(t) = \Delta_{2,m}(t)$. □

As a result of the proof one obtains the following.

Corollary 4.13. *The characteristic polynomial $\Delta(t)$ of the monodromy of a cover of S associated with (d, D, H) as above factorizes as $\Delta(t) = \frac{\Delta_1(t)\Delta_2(t)}{\Delta_0(t)} = \Delta_1(t)\Delta_{2,h}(t)$, where $\Delta_1(t)$ and $\Delta_{2,h}(t)$ are the Alexander polynomials of covers of $\mathbb{P}^1$.*

Proof: As a consequence of the proof of Theorem 4.12, the final Alexander polynomial is $\frac{\Delta_1(t)\Delta_2(t)}{\Delta_0(t)} = \frac{\Delta_1(t)\Delta_{2,m}(t)\Delta_{2,h}(t)}{\Delta_0(t)}$. At the end of the proof it is shown that $\Delta_0(t) = \Delta_{2,m}(t)$, which completes the proof. □

5. Examples and applications

5.1. Reducible normal fake quadrics of type $(d_i, q_i) = (3, 1)$, $r = 3$. Following Remark-Definition 2.7 consider S the surface associated with $(d_i, q_i) = (3, 1)$, $i = 1, 2, 3$. In this case $\alpha := \frac{1}{3} + \frac{1}{3} + \frac{1}{3} = 1$ and $\chi^{\mathrm{orb}} = 0$ (see (2.2)). Note that $\kappa := \operatorname{lcm}(d_1, d_2, d_3) = 3$ and $\operatorname{Cl}(S) \cong \mathbb{Z}^2 \oplus (\mathbb{Z}/3)^2$ by Proposition 2.10.

Example 5.1. This example will highlight the relevance of the choice of the torsion class H in the cohomology of the covering of S associated with (d, D, H) as well as the importance of the *greatest common vertical cover* introduced in Definition 4.6. We present different horizontal coverings associated with different torsion divisors whose cohomology invariants are different as well as their greatest common vertical covers.

Let $\sigma_a\colon \hat{S}_a \to S$ be the cover associated with $(3, D, H_a)$, where $D := G$, $H_a := C - aT$, $a = 0, 1, 2$ (recall $T := E - C$). Note that σ_a is a horizontal cover. One has

$$L^{(l)} = -lH_a + \left\lfloor \frac{l}{3} \right\rfloor G = -lH_a = laE - l(a+1)C.$$

Applying Reduction 4.4, it is enough to consider the covering of S associated with $(3, D'_a, 0)$, where

$$D'_a = D - 3H_a = -3(a+1)C + 3aE + G \sim 0.$$

By Proposition 4.8 one has

$$\begin{aligned} h^1(S_a, \mathcal{O}_{S_a}(L^{(l)})) &= -1 + \left\{ -\frac{l(a+1)}{3} \right\} + \left\{ \frac{la}{3} \right\} + \left\{ \frac{l}{3} \right\} \\ &= \left\lfloor -\frac{l(a+1)-1}{3} \right\rfloor - \left\lfloor -\frac{la}{3} \right\rfloor = \begin{cases} 1 & \text{if } l = 2,\ a = 1, \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

As a consequence,

$$h^1(\hat{S}_a, \mathcal{O}_{\hat{S}_a}) = \begin{cases} 1 & \text{if } a = 1, \\ 0 & \text{otherwise.} \end{cases}$$

Let us consider the composition π of the $(1, 2)$-weighted blow-ups of the points $A_i \cap C$ and the $(1, 1)$-weighted blow-ups of $A_i \cap E$. We will denote by C_i (resp. E_i) the exceptional component resulting after the blow-up of $A_i \cap C$ (resp. E_i). Then,

$$\pi^*(D'_a) = G - 3(a+1)C + 3aE + \sum_{i=1}^{3} (aE_i - (a+1)C_i).$$

The restriction of the covering to a generic fiber F ramifies at the three intersection points with G and thus $\pi^{-1}(F)$ is a curve of genus 1. The restriction of the covering to A_i ramifies at the point $A_i \cap G$ with index 1. The other ramification points are $A_i \cap C_i$ (with index $2 - a$) and $A_i \cap E_i$ (with index a).

In particular, for $a = 0, 2$, the cover of A_i is rational while for $a = 1$ it is a curve of genus 1. Note that $\tau = 3$, hence the three restrictions coincide with their *greatest common vertical covering*, and the monodromy of the covering of S coincides with the monodromy of the covering of each A_i.

On the other hand, if one replaces H_a by $H'_a := H_a + A_1 - A_2$, then $\tau' = 1$ and the *new greatest common vertical covering* is the identity, i.e., $H^1(\hat{S}'_a, \mathcal{O}_{\hat{S}'_a}) = 0$, where $\hat{S}'_a$ denotes the respective covering. Note that the restrictions of both coverings over$F$ do not change.

Example 5.2. This example shows the effect of the *mixed* part, see (4.10), in a horizontal covering. It also shows an example of a non-connected covering.

Consider $\sigma\colon \hat{S} \to S$, where S is as at the beginning of Subsection 5.1, with $d = 6$, $D = 3(C + G)$, $H = 2C$. After applying Reduction 4.4 one has a horizontal covering of S associated with $(6, D', 0)$, where $D' = 3(G - 3C) \sim 0$ is a horizontal divisor and $n = 3$. The induced covering over a generic fiber has three connected components, each one being a genus 1 curve obtained as a double covering of $\mathbb{P}^1$ ramified at four points (the three intersections with G and the intersection with C). In order to study the covering over A_i we need to use π as in Example 5.1, to obtain

$$\pi^*(D') = 3\left(G - 3C - \sum_{i=1}^{3} E_{P_i}\right),$$

i.e., the coverings of A_i consist of three copies of $\mathbb{P}^1$. In fact, $\hat{S}$ has three components with vanishing 1-homology.

Replacing the H divisor class by $H = 2E$, the situation over F does not change. Applying Reduction 4.4 one has $D' = 3(G + C - 4E)$, $n = 3$, $d_n = 2$, and

$$L^{(2)} = G + C - 4E \sim 4(C - E) = -4T \sim 2T, \quad L^{(4)} \sim T.$$

Hence the *mixed* component of the covering corresponds to a 3-cover ramified over $D = 0$ and $H = -T$, i.e., of type $(3, 0, 2T) \equiv (3, -6T, 0)$.

Since $h^1(S, \mathcal{O}_S(T)) = 0$, and $h^1(S, \mathcal{O}_S(2T)) = 1$ (see (3.12) in Proposition 3.8), which correspond to the cohomology of a genus 1 curve as a 3-cover of $\mathbb{P}^1$. In fact,

$$\pi^*(D') = 3G + C - 4E + \sum_{i=1}^{3} E_{P_i} - 4\sum_{i=1}^{3} E_{Q_i},$$

and then the covers over A_i are 6-covers of $\mathbb{P}^1$ ramified along three points with ramification indices 3, 1, 2, i.e., a curve of genus 1. The final characteristic polynomial is $(t^2 - t + 1)(t^2 + t + 1)$.

Finally, replacing the H divisor by $H = 2E + A_1 - A_2$, one obtains $L^{(2)} = G + C - 4E - 2A_1 + 2A_2 \sim -T - 2A_1 + 2A_2 \sim A_2 - A_3$, $L^{(4)} \sim A_3 - A_2$.

The mixed component, that is, the invariant part coming from the greatest common vertical covering has trivial monodromy. In this case $\tau = 3$, and the horizontal part is a 2-covering of $\mathbb{P}^1$ ramified in principle at three points with ramification indices $(3, 1, 2) \equiv (1, 1, 0)$, i.e., the covering is a $\mathbb{P}^1$ and thus its monodromy is trivial.

5.2. A general example of cyclic coverings of reducible normal fake quadrics. This will describe a general example of a cyclic covering of a reducible normal fake quadric. The constructive proof of Theorem 4.12 will be applied to this case in order to explicitly exhibit the vertical and horizontal $\mathbb{P}^1$-coverings whose H^1-eigenspace invariant subspaces reconstruct the first cohomology of the cyclic covering.

Let us consider the reducible normal fake quadric S associated with the numerical data $(d_1, d_2, d_3) = (6, 9, 18)$ and $(q_1, q_2, q_3) = (5, 1, 1)$ as described in Remark-Definition 2.7. Note that $\alpha = 1$, $\kappa = 18$, and $\chi^{\text{orb}} = -\frac{2}{3}$.

We consider the divisors

$$\begin{aligned} D &:= 90(C + E) + 15G_1 + 165G_2 + 18(2A_1 + A_2 + 2A_3), \\ H &:= E + G_2 + 2A_1 - 3A_2 + A_3. \end{aligned}$$

Since

$$D - 180H = 90(C - E) + 15(G_1 - G_2) + 18(-18A_1 + 31A_2 - 8A_3) \sim (-54 + 62 - 8)F = 0,$$

the cyclic covering $\pi_{180}\colon X \to S$ associated with $(180, D, H)$ is well defined.

Following the proof of Theorem 4.12 one has to determine the *vertical* component of the covering, which is given by (μ_1, D_v, H_v), where

$$\mu_1 = \gcd(180, 90, 90, 15, 165) = 15, \quad D_v = 18(2A_1 + A_2 + 2A_3),$$

and

$$\begin{aligned} H_v &= 12H - 6(C+E) - G_1 - 11G_2 \sim 6(E-C) + 24A_1 - 36A_2 + 12A_3 \\ &\sim 6(5A_1 + A_2 + A_3 - F) + 24A_1 - 36A_2 + 12A_3 \\ &= 54A_1 - 30A_2 + 18A_3 - 6F \sim F - 3A_2. \end{aligned}$$

Applying Reduction 4.4, this vertical covering is associated with $(\mu_1, D_v', 0)$, where

$$D_v' = 18(2A_1 + A_2 + 2A_3) - 15F + 45A_2 = 36A_1 + 63A_2 + 36A_3 - 15F.$$

Using Proposition 4.1 we see that the monodromy of the *vertical cover* π_{15} coincides with the monodromy of the covering restricted to the preimage of E, associated with the divisor

$$E_C = 6\langle\gamma_1\rangle + 7\langle\gamma_2\rangle + 2\langle\gamma_3\rangle - 15\langle\infty\rangle;$$

the characteristic polynomial of the action of the monodromy on the first cohomology group of the structure sheaf of the cover is

$$\prod_{j\in\{2,4,6,7,12,14\}} \left(t - \exp\frac{2\pi j\sqrt{-1}}{15}\right);$$

in particular, the characteristic polynomial of the covering is the product of the cyclotomic polynomials for 5 and 15. Analogously, for the horizontal part, the covering is associated with (μ_2, D_h, H_h), where

$$\mu_2 = \gcd(180, 36, 18, 36) = 18, \quad D_h = 90(C+E) + 15G_1 + 165G_2,$$

and

$$H_h = 10H - (2A_1 + A_2 + 2A_3) \sim 10(E+G_2) + 18A_1 - 31A_2 + 8A_3 \sim 10(E+G_2) + 4(2A_3 - A_2),$$

which is the expression as in Lemma 4.3, where $T' = 4(2A_3 - A_2)$ is a torsion divisor of order 9 whose least common multiple with $T = 5A_1 + A_2 + A_3 - F$ is 0. According to Reduction 4.4 note that $d_\tau = 9$, $\tau = 2$, and

$$n = \gcd(18, 90, 15, 165) = 3.$$

In other words, one is interested in studying the horizontal covering associated with $(18, D_h', T')$, where

$$D_h' = D_h - 18H_h = 90(C-E) + 15(G_1 - G_2). \tag{5.1}$$

The H^1-eigenspaces of this covering, according to Proposition 4.9, can be recovered from its *mixed* part and its greatest common vertical covering. The mixed part has already been considered in the vertical component, that is, in (μ_1, D_v, H_v) above, as for the greatest common vertical 2-fold covering of the restrictions to the special fibers A_i. From (5.1), the double covering of $\mathbb{P}^1$ is ramified in principle at four distinct points with indices $\left(\frac{90}{18}, -\frac{90}{18}, 15, -15\right)$ which are congruent with $(1,1,1,1)$ mod 2. Then the characteristic polynomial of this part is $\Delta_{2,h}(t) = t + 1$ (the cover is an elliptic curve).

5.3. Application to the cohomology of surfaces which are quotients of a product of curves. Let us consider a horizontal covering $\pi\colon X \to S$ of a reducible normal fake quadric S associated with $(d, D, 0)$, where

$$D = \kappa(cC + eE) + \sum_{j=1}^{s} g_j G_j, \quad \gcd(\kappa c, \kappa e, g_1, \dots, g_s) = 1.$$

The condition on the greatest common divisor implies that the covering $\pi_F\colon F_{(d)} \to F$, the restriction of π on the generic fiber F, is connected, where $F_{(d)} := \pi^{-1}(F)$ is a smooth projective curve. Consider the following diagram:

$$\begin{tikzcd} Y \arrow[r] \arrow[d, "\tilde{\pi}"] & X \arrow[d, "\pi"] \\ G \times \mathbb{P}^1 \arrow[r, "\tau_2"] & S \end{tikzcd}$$

The covering $\tilde{\pi}$ is associated with a horizontal divisor of $G\times\mathbb{P}^1$ where the sequence of ramification indices is c (at each point in $\tau_2^{-1}(F\cap C)$), e (at each point in $\tau_2^{-1}(F\cap E)$), and g_i (at each point in $\tau_2^{-1}(F \cap G_i)$), $i = 1, \dots, s$. Actually $Y = G \times F_{(d)}$ and $\tilde{\pi} = 1_G \times \pi_F$. As a consequence, X is the quotient of $G \times F_{(d)}$ and its cohomology can be computed from the formulas in this work. The action on $G \times F_{(d)}$ is not free, but X admits structures of isotrivial fibrations over orbifolds.

6. Lê–Yomdin singularities

A useful application of cyclic covers of reducible normal fake quadrics is given for the semistable reduction of (weighted) Lê–Yomdin surface singularities introduced in [**7**]. Let $(V, 0) := \{F = 0\} \subset (\mathbb{C}^3, 0)$ be a hypersurface singularity where $F := f_m + f_{m+k} + \cdots$ is the weighted homogeneous decomposition of the analytic germ $F \in \mathbb{C}\{x, y, z\}$ for some weights $\theta = (\theta_x, \theta_y, \theta_z)$. We say $(V, 0)$ is a (θ, k)*-weighted Lê–Yomdin singularity* if

$$V(\operatorname{Jac}(f_m)) \cap V(f_{m+k}) = \emptyset,$$

where $V(\operatorname{Jac}(f_m)), V(f_{m+k}) \subset \mathbb{P}^2_\theta$ are algebraic varieties in the weighted projective plane $\mathbb{P}^2_\theta = \operatorname{Proj}\mathbb{C}[x, y, z]_\theta$, $\mathbb{C}[x, y, z]_\theta$ is the θ-graded polynomial algebra over $\mathbb{C}$, and $\theta = (\deg x, \deg y, \deg z) = (\theta_x, \theta_y, \theta_z)$.

The term Lê–Yomdin singularity comes from the original papers by Yomdin [**19**] and Lê [**20**] on hypersurface singularities (see [**8**] for more details). The Milnor fiber of a (θ, k)-weighted Lê–Yomdin singularity can be studied by means of Steenbrink's spectral sequence [**31**] associated with the semistable reduction [**27, 31, 25**] of a $\mathbb{Q}$-resolution of singularities of $(V, 0)$; see [**24**] for the non-weighted case.

This resolution is obtained with two types of blow-ups. The first one is the θ-weighted blow-up of $(\mathbb{C}^3, 0)$. This first exceptional divisor and its cyclic cover in the semistable reduction deserve special treatment and they are completely determined by the projectivized tangent cone $V(f_m)$ as a curve in the resulting exceptional divisor, which is the weighted homogeneous plane $\mathbb{P}^2_\theta$; see [**5**, Proposition 5.13].

The other blow-ups are those required for the minimal $\mathbb{Q}$-embedded resolution of the pair $(\mathbb{P}^2_\theta, V(f_m))$; each weighted blow-up of this embedded resolution yields a weighted blow-up of the surface singularity as explained in [**25**, §6.2]. Each one of these blow-ups contributes with an exceptional divisor which produces (as cyclic covers) certain divisors in the semistable reduction. Most of the invariants appearing in Steenbrink's spectral sequence are birational invariants. In particular, instead of studying the cyclic covers at the end of the resolution, one can study them at this first stage, where they are cyclic covers of quotients of weighted projective planes with a very precise ramification locus.

For the particular case of superisolated singularities, the strategy in [**2**] includes performing an additional birational transformation such that the final result is a cyclic cover of $\mathbb{P}^1 \times \mathbb{P}^1$ ramified along some fibers and sections. The parallel strategy for weighted Lê–Yomdin singularities replaces $\mathbb{P}^1 \times \mathbb{P}^1$ by a reducible normal fake quadric, the fibers by regular and exceptional fibers, and the sections by curves C, E, G; see [**26**]. In the upcoming sections computations will be carried out for the particular case of the second type of weighted blow-ups, namely where the blown-up point P is smooth in the partial resolution and where the weights (w_x, w_y, w_z) of ω are pairwise coprime. Following [**25**, §6.2], an embedded $\mathbb{Q}$-resolution of $V(f_m)$ will be described first.

6.1. Semistable normalization in dimension 2. In this section, one step of the $\mathbb{Q}$-resolution of a point $V(f_m)$ will be described, namely, a (w_x, w_y)-blow-up of a smooth point P_0 in a 2-dimensional chart where the local equation of the total transform of the singularity is

$$x^{km_x} y^{km_y} h(x,y) = 0;$$

the (w_x, w_y)-multiplicity of $h(x,y)$ is denoted by ν. The condition on the exponents of x, y is given by the smoothness in dimension 3. The ν-form of $h(x,y)$ is

$$\left(x^{a_x} y^{a_y} \prod_{i=1}^{r} (y^{w_x} - \gamma_i x^{w_y})^{e_i} \right)^s, \tag{6.1}$$

where $\gamma_1, \dots, \gamma_r \in \mathbb{C}^*$ are pairwise distinct, $\kappa := \frac{k}{s} \in \mathbb{Z}$, and $\gcd(\kappa, a_x, a_y, e_1, \dots, e_r) = 1$. We denote $e := \sum_{i=1}^r e_i$ and $\nu_0 := w_x a_x + w_y a_y + w_x w_y e = \frac{\nu}{s}$. Moreover, this is related to the third weight of ω by $w_z = \frac{\nu_0}{\kappa} = \frac{\nu}{k} \in \mathbb{Z}$.

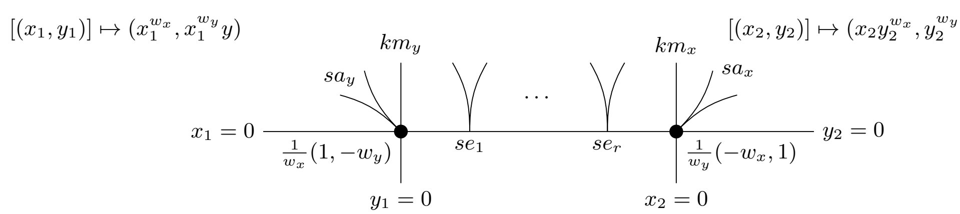

FIGURE 7. Weighted (w_x, w_y)-blow-up at P_0.

The multiplicity of the exceptional divisor is

$$k(w_x m_x + w_y m_y) + \nu = k\,\overbrace{(w_x m_x + w_y m_y + w_z)}^{m_\omega}.$$

Let $u, v \in \mathbb{Z}$ such that $uw_x + vw_y = 1$. Let us consider the weighted blow-up of type $(Nw_x - v, 1)$ at the quotient singularity of order w_x, where $N \gg 0$. The multiplicity of this auxiliary exceptional divisor (mod km_ω) is

$$\begin{aligned} \frac{(Nw_x - v)km_\omega + km_y + sa_y}{w_x} &\equiv \frac{km_y + sa_y - kv(w_x m_x + w_y m_y + w_z)}{w_x} \\ &\equiv k\underbrace{(um_y - vm_x)}_{-c} + s\frac{a_y - v(w_x a_x + w_y a_y + w_x w_y e)}{w_x} \\ &\equiv s(\underbrace{-\kappa c + ua_y - va_x - vw_y e}_{e_\infty}). \end{aligned} \tag{6.2}$$

We proceed in a similar way with a weighted blow-up of type $(1, Nw_y - u)$ at the quotient singularity of order w_y. The multiplicity of this auxiliary exceptional divisor (mod km_ω) is

$$s(\underbrace{\kappa c + va_x - ua_y - uw_x e}_{e_0}). \tag{6.3}$$

Hence the gcd of the multiplicity at each new divisor and its neighboring divisors (which can be replaced by the multiplicities computed above) is

$$s\gcd(\kappa m_\omega, e_0, e_\infty, e_1, \ldots, e_r) = s\underbrace{\gcd(m_\omega, e_0, e_\infty, e_1, \ldots, e_r)}_{\delta_\omega}. \tag{6.4}$$

Note that $\gcd(\delta_\omega, \kappa) = 1$ and there exist $\beta_1, \beta_2 \in \mathbb{Z}$ such that $\beta_1\delta_\omega + \beta_2\kappa = 1$.

Let us summarize the notation and some formulas introduced in this section.

($\mathcal{C}1$) $\kappa = \frac{k}{s}$ with the condition $\gcd(\kappa, a_x, a_y, e_1, \ldots, e_r) = 1$.
($\mathcal{C}2$) $e = e_1 + \cdots + e_r$.
($\mathcal{C}3$) $\nu_0 = w_x a_x + w_y a_y + w_x w_y e = \frac{\nu}{s}$.
($\mathcal{C}4$) $w_z = \frac{\nu_0}{\kappa} = \frac{\nu}{k}$.
($\mathcal{C}5$) $m_\omega = w_x m_x + w_y m_y + w_z$.
($\mathcal{C}6$) $uw_x + vw_y = 1$.
($\mathcal{C}7$) $e_0 = va_x - ua_y - uw_x e + \kappa c$.
($\mathcal{C}8$) $e_\infty = ua_y - va_x - vw_y e - \kappa c$.
($\mathcal{C}9$) $\delta_\omega = \gcd(m_\omega, e_0, e_\infty, e_1, \ldots, e_r)$.
($\mathcal{C}10$) $\beta_1\delta_\omega + \beta_2\kappa = 1$.

6.2. Semistable normalization in dimension 3. In this section we use the blow-up performed in Subsection 6.1 at the embedded surface singularity in dimension 3. In particular, consider a chart where P is the origin and the local equation of the total transform of the surface singularity V is of the form

$$(x^{m_x} y^{m_y})^{m+k} z^m (z^k - h(x,y)) = 0.$$

The ν-form of $z^k - h(x,y)$ is

$$\prod_{j=1}^{s}\left(z^\kappa - \zeta_s^j x^{a_x} y^{a_y} \prod_{i=1}^{r}(y^{w_x} - \gamma_i x^{w_y})^{e_i}\right) = z^k - x^{sa_x} y^{sa_y} \prod_{i=1}^{r}(y^{w_x} - \gamma_i x^{w_y})^{se_i}.$$

The exceptional component of the blow-up is isomorphic to $\mathbb{P}^2_\omega$. The multiplicity of this divisor equals

$$d := (w_x m_x + w_y m_y + w_z)\overbrace{(m+k)}^{\delta} = m_\omega \delta, \tag{6.5}$$

where m_ω has been introduced in ($\mathcal{C}5$). Since the total transform V' of V is linearly equivalent to the zero divisor, this is also the case for its intersection with $E \cong \mathbb{P}^2_\omega$. This intersection can be decomposed into two terms: the divisor obtained by the intersection of V' with E and with the components of V' which are different from E. The latter can be described as

$$D := \delta(m_x X + m_y Y) + mZ + \sum_{j=1}^{s} G_j,$$

where

$$X : x{=}0, \quad Y : y{=}0, \quad Z : z{=}0, \quad G_j : z^\kappa - \zeta_j x^{a_x} y^{a_y} \prod_{i=1}^{r}(y^{w_x} - \gamma_i x^{w_y})^{e_i} {=} 0, \tag{6.6}$$

and the divisors G_j are irreducible. The self-intersection E^2 is linearly equivalent to H, where H is any divisor in $\mathbb{P}^2_\omega$ of degree 1, e.g. $H \sim uX + vY$ as $\deg_\omega H = 1$; see ($\mathcal{C}6$). Hence the intersection is $D - dH \sim 0$, the support of a meromorphic function. From the point of view of a cyclic cover of degree D the geometric ramification occurs mod d and this is why we have provided the decomposition $D - dH$. From these data one can choose a particular presentation of $\mathrm{Cl}(\mathbb{P}^2_\omega)$ as

$$\langle X, Y, Z, F_1, \ldots, F_r, F \mid w_z X \sim w_x Z, w_z Y \sim w_y Z, w_y X \sim w_x Y \sim F \sim F_i,\ 1 \le i \le r\rangle,$$

where $G_1 \sim \cdots \sim G_s \sim \kappa Z$, F_i is the *line* joining $[0:0:1]_\omega$ and $[1 : \gamma_i^{\frac{1}{w_x}} : 0]_\omega$, and F is a generic *line* through $[0:0:1]_\omega$.

Hence, it makes sense to consider the cyclic covering ramified along (d,D,H). Equivalently, using Reduction 4.4, one can consider the covering associated with $(d, D', 0)$, where

$$D' = \delta(m_x - um_\omega)X + \delta(m_y - vm_\omega)Y + mZ + \sum_{j=1}^{s} G_j.$$

In order to express this cover as that of a $\mathbb{Q}$-normal crossing divisor, one needs to perform some birational transformations. This is where the reducible normal fake quadrics come into play. Let us start with the (w_x, w_y)-blow-up of $[0:0:1]_\omega$. The new surface will be denoted by Σ and the exceptional component of this blow-up by E. For simplicity, we keep the notation for the strict transforms in Σ. Taking into account that the total transform of X (resp. Y) in Σ is $X + \frac{w_x}{w_z}E$ (resp. $Y + \frac{w_y}{w_z}E$), it is not hard to check that

$$\begin{aligned}\mathrm{Cl}(\Sigma) = \langle X, Y, Z, F_1, \ldots, F_r, F, E \mid\ & w_z(uX + vY) = Z - E,\\ & w_y X = w_x Y = F_i = F,\ 1 \le i \le r\rangle.\end{aligned}$$

The multiplicity of E as a component of the total transform of D' is

$$\delta\frac{(m_x - um_\omega)w_x + (m_y - vm_\omega)w_y}{w_z} = -\delta;$$

see ($\mathcal{C}4$) and ($\mathcal{C}5$) on page 399. By the previous calculations, note that the new ramification divisor D_Σ can be described as

$$D_\Sigma = \delta(m_x - um_\omega)X + \delta(m_y - vm_\omega)Y - \delta E + mZ + \sum_{j=1}^{s} G_j.$$

Next, one can perform generalized Nagata transformations on the fibers over the following points of $E \equiv \mathbb{P}^1$: $0 \equiv [0:1]$, $\infty \equiv [1:0]$, $\gamma_i \equiv [\gamma_i : 1]$, $i = 1, \ldots, r$. The goal of these transformations is to separate the divisors G_j from Z keeping them away from E, which can be done using the following weights:

$$\frac{1}{\gcd(\kappa, a_x)}(\kappa, a_x) \text{ at } 0, \quad \frac{1}{\gcd(\kappa, a_y)}(\kappa, a_y) \text{ at } \infty, \quad \frac{1}{\gcd(\kappa, e_i)}(\kappa, e_i) \text{ at } \gamma_i.$$

Moreover we obtain a reducible normal fake quadric S as in Subsection 2.3, as one can check by computing the self-intersection of the strict transform of Z in S:

$$(Z\cdot Z)_S = (Z\cdot Z)_\Sigma - \sum_{i=1}^{r} \frac{e_i}{\kappa} - \frac{a_x}{\kappa w_y} - \frac{a_y}{\kappa w_x} = \frac{w_z}{w_x w_y} - \frac{a_x w_x + a_y w_y + w_x w_y e}{\kappa w_x w_y} = 0. \tag{6.7}$$

To obtain a full description of S, let us describe its cyclic quotient singular points. If the new exceptional divisors are denoted by $A_0, A_\infty, A_1, \ldots, A_r$, then there are two singular points on each A_i, $i \in I = \{0, 1, \ldots, r, \infty\}$, whose combinatorial data (d_i, q_i), as described in Subsection 2.2, can be calculated using [**9**, Theorem 4.3] in terms of the invariants introduced in (6.1), (6.2), and (6.3) as, see also ($\mathcal{C}1$), ($\mathcal{C}7$), and ($\mathcal{C}8$):

$$d_i = \frac{\kappa}{\gcd(\kappa, e_i)}, \quad q_i = \frac{e_i}{\gcd(\kappa, e_i)}, \quad i \in I. \tag{6.8}$$

Note that

$$\operatorname{lcm}_{i\in I} d_i = \frac{\kappa}{\gcd(\kappa, e_i, i \in I)} = \kappa. \tag{6.9}$$

The first equality in (6.9) is a consequence of (6.8) and (A.4) in the appendix. For the second equality one needs to show

$$\gcd(\kappa, e_i, i \in I) = 1. \tag{6.10}$$

Since $\mathbb{Z} = \mathbb{Z}\langle \kappa, a_x, a_y, e_1, \dots, e_r\rangle$, it is enough to show that $a_x, a_y \in \mathbb{Z}\langle \kappa, e_i, i \in I\rangle$; see ($\mathcal{C}1$). This is a consequence of

$$\begin{pmatrix} v & -u \\ w_x & w_y \end{pmatrix}\begin{pmatrix} a_x \\ a_y \end{pmatrix} \equiv \begin{pmatrix} 0 \\ 0 \end{pmatrix} \bmod \mathbb{Z}\langle \kappa, e_i, i \in I\rangle \text{ and } \begin{vmatrix} v & -u \\ w_x & w_y \end{vmatrix} = 1.$$

As a word of caution, note that we are not imposing that the values q_i be in $\{1, \dots, d_i - 1\}$ (see also Remark 2.5), only $\gcd(d_i, q_i) = 1$. There could be other choices of q_i corresponding to different generalized Nagata transformations; their remainder classes mod d_i, however, remain unchanged. According to our construction and (6.7), one can check that $\alpha = \sum_{i\in I} \frac{q_i}{d_i} = 0$.

Finally, let us compute the total transform D_S of D_Σ in S. First, the multiplicity of A_i, $i = 1, \dots, r$, as a component of D_S is given by

$$m\frac{e_i}{\gcd(\kappa, e_i)} + s\frac{e_i\kappa}{\gcd(\kappa, e_i)} = \delta q_i.$$

On the other hand, the multiplicity of A_0 as a component of D_S is

$$\begin{aligned}
\delta\frac{a_x + \kappa(m_x - um_\omega)}{w_y \gcd(\kappa, e_0)} &= \delta\frac{a_x + \kappa m_x(1 - uw_x) - u\kappa(m_y w_y + w_z)}{w_y \gcd(\kappa, e_0)} \\
&= \delta\frac{a_x + \kappa m_x v w_y - u\kappa m_y w_y - u(w_x a_x + w_y a_y + w_x w_y e)}{w_y \gcd(\kappa, e_0)} \\
&= \delta\frac{a_x v + \kappa c - u(a_y + w_x e)}{\gcd(\kappa, e_0)} = \delta q_0.
\end{aligned}$$

Analogously, the multiplicity of A_∞ is δq_∞. Then, the ramification divisor D_S is

$$D_S = \delta\sum_{i\in I} q_i A_i + mZ - \delta E + \sum_{j=1}^{s} G_j.$$

Finally, note that

$$\operatorname{Cl}(S) = \left\langle A_0, A_\infty, A_1, \dots, A_r, F, Z, E \mid d_i A_i \sim F,\, i \in I,\, E - Z \sim \sum_{i\in I} q_i A_i \right\rangle.$$

The last step is to apply the contents of Subsection 4.3 to the covering of S associated with $(d, D_S, 0)$. Following the constructive proof of Theorem 4.12 and its notation, the subset of indices J_1 corresponds to the irreducible divisors $Z, E, G_1, \dots, G_s$ and then $\mu_1 = 1$ and also $\mu_0 = 1$. Then, there is no contribution from the vertical part (a) and from the mixed part (c). Let us check the horizontal part (b). The irreducible components of D_S involved in J_2 are A_i, $i \in I$, and hence, using (A.3), one obtains

$$\begin{aligned}\gcd(\delta m_\omega, \delta q_i, i \in I) &= \delta \gcd\left(m_\omega, \frac{e_i}{\gcd(\kappa, e_i)}, i \in I\right) \\ &= \delta \gcd\left(m_\omega, \frac{\gcd(e_i, i \in I)}{\gcd(\kappa, e_i, i \in I)}\right) = \delta\delta_\omega.\end{aligned}$$

The last equality is a consequence of (6.3) and (6.10); see also ($\mathcal{C}$9). Recall that $d = \delta m_\omega$ by (6.5) and $\delta_\omega | m_\omega$ by (6.4). Thus the monodromy of the original d-cover coincides with the monodromy of a horizontal $(\delta\delta_\omega)$-cover associated with $(\delta\delta_\omega, D_h, H_h)$, where (see ($\mathcal{C}$10))

$$D_h = mZ - \delta E + \sum_{j=1}^{s} G_j, \quad H_h = -\frac{1}{\delta_\omega}\sum_{i\in I} q_i A_i \sim -\beta_1 \sum_{i\in I} q_i A_i \sim \beta_1(Z - E)$$

(one can easily check that the relation $D_h - \delta\delta_\omega H_h \sim 0$ holds).

Finally, after applying the process described in Subsection 4.2, the horizontal cover satisfies $n = 1$ and $\tau = \delta\delta_\omega$.

As a consequence of Theorem 4.12 and the discussion above, one has the following.

Proposition 6.1. *Under the conditions and notation described above, the invariant subspaces of the monodromy action on the semistable reduction S of* (6.6) *coincide with the* greatest common $(\delta\delta_\omega)$-covering *of the restrictions to the special fibers A_i, $i \in I$.*

Let us re-prove the above result without the use of the torsion arguments. Let us consider the restriction $\pi_F \colon F_{\delta\delta_\omega} \to F$ of the $(\delta\delta_\omega)$-subcover to a general fiber F. The ramification happens at $Z \cap F$ (with multiplicity m), $E \cap F$ (with multiplicity $-\delta$), and $G_j \cap F$ (κ points with multiplicity 1). The following congruences mod $(\delta\delta_\omega)$ hold:

$$m \equiv m - \delta\delta_\omega\beta_1 \equiv m - \delta(1 - \beta_2\kappa) \equiv \kappa\,\overbrace{(\beta_2\delta - s)}^{\hat{m}}.$$

Then, there is a pull-back diagram

$$\begin{tikzcd} F_{(d)} \arrow[r] \arrow[d, "\pi_F"] & X \arrow[d, "\pi_X"] \\ F \arrow[r] & \mathbb{P}^1 \\ z \arrow[r, mapsto] & z^\kappa \end{tikzcd} \tag{6.11}$$

where the covering π_X has ramification indices $\hat{m}$ (at 0), 1 (at s points in $\mathbb{C}^*$), and $-(s + \hat{m})$ (at ∞).

A similar diagram exists if we replace F by A_i and κ by $\frac{\kappa}{d_i}$, for $i \in I$. In order to see this, one needs to know the multiplicity of the ramification divisor at $Z \cap A_i$, which is obtained by performing a $(q_i', 1)$-blow-up as in the proof of Proposition 4.8, i.e., $q_i' q_i = h_i d_i - 1$. The computed multiplicity is

$$\frac{\delta q_i q_i' + m}{d_i} = \delta h_i - s\frac{\kappa}{d - i}.$$

Also, one needs to check the following congruence mod $\delta\delta_\omega$:

$$\tfrac{\kappa}{d_i}(\beta_2\delta - s) \equiv \delta h_i - s\tfrac{\kappa}{d_i} \iff \tfrac{\kappa}{d_i}\beta_2\delta \equiv \delta h_i,$$

which is equivalent to $\frac{\kappa}{d_i}\beta_2 \equiv h_i \mod \delta_\omega$. Since $\gcd(d_i, q_i) = 1$ and δ_ω is a divisor of q_i the above congruence is equivalent to $1 - \beta_1\delta_\omega = \kappa\beta_2 \equiv d_i h_i \equiv 1 + q_i q_i' \mod \delta_\omega$, which is obviously true. Then the following diagram also holds:

$$\begin{array}{ccc} A_{i,(\delta\delta_\omega)} & \longrightarrow & X \\ \big\downarrow{\scriptstyle \pi_{A_i}} & & \big\downarrow{\scriptstyle \pi_X} \\ A_i & \longrightarrow & \mathbb{P}^1 \\ z & \longmapsto & z^{\frac{\kappa}{d_i}} \end{array}$$

This provides an alternative proof of the result since it means that $\delta\delta_\omega$ is the maximal degree where the diagram (6.11) holds. This describes the curve cover that contains all the necessary information.

Corollary 6.2. *The characteristic polynomial of the monodromy action on the semistable reduction S of* (6.6) *is*

$$\frac{(t-1)^{2-s}(t^{\delta\delta_\omega}-1)^s}{(t^\delta - 1)(t^{\gcd(m,s\delta_\omega)}-1)}.$$

Proof: From Proposition 6.1 we know that it coincides with the characteristic polynomial of π_X from (6.11). We use a zeta-function argument. The covering admits a stratification with a dense strata of Euler characteristic $-s$, where each point has $\delta\delta_\omega$ preimages (the unramified part). There are s points with only one preimage. For the other two remaining points the number of preimages are $\gcd(\delta\delta_\omega, \hat{m})$ and $\gcd(\delta\delta_\omega, -\hat{m}-s) = \gcd(\delta\delta_\omega, \beta_2\delta) = \delta$. Since $\hat{m} = \beta_2\delta - s = m\beta_2 - \beta_1\delta_\omega s$ we have

$$\begin{pmatrix} \hat{m} \\ \delta\delta_\omega \end{pmatrix} = \begin{pmatrix} \beta_2 & -\beta_1 \\ \delta_\omega & \kappa \end{pmatrix} \begin{pmatrix} m \\ s\delta_\omega \end{pmatrix},$$

and as the square matrix is unimodular then $\gcd(\delta\delta_\omega, \hat{m}) = \gcd(s\delta_\omega, m)$. Hence

$$\frac{(t-1)^2}{\Delta_1(t)} = \frac{\Delta_2(t)\Delta_0(t)}{\Delta_1(t)} = \zeta(t) = (t-1)^s(t^\delta - 1)(t^{\gcd(s\delta_\omega,m)}-1)(t^{\delta\delta_\omega}-1)^{-s}. \qquad \square$$

Appendix A. Basic arithmetic

Some basic well-known properties of the floor and ceiling functions, the greatest common divisor, and the least common multiple are set out here for completeness. The purpose is to recall some of the most frequently used properties in this paper so the reader can be referred to them, simplifying the overall exposition.

A.1. Floor and ceiling. Given a real number $a \in \mathbb{R}$, let us denote by $\lfloor a \rfloor$ its integral part (or floor) and by $\{a\}$ its decimal part so that one can write

$$a = \lfloor a \rfloor + \{a\},$$

where $\lfloor a \rfloor$ is the unique integer satisfying $a - 1 < \lfloor a \rfloor \leq a$ and $0 \leq \{a\} < 1$. The ceiling (or roundup) is denoted by $\lceil a \rceil$, it satisfies $\lceil a \rceil = -\lfloor -a \rfloor$, and it is the unique integer such that $a \leq \lceil a \rceil < a + 1$.

Two more useful properties are used throughout this paper. Let $a \in \mathbb{R}$ and $m, n \in \mathbb{Z}$; then

$$\left\lfloor \frac{\lfloor a \rfloor}{n} \right\rfloor = \left\lfloor \frac{a}{n} \right\rfloor \quad \text{and} \quad \left\lceil \frac{m+1}{n} \right\rceil = \left\lfloor \frac{m}{n} \right\rfloor + 1, \tag{A.1}$$

and, consequently,

$$\left\lfloor \frac{-1-m}{n} \right\rfloor = -1 - \left\lfloor \frac{m}{n} \right\rfloor. \tag{A.2}$$

A.2. Greatest common divisor and least common multiple. Let $m_1, \dots, m_r, k \in \mathbb{Z}$; then

$$\gcd\left(\frac{m_1}{\gcd(k,m_1)}, \dots, \frac{m_r}{\gcd(k,m_r)}\right) = \frac{\gcd(m_1,\dots,m_r)}{\gcd(k,m_1,\dots,m_r)}. \tag{A.3}$$

If, in addition, $m \in \mathbb{Z}$ is a multiple of all the m_i's, then

$$\operatorname{lcm}\left(\frac{m}{m_1}, \dots, \frac{m}{m_r}\right) = \frac{m}{\gcd(m_1,\dots,m_r)}. \tag{A.4}$$

Let $d_1, \dots, d_r \in \mathbb{Z} \setminus \{0\}$ such that there exist $q_1, \dots, q_r \in \mathbb{Z} \setminus \{0\}$, $\gcd(q_i, d_i) = 1$, for which

$$\sum_{i=1}^{r} \frac{q_i}{d_i} \in \mathbb{Z};$$

then

$$\operatorname{lcm}(d_1,\dots,d_r) = \frac{\gcd\left(\left\{\frac{d_1\cdot\ldots\cdot d_r}{d_i} \mid i=1,\dots,r\right\}\right)}{\gcd\left(\left\{\frac{d_1\cdot\ldots\cdot d_r}{d_i d_j} \mid i,j=1,\dots,r,\ i\neq j\right\}\right)}. \tag{A.5}$$

The first two results are classical; let us proof this last one (the technique can easily be adapted by the reader for the other proofs). The hypothesis implies that for any i, d_i divides the lcm of all the d_j's but d_i.

Fix a prime number p and consider the valuation ν_p associated with this prime. Let $n_i := \nu_p(d_i)$. For the sake of simplicity, let us order the numbers $d_1, \dots, d_r$ such that $n_1 \leq \cdots \leq n_r$. The condition that d_r divides $\operatorname{lcm}(d_1, \dots, d_{r-1})$ implies that $n_r \leq \max(n_1, \dots, n_{r-1}) = n_{r-1} \leq n_r$, i.e., $n_r = n_{r-1}$. The p-valuation of the left-hand side of (A.5) equals $\max(n_1, \dots, n_r) = n_r$. The p-valuation of the numerator of the right-hand side equals

$$\min_{1\leq i\leq r}\left((n_1+\cdots+n_r)-n_i\right) = (n_1+\cdots+n_r) - \max_{1\leq i\leq r}(n_i) = n_1+\cdots+n_{r-1}.$$

The p-valuation of the denominator of the right-hand side equals

$$\min_{1\leq i<j\leq r}\left((n_1+\cdots+n_r)-n_i-n_j\right) = (n_1+\cdots+n_r) - \max_{1\leq i<j\leq r}(n_i+n_j) = n_1+\cdots+n_{r-2}$$

and we obtain that both sides have the same p-valuation. Since it is valid for all primes, the equality follows.

Appendix B. Coverings of the projective line

The following proof was inspired by the calculations in [**31**, p. 547].

Lemma B.1. *Let $F\colon Y \to X$ be a cyclic branched covering of e sheets between two orientable compact surfaces. Denote by Δ the characteristic polynomial of the monodromy of F acting on the cohomology groups. Assume Y has r connected components. Then,*

$$\Delta_{H^1(Y)}(t) = \frac{(t^r-1)^2\cdot(t^e-1)^{-\chi(\check{X})}}{\prod_{Q\in\mathcal{R}}(t^{\mu(Q)}-1)},$$

where $\mathcal{R}$ is the ramification set of F, $\check{X} := X \setminus \mathcal{R}$, and $\mu(Q)$ denotes the number of preimages of Q.

Proof: Let us write $\check{Y} := Y \setminus F^{-1}(\mathcal{R})$. Since $F_| : \check{Y} \to \check{X}$ is a topological covering of e sheets, one gets that $\Delta_{H^0(\check{Y})}(t) \cdot \Delta^{-1}_{H^1(\check{Y})}(t) = (t^e - 1)^{\chi(\check{X})}$. On the other hand, by virtue of Mayer–Vietoris, there is an exact sequence

$$0 \longrightarrow H^1(Y) \longrightarrow H^1(\check{Y}) \longrightarrow H^0(Y \setminus \check{Y}) \longrightarrow H^2(Y) \longrightarrow 0.$$

Therefore

$$\Delta_{H^1(Y)}(t) = \frac{\Delta_{H^0(\check{Y})}(t) \cdot (t^e - 1)^{-\chi(\check{X})} \cdot \Delta_{H^2(Y)}(t)}{\Delta_{H^0(Y \setminus \check{Y})}(t)}.$$

The monodromy on $H^0(\check{Y}) \cong H^2(Y)$ is the rotation of the corresponding r irreducible components. Hence its characteristic polynomial is $t^r - 1$. Finally, every point $Q \in \mathcal{R}$ gives the factor $t^{\mu(Q)} - 1$ in $\Delta_{H^0(Y \setminus \check{Y})}(t)$. $\square$

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Departamento de Matemáticas, IUMA, Universidad de Zaragoza, C. Pedro Cerbuna 12, 50009, Zaragoza, Spain

E-mail address: artal@unizar.es

E-mail address: jicogo@unizar.es

E-mail address: jorge@unizar.es

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