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IRREDUCIBLE MODULES OVER GROUP RINGS

Talk delivered at Universitat Autònoma de Barcelona, 19 April 1979, by B. Hartley.

Throughout, K denotes a field and G a group. We begin with:

Maschke's Theorem If G is finite and char K $\nmid |G|$, then every KG-module is completely reducible.

Thus, if V is a KG-module and U a submodule, then U is complemented in V; in particular this holds for irreducible submodules U.

Definition If R is a ring and U an R-module, then U is *injective* if U is a direct summand of every larger R-module.

(Conventions: all rings have identity element 1, all modules are right modules, and 1 acts as the identity on all R-modules). Thus, from Maschke's Theorem:

Proposition 1 If G is finite and char K $\nmid |G|$, then every irreducible KG-module is injective.

If U is any irreducible KG-module, we have an exact sequence

$$0 \to U' \to KG \to U \to 0$$

of right modules, and by Maschke's Theorem $KG \simeq U' \oplus U$, if G is finite and char K $\nmid |G|$. Thus

Proposition 2 If G is finite and char K $\nmid |G|$, then every irreducible right KG-module is isomorphic to a minimal right ideal of KG.

We will discuss what happens to these propositions when G is allowed to be infinite.

I. Minimal right ideals

If G is infinite and $0 \neq \alpha \in KG$, then $\alpha = \lambda_1 g_1 + \ldots + \lambda_n g_n$ $(0 \neq \lambda_i \in K,\ g_i \in G)$. We can choose $g \in G$ such that $g_1 g \notin \{g_1, g_2, \ldots, g_n\}$, and then αg is not a scalar multiple of α. Thus

Lemma 1.1 If G is infinite then KG does not contain a 1-dimensional right ideal. In particular the trivial KG module K is not isomorphic to a minimal right ideal of KG.

Thus Proposition 2 is false whenever G is infinite, so we modify our question: which irreducible KG-modules are isomorphic to minimal right ideals of KG? When does KG contain a minimal right ideal?

This is mostly work of J. S. Richardson (Proc. London Math. Soc. (3) 35 (1977)).

Conjecture: If KG possesses a minimal right ideal, then G is locally finite.

This is known to be true under fairly mild restrictions on G, but the general case seems difficult.

We will restrict ourselves to the locally finite case.

Example of an infinite group G such that QG contains a minimal right ideal.

Let

$$G = C_{p^\infty} = \langle x_1, x_2, \ldots : x_1^p = 1,\ x_{i+1}^p = x_i \rangle$$

Thus $G \simeq H = \langle \zeta_1, \zeta_2, \ldots \rangle$ where $\zeta_m = e^{2\pi i/p^m}$

We have a homomorphism $q: QG \to L = Q(\zeta_1, \zeta_2, \ldots) \leq \mathbb{C}$, induced by $x_m \to \zeta_m$. Let $e = \frac{1}{p}(1 + x_1 + \ldots + x_1^{p-1})$. Clearly $e \in \ker q$. Conversely, if $e \in \ker q$, then $\alpha \in Q < x_m >$ for some m, and we can write $\alpha = f(x_m)$ where $f \in Q[X]$ and f has degree at most $p^m - 1$. Since $f(\zeta_m) = 0$, we have $f(X) = g(X)\Phi(X)$ where $g(X) \in Q(X)$ and $\Phi(X) = 1 + X^{p^{m-1}} + X^{2p^{m-1}} + \ldots + X^{(p-1)p^{m-1}}$. Thus, substituting x_m for X, $\alpha = g(x_m)(1 + x_1 + \ldots + x_1^{p-1}) \in eQG$. It follows that $\ker q = eQG$, and if $f = 1 - e$, then as e is an idempotent, $QG = fQG \dotplus eQG$, and $fQG \approx L$. Since L is a field, fQG is a minimal ideal of QG.

This example is fairly representative of the general situation. Note that if G is the above group, then $\mathbb{C}G$ contains no minimal right ideals. This is because as $\mathbb{C}$ is algebraically closed, every irreducible $\mathbb{C}G$-module is one-dimensional, and then Lemma 1.1 applies.

Theorem 1.2 (B. Hartley, J.S. Richardson, J. London Math. Soc. 1977). Let G be locally finite. Then KG contains a minimal right ideal if and only if

(i) G contains a normal subgroup H of finite index such that $H \simeq C_{p_1^\infty} \times \ldots \times C_{p_t^\infty}$, where the primes $p_1, \ldots, p_t$ are all distinct and different from char K.

(ii) $[K_0(H) \cap \bar{K} : K_0] < \infty$

Here K_0 is the prime field of K, $\bar{K}$ is an algebraic closure of K, and $K_0(H)$ is the field generated over K_0 by the primitive p_i^n-the roots of 1 for $1 \leq i \leq t$, $n = 1,2,3,\ldots$, in other words, by the primitive roots of 1 corresponding to the orders of the elements of H.

The proof of this result is quite involved and in fact depends on the Feit - Thompson Theorem via work of Šunkov.

When KG contains a minimal right ideal, there are a number of strong consequences for the structure of KG. For example, suppose further that G has no elements of order p = char K.

Define the socle of KG : $S_1(KG)$ is the submodule generated by the minimal submodules (i.e. minimal right ideals) of KG, and $S_{i+1}(KG)/S_i(KG) = S_1(KG/S_i(KG))$. Then with the hypotheses and notation of Theorem 1.2,

$$S_{\ell+1}(KG) = KG$$

and each factor $S_{i+1}(KG)/S_i(KG)$ is a ring, in general without 1, which is a direct sum of matrix rings over division rings, each generated by a centrally primitive idempotent in $KG/S_i(KG)$.

If V is an arbitrary irreducible KG-module, then if C is the kernel of the corresponding representation of G, let i be the number of groups $C_{p_1^\infty}, \ldots, C_{p_t^\infty}$ contained in C. Then V is isomorphic to a submodule of $S_i(KG)/S_{i-1}(KG)$.

In particular, if V is faithful, then V is isomorphic to a minimal right ideal of KG.

For the proof of these and other results, see the paper of Richardson mentioned above.

II Injective modules.

Now we ask: which irreducible KG-modules are injective? and how far can an irreducible KG-module depart from injectivity?

Essential extensions Let U be a submodule of an R-module N. We say that N is an essential extension of U, if $M \cap U \neq 0$ for every non-zero submodule M of N.

The injective hull of an R-module U can be characterized as a minimal injective module containing U, or a maximal essential extension of U. It is known that every R-module U possesses an injective hull $\bar{U}$, which is unique up to isomorphism. The complexity of $\bar{U}$ is in some sense a measure of how far U departs from being injective.

Theorem 2.1 (Farkas and Snider (1974); B. Hartley, Quarterly J. Math. (1977)) Let G be a countable group. Then every irreducible KG-module is injective if and only if (i) G is a locally finite p'-group (p = char K $\geq$ 0) (ii) G has an abelian subgroup of finite index.

A p'-group is one which has no elements of order p. If p = 0 this is no restriction.

The proof of the sufficiency of (i) and (ii) is quite easy. The proof of necessity has two stages:

(a) G is a locally finite p'-group. This just uses the fact that the trivial module is injective.

(b) If V is any irreducible module for R = KG and R_0 is the annihilator of V, then R/R_0 is simple artinian, and so if $E = \mathrm{End}_{KG} V$, then $\dim_E V < \infty$. Of course, E is a division ring.

The proof is concluded by

Theorem 2.2 If G is a locally finite p'-group (p = char K) then every irreducible KG-module has finite dimension over its endomorphism ring if and only if G has an abelian subgroup of finite index.

An account of this work can also be found in Passman's book "The Algebraic Structure of Group Rings". In Theorem 2.2, if we drop the restriction that G is a p'-group, we can conclude that $G/O_p(G)$ has an abelian subgroup of finite index (B. Hartley unpublished).

Musson (Math. Proc. Cambridge Philos. Soc. 1978) has shown that the conclusions of Theorem 2.1 hold under weaker hypotheses on the irreducible modules; for example if G is locally finite and the injective hull of every irreducible KG-module has countable dimension, then G has an abelian p'-subgroup of finite index (p = char K). More results of this kind can be found in his paper.

Next we consider the question: How bad can essential extensions of irreducible modules be?

Theorem 2.3 Let K be a field of characteristic $p \geq 0$, G be a countable locally finite p'-group, and V be an irreducible KG-module, with $E = \mathrm{End}_{KG} V$. Then there are two possibilities:

(i) $\dim_E V < \infty$, and V is injective

(ii) $\dim_E V = \infty$, and there is a non-split exact sequence $0 \to V \to W \to U \to 0$, where U is a direct sum of $\aleph_1$ irreducible modules.

Thus the departure from injectivity is very wild. By Theorem 2.1, if G does not have an abelian subgroup of finite index, then (ii) occurs for some V. A version of this with $\aleph_1$ replaced by 1 is in B. Hartley, Quarterly J. Math. 1977.

Also it can happen, with the notation of 2.3, that V is a submodule of a module W whose proper submodules, under inclusion, form a well ordered set whose order type is the first uncountable ordinal (B. Hartley, Proc. London Math. Soc. 1977). The exact conditions under which this happens are not clear, but for example if G is the direct product of an infinite number of dihedral groups of order 8 and char $k \neq 2$, then this kind of behaviour occurs.

Thus, except under rather strong restrictions, irreducible modules are a long way from being injective. But here is a positive result:

Theorem 2.4 Let G be a polycyclic -by- finite group and k be the integers or an algebraic extension of a finite field. Then

(i) Every irreducible kG-module has finite dimension (over k, or over $\mathbb{Z}/p\mathbb{Z}$ for some p, if $k = \mathbb{Z}$) (Roseblade, J. Pure Applied Algebra 1973).

(ii) If V is an irreducible kG-module, then the injective hull of V is artinian (Musson, Jategaonkar (not yet published)).

I understand that S. Donkin (Warwick) has extended (ii) to the case when k has characteristic zero and V is finite dimensional. The proof of (ii) in the above case uses quite complicated ring theoretic methods and involves an interesting application of Morita duality.