THE NORMAL AND MACKEY TOPOLOGIES
ON CO-ECHELON SPACES

by

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Abstract. A necessary and sufficient condition for the coincidence of the normal and Mackey topologies on a co-echelon space of order one is studied.

Introduction. Let \( a_n^{(k)} \), \( n=0,1,..., k=1,2,... \) be such that the following conditions are satisfied.

a) \( a_n^{(k)} > 0 \), for each \( k \) and \( n \).

b) \( a_n^{(1)} < a_n^{(2)} < a_n^{(3)} ... n=0,1,... \)

Let \( E \) and \( E^X \) be the echelon and co-echelon spaces respectively, corresponding to the steps \( a_n^{(k)} \) [1, p.419].

In this paper, the following theorem is proved.

Theorem. The normal and Mackey topologies in \( E^X \) coincide if and only if \( E \) with the normal topology is nuclear.

It is known that the normal and Mackey topologies in \( E^X \) coincide if and only if for every sequence \( (x_n) \) satisfying \( \lim_{n \to \infty} x_n \), \( a_n^{(k)} = 0 \) for \( k=1,2,..., \) it follows that \( \sum_{n=0}^{\infty} |x_n| a_n^{(k)} \to 0 \), for every \( k=1,2,... \) [3].
On the other hand, the Grothendieck-Pietsch criterium establishes that $E$ with the normal topology is nuclear if and only if for every $k$, there exists an $N(k)$ such that $\sum_{n=0}^{\infty} \frac{a_n^{(k)}}{a_n^{(N(k))}} < 2$, [2, p.98].

The previous theorem is, then, an immediate consequence of the following proposition.

**Proposition.** Let $k_0$ be such that for each $j=1,2,\ldots$, we have $\sum_{n=0}^{\infty} \frac{a_n^{(k_0)}}{a_n^{(j)}} = \infty$. There exists a sequence $(a_n^k)$, $a_n > 0$, $n=1,2,\ldots$, such that $\lim_{n \to \infty} a_n a_n^{(k_0)} = 0$, $k=1,2,\ldots$ while $\sum_{n=0}^{\infty} a_n a_n^{(k_0)} = \infty$.

**Proof.** $\sum_{n=0}^{\infty} \frac{a_n^{(k_0)}}{a_n^{(j)}} = \infty$, $j=1,2,\ldots$ implies that there exists $0 < n_1 < n_2 < \ldots$ such that

$$\sum_{n=0}^{n_1} \frac{a_n^{(k_0)}}{a_n^{(1)}} \geq 2$$

$$\sum_{n=n_1+1}^{n_1+1} \frac{a_n^{(k_0)}}{a_n^{(i+1)}} \geq 2^{i+1}, \quad i=1,2,\ldots$$

Consider, now, the sequence

$$a_n = \begin{cases} \frac{1}{2} (a_n^{(1)})^{-1}, & 0 \leq n \leq n_1 \\ \frac{1}{2^{i+1}} a_n^{(i+1)}^{-1}, & n_i + 1 \leq n \leq n_{i+1}, \quad i=1,2,\ldots \end{cases}$$
It is obvious that \( \lim_{n \to \infty} a_n a^{(k)}_n = 0, \ k=1,2,\ldots \) because given \( k \), we have

\[
a^{(j)}_n a^{(k)}_n \leq 1, \ j \geq k, \ n=1,2,\ldots
\]

However, \( \sum_{n=0}^{\infty} a_n a^{(k)}_n = \omega \).

**REFERENCES**


**Appendix.** The referee has kindly pointed out to us that it is not necessary to take \( a^{(k)}_n > 0 \) for each \( k \) and \( n \). Supposing \( a^{(k)}_n \geq 0 \) for each \( k \) and \( n \), and that for each \( k \in \mathbb{N} \) there exists an \( n \in \mathbb{N} \) such that \( a^{(k)}_n \neq 0 \), then we may proceed in an analogous way finding the same results.