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ON LATTICE-DILATIONS AND CONTRACTIONS IN f-RINGS

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ABSTRACT. This paper splits broadly into two related parts, concerned respectively with generalized idempotents (associated to superunities and subunities) and with lattice-dilations and contractions in an f-ring A. If u is a superunity, we characterize the mappings $F:A\rightarrow A$ satifying $|F(x)-F(y)| = u|x-y|$ (u-dilations) as the mappings of the form $F(x)= x(u-2e)+b$, with e being a generalized idempotent, and obtain an analogous result for lattice-contractions. The set of the homogeneous ones (both cases) are proved to be Boolean algebras.

Our terminology and notations are mostly standard, and follow widely [1]. Recall that in an ℓ-ring A, u is a superunity [3] if $ux\wedge xu \geqslant x$ holds for every $x\geqslant 0$, and s is a subunity [6] if $0<sx\leqslant x$, $0<xs\leqslant x$ for every $x>0$. Now, it follows a summary of results without proofs.

1. Generalized idempotents. In an ℓ-ring A, we introduce the following definitions: a) if u is a superunity, then $e\in A$ is a u-idempotent if $eu = ue = e^2$. b) if s is a subunity, then $e\in A$ is an s-idempotent if $es = se = e^2$. The respective sets will be denoted by I(u) and I(s).

We first show that in an f-ring A, $I(p) =\{x\in A \mid x\wedge(p-x) = 0\}$ if p is a subunity or a superunity, and obtain as a consequence:

Theorem 1. I(p) is a Boolean algebra with the ordering of the ring.

2. Boolean algebras of generalized idempotents. On account of I(p) having no special property which an arbitrary Boolean algebra need not have, we have analized its boolean properties in connection with lattice and algebraic-theoretic properties of the ring. Since I(p) is closed by taking arbitrary suprema and infima, it is easy to relate the order completeness properties of A with completeness properties of I(p).

We pay considerable attention to projectable f-rings, that is, those for which $a^{\perp} \oplus a^{\perp\perp} = A$ holds for every $a \in A$. They are specially interisting in this context in view of the following result for f-rings with a superunity u:

Theorem 2. A is projectable if and only if the polar of every element is the polar of a unique u-idempotent.

For the "only if" part of Th.2, it suffices proving that if $a \in A$, then $a^{\perp} = e^{\perp}$, being e the projection of u onto $a^{\perp\perp}$.

Some completeness and projection properties imply that the f-ring A be projectable. For instance, those of the main inclusion theorem [4], and others. We have proved here that if I(u) is convex, then A is projectable. From Th.2 we obtain that if A is projectable and non totally ordered, then I(u) is non trivial.

Some results suggest the interest of studying the subset $P_u(A) = \{ e^{\perp} \mid e \in I(u)\}$ of all the polars P(A) of the ring. In this connexion we prove:

Theorem 3. With the ordering of P(A), $P_u(A)$ is a Boolean subalgebra of P(A), isomorphic to the algebra I(u). Moreover, $P_u(A)$ is a sub-lattice of PP(A), the lattice of all principal polars, and a sualgebra of the Boolean algebra of direct summands. If P(A) is superatomic,then I(u) is finite. If u' is another superunity, then I(u) and I(u') are isomorphic in the following cases: a) A is Dedekind-complete; b) I(u) and I(u') are complete Boolean algebras and are convex in A.

The proof of Th.3 uses mainly the decomposition $A = e^{\perp} \oplus (u-e)^{\perp}$ if $e \in I(u)$, and the fact that if $e_1, e_2 \in I(u)$ and $e_1^{\perp} = e_2^{\perp}$,then $e_1 = e_2$. The isomorphism of the statement is given by $I(u) \to P_u(A)$, $e \mapsto (u-e)^{\perp}$.

By using Th.3 in the projectable case, we obtain:

Theorem 4. Let A be a projectable f-ring. a) I(u) is atomic in the following cases: 1) P(A) is atomic; 2) A is basic or completely distributive. b) If A has a finite basis, then I(u) is finite.

3. Boolean algebras of lattice-dilations. If u is a central superunity of the f-ring A, we generalize the notion of ℓ-isometry ([2] , [5] , [6]) by considering the mappings $F: A \to A$ that satisfy $|F(x)-F(y)| = u|x-y|$ for every $x,y \in A$. They will be called lattice-dilations or u-dilations, since $|F(x)- F(y)| \geqslant |x-y|$. We denote by $H_u(A)$ the set of all homogeneous u-dilations, that is, those for which F(0) = 0. On the other hand, if $e \in I(u)$, we consider the mapping $\sigma_e : A \to A$, $\sigma_e(x) = x(u-2e)$ and set $\mathcal{D}_u(A) = \{\sigma_e \mid e \in I(u) \}$.

The fundamental result we have obtained now follows:

Theorem 5. a) $H_u(A) = \mathcal{D}_u(A)$. b) Every u-dilation F is of the form $F(x) = x(u-2e) + b$, being $b \in A$ and $e \in I(u)$. c) Every $F \in H_u(A)$ is a homotecy of ratio a, with $|a| = u$.

Part a) of Th.5 has been proved by means of a suitable representation of A as a subdirect product of totally ordered rings, and the fact that for a totally ordered ring with a superunity u, the only u-dilations are $\sigma_0(x) = ux$ for every x, and $\sigma_u(x) = -ux$ for every x. Part c) follows on account of $e \in I(u)$ being a component of u.

Now, if we consider the Boolean ring structure of the Boolean algebra I(u), then Th.5 enables us to endow $H_u(A)$ with a Boolean structure:

Theorem 6. With the operations $(\sigma_e \tilde{\oplus} \sigma_{e'})(x) = x(u-2|e-e'|)$ and $(\sigma_e \tilde{\ast} \sigma_{e'})(x) = x(u-2(e \wedge e'))$, $(H_u(A), \tilde{\oplus}, \tilde{\ast})$ is a Boolean ring with unity, isomorphic to the Boolean ring I(u).

Now in view of the isomorphism $H_u(A) \simeq I(u)$ and the theorems 3 and 4, we have derived the corresponding properties of the Boolean algebra $H_u(A)$, but we shall not explicitly mention them here. However, it is worth noting that if A is projectable and non totally ordered, then there exist non trivial u-dilations.

The u-dilations σ_0 and σ_u are interesting since we have:

Theorem 7. If $F \in H_u(A)$, then there exists a unique decomposition $A = B \oplus C$, with B,C being ℓ-ideals, for which $F|_B = \sigma_0$ and $F|_C = \sigma_u$.

Indeed, by Th.5 $F = \sigma_e$, for some $e \in I(u)$, and it suffices taking $B = e^{\perp}$ and $C = (u-e)^{\perp}$.

Theorem 7 enables us to give some geometric interpretation of homogeneous u-dilations, especially by means of the concept of lattice axial symmetry. Recall from [6] that if $a \in A$, then $f: A \to A$ is a lattice axial symmetry of axis a if: 1) f is a group homomorphism; 2) $A = \langle a \rangle \oplus a^{\perp}$ and 3) $f|_{\langle a \rangle} = I$, $f|_{a^{\perp}} = -I$, with I being the identity mapping. Then we can partially rephrase Th.7: Every homogeneous u-dilation σ_e is a homotecy of ratio u on orthogonal directions, followed by a lattice axial symmetry with respect to one of that directions (of axis u-e), or, which is the same: it is a lattice axial symmetry followed by a homotecy of ratio u. Conversely, every lattice axial symmetry, followed by a homotecy of ratio u is a homogeneous u-dilation.

In the course of our study the set B of all square roots of u^2 has naturally arisen. We have **proved** that $B = \{a \mid |a| = u\}$. Moreover,

Theorem 8. With the same ordering of the ring, B is a Boolean algebra, that is isomorphic to the Boolean algebra I(u). Hence isomorphic to $H_u(A)$.

The preceding isomorphism is given by $I(u) \longrightarrow B$, $e \longmapsto 2e-u$. It is possible now to transfer to B many of the properties that could be asserted for the Boolean algebra I(u).

4. Lattice-contractive mappings. If s is a central subunity of the f-ring A, we can define "mutatis mutandis" the concept of lattice-contraction (s-contraction) and homogeneous s-contraction by only interchanging u by s in the definition. With certain additional assumptions on A , it is possible to develop a theory for s-contractions, that is parallel to that of u-dilations, though less satisfactory in some aspects. The difficulty appears when some of the properties that are valid for u-idempotents do not hold for s-idempotents. For instance, the decomposition $A = e^{\perp} \oplus (u-e)^{\perp}$ is no more valid for every $e \in I(s)$. It remains valid however if A is Dedekind-complete or if s is a formal unity. Other properties still hold in absence of nonzero nilpotent elements.

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