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FLAT SUBMERSIONS AND HORIZON IMMERSIONS

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Abstract

The notion of flat submersion is introduced. We prove that no covering map, with order greater than 1, is 1-flat, relate 2-flat submersions $M^{n+1} \to S^n$ to horizon immersions and show that k-flat submersions $M^m \to S^n$ give rise to embeddings $M^m \to RP^{n+k}$.

1. Introduction

Throughout this note all the manifolds are C^∞, Hausdorff, compact, connected and boundaryless. All the maps are C^∞, unless otherwise stated. If M and N are manifolds, of dimensions m and n respectively, we shall denote them by M^m and N^n.

Let $\Pi: M^m \longrightarrow N^n$ be a submersion.

Definition 1:

We say that Π is a k-flat submersion $(k>m-n)$ if there exists a map $F:M^m \longrightarrow R^k$ such that, for every $y \in N^n$, $F|\Pi^{-1}(y)$ is an embedding.

Proposition 1:

If Π is a k-flat submersion then $(\Pi,F):M^m \longrightarrow N^n \times R^k$ is an embedding.

Proof:

It is clear that (Π,F) is C^∞ and injective. The only thing to worry about is the rank of (Π,F). Let x be any point in M^m. We prove that the kernel of $(\Pi,F)_{*x}: T_xM^m \longrightarrow T_{\Pi(x)} N^n \times T_{F(x)}R^k$ is 0-dimensional. As $\ker(\Pi,F)_{*x} = \ker \Pi_{*x} \cap \ker F_{*x}$, if $v \in \ker(\Pi,F)_{*x}$ we have $F_{*x}(v) = 0$. On the other hand

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$v \in \ker \Pi_{*x}, \ker \Pi_{*x} = i_{*x}(T_x \Pi^{-1}(\Pi(x)))$, where i is the inclusion $\Pi^{-1}(\Pi(x)) \to M$, and $F|\Pi^{-1}(x)$ is an embedding, therefore v must be zero.

2. Flat covering maps

In this section we deal with the case $m=n$. We shall consider a more general situation. Let M be a compact, path-connected topological space and $\Pi: M \to N$ be a covering map, where N is also a topological space.

Proposition 2:

Let $\Pi: M \longrightarrow N$ be a covering map such that, for any $y \in N$, $\# \Pi^{-1}(y) > 1$. If $f: M \to R$ is a continuous map then there exist x and y in the same fibre such that $f(x) = f(y)$.

Proof:

We use some facts about covering spaces [4]. Let x_1 be an absolute maximum and y_1 be an absolute minimum of f. Take a continuous path $\gamma: [0,1] \to M$ such that $\gamma(0) = x_1$ and $\gamma(1) = y_1$. Choose $x_2 \neq x_1$, with $\Pi(x_2) = \Pi(x_1)$ and consider the lift γ' of $\Pi o \gamma$ such that $\gamma'(0) = x_2$. We define the map $\phi: [0,1] \to R$ by taking $\phi(t)$ to be $fo\gamma(t) - fo\gamma'(t)$. If $\phi(0)$ or $\phi(1)$ is zero then the proposition is proved. If not, ϕ must have a zero t_0, because $\phi(0) > 0$ and $\phi(1) < 0$, and the proposition follows.

An immediate consequence is

Corollary 1:

If $f: S^n \to R$ is continuous then there exists $x \in S^n$ such that $f(x) = f(-x)$.

Of course corollary 1 can also be deduced from the Borsuk-Ulam theorem [4].

3. Flat submersions and horizon immersions

We start with the definition of horizon immersion [2]. Let $f: M \longrightarrow R^{m+1}$ be an immersion and L denote an affine line in R^{m+1}.

Definition 2:

We say that $f: M^m \to R^{m+1}$ is a horizon immersion with base-line L if, for every $x \in M^m$, the affine tangent space to $f(M)$ at $f(x)$ does not contain L.

In what follows we shall be considering submersions $\Pi: M^{n+1} \to S^n$ and we shall use the following chain of diffeomorphisms

$$S^n \times R^2 \xrightarrow{\lambda_1} S^n \times R^+ \times R \xrightarrow{\lambda_2} R^{n+1} \setminus \{0\} \times R$$

where $\lambda_1(x,t_1,t_2) = (x, e^{t_1}, t_2)$, $\lambda_2(x,t_1,t_2) = (t_1 x, t_2)$ and R^+ denotes the positive reals. The inclusion $i: R^{n+1} \setminus \{0\} \times R \to R^{n+2}$ will also be used and the composition $i \circ \lambda_2 \circ \lambda_1$ will be denoted by λ. From now on L will denote the affine line $\{(0,\dots,0,t) \in R^{n+2} : t \in R\}$.

Proposition 3:

a) If $\Pi: M^{n+1} \longrightarrow S^n$ is a 2-flat submersion then $\lambda O(\Pi,F)$ is a horizon embedding, with base-line L, such that $(\lambda O(\Pi,F))^{-1}(L) = \phi$.

b) If $f: M^{n+1} \longrightarrow R^{n+2}$ is a horizon embedding with base-line L such that $f^{-1}(L) = \phi$ then $\Pi: M^{n+1} \longrightarrow S^n$ given by $\Pi(x) = (f_1(x),\dots,f_{n+1}(x)) / \|(f_1(x),\dots,f_{n+1}(x))\|$ is a 2-flat submersion. ($\|\ \|$ denotes the standard norm in R^{n+2}).

Proof:

a) If Π is 2-flat then (Π,F) is transverse to $\{y\} \times R^2$, for every $y \in S^n$. It is easy to check that $\lambda O(\Pi,F)$ is also transverse to each 2-plane containing L and therefore no affine tangent $(n+1)$-plane to $\lambda O(\Pi,F)(M)$ contains L. That is to say, $\lambda O(\Pi,F)$ is a horizon embedding.

b) We replace f by $f': M^{n+1} \longrightarrow R^{n+1} \setminus \{0\} \times R$. Using λ_1^{-1} and λ_2^{-1} we obtain a map $M \to S^n \times R^2$ of which the first component is Π and the second one has the required property for 2-flatness. We remark that Π has no critical points because no tangent $(n+1)$ - plane to $f(M)$ contains L.

Corollary 2:

If $\Pi: M^{n+1} \longrightarrow S^n$ is a 2-flat submersion then M is diffeomorphic to $S^n \times S^1$.

Proof:

Robertson and Chillingworth have shown [2] [3] that if $f: M^{n+1} \to R^{n+2}$, $n \geq 1$, is a horizon immersion with base-line L, such that $f^{-1}(L) = \phi$ then M is diffeomorphic to the Klein bottle or $S^n \times S^1$. However the Klein bottle cannot admit a 2-flat submersion because it cannot be embedded in 3-space.

4. Embeddings in real projective n-space RP^n.

In [1] it was shown that if $\Pi: S^1 \times S^1 \longrightarrow S^1$ is 2-flat it is possible to construct an embedding $S^1 \times S^1 \longrightarrow RP^3$, related to the Hopf map $h: S^3 \longrightarrow S^2$. In this section we show how to obtain an embedding $M^m \longrightarrow RP^{n+k}$ if we have a k-flat submersion $\Pi: M^m \to S^n$. Let us take $F: M^m \longrightarrow R^k$, associated with Π

If necessary, we replace F by $F':M^m \longrightarrow R^k$ such that $F'(M) \subset R^{+k}$. We take $\Psi: S^n \times R^k \longrightarrow RP^n$ defined by $\Psi(x,y) = [x,y]$, where $[x,y]$ is the 1-dimensional subspace of R^{n+k+1} determined by (x,y). The rank of Ψ is $n+k$ and it is straightforward to check that $\Psi \circ (\Pi, F)$ is an embedding.

Thanks are due to Stewart A.Robertson for suggesting the proof of proposition 2, much simpler than our previous one.

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