Pub. Mat. UAB
Nº 22 Nov. 1980
Actes VII JMHL

THE CONDITIONAL INDEPENDENCE PROPERTY IN FILTRATIONS ASSOCIATED TO STOPPING LINES

David Nualart, Marta Sanz

Facultat de Matemàtiques. Universitat de Barcelona
Secció de Matemàtiques. Universitat Autònoma de Barcelona

This paper studies filtrations associated in a normal way to stopping lines, and in particular the conditional independence property, usually known as the F4 property. Stopping lines with respect to some special filtrations are studied by means of a characterization of predictable processes.

Preliminaries and notation. In R_+^2 we consider the partial order

$$(s,t) \leq (s',t') \Leftrightarrow s \leq s' \text{ and } t \leq t',$$

and a strengthening of it

$$(s,t) < (s',t') \Leftrightarrow s < s' \text{ and } t < t'.$$

Let $z = (s,t)$ be a generic point of R_+^2, we will denote

$$R_z = \{z' \in R_+^2,\ z' \leq z\} = [0,z], \quad \mathring{R}_z = \{z' \in R_+^2,\ z' < z\} = [0,z),$$

$$U_z = \{z' \in R_+^2,\ z' > z\} = (z,\infty).$$

If A is a subset of R_+^2 we define the following sets:

$$H_A = \bigcup_{z \in A} R_z, \quad H_A^- = \bigcup_{z \in A} \mathring{R}_z, \quad H_A^+ = \bigcup_{z \in A} U_z.$$

DEFINITIONS. (1) $A \subset R_+^2$ is a separation line if A is a non empty, closed set satisfying the following two properties:

(i) $H_A \cap H_A^+ = \emptyset$

(ii) $H_A^- \cup H_A^+ \cup A \cup (0,\infty)^c = R_+^2$, where 0 denotes the origin of R_+^2

The set of all the separation lines will be denoted by S.

Let $(\Omega,\mathcal{F},P)$ be a complete probability space and let $(\mathcal{F}_z)_{z\in\mathbb{R}^2_+}$ be a filtration on it. $(\mathcal{F}_z)_{z\in R^2_+}$ has the F4 property if, for each z, $\mathcal{F}^1_z = \bigvee_{y\geq 0} \mathcal{F}_{sy}$ and $\mathcal{F}^2_z = \bigvee_{x\geq 0} \mathcal{F}_{xt}$ are conditionally independent given $\mathcal{F}_z$.

(2) $\lambda: \Omega \to S\cup\{\infty\}$ is a <u>stopping line</u> (s.l.) with respect to $(\mathcal{F}_z)_{z\in R^2_+}$ if $\{1_{H_{\lambda(\omega)}}(z),\ z\in R^2_+\}$ is a measurable and adapted process.

We can define a partial order for subsets of R^2_+ in the following way:

$$A \prec B \Leftrightarrow H_A \subseteq H_B.$$

Then $\{1_{H_{\lambda(\omega)}}(z),\ z\in R^2_+\}$ is an adapted process if and only if

$$\forall z\in R^2_+,\ \{\omega | z\prec \lambda(\omega)\} \in \mathcal{F}_z$$

1. Let $\lambda: \Omega \to S\cup\{\infty\}$ be measurable, that is $\{1_{H_{\lambda(\omega)}}(z),\ z\in R^2_+\}$ is a measurable process. Our purpose is to study the smallest filtration making λ a s.l. and to verify if this filtration has the F4 property.

It is easy to see that the smallest filtration making λ a s.l. is that given by

$$\forall z\in R^2_+,\quad \mathcal{F}_z = \sigma < \{\omega,\ z' \prec \lambda(\omega)\},\quad z' \in [0,z] >.$$

EXAMPLES. (1) $\Omega = R^2_+$, $\mathcal{F} = \mathcal{B}(R^2_+)$, $\lambda(z) = \underline{\partial} z = \{(s,\beta), 0\leq\beta\leq t\} \cup \cup\{(\alpha,t), 0\leq\alpha\leq s\}$, then $\mathcal{F}_z = \sigma<[z',\infty),\ z'\in[0,z]>$. A Borel function f: $\Omega\to R$ is $\mathcal{F}_z$-measurable if and only if it is constant in $[z,\infty)$, if only depends on x in $[0,s)\times[t,\infty)$, and if only depends on y in $[s,\infty)\times[0,t)$

$(\mathcal{F}_z)_{z\in R^2_+}$ is a product filtration, so it satisfies (F4) with respect to any product probability.

(2) $\Omega = R^2_+$, $\mathcal{F} = \mathcal{B}(R^2_+)$, $\lambda(z) = \bar{\partial} z = \{(s,\beta),\ t\leq\beta\leq\infty\} \cup \{(\alpha,t),\ s\leq\alpha\leq\infty\}$, then $\mathcal{F}_z = \sigma<\mathcal{B}[0,z),\ [0,z)^c>$ It might be proved that this filtration satisfies (F4) if and only if P only charges increasing curves.

(3) Let G be the group of transformations of R^2_+ generated by $g_+(s,t) = (t,s)$ and $g_a(s,t) = (as, \frac{t}{a})$, $a>0$. It is known that if $\{W_z,\ z\in R^2_+\}$ is a Wiener process then $\{W_{g(z)}, z\in R^2_+\}$, $\forall g\in G$, is another one (see [6]).

Let us consider $\Omega = R^2_+$, $\mathcal{F} = \mathcal{B}(R^2_+)$, $\lambda(z) = \underline{\partial} g(z)$, then

$\mathcal{F}_z^{(g)} = \sigma\langle[z',\infty),\ z' \in [0,\ g^{-1}(z)]\rangle$. In comparing $\mathcal{F}_z^{(g)}$ with the $\mathcal{F}_z$ given in example one, we observe that $\mathcal{F}_z^{(g)} = \mathcal{F}_{g^{-1}(z)}$, $\mathcal{F}_z^{(g)1} = \mathcal{F}^1_{g^{-1}(z)}$ and $\mathcal{F}_z^{(g)2} = \mathcal{F}^2_{g^{-1}(z)}$ therefore $(\mathcal{F}_z^{(g)})_{z \in R_+^2}$ verifies (F4) with respect to any product probability.

Example 3 lead us to consider filtrations related to changing time for two-parameter Wiener processes.

Denote by $\mathcal{H}$ the set of continuous, one to one, Lebesgue-measure preserving functions $f: R_+^2 \to R_+^2$ such that $z_1 \in f(R_{z_2}) \Rightarrow R_{z_1} \subset f(R_{z_2})$.
It is proved in [6] that if $\{W_z,\ z \in R_+^2\}$ is a Wiener process, then $\{W(f(R_z)),\ z \in R_+^2\}$ is another one.

THEOREM. Let $f \in \mathcal{H}$. The following statements are equivalent:

(i) $f \in G$. (ii) the filtration associated to $\lambda(\omega) = \partial H_{f(R_\omega)}$ verifies (F4) with respect to any product probability.

Proof. Example 3 shows (i) $\Rightarrow$ (ii) it can be proved that $\mathcal{F}_z = \sigma\langle[f^{-1}(z'),\infty),\ z' \in [0,z]\rangle$. We also have (ii) $\Rightarrow$ $f^{-1}(R_z)$ is a rectangle, $\forall z \in R_+^2$. The result follows from the equivalence $f \in G \Leftrightarrow f^{-1}(R_z)$ is a rectangle for every $z \in R_+^2$.$\square$

2. In this section we consider the filtration of examples 1 and 2. They might be transformed into right continuous filtrations by considering $\mathcal{F}_z^{\circ} = \bigcap_{z' > z} \mathcal{F}_{z'}$, then we respectively obtain:

$$\mathcal{F}_z^{\circ} = \sigma\langle(z',\infty),\ z' \in [0,z]\rangle, \tag{2.1}$$

$$\mathcal{F}_z^{\circ} = \sigma\langle\mathcal{B}([0,z]),\ [0,z]^c\rangle. \tag{2.2}$$

Notice that if λ is an s.l. the measurable and adapted process $(1_{H^{\cdot}_{\lambda(\omega)}}(z))_{z \in R_+^2}$ also is continuous to the left, so it must be predictable.

For the case (2.1) we have (see [4]): $(X_z)_{z \in R_+^2}$ is a predictable process if and only if there exist measurable functions H and h such that

$$X_z(\omega) = H(\omega,z)\ 1_{\{\omega<z\}} + h(z)\ 1_{\{\omega<z\}^c}.$$

Then if λ is an s.l. with $D_\omega = H_{\lambda(\omega)}$, $1_{D_\omega}(z) = H(\omega,z)\ 1_{\{\omega<z\}}(\omega) + h(z)1_{\{\omega<z\}^c}(\omega)$ where $h = 1_D$, $H(\omega,z) = 1_{A(\omega)}(z)$, and D is s stopping set

associated to a separation line L.

It can be proved that:

$D_\omega = D$ if $\omega \notin D$, that is $\lambda(\omega) = L$ if $L \prec \bar{\partial}\omega$

$D_\omega = (D \cap Q_\omega) \cup A(\omega)$ if $\omega \in D$, $(Q_\omega = (\omega,\infty)^c)$, that is

$$(L \wedge \bar{\partial}\omega) \prec \lambda(\omega) \prec (L \vee \underline{\partial}\omega) \text{ if } \underline{\partial}\omega \prec L$$

In the case (2.2), $(X_z)_{z \in R_+^2}$ is a predictable process if and only if there exist measurable functions H,F,G and h such that

$$X_z(\omega) = H(\omega,z)1_{\{\omega<z\}}(\omega) + F(\omega_2,z)1_{D_z^1}(\omega) + G(\omega_1,z)1_{D_z^2}(\omega) + h(z)1_{[z,\infty)}(\omega),$$

where: $\omega = (\omega_1,\omega_2)$, $D_z^1 = \{\omega | \omega_1 \geq s,\ 0 \leq \omega_2 < t\}$, $D_z^2\ \{\omega | 0 \leq x < s,\ t \leq y\}$.

Let λ be a separation line. By the same arguments as before we obtain: there exist a separation line L such that:

$$\lambda(\omega) = L \text{ if } D = H_L \subset [0,\omega], \text{ that is if } L \preccurlyeq \partial\omega.$$

If $L \not\preccurlyeq \underline{\partial}\omega$ and $L \not\succ \underline{\partial}\omega$, $\lambda(\omega)$ is any separation line (measurable with respect to ω) lying between $\underline{\partial}\omega \wedge L$ and $\bar{\partial}\bar{\omega} \vee L$; $\lambda(\omega)$ restricted to $[(\omega_1,0),\infty)$ only depends on ω_1, $\lambda(\omega)$ restricted to $[(0,\omega_2),\infty)$ only depends on ω_2. $(\bar{\omega} = (\omega_1,\bar{\omega}_2) \wedge (\bar{\omega}_1,\omega_2)$, where $(\omega_1,\bar{\omega}_2),(\bar{\omega}_1,\omega_2)$ are points on L)

If $\underline{\partial}\omega \prec L, \lambda(\omega)$ is any separation line, $\lambda(\omega) \succ \underline{\partial}\omega$, when restricted to $[\omega_1,\infty) x [0,\omega_2]$ of only depends on ω_1, and restricted to $[0,\omega_1] x [\omega_2,\infty)$ it only depends on ω_2.

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