Pub. Mat. UAB
Nº 22 Nov. 1980
Actes VII JMHL

SOME EXPERIMENTS WITH INTERVAL SHOOTING METHODS FOR TWO POINT BOUNDARY VALUE PROBLEMS IN ORDINARY DIFFERENTIAL EQUATIONS.

María Raquel G.P. Valença

Area de Matemática
Universidade do Minho

Abstract

The purpose of this communication is to give some discussion, supported by numerical results of interval shooting techniques for 2 ptB.V.P. with a commentary on the relative advantages and disavantages of these methods.

1. The Aim of Interval Analysis

The aim of interval analysis is the simultaneous machine computation of guaranteed bounds on the exact solution of a problem.

Quantities are represented by intervals containing their exact value and techniques are developed such that throughout computation round-off, truncation and propagation of initial error are all accounted for.

2. Interval Arithmetic

In the sequel we use the notation x^I for an interval with lower and upper bounds $\underline{x}$, $\overline{x}$ such that

$$x^I = [\underline{x} , \overline{x}] \quad , \quad \underline{x} \leq \overline{x} \quad , \quad \underline{x}, \overline{x} \in \mathcal{R}$$

The width of an interval is denoted by $W(x^I) = \overline{x} - \underline{x}$.

Arithmetic operations with intervals are defined in such a way that the resulting interval will always contain the exact result of the operation, Moore (3) .In various contexts we shall have to compute intervals which are functions of intervals, Moore (3).

3. Shooting Methods for Linear Problems

We consider first the Linear problem of second order given by

$$y''=p(x)y'+q(x)y+r(x), x \in [a,b] \quad (3.1)$$

together with boundary conditions

$$a_1 y(a)-a_2 y'(a) = \alpha, \quad b_1 y(b)+b_2 y'(b)=\beta \quad (3.2)$$

where a_i, b_i, $i=1,2$, $y'(a)$, $y'(a)$, $y(b)$, $y'(b)$, α, β may denote intervals.

We assume that $p(x)$, $q(x)$ and $r(x)$ are continuous on (a,b) and that the homogeneous problem derived from (3.1), (3.2) has only the trivial solution $y(x) = 0$ in which case (3.1), (3.2) has a unique solution.

In theory we can find this by combining the solutions of two initial value problems to satisfy the pair of boundary conditions.

A straight forward interval extension of the classical shooting technique was developed, Valença (5). The associated initial value problems are solved by interval initial value methods, Moore (3), Valença (5), Hansen (2).

An estimate of the order of accurary obtained is found Valença (5) in terms of the order of the interval initial value methods used.

$$W(y^I(x)) = Nh^k$$

Where N is a constant, h the integration step and K the order of the initial value methods.

Example 3.1.

$$y''= \frac{2}{x^2} \quad y- \frac{1}{x} \qquad x\in[2,3]$$

$$y(2) =0, y(3)=0.$$

A fourth order interval initial value method produced with

$h=0.1$ max $w(y^I) < 2.0\times10^{-5}$ max $y(x) \simeq 0.55$

The corresponding results with h=0.2 had a width which were a multiple very close to 14.6 of those for h=0.1. This gives reasonable proof of $O(h^4)$ accuracy.

In certain circunstances, however the computed interval solutions may largely overestimate the error in the solution.

The method is unstable if the initial value problems have solutions that increase rapidly in absolute value while the solution of the boundary value problem remains almost stationery. Example in Valença (5).

If the initial value problems have oscillatory solutions a technique to reduce the so called "wrapping effect", Moore (3), must be used. We note however that this is somewhat unusual for well conditioned boundary value problems.

4. Shooting Methods for NonLinear Problems.

We consider the nonlinear differential equation of second order

$$y''=f(X,y,y' \qquad (4.1)$$

together with linear boundary conditions (3.2)

We assume f continuous and Lipschitzian on $\mathcal{R}\{a \leq x \leq b, y^2+y'^2 < \infty, \frac{\partial f}{\partial y} > 0,$

$$\left|\frac{\partial f}{\partial y'}\right| < M \text{ on } \mathcal{R};\ a_i \geq 0, b_i \geq 0, i=1,2,\ a_1+a_2 > 0,\ b_1+b_2 > 0,\ a_1+b_1 > 0.$$

An iterative scheme which is essentially an interval extension of the classical shooting technique was develops Valença (5) and conditions for convergence stated.

Example 4.1

$$y''=-1-0.49y'^2 \ ,\ y(0)=0, y(1)=0.$$

With starting intervals of width of order 10^{-1} and $h = 0.1$ using a fourth order interval initial value method the algorithm converges rapidly producing after three "steps" an interval solution of max $w(y^I) < 3.5\times10^{-5}$, max $y(x) \approx 0.13$, $x=0.5$.

5. Conclusions

Interval shooting methods are naturally dependent on the particularities of interval initial value methods. We were able to compute accurate bounding solutions for some problems whereas in other cases very pessimistic bounds can be obtained. One of the causes of the bad results produced in these cases occurs in the classical method as well. Precautions must also be taken when the associated initial value problems are such that the influence of the "wrapping effect" is considerable. We note that more restrictions are imposed with interval shooting methods than with the corresponding classical methods but this after all may be the price for the knowledge of guaranteed error bounding solutions.

BIBLIOGRAPHY:

(1)

Fox, L, and Valença M.R "Some experiments with Interval Methods for 2 pt B.V.P. in O.D.E." to be published

(2)

Hansen, E. (ed) "Topics in Interval Analysis" Clarendon Press.

(3)

Moore, R. E. "Interval Analysis" Prentice Hall

(4)

Nickel, K. "The application of Interval Analysis to the Numerical Solution of Differential Equations" - Interner Bericht des Inst. F. Informatik 69/9, Universitaet Karlsruhe

(5)

Valença, M. R. "Interval Methods for Ordinary Differential Equations" D.Phil Thesis, Oxford (1978).

A participação da autora nas VII Jornadas Matemáticas foi subsidiada pelo Instituto Nacional de Investigação Científica,Portugal.