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MATRICIAL NORMS: SOME APPLICATIONS AND OPEN QUESTIONS

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Abstract - We mention some theoretical aspects of matricial norms, including some recent developments [Robert, Deutsch, Bauer, Barker, Meixner, Coimbra]. Then we refer several applications of matricial norms: convergence of iterative procedures, weak contraction in vectorial norm/fixed point questions [Robert]; approximation in v-metric spaces [Coimbra]; localization of zeros of polynomials [Deutsch, Vitória]; bounds for latent roots of lambda-matrices [Vitória]; vibrating systems [Clímaco+Vitória]; lower bounds of linear mappings [Bode]. Finally, we include some open questions concerning both theoretical and practical aspects.

1. Applications of matricial norms

In this section we refer some applications of properties of vectorial [matricial] norms.

1.1. Convergence of linear iterations

One utilizes properties of matricial norms to control the convergence of iterative procedures - speed and error. Block-iterations: Gauss-Seidel, Jacobi, over-relaxation.

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1.2. "Série-paralèle" iterative methods.

One constructs algorithms for linear iterations fitted to automatic calculations "in parallel" - several arithmetic units, working simultaneously, branched to a common memmory. These algorithms can be placed at an intermediary level between the Gauss-Seidel method — sequential — and the Jacobi method — parallel.

1.3. Weak contraction in vectorial norm/fixed point problems

Perron-Frobenius theory for matrices partitioned into blocks.

1.4. Convergence of matrix sequences

There are results, using (scalar) norms, on convergence of matrix sequences. Such results can be generalized by using vectorial norms.

1.5. Bounds for the zeros of polynomials

One takes the companion matrix C of a polynomial $p(z) = a_0 z^n + a_1 z^{n-1} + \dots + a_{n-1} z + a_n$, $a_i \in \mathbb{C}$, $(i=0,1,\dots,n)$ and constructs several adequated matricial norms related to matrix C. So we get upper bounds for the spectral radius of matrix C, i.e. to the absolute values for the roots z of $p(z)$.

1.6. Bounds for the latent roots of lambda-matrices

We consider the lambda-matrix $M(\lambda) = I\lambda^n + A_1\lambda^{n-1} + \dots + A_{n-1}\lambda + A_n$, where $A_i, I \in M(C)_{p,p}$, $(i=1,\dots,n)$. We need to know the zeros of $\det M(\lambda) = |I\lambda^n + A_1\lambda^{n-1} + \dots + A_{n-1}\lambda + A_n| = 0$, that is to say, we need to know the *latent roots* λ to the lambda-matrix $M(\lambda)$. The calculation of the latent roots may become a formidable task, this depending on n and p. Instead of calculating them, we look for localization domains of the latent roots.

We associate to the lambda-matrix its block-companion matrix M. The eigenvalues of block-companion matrix M are the latent roots of the lambda-matrix. Then we construct several matricial norms related to the matrix M, and get upper bounds for $|\lambda|$. Incidentally, we obtain known (and new) results for zeros of (scalar) polynomials.

1.7. Approximation in v-metric spaces

Generalizations to v-metric spaces [pseudo-metric spaces] of results in metric spaces. Generalization of Newton's method to super-v-metric spaces, the vectorial metric being induced by a vectorial norm.

1.8. Lower bounds of linear mappings

Given a (regular) vectorial norm Ψ of order k on K^n. Lower bound of a matrix [operator] $A \in M_{n,n}(\mathbf{K})$ with respect to the vectorial norm Ψ is a matrix $N \in M_{k,k}(\mathbf{R})$ such that $\forall x \in \mathbf{K}^n$, $N\Psi(x) \leq \Psi(Ax)$.

We can apply these lower bounds to the estimation (in vectorial norm) of the obtained approximation to the solution of a linear system.

1.9. Eigenvalues in block-partitioned matrices

Generalization of Gerschgorin results to matrices partioned into blocks.

1.10. Vibrating systems

For studying the motion equations, one can use (classical) Lagrange method. The solution of such equations will represent a sinusoidal motion, so we look for solutions of an equation of type $A\ddot{\vec{p}}+B\dot{\vec{p}}+C\vec{p}=\vec{P}$. For solving these equations we need to know the latent roots of the lambda-matrix $M(\lambda)=A\lambda^2+$ $+B\lambda+C$. But in certain situations we do not need to calculate the latent roots of lambda-matrix $M(\lambda)$. It suffices to know a localization domain of the latent roots. In this case are well suited the results referred to in 1.6.

2. Open questions

In this section we mention some seemingly open questions, involving matricial norms:

2.1. Deviation of a matrix from singularity

2.2. Deviation of a matrix from normality

2.3. Spectral variation of two matrices

2.4. Generalizations of a fundamental inequality on matricial norms

(*) $\rho(A) \leq \rho\,[\Phi(A)]$, $\rho(M)$: spectral radius of a matrix M;

$\Phi(M)$: matricial norm of a matrix M.

We raise the following questions:

a) It is possible to consider a relation "similar" to (*) in the context of matrices whose elements are intervals?

b) How "seems" relation (*) if the elements of matrix A belong to a p.i.r. (principal ideal aing)?

c) Let $\lambda_1,\lambda_2,\dots,\lambda_n$ be the eigenvalues of $A \in M_{n,n}(\mathbf{K})$ and $\mu_1,\mu_2,\dots,\mu_k$ be the eigenvalues of its matricial norm $\Phi(A) \in M^+_{k,k}(R)$. Suppose that

$$|\lambda_1| \geq |\lambda_2| \geq \dots \geq |\lambda_n| \quad \text{and} \quad |\mu_1| \geq |\mu_2| \geq \dots \geq |\mu_k|$$

From (*) we have

$$|\lambda_1| = \rho(A) \leq \rho\ (\Phi(A)) = |\mu_1|$$

What can we say about matrices such that

$$|\lambda_2| \leq |\mu_2| \quad |\lambda_3| \leq |\mu_3| \ , \quad \text{etc.?}$$

Remark

A complet version of this paper will appear elsewhere.

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